Multiscale 2-Mapper – Exploratory Data Analysis Guided by the First Betti Number

Halley Fritze

Topological Data Visualization Workshop

June 9th , 2025

Mapper

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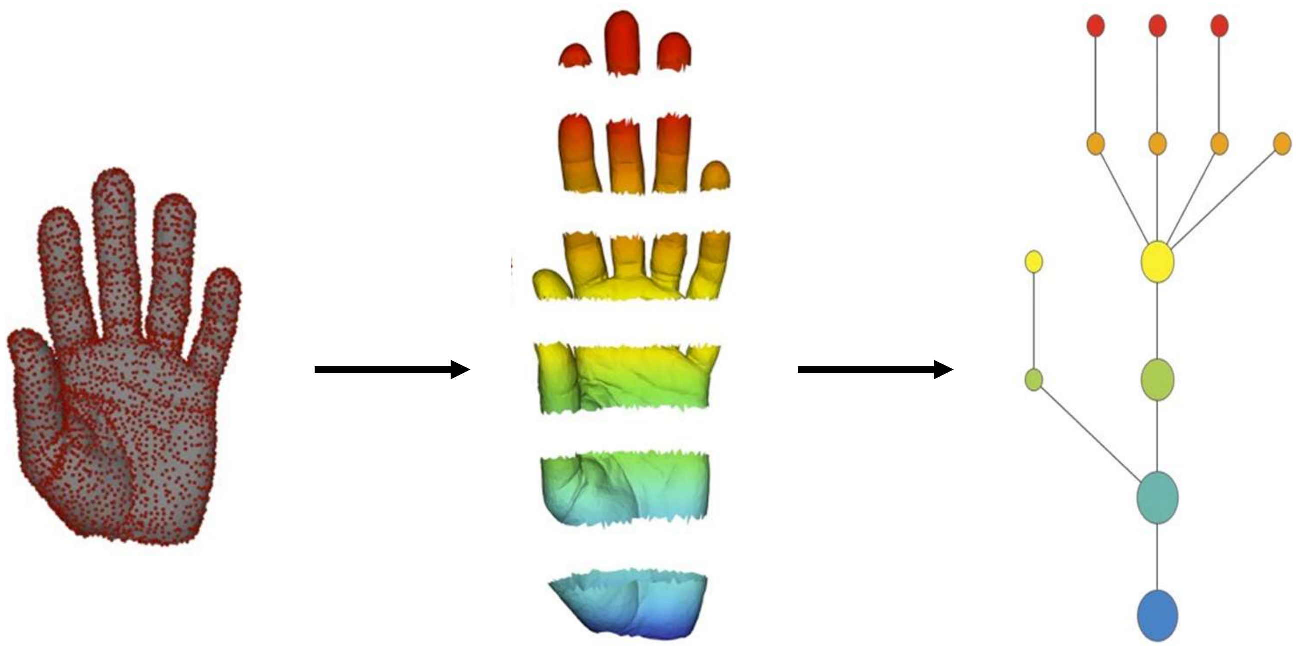

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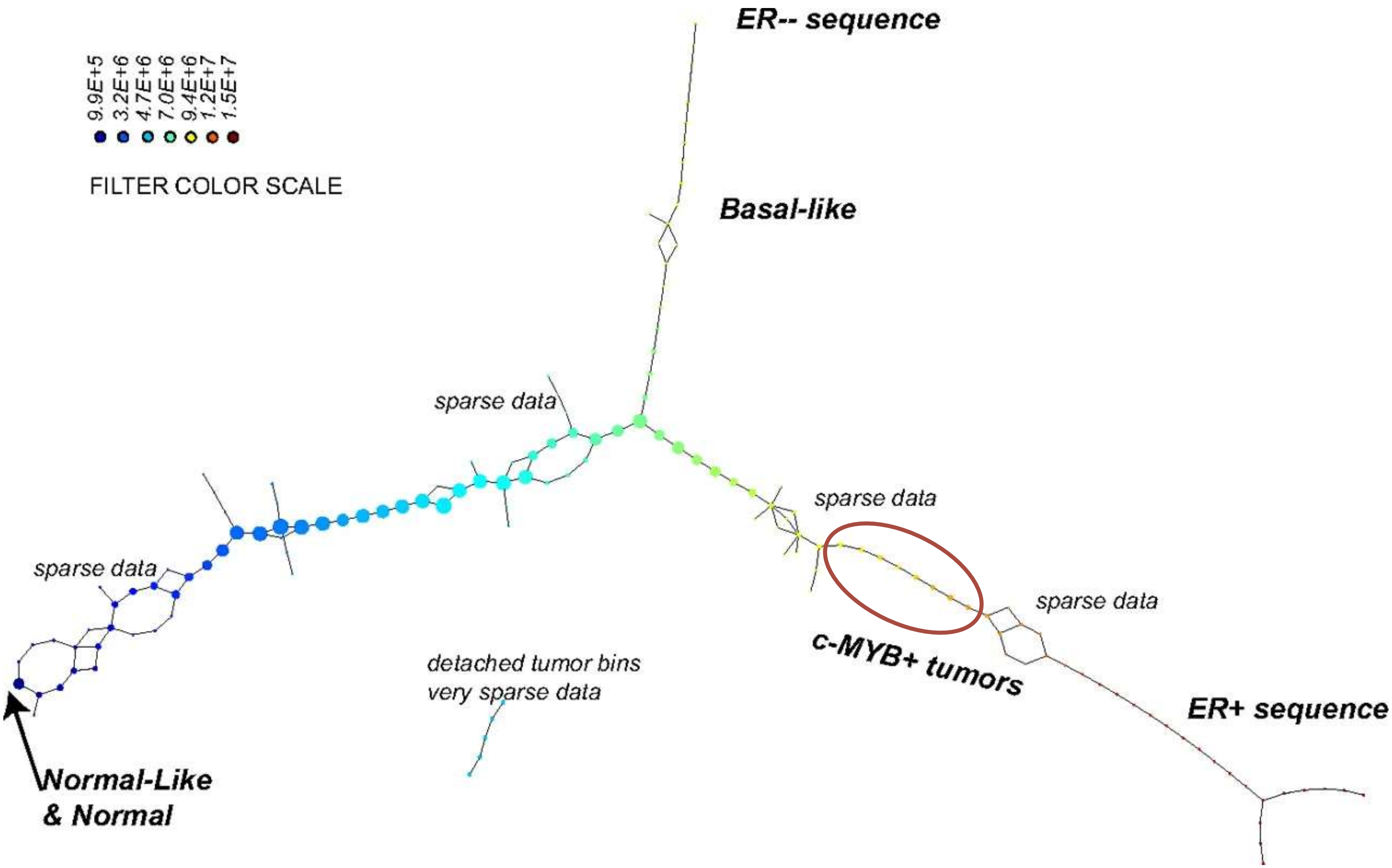

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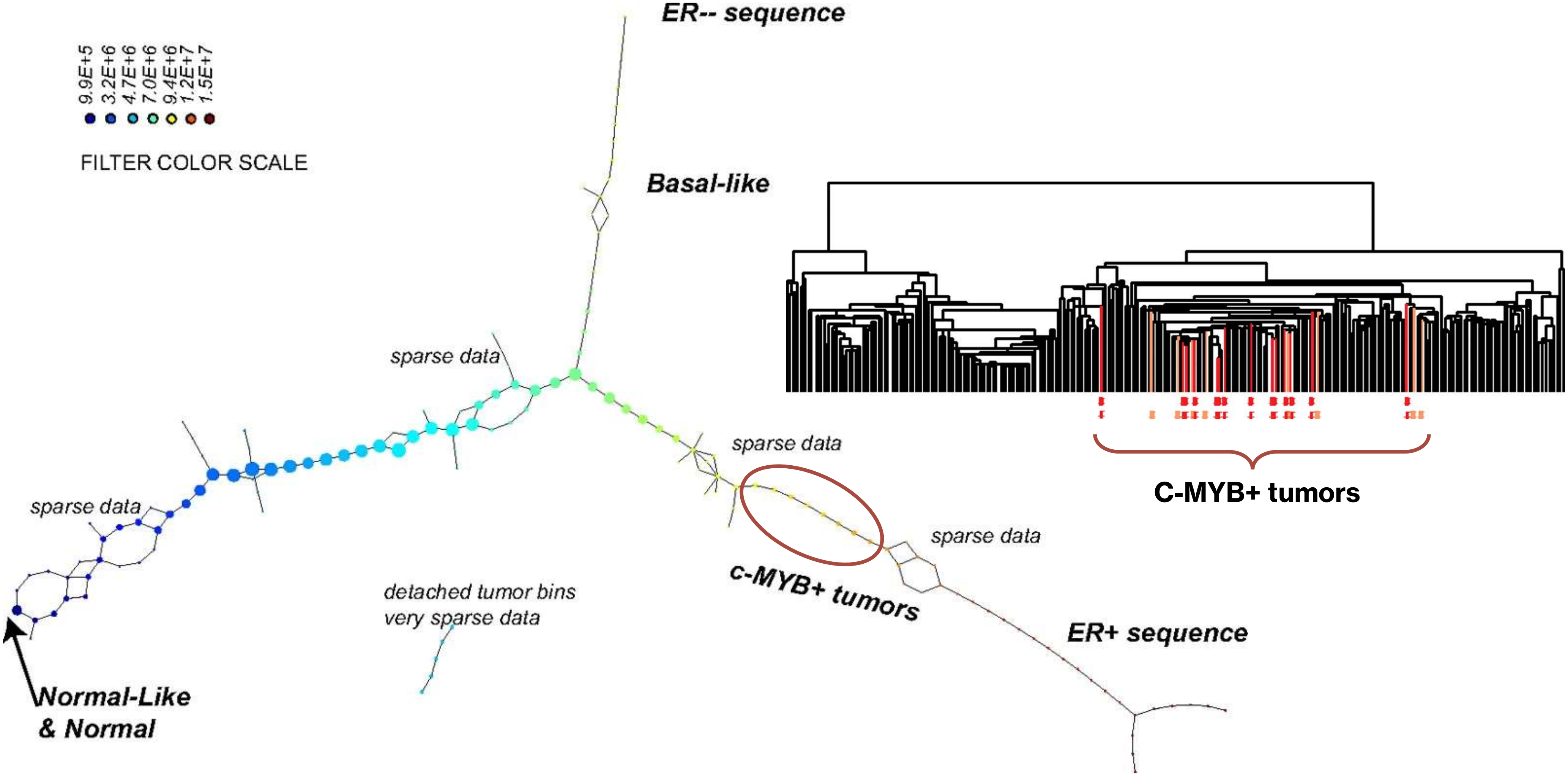

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Definition. Let $X \subseteq \mathbb{R}^d$ be a data set. Given a continuous lens $f: X \to \mathbb{R}^n$, cover $\mathcal{U} = \{U_i\}_{i \in I}$ of $f(X)$, and clustering algorithm we define the *Mapper graph* as

$$M(f, \mathcal{U}) \coloneqq \mathcal{N}^1(f^{-1}(\mathcal{U}))$$

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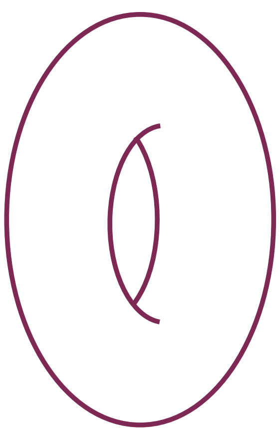

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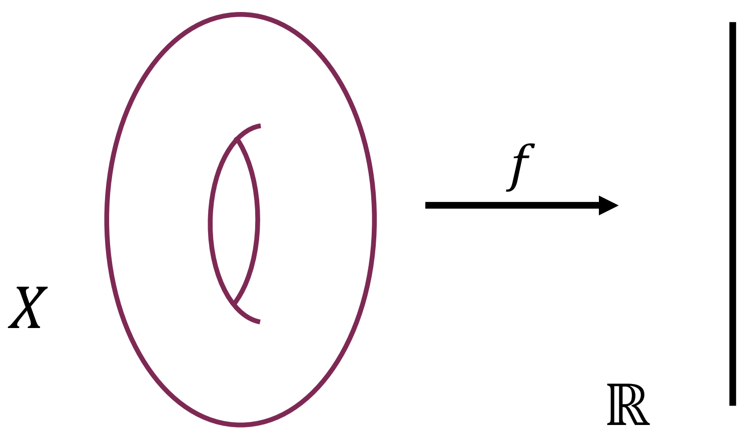

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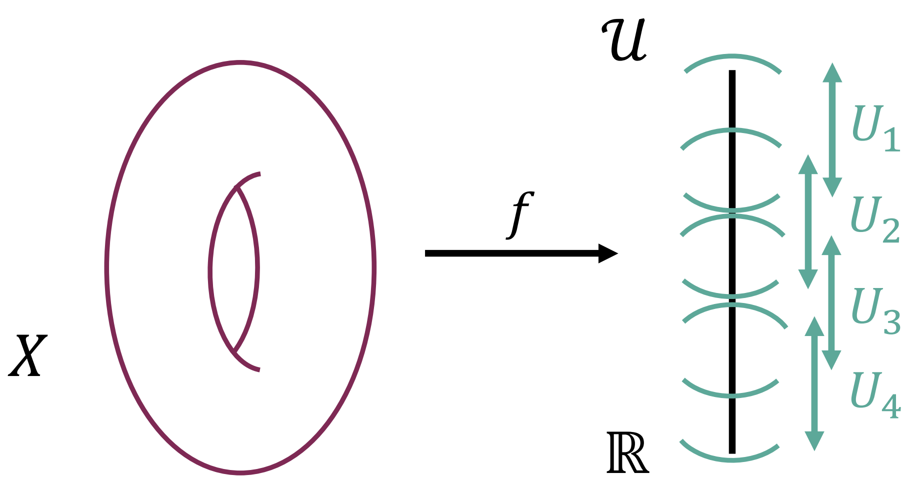

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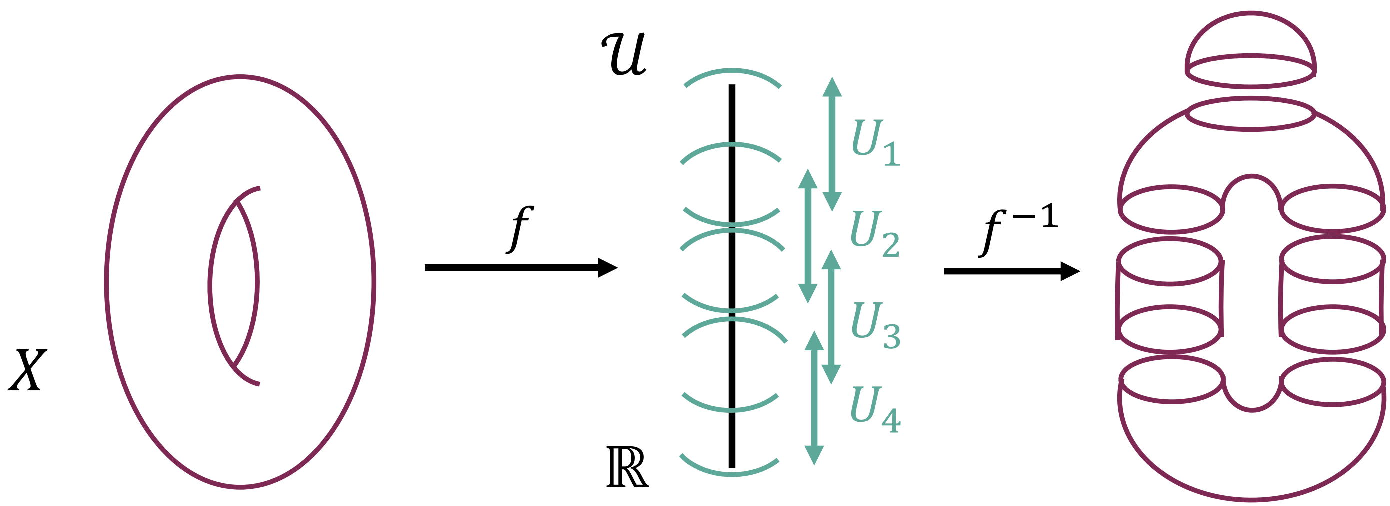

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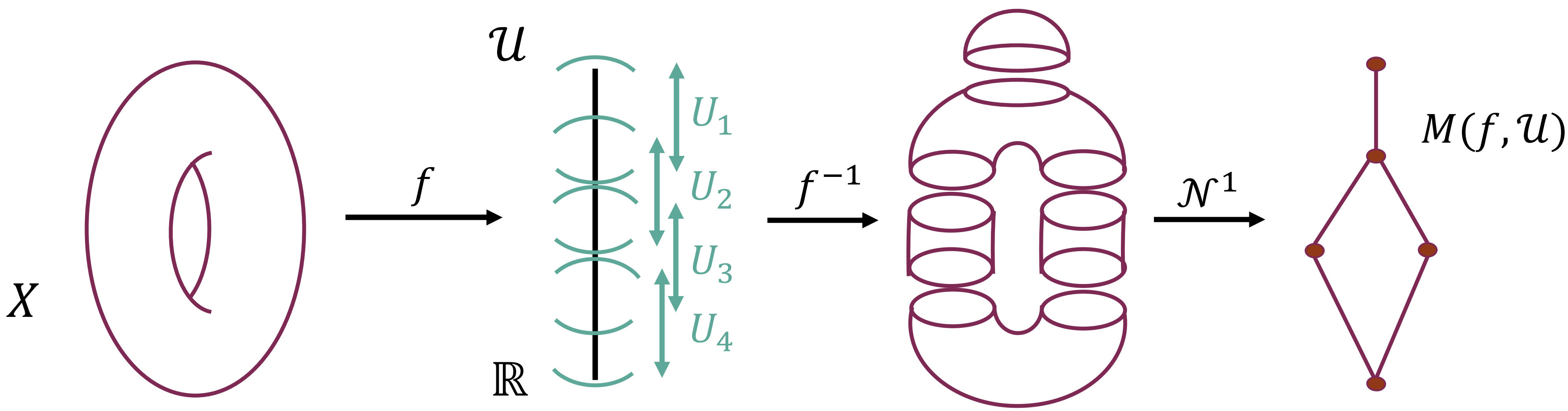
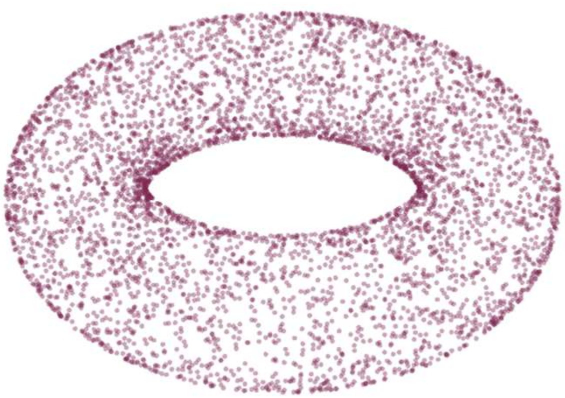


Example.

Let $X \subset \mathbb{R}^3$ be a point cloud of a torus.

- Choose $f: X \to \mathbb{R}^2$ to be the coordinate projection to $\mathbb{R}^2$.



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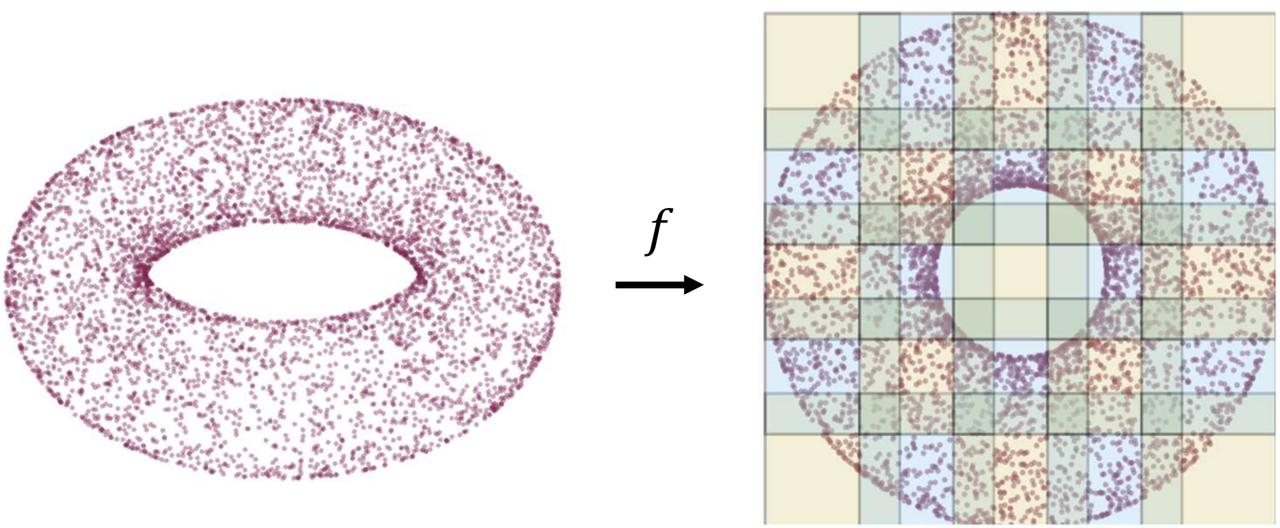
- Cover $f(X)$ with overlapping squares.



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- Choose $f: X \to \mathbb{R}^2$ to be the coordinate projection to $\mathbb{R}^2$.
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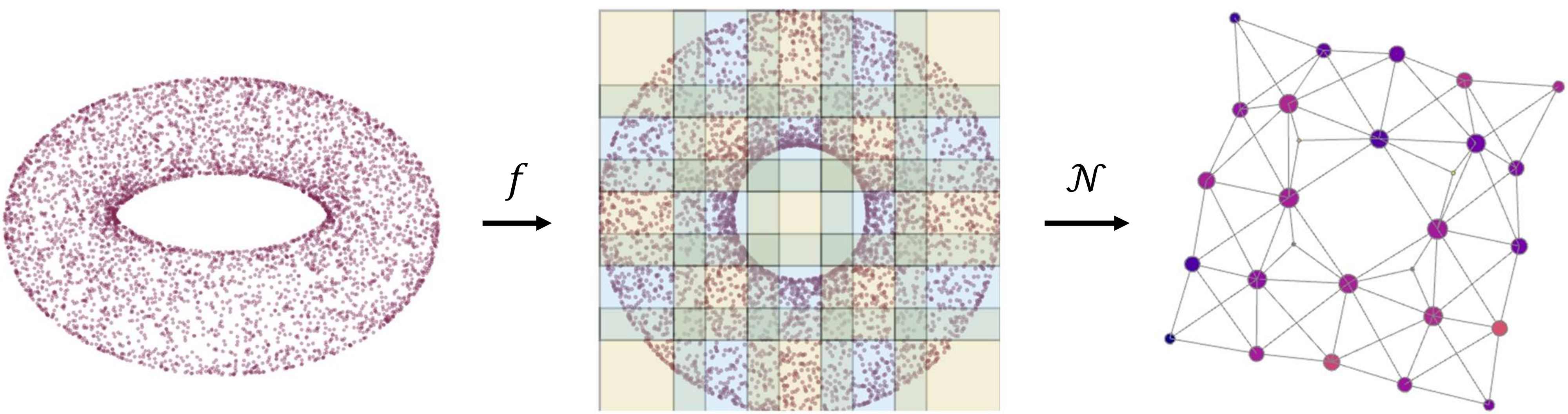
- Choose DBSCAN as the clustering algorithm.



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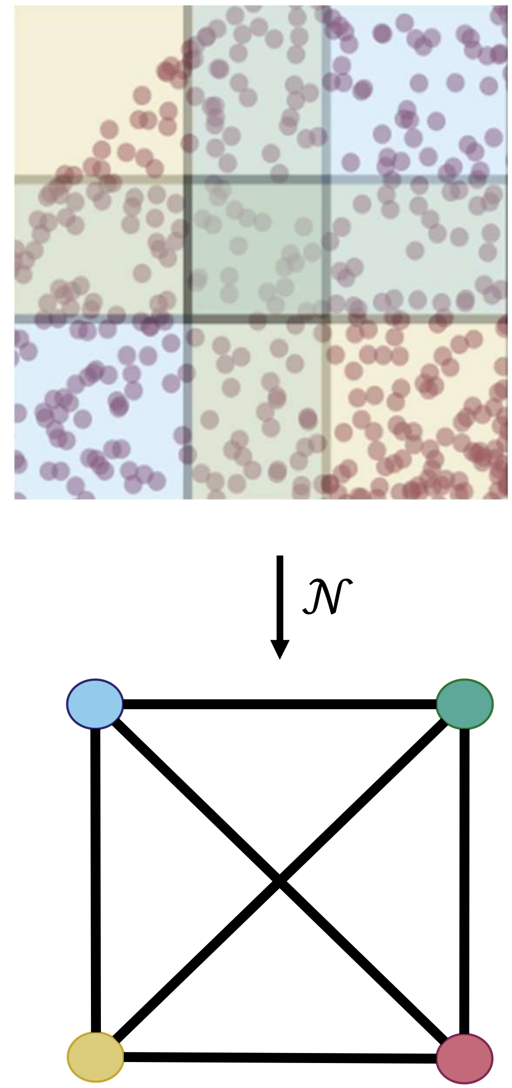
The intersection structure of a cover can yield higher dimensional nerve when $f(X) \subset \mathbb{R}^2$.



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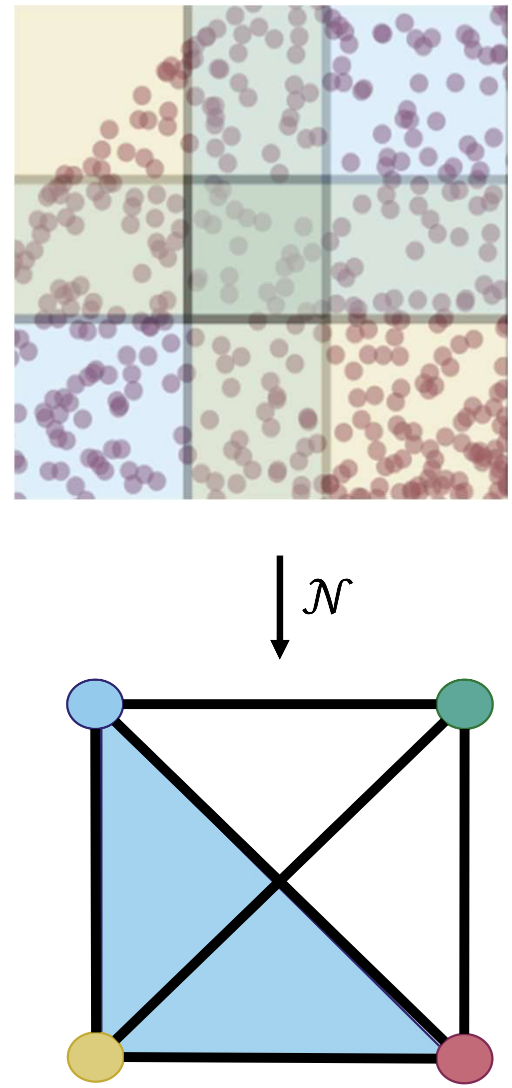

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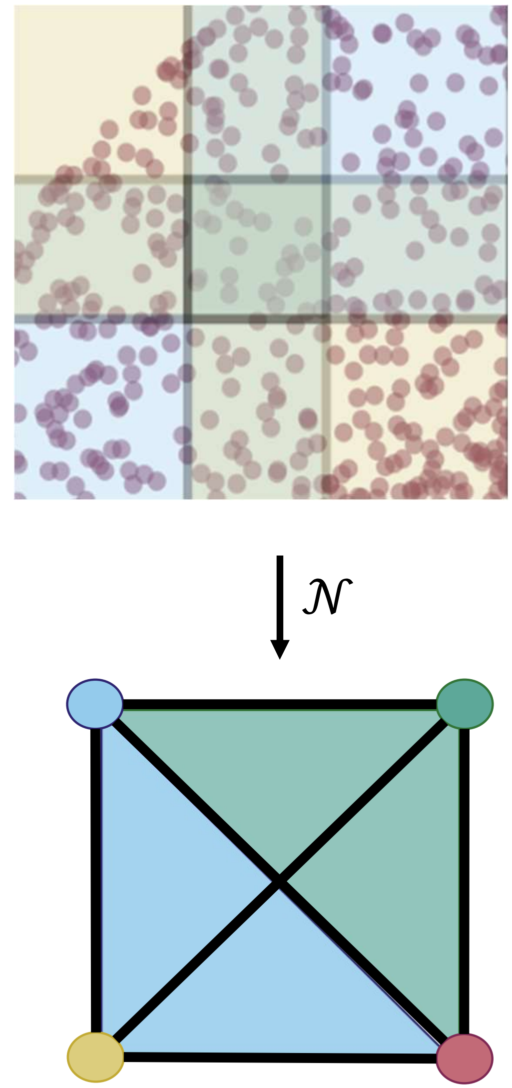

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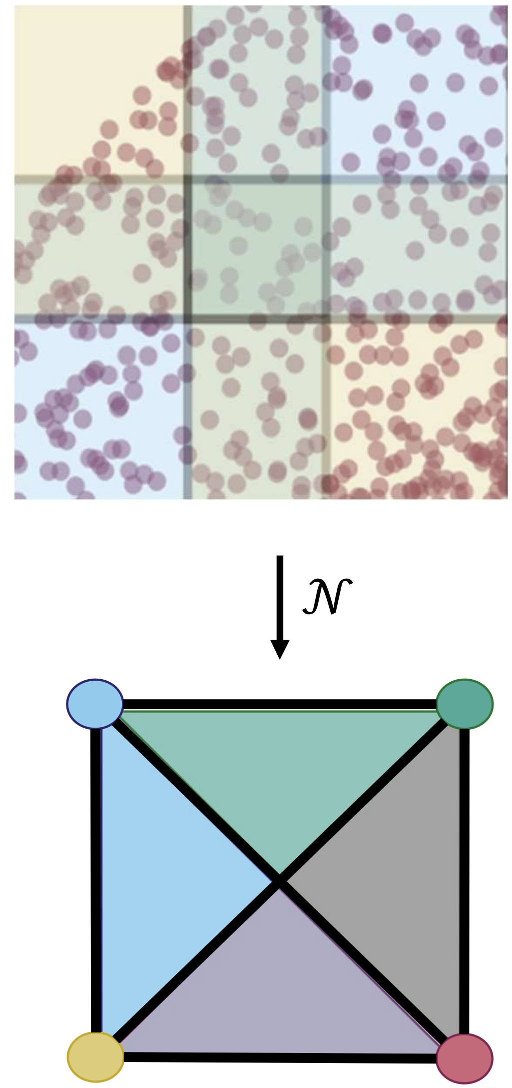

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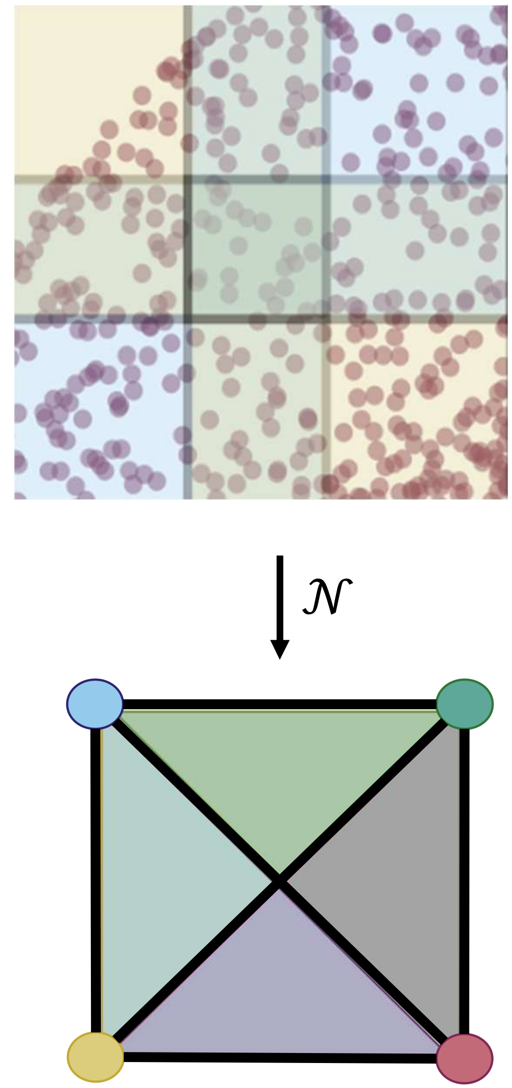

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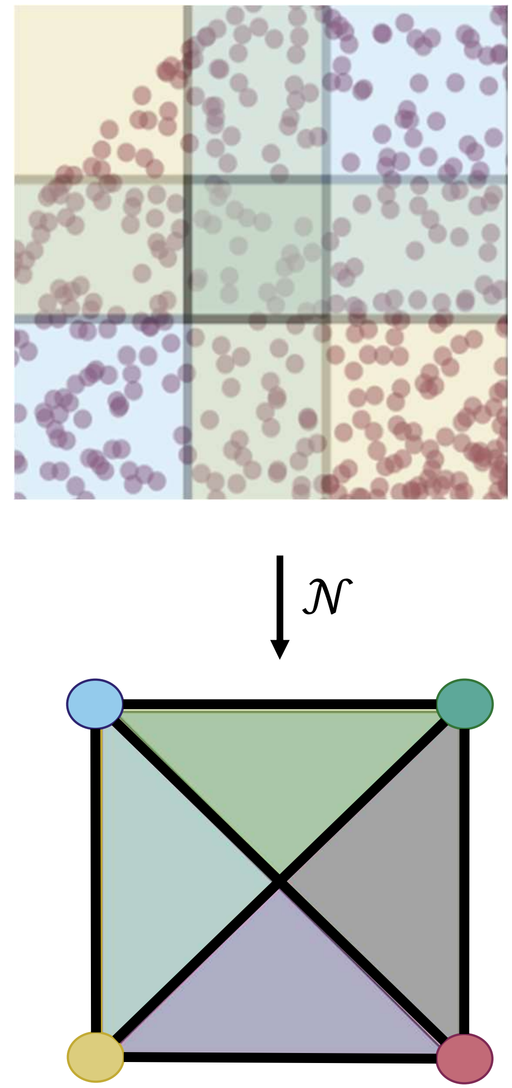
Taking this into account could yield better manifold approximations of point clouds.



2-Mapper

Definition. Given a point cloud X, continuous filter $f: X \to \mathbb{R}^n$, cover $\mathcal{U}$ and clustering algorithm, we define the *2-Mapper complex of* X as

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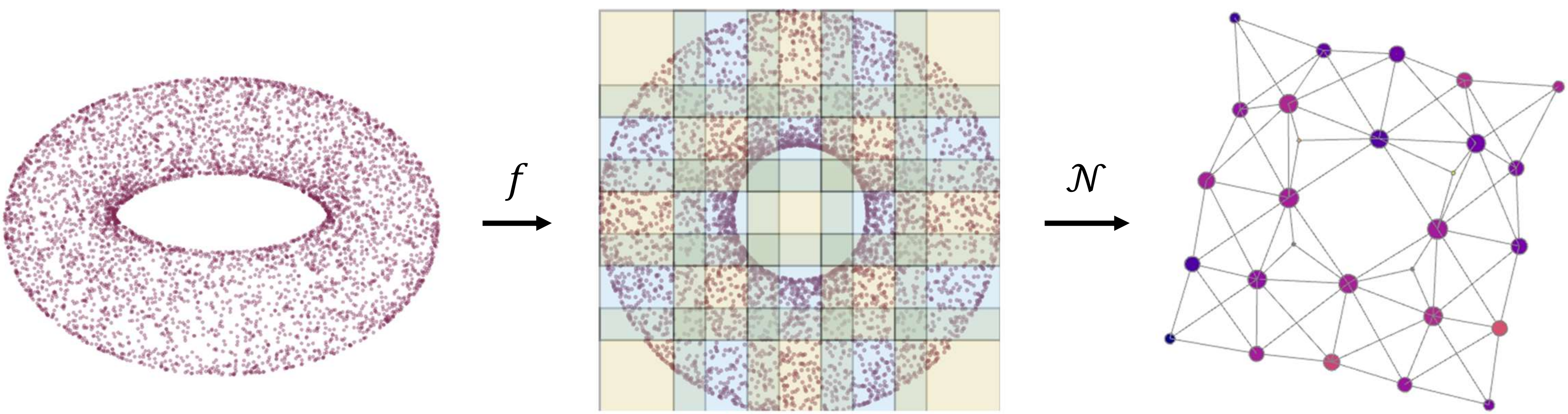

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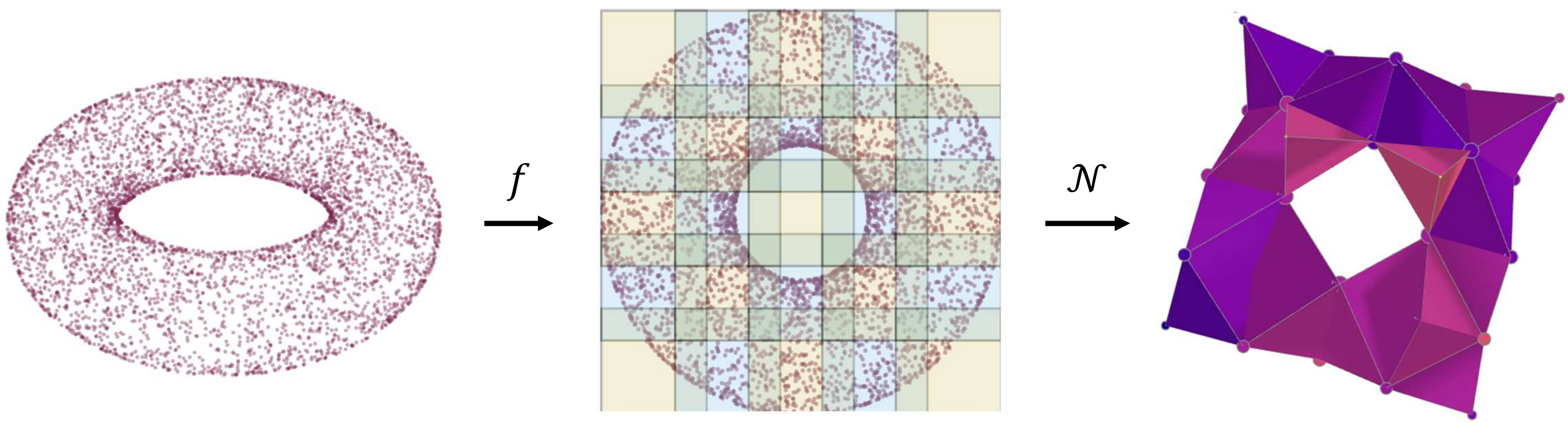
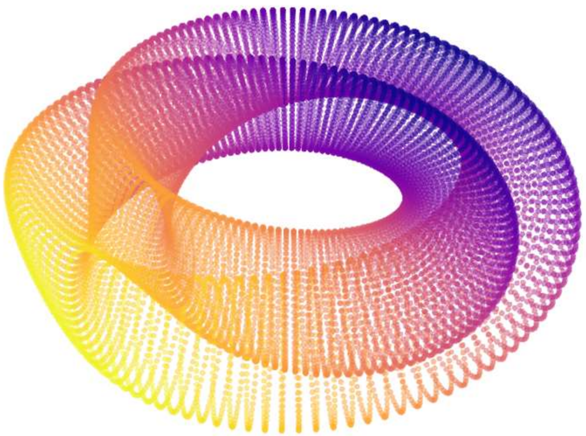


2-Mapper

Question. How do we determine parameter choices for 2-Mapper?



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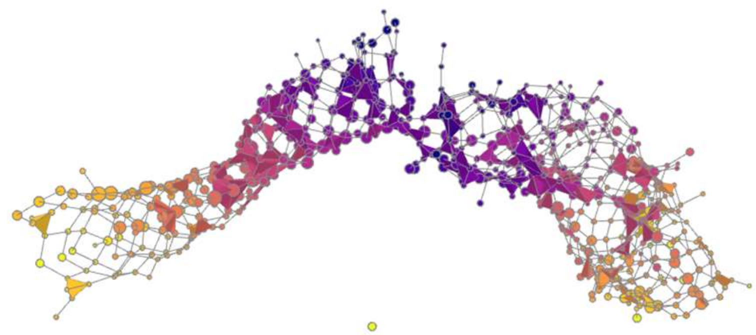

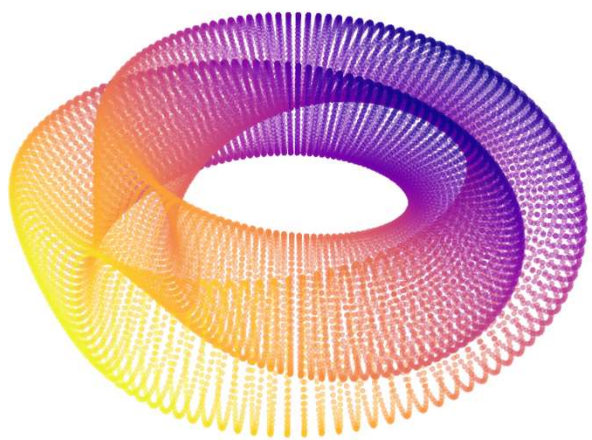

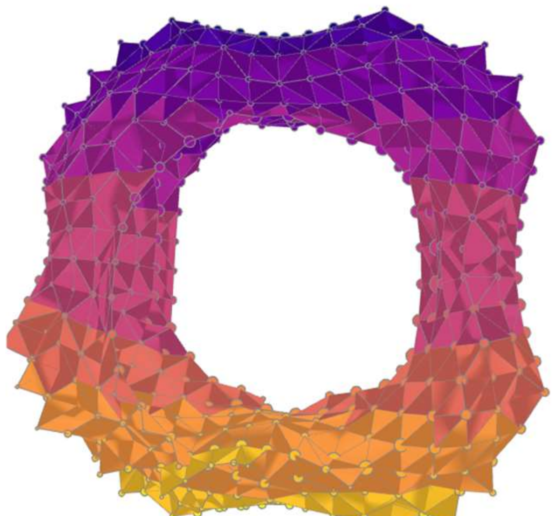

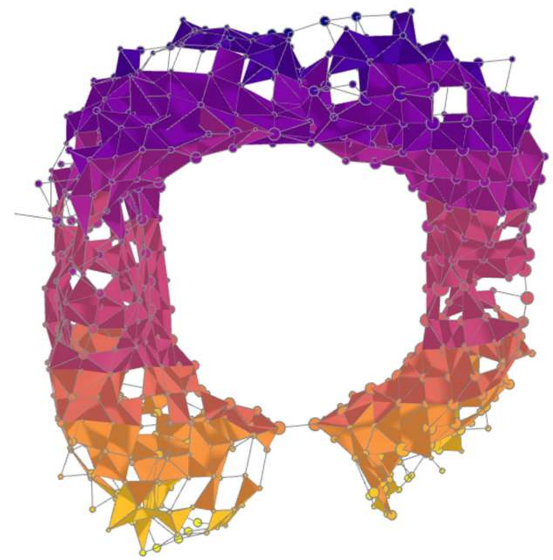

Optimal Covers

2-simplices only require a non-empty triple intersection of cover sets.

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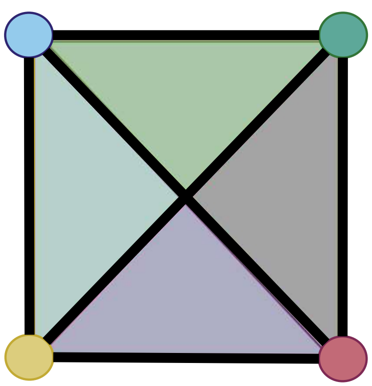
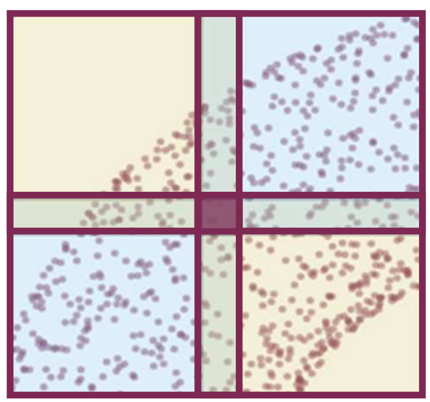
Cubical Cover





Four 2-Simplices

Optimal Covers

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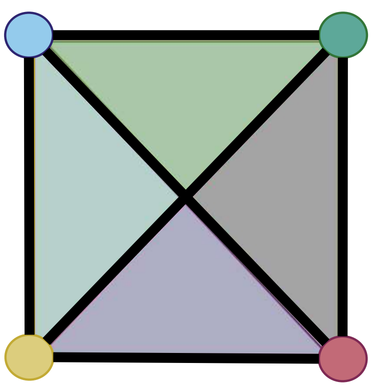
Cubical Cover

Lattice Cover



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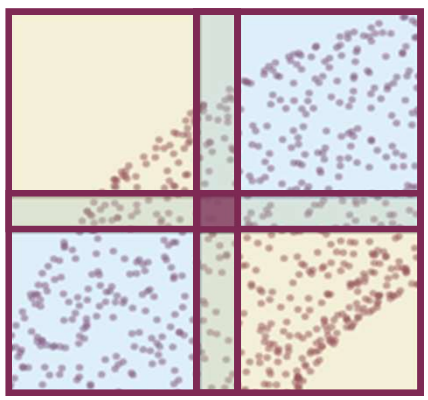
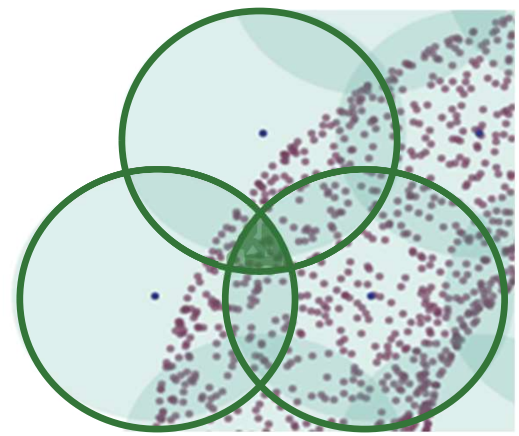
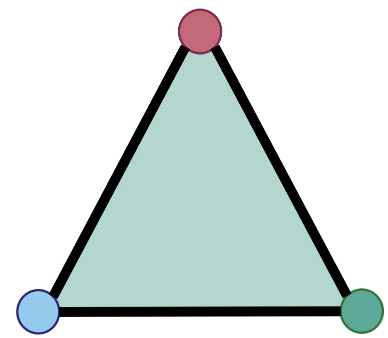






One 2-Simplex

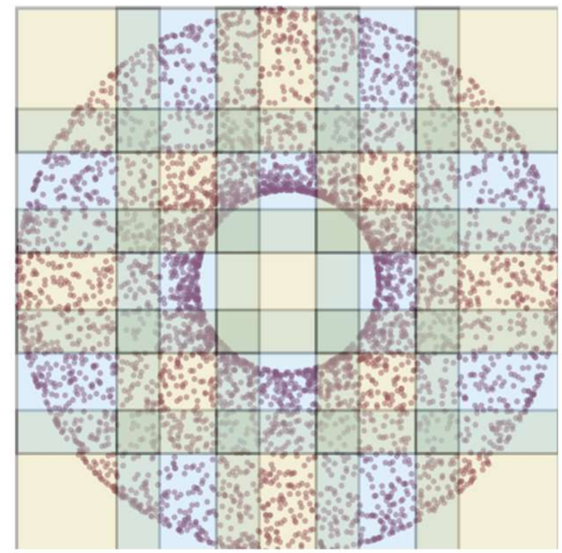
Defining Covers

Definition. Let $X \subset \mathbb{R}^d$ be a data set. The *bounding box* on X is the space $B = \prod_{i=1}^{d} [m_i, M_i]$, where $m_i = \min\limits_{x \in X} \pi_i(x)$ and $M_i = \max\limits_{x \in X} \pi_i(x)$ for coordinate projections π_i.

We define each cover with parameters k and g representing the scale for the number of cover sets and their overlap proportion, respectively.

Cubical Cover

Definition. Let $\mathrm{Z} \subset \mathbb{R}^n$ be a compact topological space with bounding box B. The cubical cover $\mathcal{U}$ on Z constructed with $k \geq 1$-intervals and overlap fraction $0 < g < 1$ is a cover of boxes $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ so that each cover set is of the form



$$U_{\alpha,s} = \prod_{i=1}^{n} \left[c_{\alpha,i} - \frac{l_i}{2}, c_{\alpha,i} + \frac{l_i}{2}\right]$$

Where $c_{\alpha,i} = m_1 + (\alpha_i - 1)(1 - g)l_i + \frac{1}{2} l_i$
and $l_i = \frac{M_i - m_i}{k-(k-1)g}$.

Cubical Lattice Cover

Definition. Let $Z \subset \mathbb{R}^n$ be a compact topological space with bounding box $B = \prod_{i=1}^{n}[m_i, M_i]$. We define the *cubical lattice cover* $\mathcal{U}$ over Z constructed with k-intervals and overlap fraction g as the cover $\mathcal{U} = \{U_\xi\}_{\xi \in B \cap \mathbb{Z}^n}$ whose cover sets are hypercubes defined

$$U_\xi = \prod_{i=1}^{n}\left[c\left(\xi_i - \frac{1}{2(1-g)}\right), c\left(\xi_i + \frac{1}{2(1-g)}\right)\right],$$

with $c = \max\limits_{1 \le i \le n} \frac{M_i - m_i}{k-(k-1)g}$

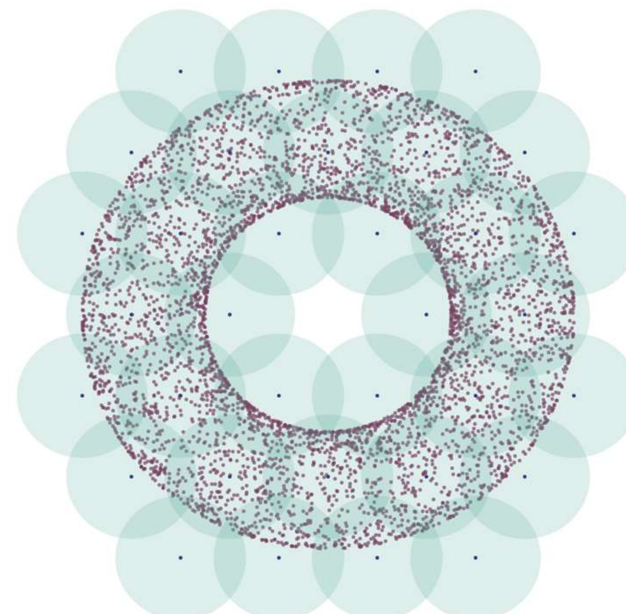
A_2^*-Lattice Cover

<u>Definition.</u> Let $Z \subset \mathbb{R}^n$ be a compact topological space with bounding box $B = \prod_{i=1}^{n}[m_i, M_i]$. Let A_2^* denote the root lattice generated with matrix

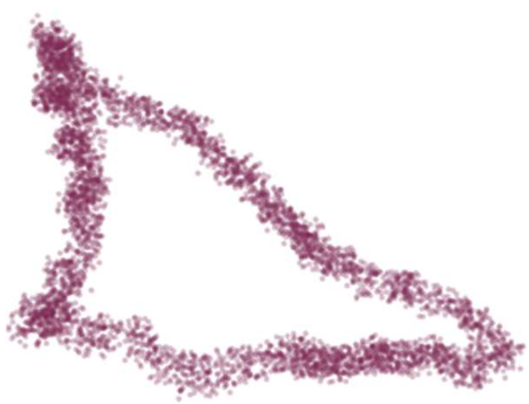
$$M_{A_2^*} = \begin{pmatrix} 1 & 0 \\ \dfrac{-1}{2} & \dfrac{\sqrt{3}}{2} \end{pmatrix}$$

The A_2^*-*lattice cover* defined with k intervals and overlap fraction g is a cover $\mathcal{L} = \left\{B_\varepsilon\left(c\xi M_{A_2^*}\right)\right\}_{c\xi \in B \cap \mathbb{Z}^2}$ with

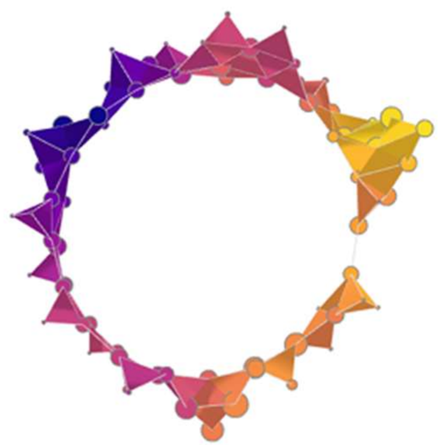
$$\varepsilon = \frac{1+g}{\sqrt{3}} \max_{i=1,2} \frac{M_i - m_i}{k}.$$



Analysis of 2-Mapper through Cover Choice



Stochastic Triangle in $\mathbb{R}^8$



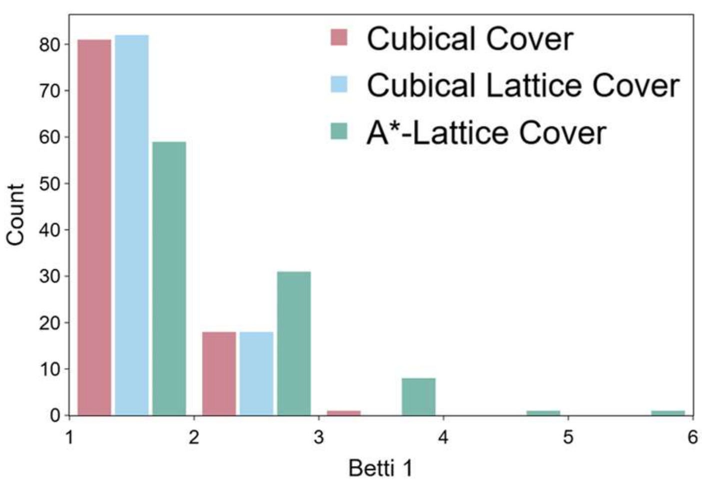
Cubical Cover

Cubical Lattice Cover

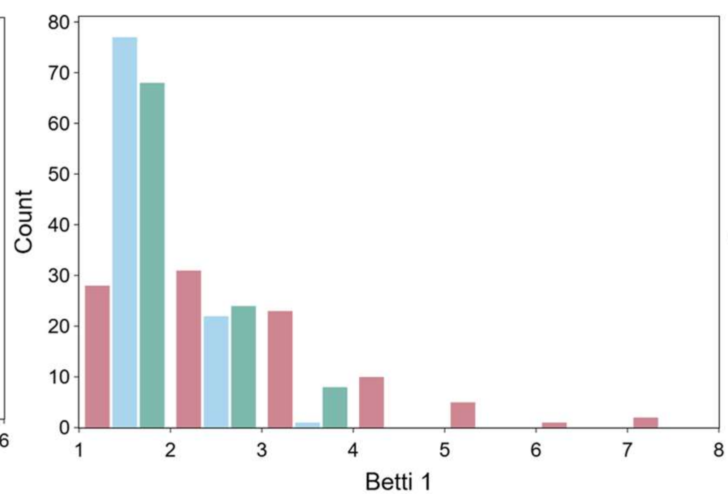
A_2^*-Lattice Cover

Analysis of 2-Mapper through Cover Choice

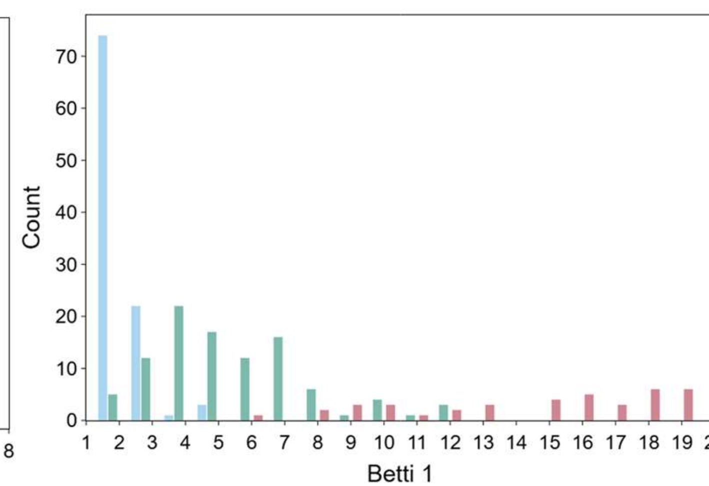
- 100 stochastic triangles generated in $\mathbb{R}^8$
- 2-Mapper complexes constructed with $k = 10$ and $g = 0.3$



2 Principal Components



3 Principal Components



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Analysis of 2-Mapper through Cover Choice

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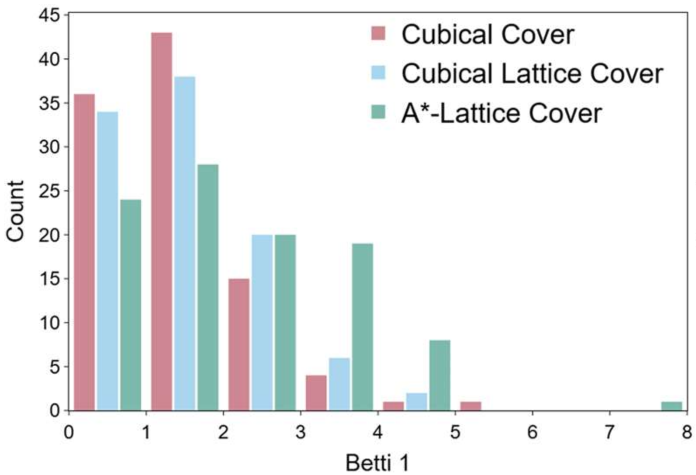
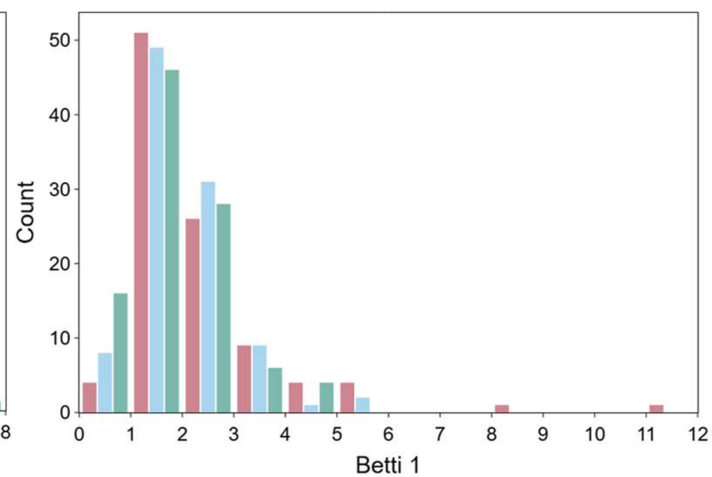
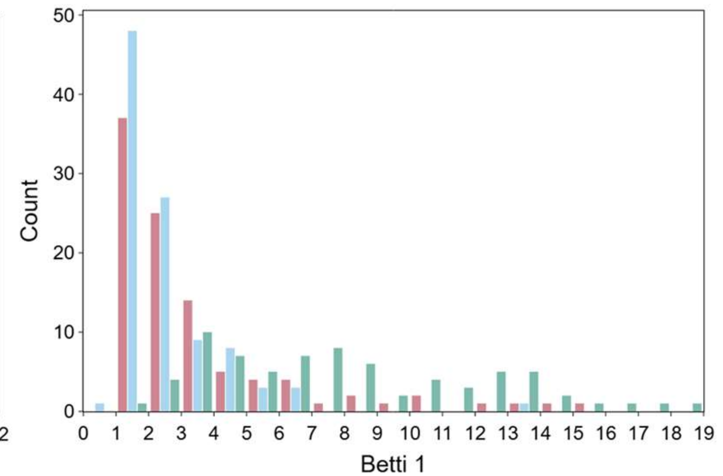


$\mathbb{R}^2$ Coordinate Projection



$\mathbb{R}^3$ Coordinate Projection

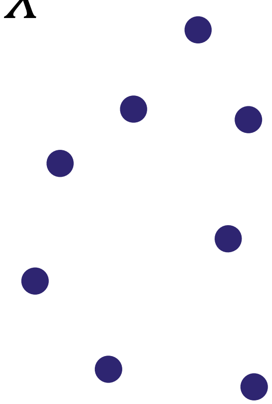


$\mathbb{R}^4$ Coordinate Projection

Persistent Homology

Construct a filtration of simplicial complexes for $r \geq 0$,

X



Persistent Homology

Construct a filtration of simplicial complexes for $r \geq 0$,

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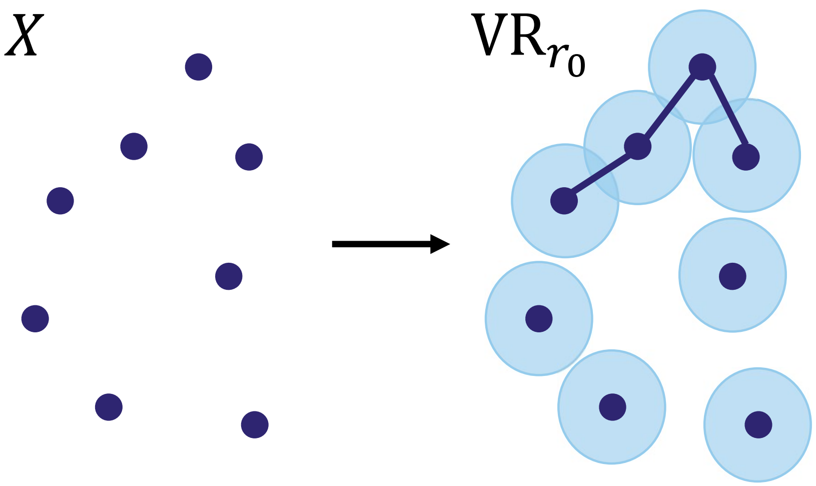
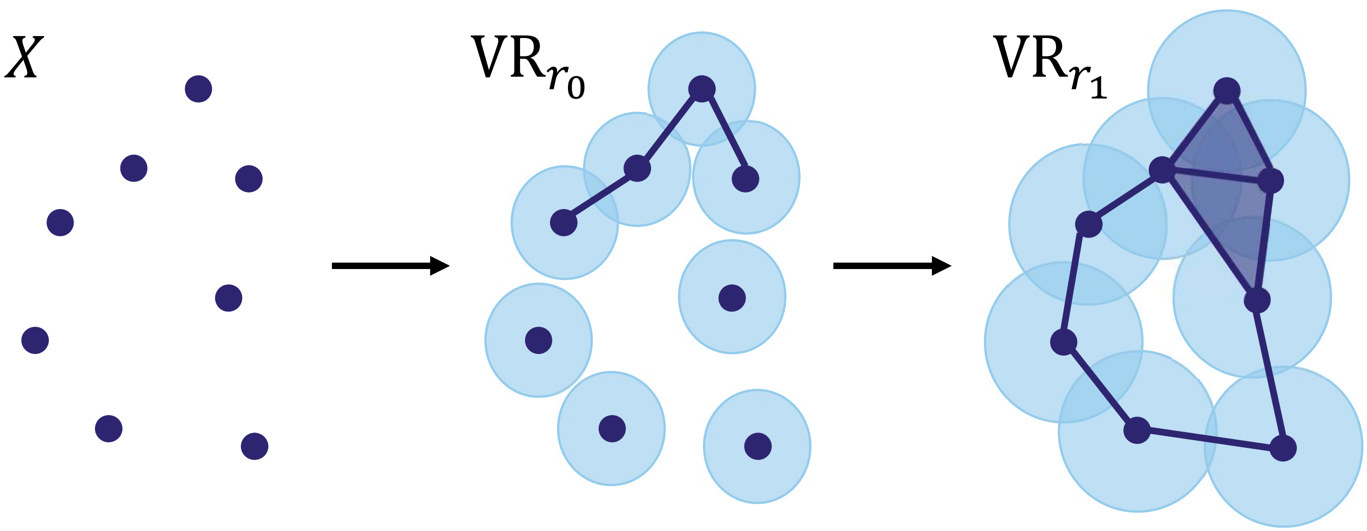


Persistent Homology

Construct a filtration of simplicial complexes for $r \geq 0$,

$$\mathrm{VR}_{r_0}(X) \subseteq \mathrm{VR}_{r_1}(X)$$



Persistent Homology

Construct a filtration of simplicial complexes for $r \geq 0$,

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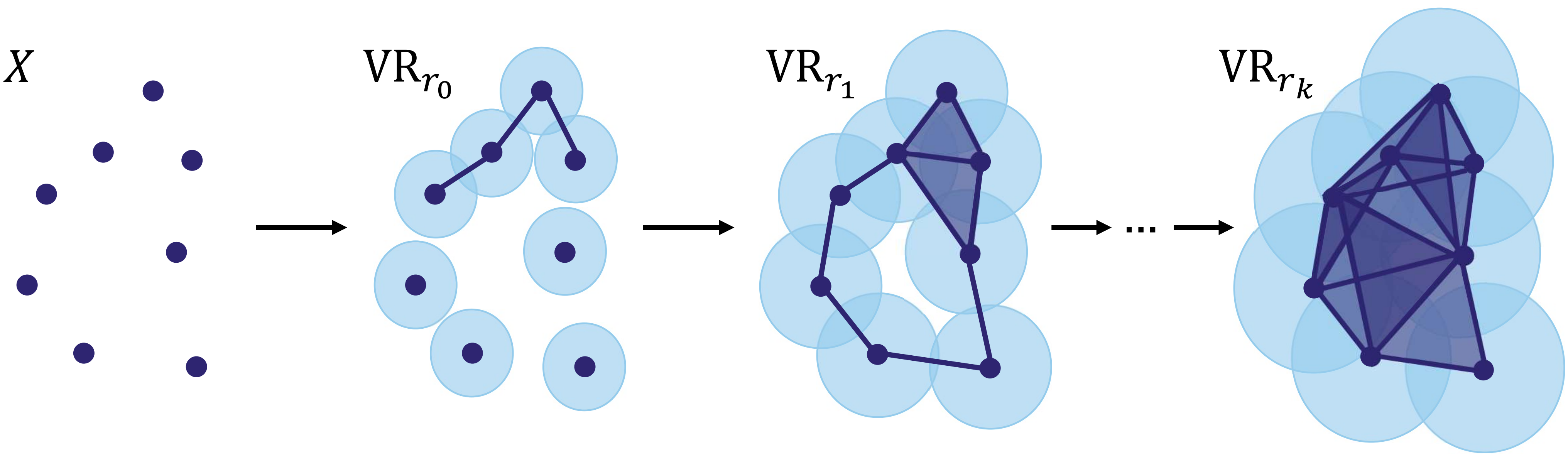
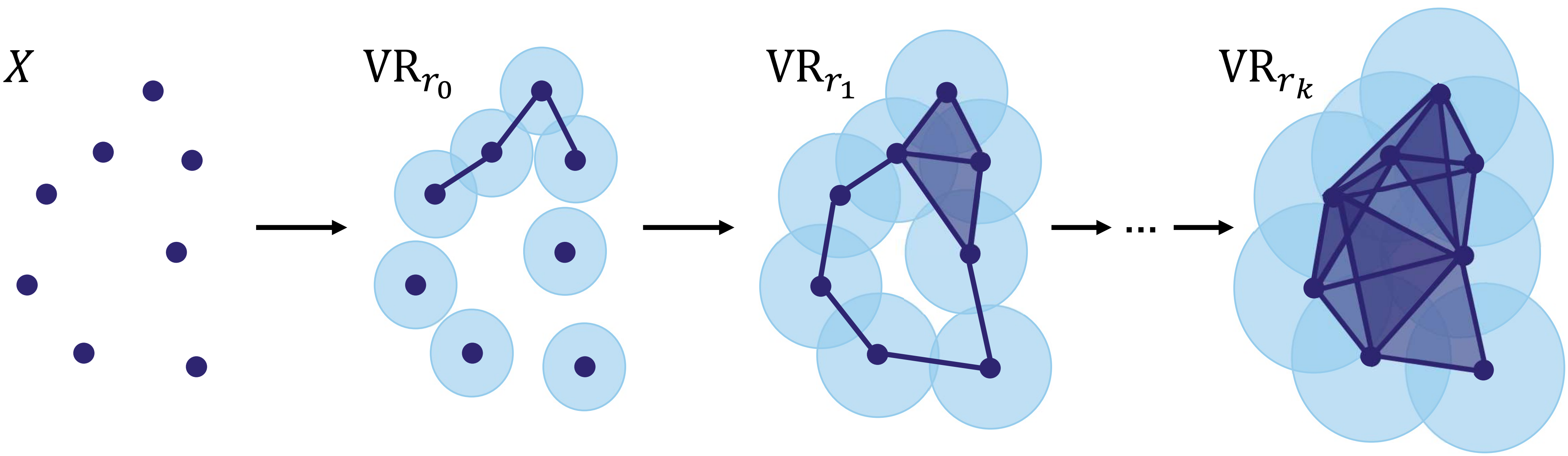


Persistent Homology

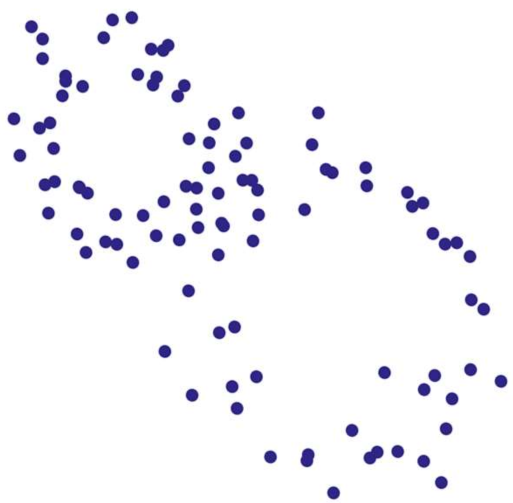
Applying homology yields a *persistence vector space:*

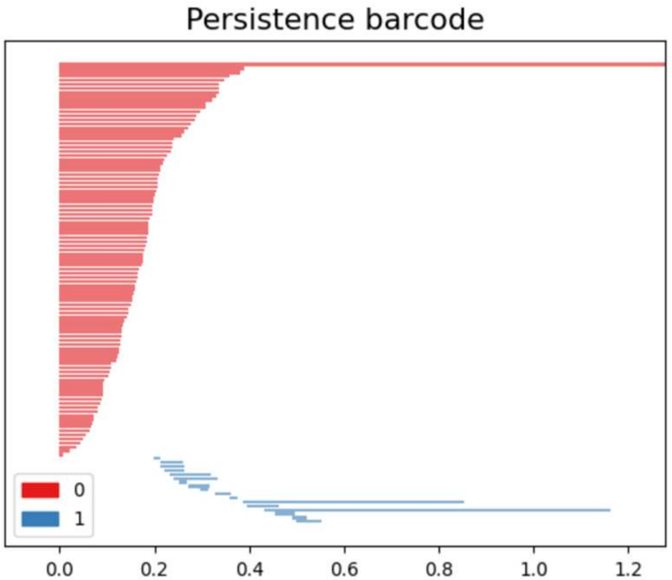
$$H_*\big(\mathrm{VR}_{r_0}(X)\,;\mathbb{Z}_2\big) \to H_*\big(\mathrm{VR}_{r_1}(X)\,;\mathbb{Z}_2\big) \to \cdots \to H_*\big(\mathrm{VR}_{r_k}(X)\,;\mathbb{Z}_2\big)$$

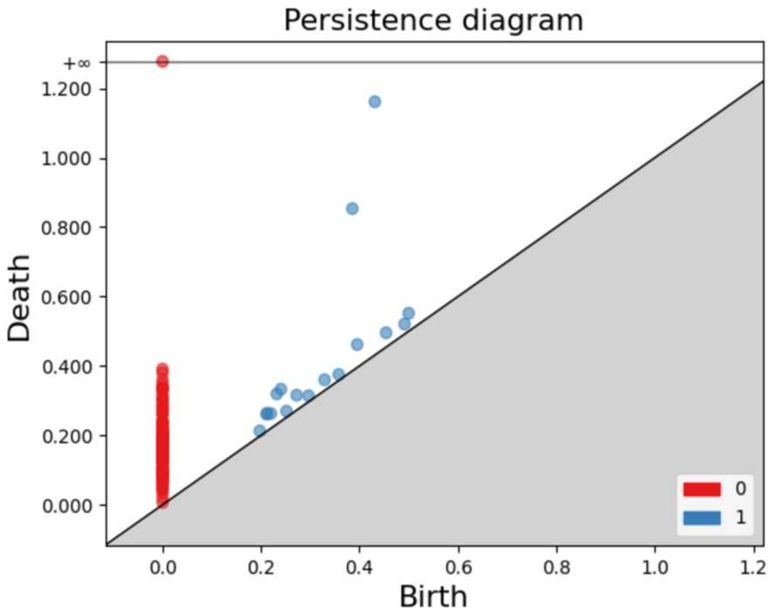


Barcodes and Persistent Diagrams

We track the birth and death times for each cycle as a persistent barcode or persistence diagram.







Distances between diagrams

Definition. Let $D_k(V)$ and $D_k(W)$ be two degree-k persistence diagrams for persistence vector spaces V and W.

Let $\Pi = \{\pi : D_k(V) \to D_k(W)\}$ be the set of bijections between their points to each other or to the diagonal $\{(x, x) : x \in \mathbb{R}_+\}$.

The *bottleneck distance* is defined as

$$d_B\big(D_k(V), D_k(W)\big) = \inf_{\pi \in \Pi} \sup_{x \in D_k(V)} \|x - \pi(x)\|_\infty$$

Multiscale Mapper

<u>Definition.</u> Let X and Z be topological spaces. For a well-behaved continuous map $f : X \to Z$ and finite open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of Z the *generalized Mapper complex* is

$$\mathrm{M}(f, \mathcal{U}) \coloneqq \mathcal{N}\big(f^*(\mathcal{U})\big)$$

Tamal K. Dey, Fecundo Mèmoli, and Yusu Wang. Multiscale Mapper: Topological Summarization via Codomain Covers. *Proceedings of the 2016 Annual ACM-SIAM Symposium on Discrete Algorithms,* pages 997-1013. 2016.

Multiscale Mapper

<u>**Definition.**</u> A *tower of covers* with *resolution* $s \in \mathbb{R}_{\geq 0}$ is a collection of covers $\mathfrak{U} = \{\mathcal{U}_\varepsilon\}_{\varepsilon \geq s}$ with maps $u_{\varepsilon,\delta}: \mathcal{U}_\varepsilon \to \mathcal{U}_\delta$ so that $u_{\varepsilon,\varepsilon} = \mathrm{Id}$ and $u_{\varepsilon,\gamma} = u_{\delta,\gamma} \circ u_{\varepsilon,\delta}$ for all $s \leq \varepsilon \leq \delta \leq \gamma$.

We write $\mathrm{res}(\mathfrak{U}) = s$ for the resolution of tower $\mathfrak{U}$.

Tamal K. Dey, Fecundo Mèmoli, and Yusu Wang. Multiscale Mapper: Topological Summarization via Codomain Covers. *Proceedings of the 2016 Annual ACM-SIAM Symposium on Discrete Algorithms,* pages 997-1013. 2016.

Multiscale Mapper

<u>Definition.</u> Let X and Z be topological spaces and $f\colon X \to Z$ be a well-behaved continuous map. Let $\mathfrak{U}$ be a tower of covers of Z.

The *Multiscale Mapper* is the tower of simplicial complexes defined

$$MM(\mathfrak{U}, f) \coloneqq \mathcal{N}(f^*(\mathfrak{U})).$$

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$$MM(\mathfrak{U}, f) \coloneqq \mathcal{N}(f^*(\mathfrak{U}).$$

For a finite sequence $\mathrm{res}(\mathfrak{U}) \leq \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_n$,

$$H_*\left(\mathcal{N}\left(f^*(\mathcal{U}_{\varepsilon_1})\right)\right) \to H_*\left(\mathcal{N}\left(f^*(\mathcal{U}_{\varepsilon_2})\right)\right) \to \cdots \to H_*\left(\mathcal{N}\left(f^*(\mathcal{U}_{\varepsilon_n})\right)\right).$$

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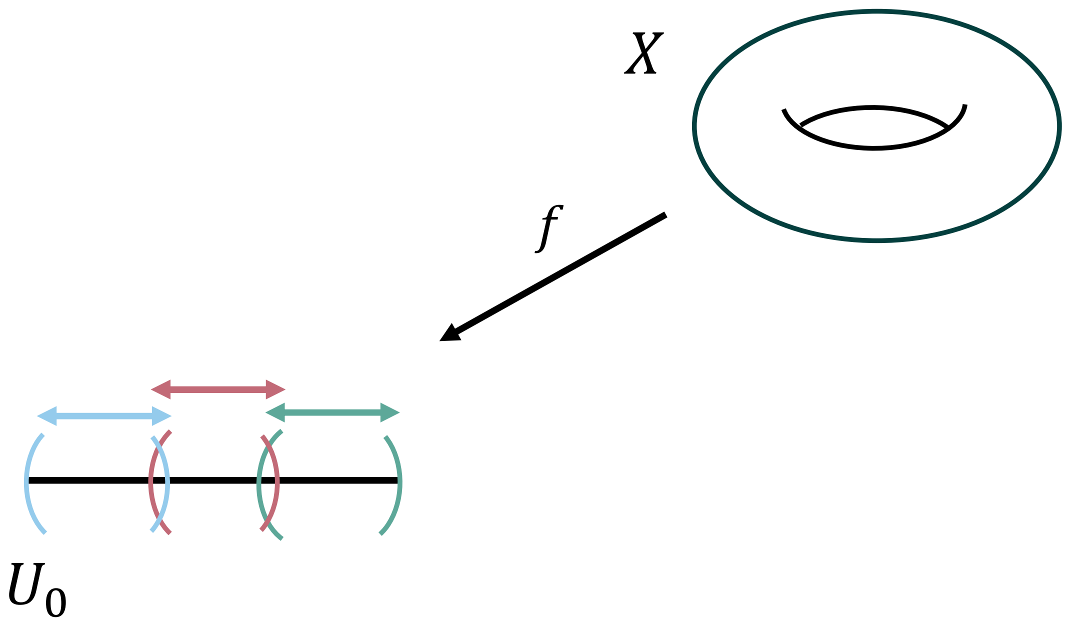
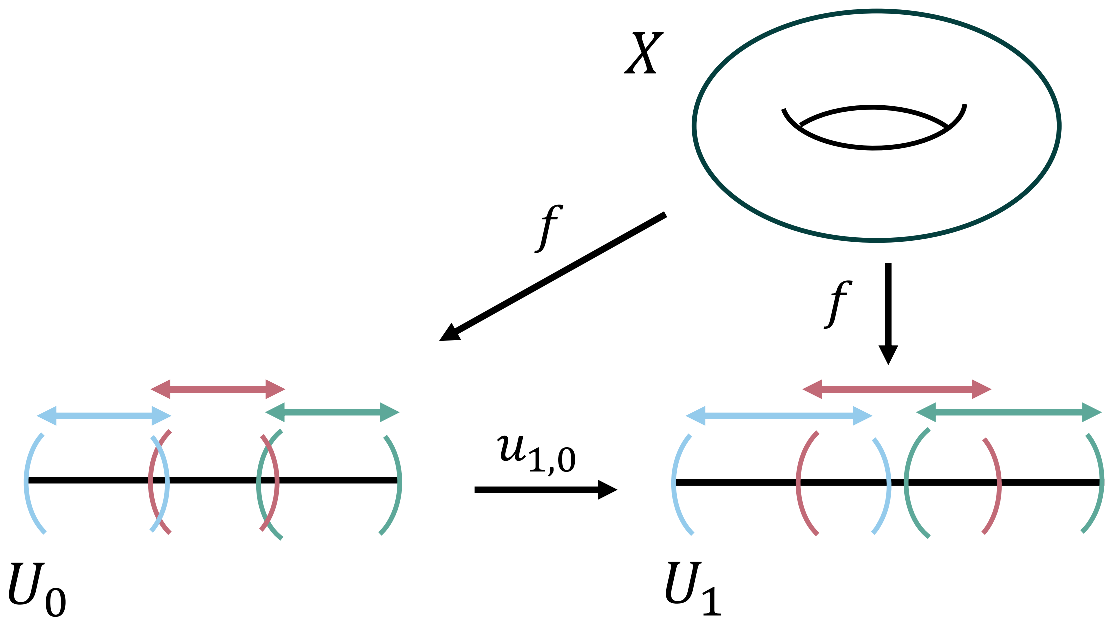
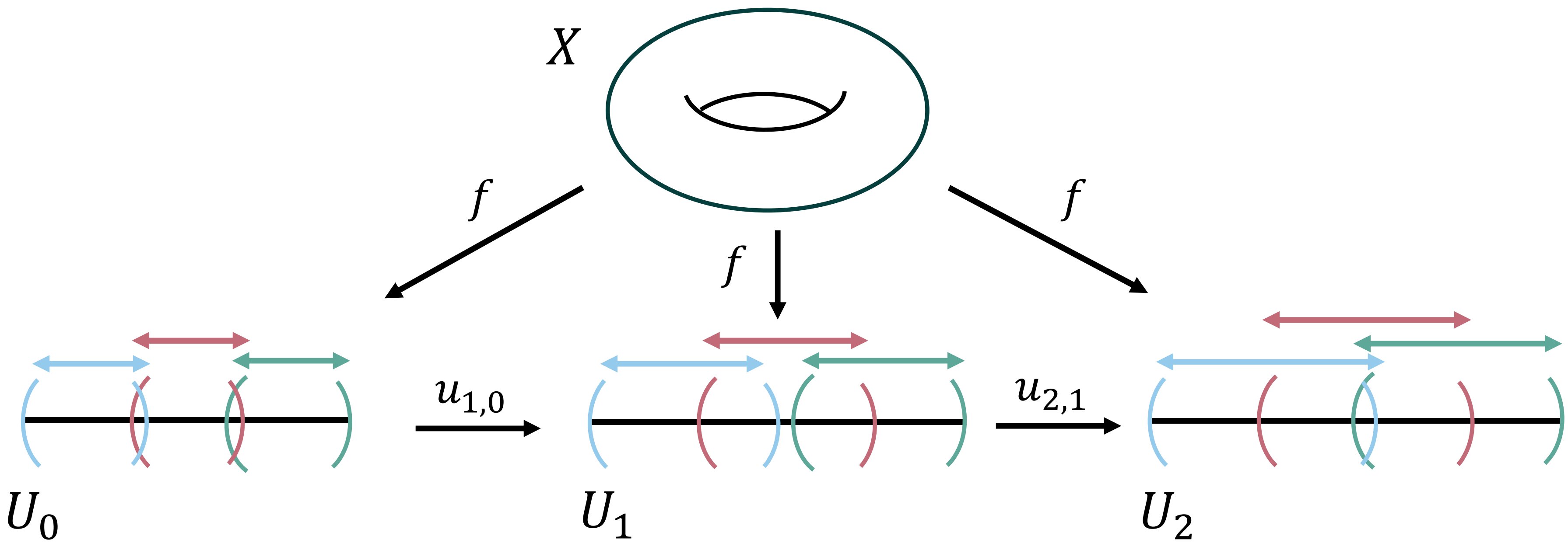
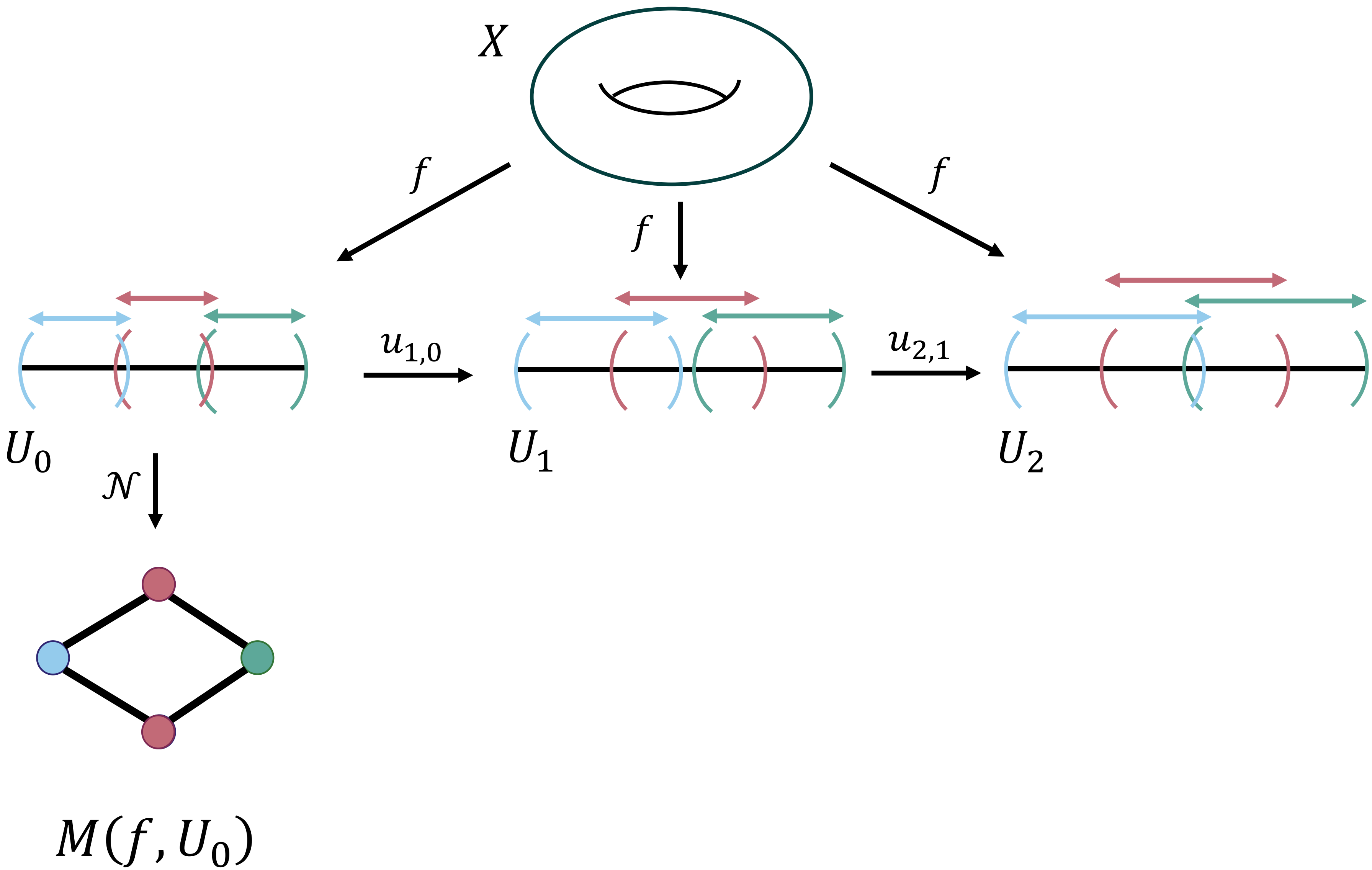

X
f
U0



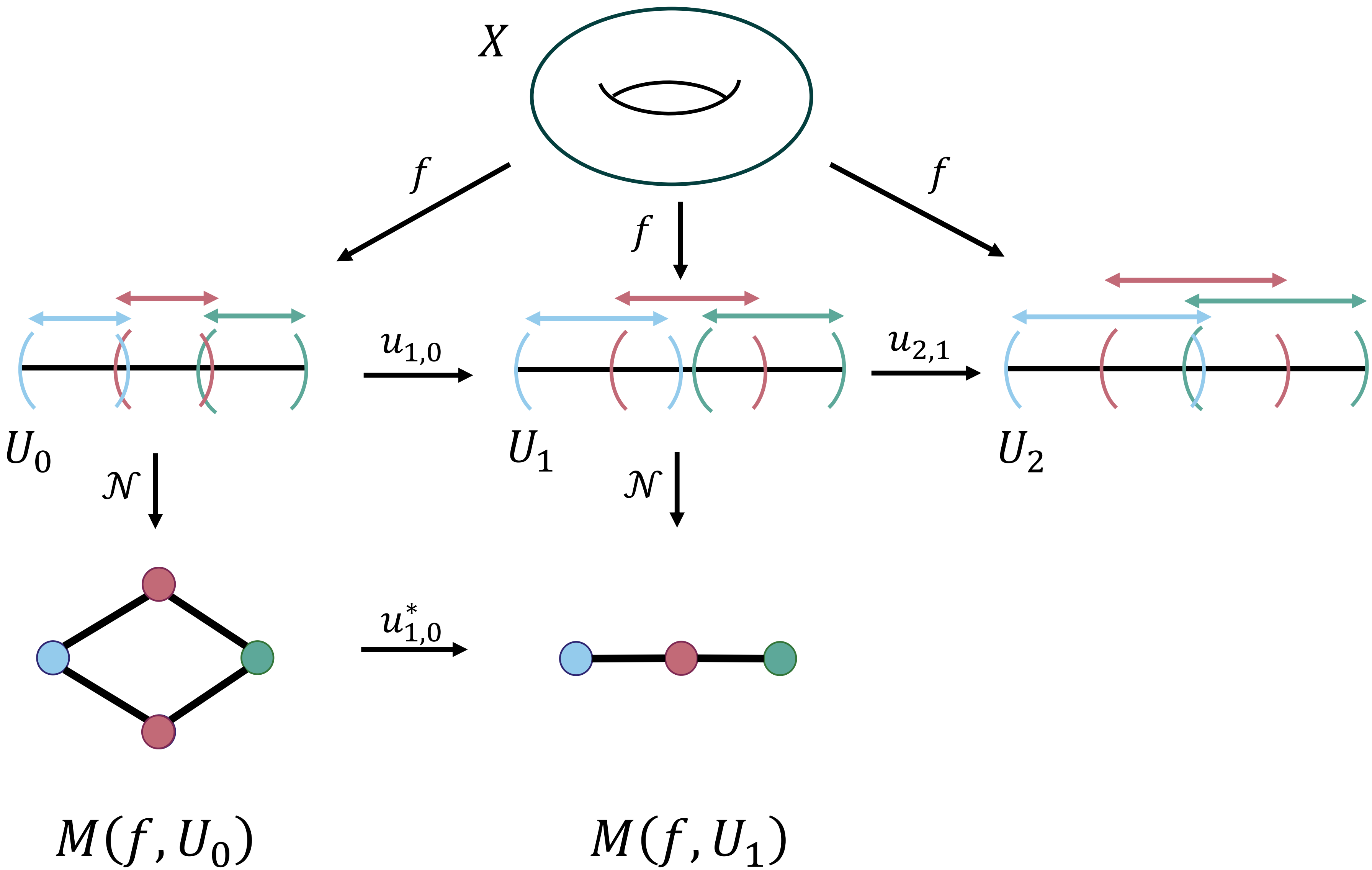
X
f
f
$u_{1,0}$
U_0
U_1

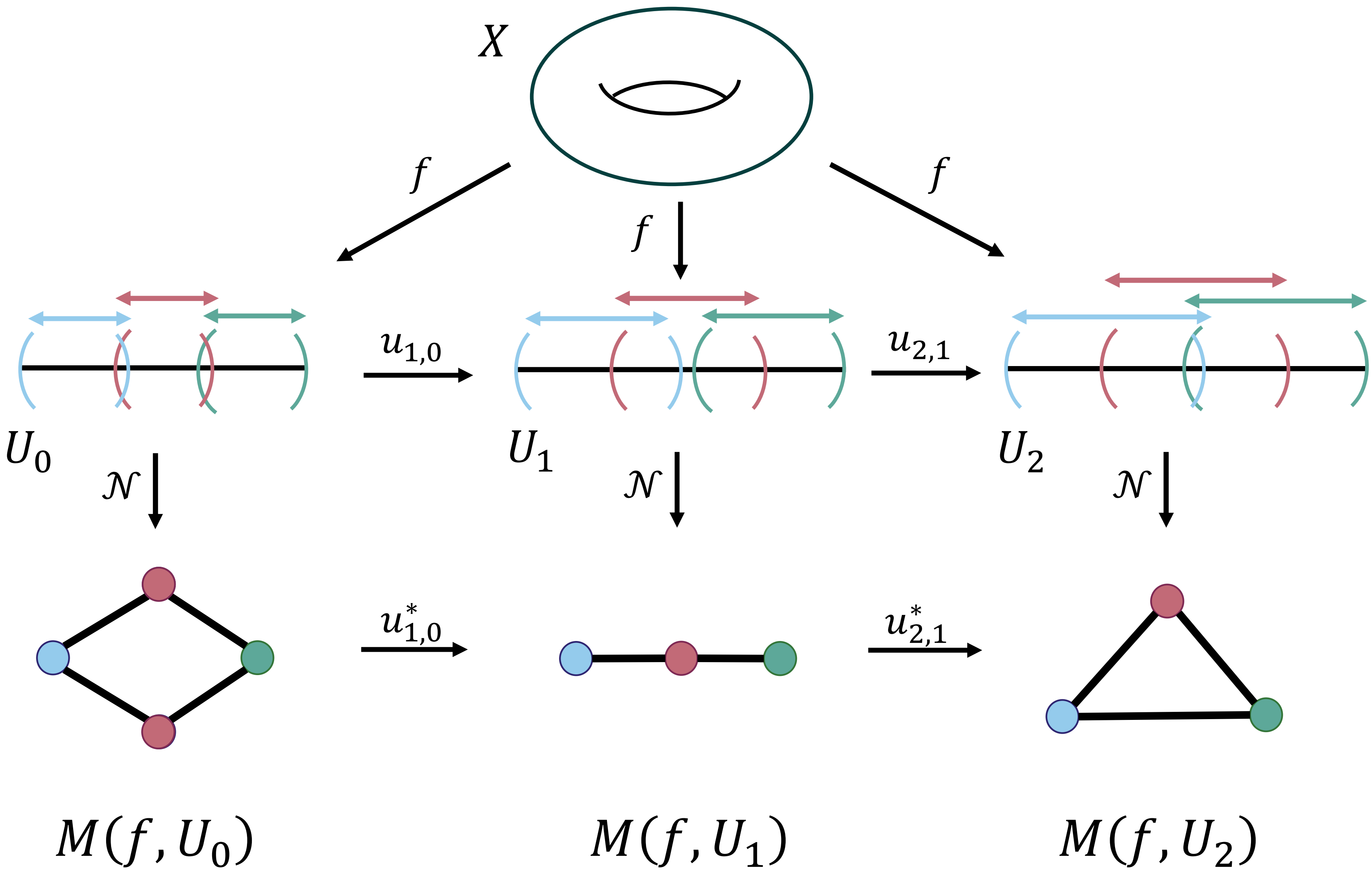


X
f
f
f
$u_{1,0}$
$u_{2,1}$
U_0
U_1
U_2



X
f
f
f
$u_{1,0}$
$u_{2,1}$
U_0
U_1
U_2
$\mathcal{N}$
$M(f, U_0)$


X
f
f
f
$u_{1,0}$
$u_{2,1}$
U_0
U_1
U_2
$\mathcal{N}$
$\mathcal{N}$
$u^*_{1,0}$
$M(f, U_0)$
$M(f, U_1)$



X
f
f
f
U_0
$u_{1,0}$
U_1
$u_{2,1}$
U_2
$\mathcal{N}$
$\mathcal{N}$
$\mathcal{N}$
$u^*_{1,0}$
$u^*_{2,1}$
$M(f, U_0)$
$M(f, U_1)$
$M(f, U_2)$

Multiscale Mapper

Definition. Let $c \geq 1$ and $s > 0$. A finite tower of covers $\mathfrak{W} = \{\mathcal{W}_\varepsilon\}$ over a compact metric space (Z, d_Z) is (c, s)-*good* if

1. $\mathrm{res}(\mathfrak{W}) = s$, and $s \leq \mathrm{diam}(Z)$
2. $\mathrm{diam}\left(W_{\varepsilon,\alpha}\right) \leq \varepsilon$ for all $W_{\varepsilon,\alpha} \in \mathcal{W}_\varepsilon$, $\varepsilon \geq s$
3. $\forall\ O \subset Z$ with $\mathrm{diam}(O) \geq s$, there exists $W \in \mathcal{W}_{c\cdot\mathrm{diam}(O)}$ such that $O \subset W$.

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Example. A tower of covers $\mathfrak{U} = \{\mathcal{U}_\varepsilon\}_{\varepsilon \geq s}$ with $\mathcal{U}_\varepsilon = \big\{B_{\varepsilon/2}(z) \mid z \in Z\big\}$ is a $(2, s)$-good tower of covers of compact metric space Z.

Tamal K. Dey, Fecundo Mèmoli, and Yusu Wang. Multiscale Mapper: Topological Summarization via Codomain Covers. *Proceedings of the 2016 Annual ACM-SIAM Symposium on Discrete Algorithms,* pages 997-1013. 2016.

Multiscale Mapper

Theorem (Dey, M., W.). For two (c, s)-good tower of covers $\mathfrak{U}, \mathfrak{V}$, the bottleneck distance between two diagrams produced from their multiscale mappers is bounded,

$$d_B\left(D_k\big(MM(f, \mathfrak{U})\big), D_k\big(MM(f, \mathfrak{V})\big)\right) \leq c.$$

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Multiscale 2-Mapper

Definition. Let $Z \subset \mathbb{R}^n$ be a compact topological space. Then for lens function $f: Z \to \mathbb{R}^m$ and tower of covers $\mathfrak{U} = \{\mathcal{U}_\varepsilon\}_{\varepsilon \geq s}$ over $f(Z)$, the *Multiscale 2-Mapper*, denoted $\mathrm{MM}_2(f, \mathfrak{U})$, for a finite sequence $s \leq \varepsilon_1 \leq \cdots \leq \varepsilon_k$ is of filtration of 2-Mapper complexes

$$\mathcal{N}^2\big(f^*(\mathcal{U}_s)\big) \to \mathcal{N}^2\left(f^*\left(\mathcal{U}_{\varepsilon_1}\right)\right) \to \cdots \to \mathcal{N}^2\left(f^*\left(\mathcal{U}_{\varepsilon_k}\right)\right).$$

Tower of Cubical Covers

Definition. Let $X \subset \mathbb{R}^n$ be a compact topological space with bounding box B and cubical cover $\mathcal{U}_s$ constructed with k-intervals and overlap fraction g so that each cover set is of the form

$$U_{\alpha,s} = \prod_{i=1}^{n} \left[c_{\alpha,i} - \frac{l_i}{2}, c_{\alpha,i} + \frac{l_i}{2} \right]$$

Where $c_{\alpha,i} = m_1 + (\alpha_i - 1)(1 - g)l_i + \frac{1}{2} l_i$ and $l_i = \frac{M_i - m_i}{k - (k-1)g}$.

Tower of Cubical Covers

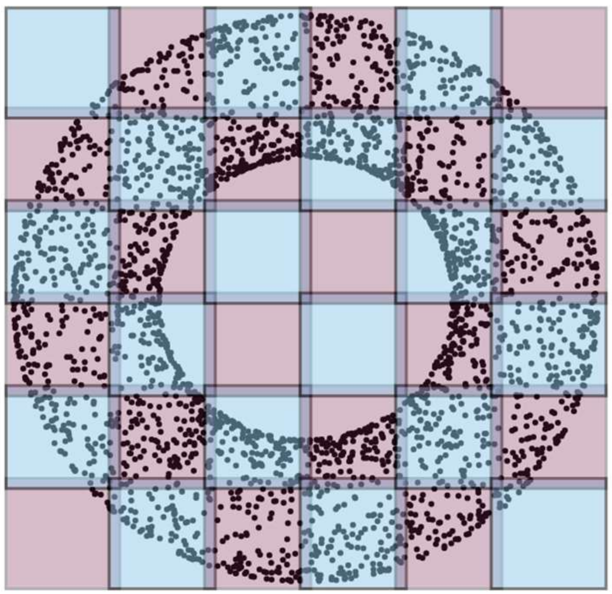
Definition. The *tower of cubical covers* $\mathfrak{U}$ on X is a tower $\mathfrak{U} = \{\mathcal{U}_\varepsilon\}_{\varepsilon \geq s}$ with resolution $\mathrm{res}(\mathfrak{U}) = s = \|(l_1, \ldots, l_n)\|_2$.
For $\varepsilon \geq s$, each cover $\mathcal{U}_\varepsilon = \left\{U_{\alpha,\varepsilon}\right\}_{\alpha \in A}$ such that for each $\alpha \in A$,

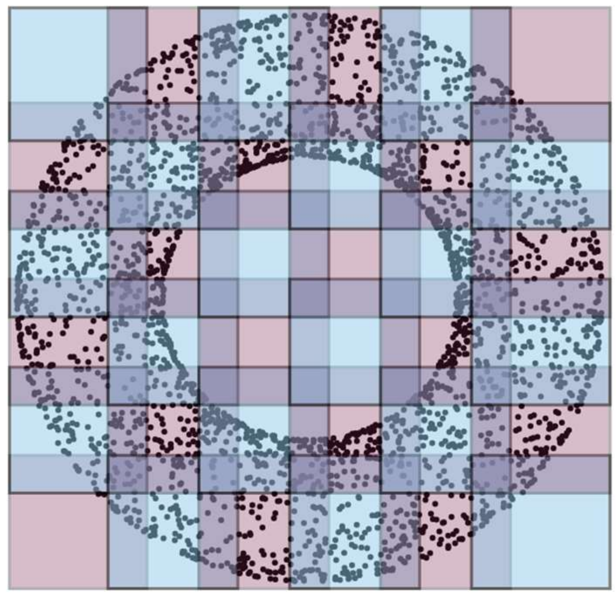
$$U_{\alpha,\varepsilon} = \prod_{i=1}^{n} \left[c_{\alpha,i} - \frac{1}{2}(l_i - \varepsilon'), c_{\alpha,i} + \frac{1}{2}(l_i + \varepsilon')\right],$$

For some $\varepsilon' \geq 0$ so that $\mathrm{diam}\left(U_{\alpha,\varepsilon}\right) = \varepsilon$.

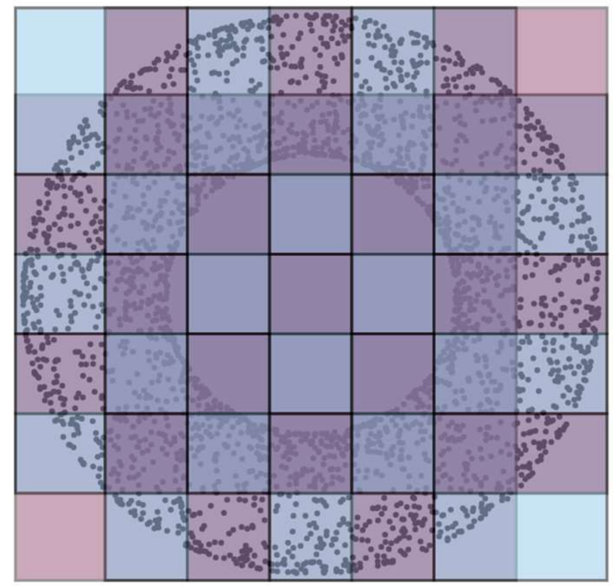
This is a filtration over the parameter g.

Tower of Cubical Covers constructed with $0.1 \leq g \leq 0.5$ and $k = 6$.





$g = 0.1$



$g = 0.3$

$g = 0.5$

Tower of Lattice Covers

Definition. For $X \subset \mathbb{R}^n$ with bounding box B, let $\mathcal{U}_s$ be a cubical lattice cover constructed with k-intervals and overlap fraction g.

The *tower of cubical lattice covers* $\mathfrak{U} = \{\mathcal{U}_\varepsilon\}_{\varepsilon \geq s}$ is a tower of covers with resolution $\mathrm{res}(\mathfrak{U}) = s = \frac{c\sqrt{n}}{1-g}$.
For each $\varepsilon \geq s$ we define each cover $\mathcal{U}_\varepsilon = \left\{U_{\alpha,\varepsilon}\right\}_{\alpha \in A}$ so that

$$U_{\alpha,\varepsilon} = \prod_{i=1}^{n} \left[c\xi_{\alpha,i} - \frac{1}{2}\left(\frac{c}{1-g} - \varepsilon'\right), c\xi_{\alpha,i} + \frac{1}{2}\left(\frac{c}{1-g} + \varepsilon'\right)\right],$$

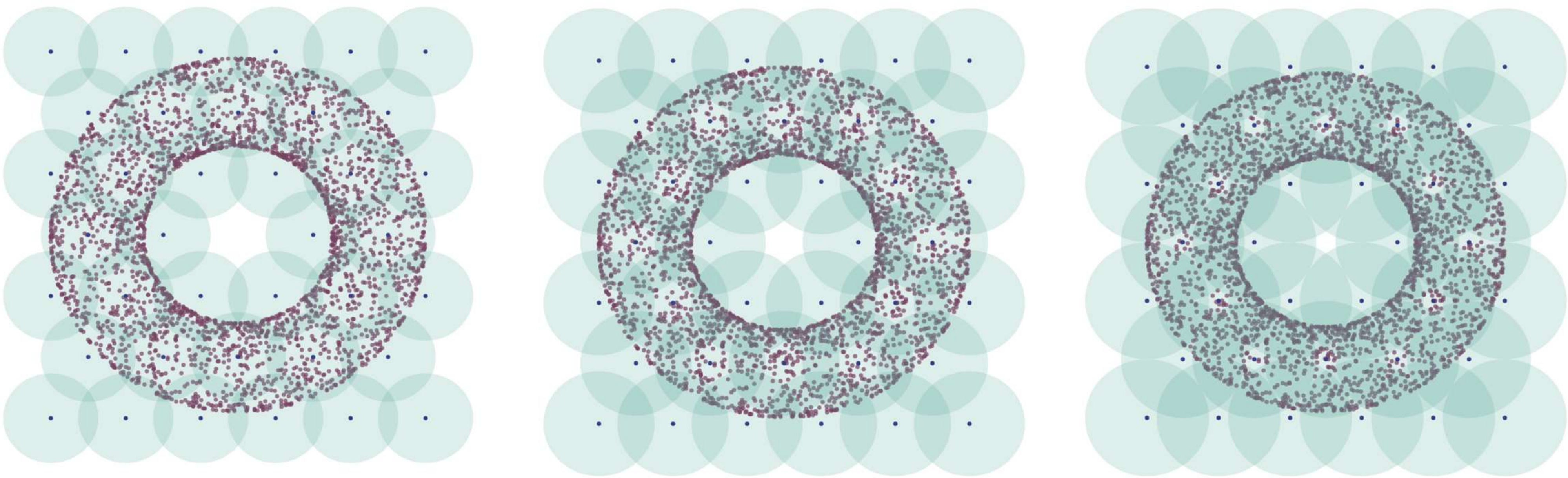
For some $\varepsilon' \geq 0$ so that $\mathrm{diam}\left(U_{\alpha,\varepsilon}\right) = \varepsilon$.

Tower of A_2^*-Lattice Covers

Definition. For $X \subset \mathbb{R}^n$ with bounding box B, let $\mathcal{L}_s$ be an A_2^*-lattice cover constructed with k-intervals and overlap fraction g.

The *tower of* A_2^**-lattice covers* $\mathfrak{L} = \{\mathcal{L}_\varepsilon\}_{\varepsilon \geq s}$ is a tower of covers with resolution $\mathrm{res}(\mathfrak{L}) = s = \frac{2c(1+g)}{\sqrt{3}}$.
For each $\varepsilon \geq s$ we define each cover $\mathcal{L}_\varepsilon = \left\{B\left(c\xi_\alpha M_{A_2^*}, \frac{1}{2}\varepsilon\right)\right\}_{\alpha \in A}$ with $\varepsilon = s + \varepsilon'$ for some $\varepsilon' \geq 0$.

Tower of A_2^*-Lattice Covers constructed with $0.1 \leq g \leq 0.5$ and $k = 6$.



$g = 0.1$ $g = 0.3$ $g = 0.5$

Stability for Covers

Theorems (F.). A tower of cubical (lattice) covers with resolution s constructed with k-intervals and overlap fraction g is $(3, s)$-good for $k \geq \sqrt{n}$.

Theorem (F.). A tower of A_2^*-lattice covers with resolution s, constructed with k-intervals and overlap fraction g is $(3, s)$-good for $k \geq \frac{2}{\sqrt{3}}(1 + g)$.

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- Two Multiscale 2-Mapper objects with these towers are close with respect to the bottleneck distance!

Multiscale Mapper with DBSCAN

<u>**Definition.**</u> *DBSCAN* is a non-parametric density-based clustering algorithm constructed with parameters ϵ and Minpts representing the search radius and minimum number of points per cluster, respectively.

W. Bungula and I. Darcy, Bi-Filtration and Stability of TDA Mapper for Point Cloud Data, arXiv: 2409.17360 [math.AT], 2024.

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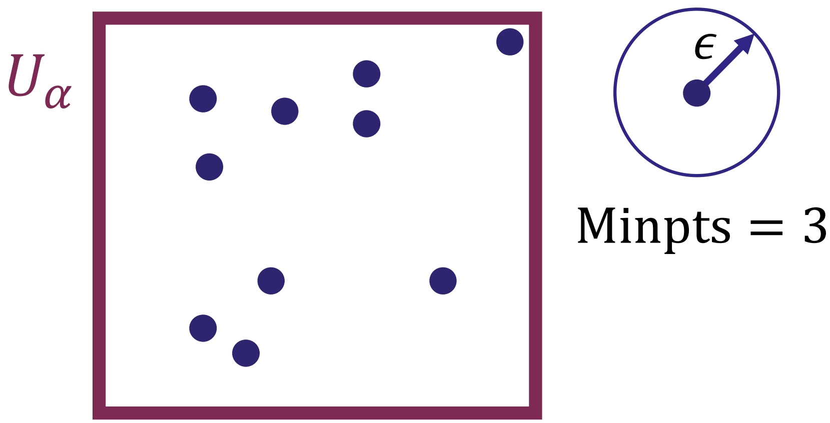

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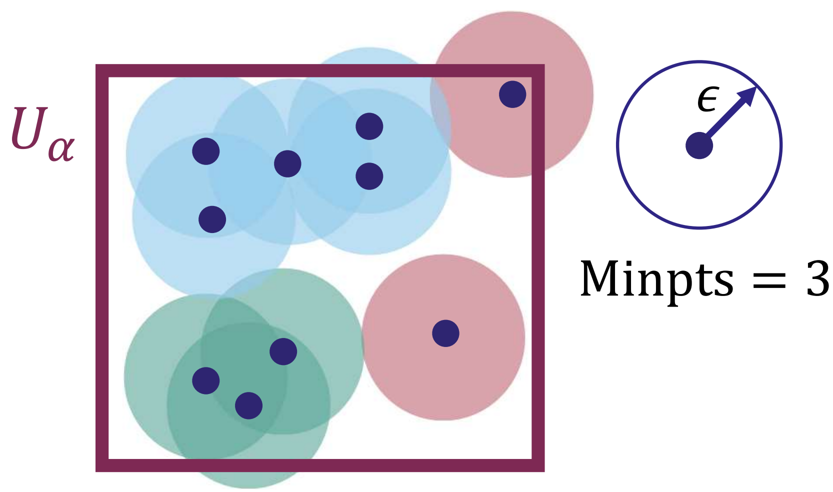

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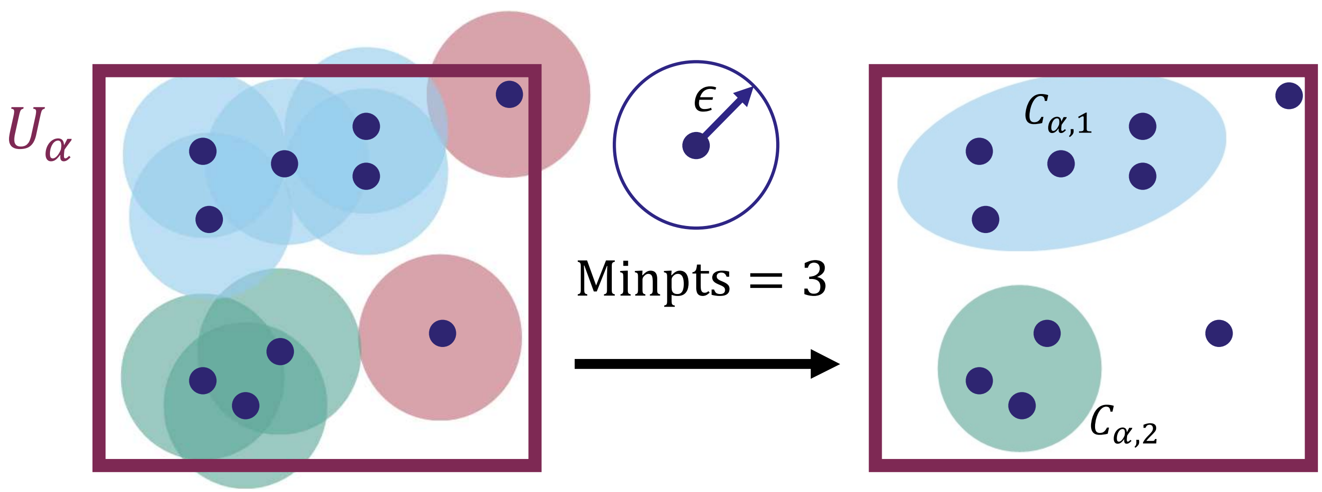


$$U_\alpha = C_{1,\alpha} \sqcup C_{2,\alpha} \sqcup N_\alpha$$

W. Bungula and I. Darcy, Bi-Filtration and Stability of TDA Mapper for Point Cloud Data, arXiv: 2409.17360 [math.AT], 2024.

Multiscale Mapper with DBSCAN

Theorem (Bungula, Darcy). Let X be a data set with tower of covers $\mathcal{U}$ and use DBSCAN to cluster X with fixed parameters ϵ and $\mathrm{Minpts} \leq 2$. Define $\mathcal{C}_\varepsilon = \left\{C_{p,\varepsilon} : p \in U_\varepsilon \; is \; a \; core \; point\right\}$. Then $\mathfrak{C} = \{\mathcal{C}_\varepsilon\}_{\varepsilon \geq s}$ is a tower of covers with maps $c_{\varepsilon,\delta} : \mathcal{C}_\varepsilon \to \mathcal{C}_\delta$ for all $\mathcal{U}_\varepsilon \subseteq \mathcal{U}_\delta$ which are closed under composition.

W. Bungula and I. Darcy, Bi-Filtration and Stability of TDA Mapper for Point Cloud Data, arXiv: 2409.17360 [math.AT], 2024.

Multiscale Mapper with DBSCAN

<u>**Theorem (Bungula, Darcy).**</u> Let X be a data set and assume DBSCAN is used to cluster X with fixed hyperparameters ϵ and $\mathrm{Minpts} \leq 2$. Then for all $\mathcal{U}_\varepsilon \subseteq \mathcal{U}_\delta$,

- There is a filtration of simplicial complexes $\phi_{\varepsilon,\delta} : \mathcal{N}(\mathcal{C}_\varepsilon) \to \mathcal{N}(\mathcal{C}_\delta)$.
- There is a filtration of homology groups $f_{\varepsilon,\delta} : H_k\big(\mathcal{N}(\mathcal{C}_\varepsilon)\big) \to H_k\big(\mathcal{N}(\mathcal{C}_\delta)\big)$ for each degree k.

W. Bungula and I. Darcy, Bi-Filtration and Stability of TDA Mapper for Point Cloud Data, arXiv: 2409.17360 [math.AT], 2024.

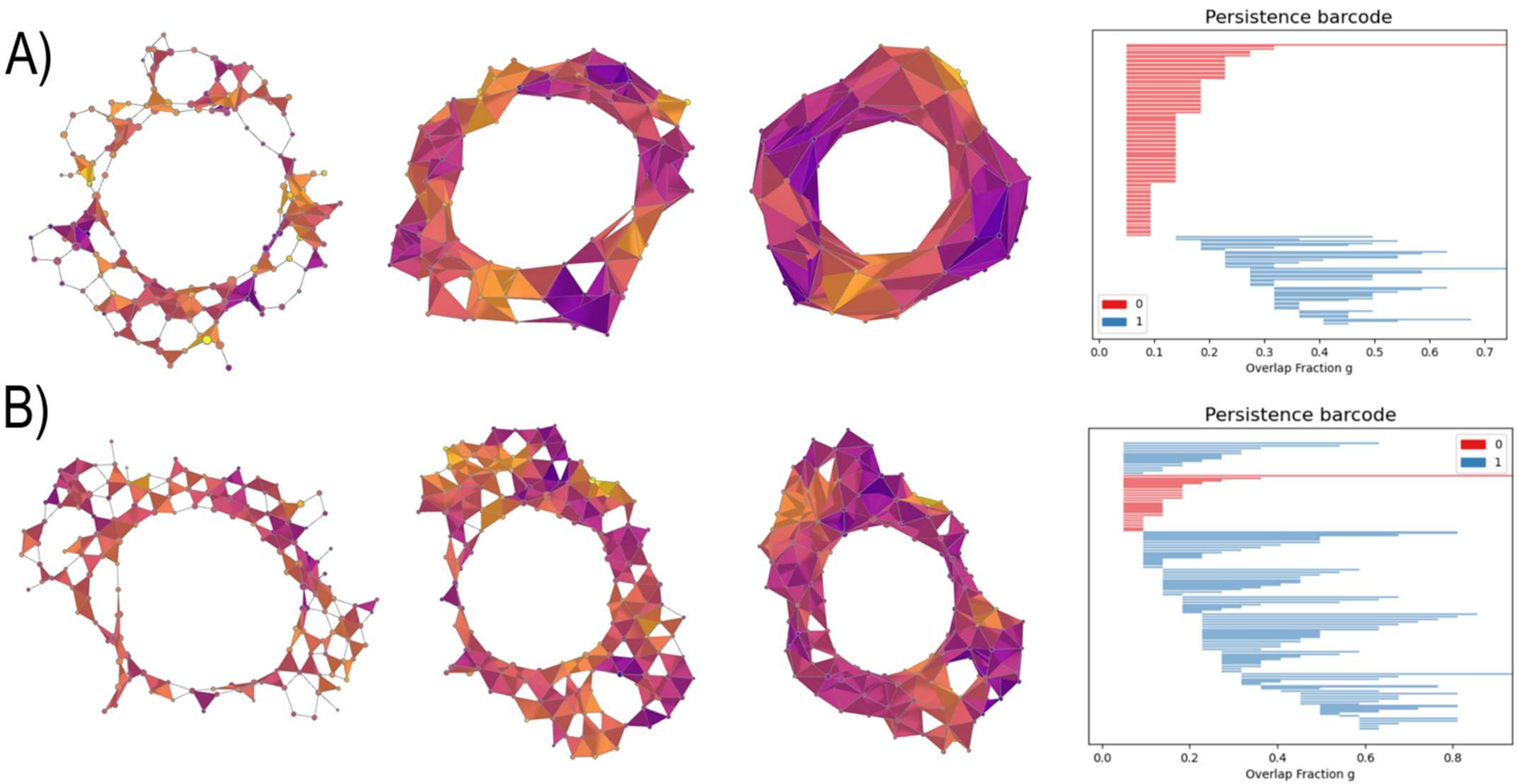
Stability for Covers with DBSCAN

Theorem (F.). Let $\mathfrak{U}$ be a tower of cubical, cubical lattice, or A_2^*-lattice covers with resolution s over a data set X. Define $\mathfrak{C} = \{\mathcal{C}_\varepsilon\}_{\varepsilon \geq s'}$ to be the cluster cover subordinate to $\mathfrak{U}$ using DBSCAN with fixed parameters ϵ and $Minpts \leq 2$. Then $\mathfrak{C}$ is a $(4, s')$-good cover where $s' \leq s$.

Stability for Covers with DBSCAN

Theorem (F.). Let $X \subset \mathbb{R}^m$ be a data set, $f : X \to \mathbb{R}^n$ be a continuous lens, and $\mathfrak{U}, \mathfrak{W}$ be two towers of covers (cubical, cubical lattice, or A_2^*-lattice). Cluster X using DBSCAN with fixed parameters ϵ and $\mathrm{Minpts} \leq 2$. Then for each $k \geq 0$,

$$d_B\big(D_k(\mathrm{MM}_2(f, \mathfrak{C}_{\mathfrak{U}})), D_k(\mathrm{MM}_2(f, \mathfrak{C}_{\mathfrak{W}}))\big) \leq 4$$



The bottleneck distances for these two barcodes is 0.0894 in degree zero and 0.1789 in degree one.

Building an Algorithm

Let $\mathrm{MM}_2(f, \mathfrak{C})$ be the Multiscale 2-Mapper computed over data set X with finite tower of covers $\mathfrak{U}$ and clustered with DBSCAN using parameters ϵ and Minpts.

Building an Algorithm

Let $\mathrm{MM}_2(f, \mathfrak{C})$ be the Multiscale 2-Mapper computed over data set X with finite tower of covers $\mathfrak{U}$ and clustered with DBSCAN using parameters ϵ and Minpts.

$$\mathrm{M}_2(\mathrm{f}, \mathcal{C}_{\varepsilon_0}) \overset{c^*_{0,1}}{\rightarrow} \mathrm{M}_2(\mathrm{f}, \mathcal{C}_{\varepsilon_1}) \overset{c^*_{1,2}}{\rightarrow} \cdots \overset{c^*_{t-1,t}}{\rightarrow} \mathrm{M}_2(f, \mathcal{C}_{\varepsilon_t})$$

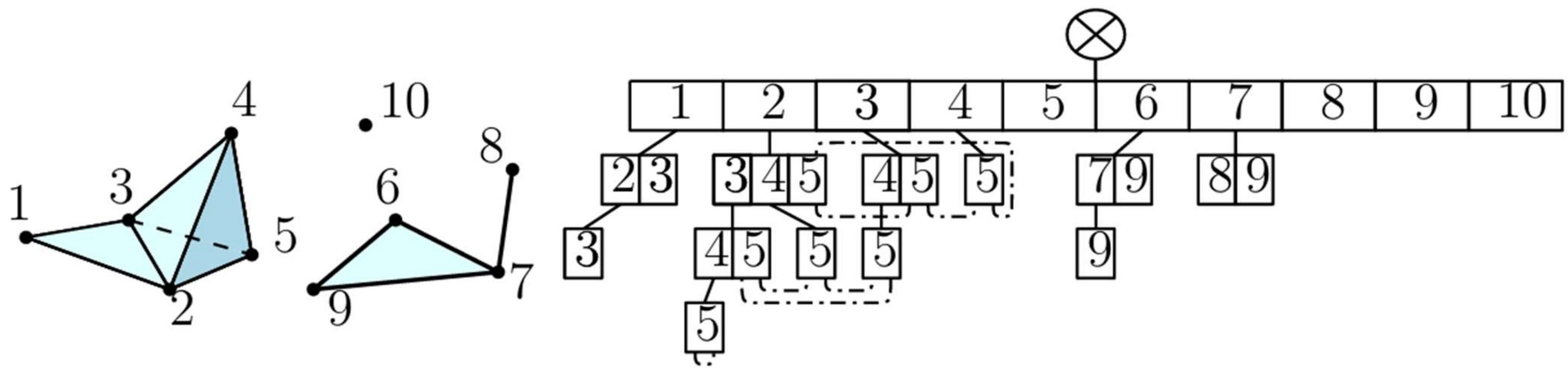
Write $M_i := \mathrm{M}_2(f, \mathcal{C}_{\varepsilon_i})$ where M_i has vertex set V_i.

For each node $n \in V_i$ we write $n = (X_n, U_{\alpha,\varepsilon_i}, C_{p_\alpha,\varepsilon_i})$

Data Cover set Cluster

Building an Algorithm

Simplex trees are used for computing with simplicial complexes and filtrations of simplicial complexes.



Each simplex σ is stored with filtration time t_σ.

J.-D. Boissonnat and C. Maria, The Simplex Tree: An Efficient Data Structure for General Simplicial Complexes, Algorithmica 70 (2014), no. 3, 406–427, doi: 10.1007/s00453-014-9887-3.

Building an Algorithm

Dream Construction. For data set X, lens $f : X \to \mathbb{R}^n$, tower of covers $\mathfrak{U}$, and DBSCAN parameters ϵ and Minpts.

- Create 2-Mapper complexes M_i for each cover in cluster cover $\mathfrak{C}$.
- Start with an empty simplex tree.
- Insert each M_i sequentially.
- Done!

Building an Algorithm

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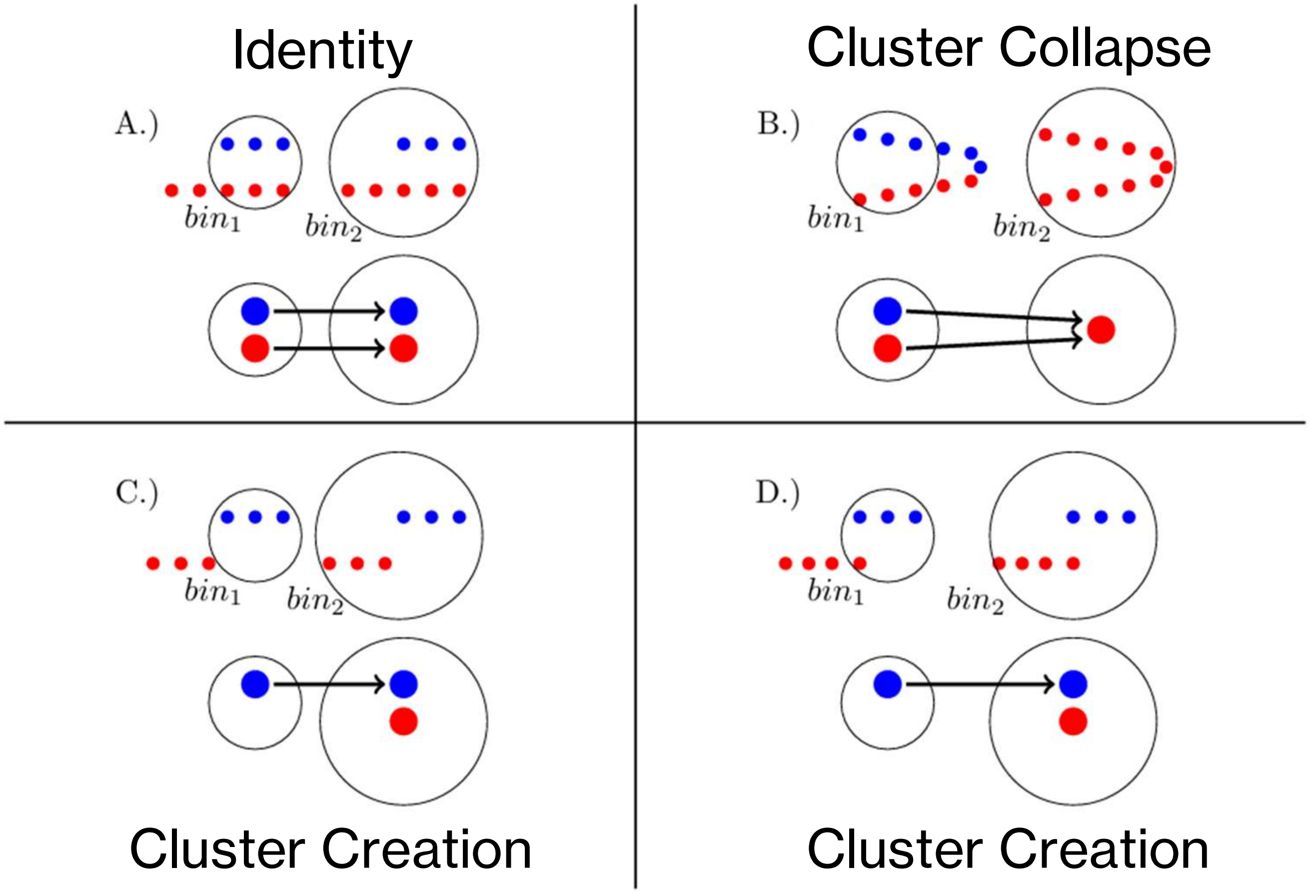
Clustering complicates our algorithm.

Building an Algorithm



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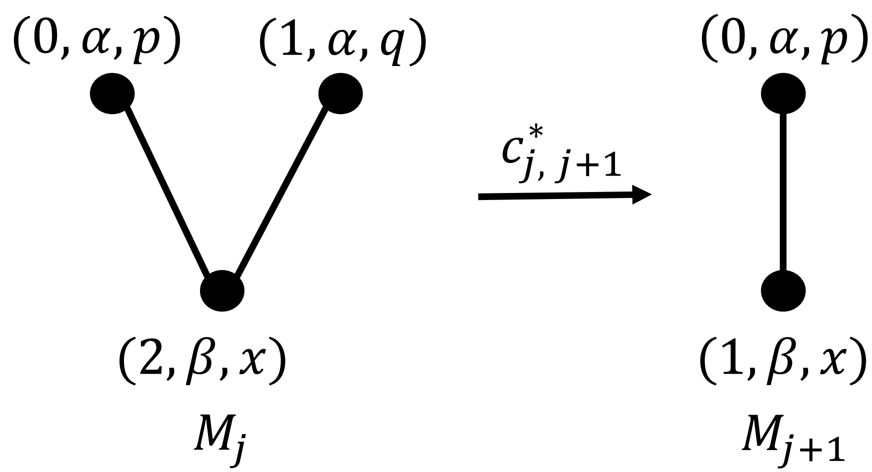
Building an Algorithm

Single Cluster Collapse



Building an Algorithm

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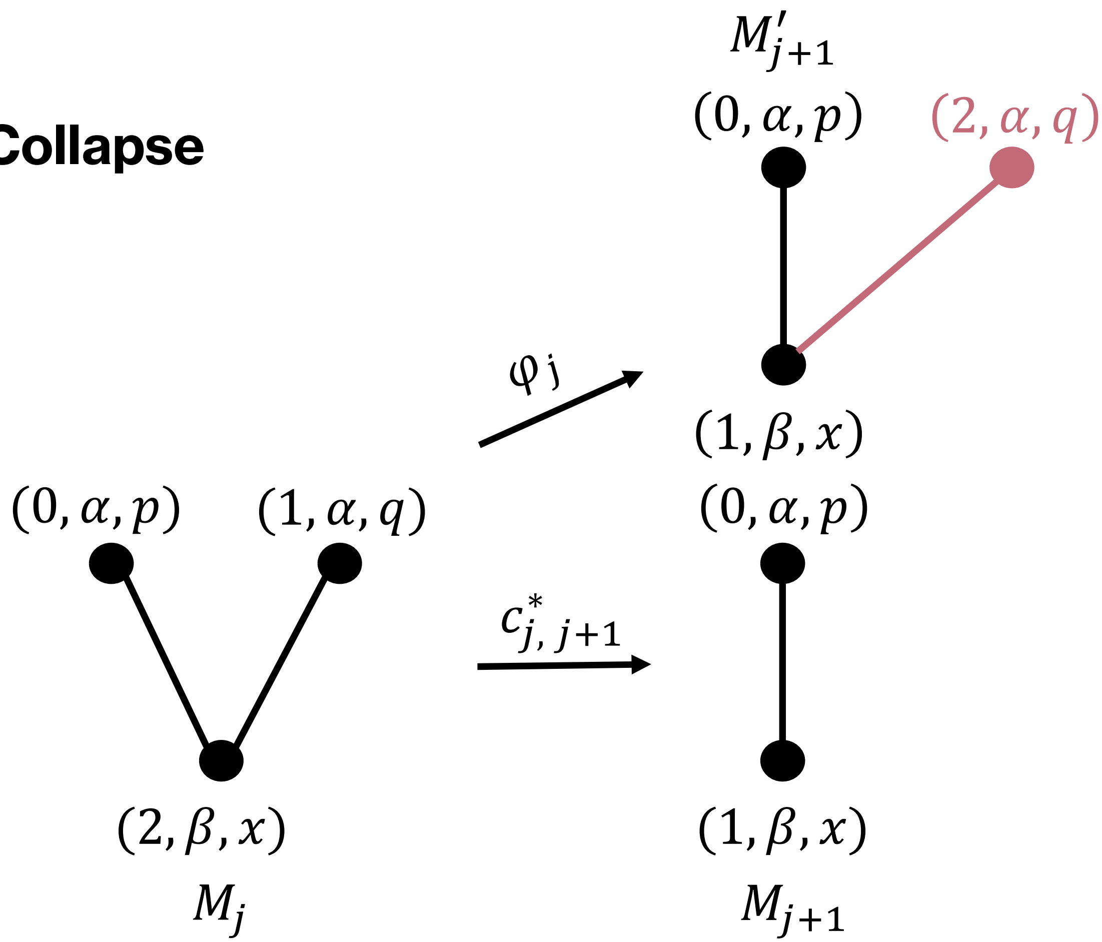

Building an Algorithm

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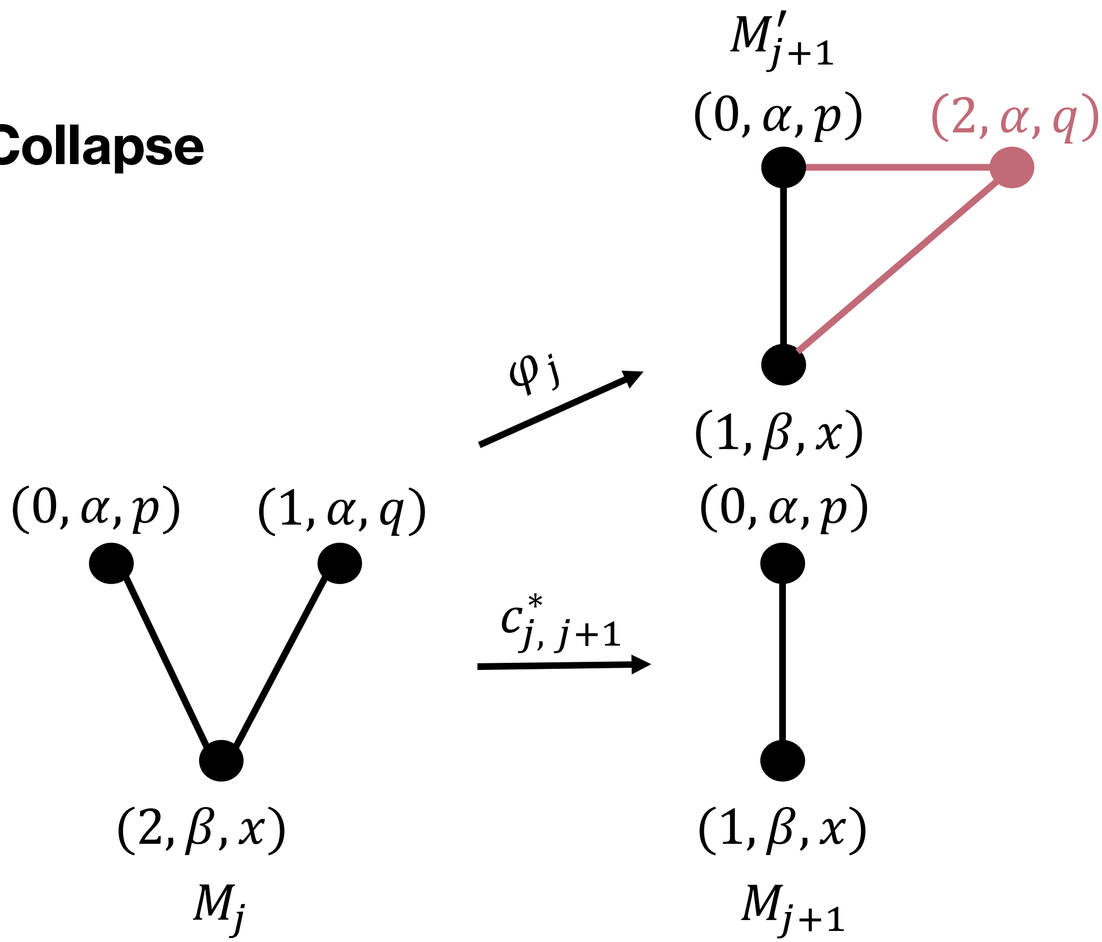

Building an Algorithm

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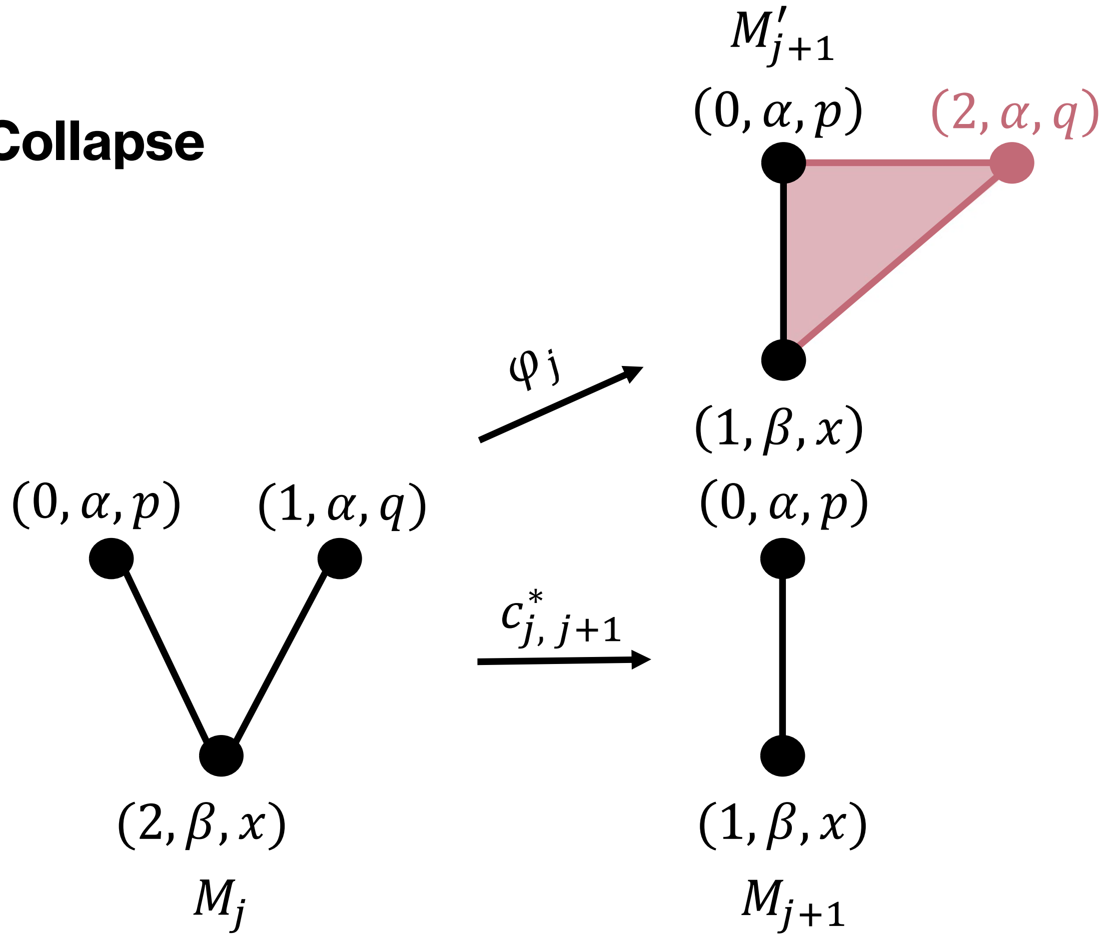
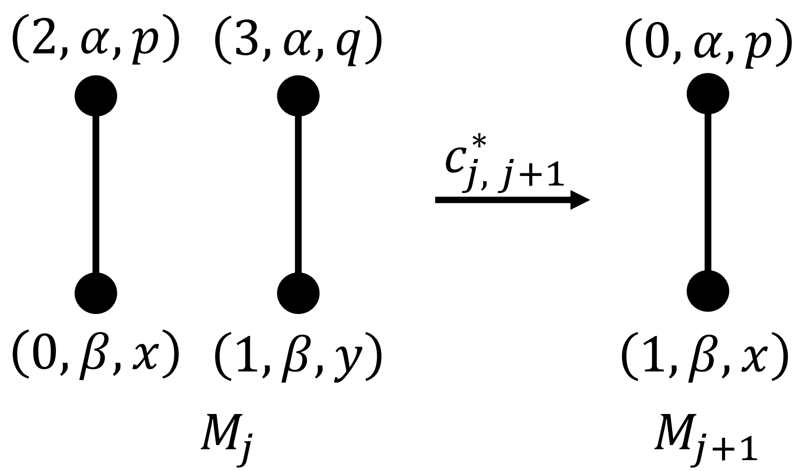


Building an Algorithm

Double Cluster Collapse



Building an Algorithm

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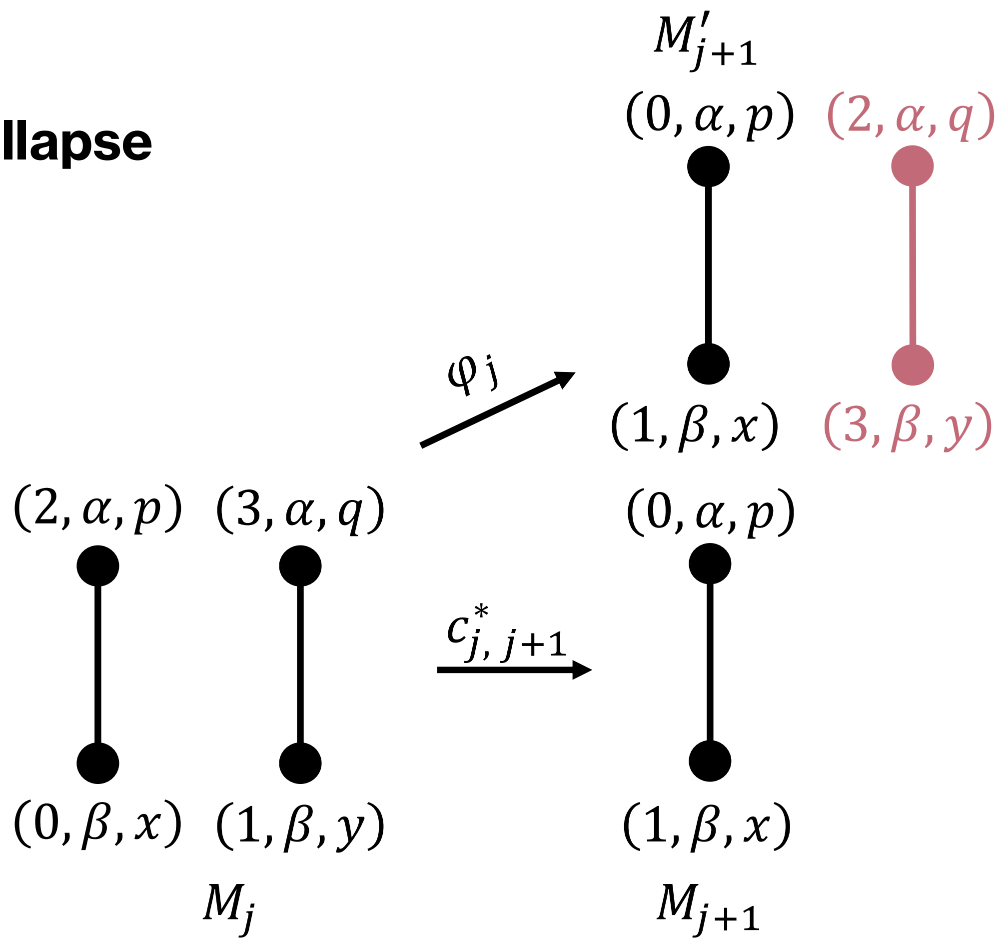
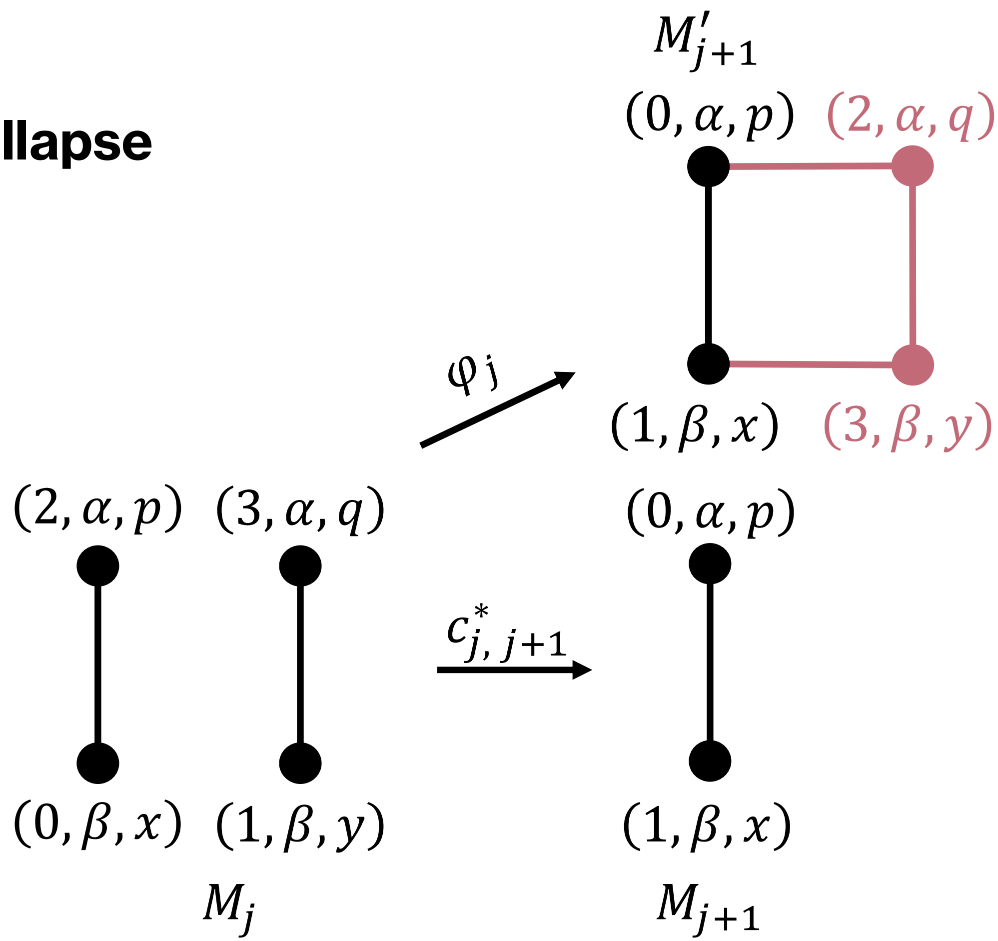


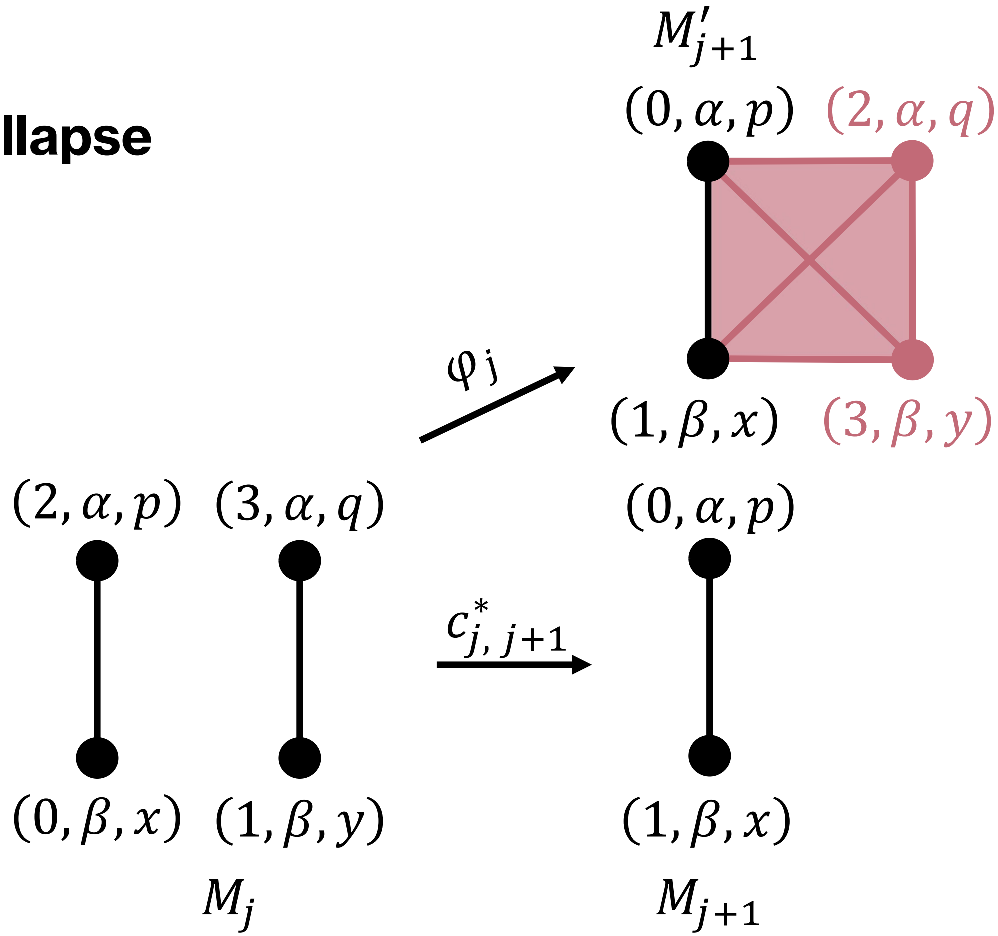
Building an Algorithm

Double Cluster Collapse



Building an Algorithm

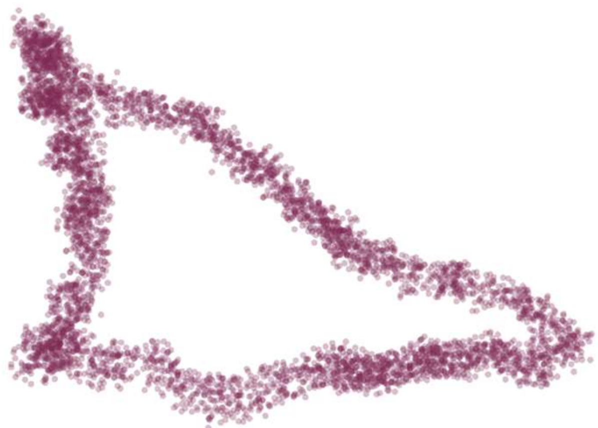
Double Cluster Collapse

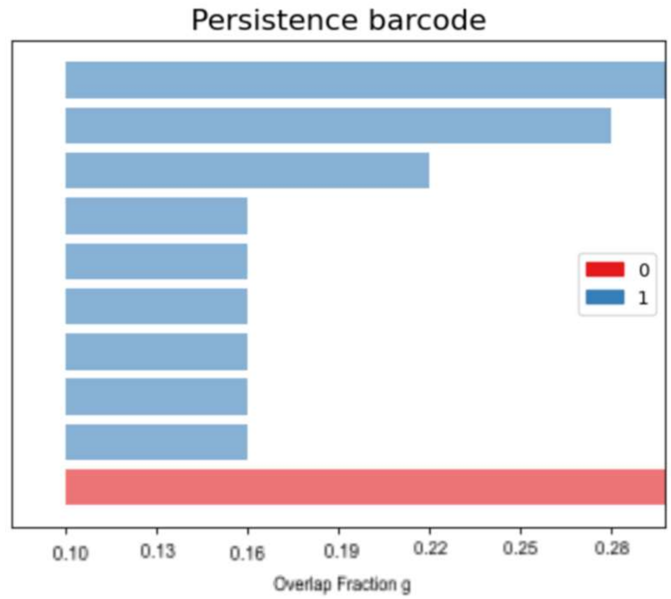


Building an Algorithm

Sketch: For data set X, lens $f : X \to \mathbb{R}^n$, tower of covers $\mathfrak{U}$, and DBSCAN parameters ϵ and Minpts.

- Create 2-Mapper complexes M_i for each cover in cluster cover $\mathfrak{C}$.
- For pairs (M_i, M_{i+1}) create maps $\varphi_i : M_i \to M'_{i+1}$ which deal with single and double collapses of clusters.
- Create Simplex Tree with 2-Mapper complexes $M_0, M'_1, \dots, M'_t$





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0.13
0.16
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0.22
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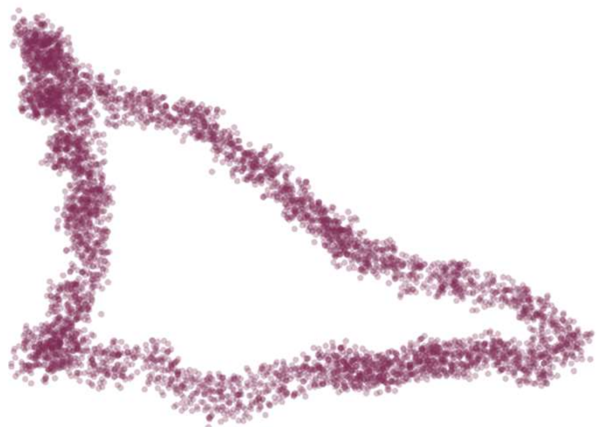
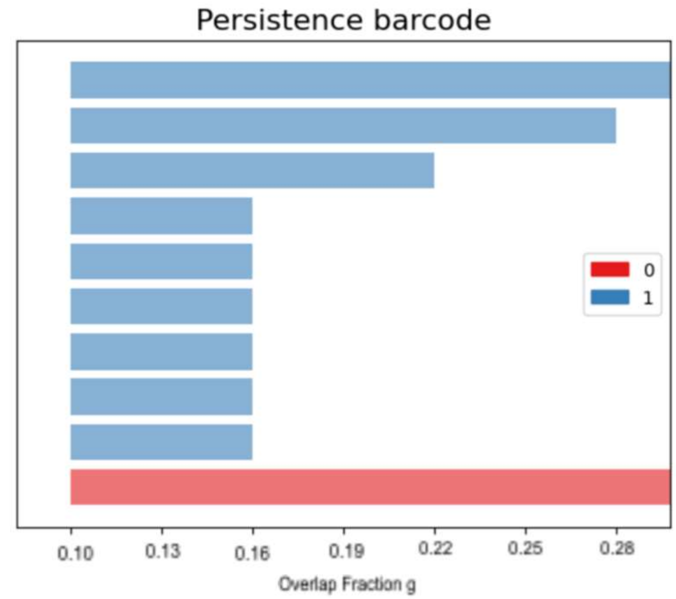
0.28
Overlap Fraction g

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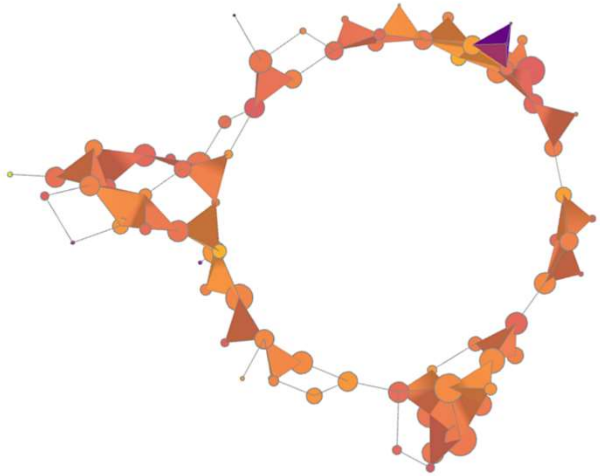
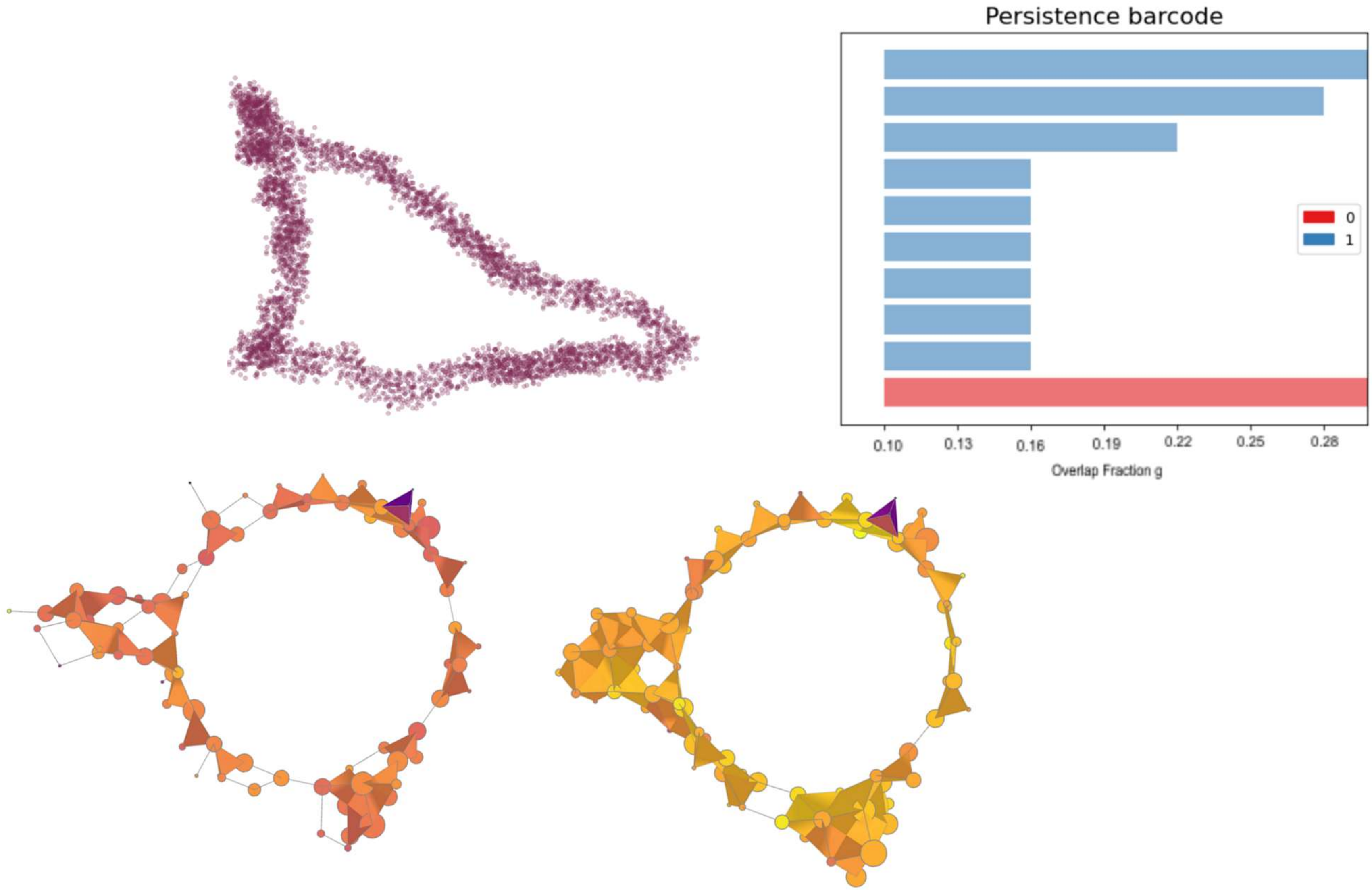
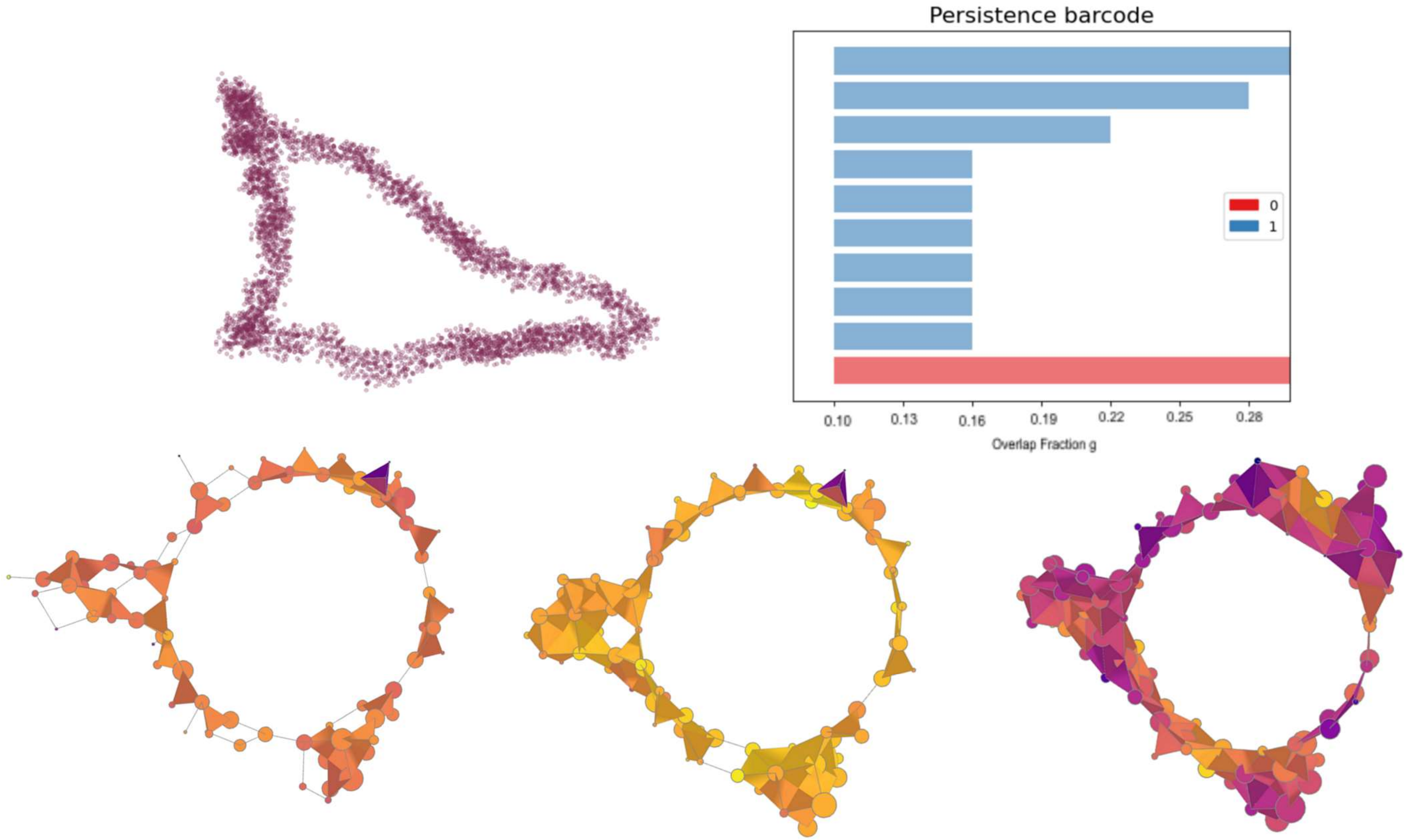


$g = 0.1$



$g = 0.1$ $\qquad$ $g = 0.2$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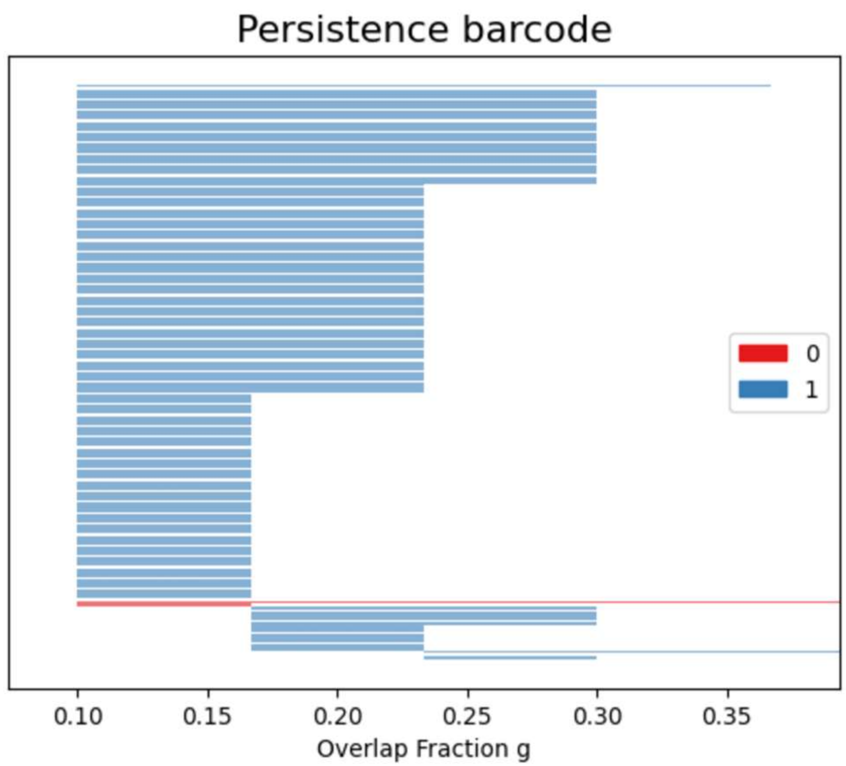
Overlap Fraction g

$g = 0.1$ $g = 0.2$ $g = 0.3$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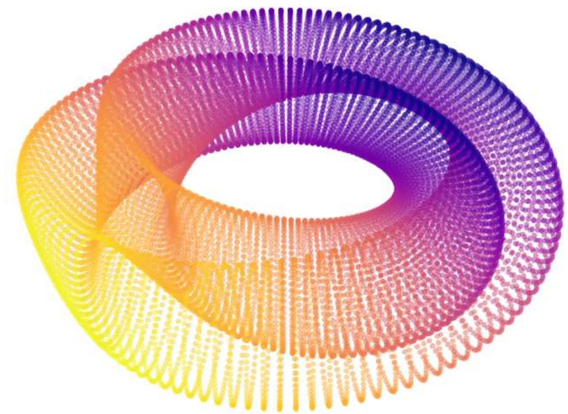
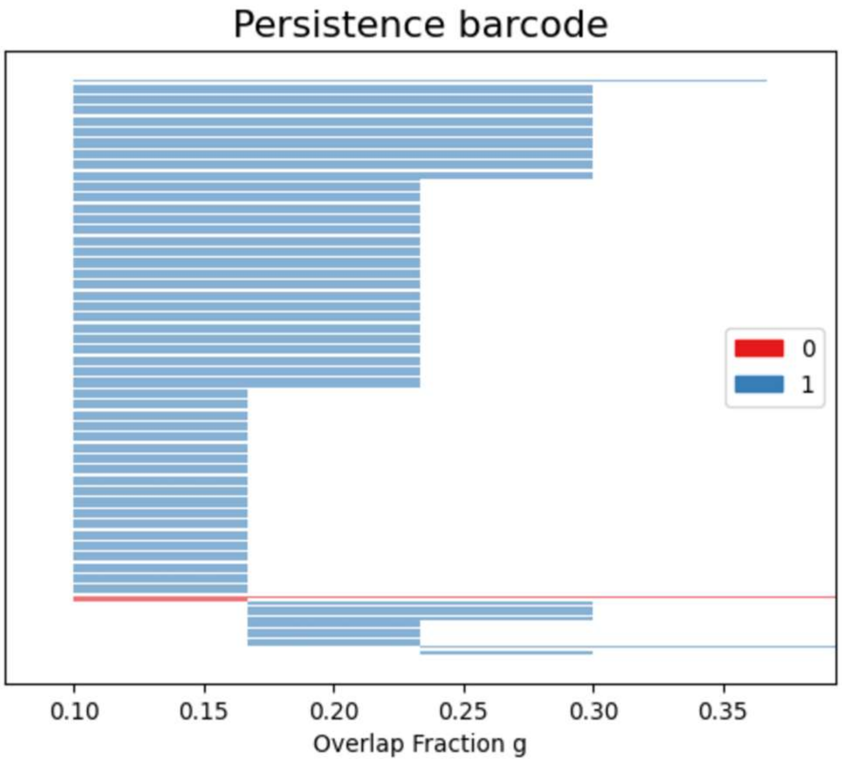
Overlap Fraction g

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Overlap Fraction g

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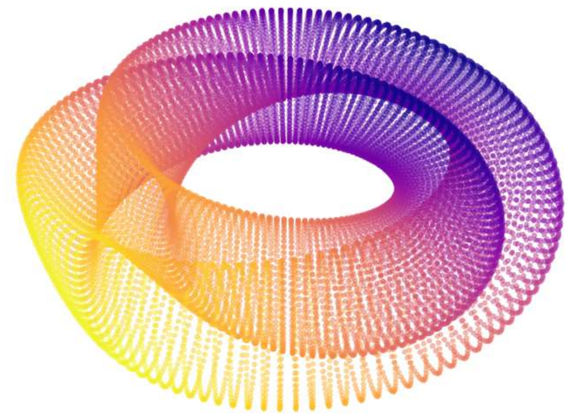

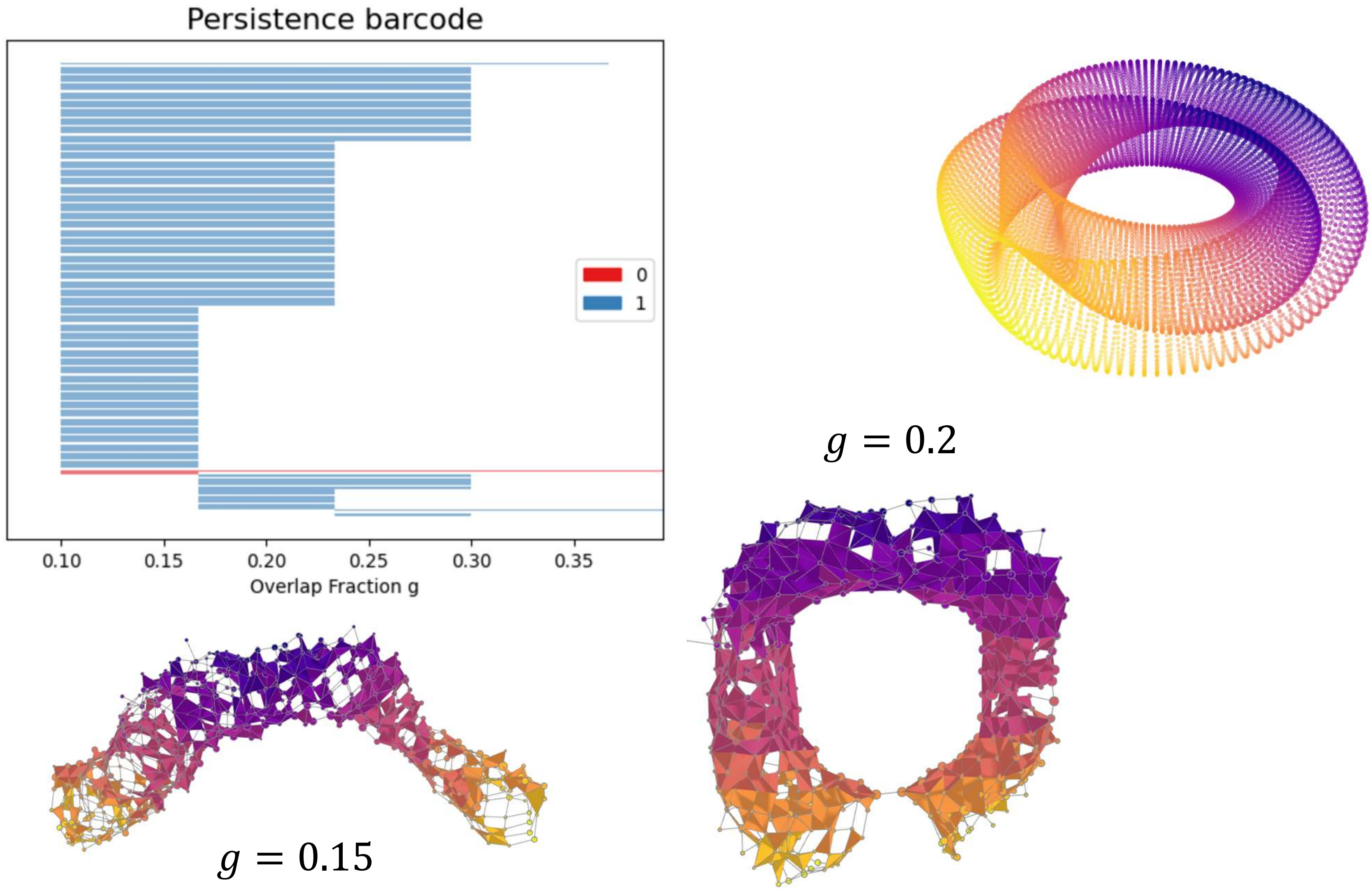
Persistence barcode
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0.20
0.25
0.30
0.35
Overlap Fraction g
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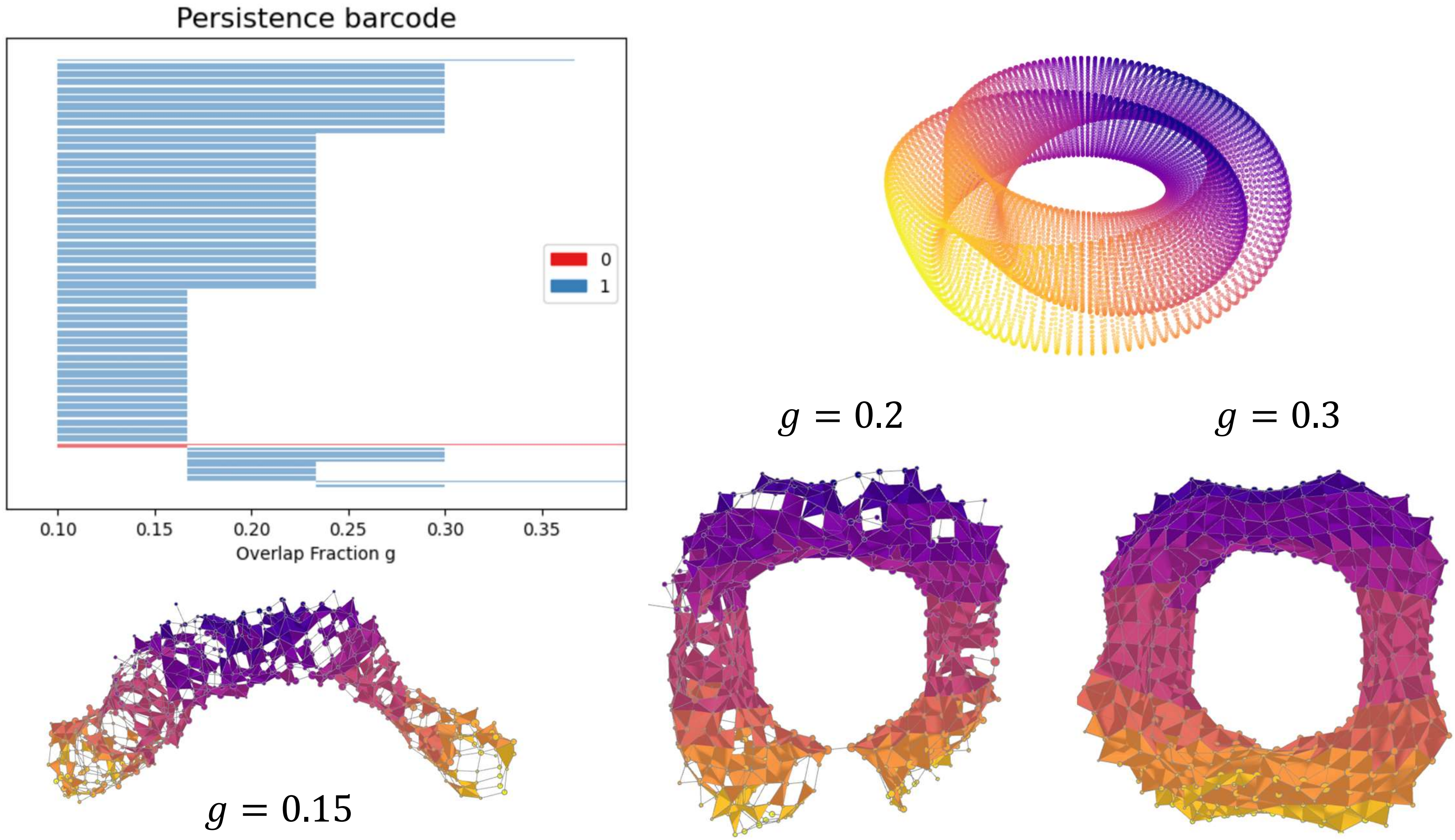
$g = 0.15$



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Overlap Fraction g
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$g = 0.3$
$g = 0.15$

Future Directions

- Estimation for likely trajectories in dynamical systems: *Trajectory Mapper.*
- *Engineering Mapper Complexes:* Computing probabilities for 2-Mapper given certain parameters.
- Visualization of clustering in genetic ancestry for human populations