Exotic Crossed Products and Coaction Functors

S. Kaliszewski
Arizona State University

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Abstract

When a locally compact group G acts on a C^*-algebra A, we have both full and reduced crossed products, each carries a dual coaction of G, and each has its own version of crossed-product duality. Inspired by work of Brown and Guentner on new C^*-completions of group algebras, we have begun to understand what we call "exotic" crossed products — C^*-algebras that lie between the familiar full and reduced crossed products — and more generally, "exotic coactions".

Some of these coactions satisfy a corresponding exotic crossed product duality, intermediate between full and reduced duality, and this leads us to introduce and study "coaction functors" induced by ideals of the Fourier-Stieltjes algebra of G. These functors are also related to the crossed-product functors used recently by Baum, Guentner, and Willett in a new approach to the Baum-Connes conjecture.

This is joint work with Magnus Landstad and John Quigg.

Reduced Crossed Products

Let (B, G, α) be a C^*-dynamical system:

B is a C^*-algebra

G is a locally compact group

α is an action of G on B:

$\alpha\colon G \to Aut(B)$ is a (strongly continuous) homomorphism

The *reduced crossed product* C^*-algebra can be defined as

$$B \rtimes_{\alpha,r} G = \overline{span}\{(\mathrm{id} \otimes \mathcal{M})(\alpha(b))(1 \otimes \lambda)(f) \mid b \in B, f \in C^*(G)\} \subseteq M(B \otimes \mathcal{K}(L^2(G))),$$

where

$\alpha(b)$ is the function $s \mapsto \alpha_{s^{-1}}(b)$ in $C_b(G, B) \subseteq M(B \otimes C_0(G))$

$\mathcal{M}\colon C_0(G) \to \mathcal{B}(L^2(G))$ is pointwise multiplication

$\lambda\colon G \to \mathcal{U}(L^2(G))$ is the left regular representation

The Baum-Connes

Conjecture: For any C^*-dynamical system (B, G, α) with second-countable G, the *reduced assembly map*

$$\mu_{\mathrm{red}}\colon K_*^{top}(G; B) \to K_*(B \rtimes_{\alpha,r} G)$$

is an isomorphism.

"Counterexamples. . . are closely connected to failures of *exactness*."
(*Baum-Guentner-Willett*)

The reduced crossed product is not (in general) exact.

Full Crossed Products

The *full crossed product* $B \rtimes_\alpha G$ is the universal C^*-algebra for *covariant representations* (π, U) of (B, G, α):

C is a C^*-algebra

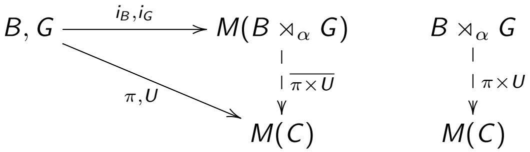
$\pi\colon B \to M(C)$ is a $*$-homomorphism

$U\colon G \to UM(C)$ is a (continuous) homomorphism such that:

$$\pi(\alpha_s(b)) = U_s \pi(b) U_s^*$$

for all $b \in B$ and $s \in G$.

In other words,



where (i_B, i_G) is the *canonical covariant representation*.

Full *vs* Reduced

The *regular covariant representation* (π, U) of (B, G, α) is defined by:

$$\pi = (\mathrm{id} \otimes \mathcal{M}) \circ \alpha \quad U = 1 \otimes \lambda \quad C = B \otimes \mathcal{K}(L^2(G))$$

This gives us the *regular representation* $\Lambda = \pi \times U$:

$$\begin{array}{ccc} B, G & \xrightarrow{\;i_B, i_G\;} & M(B \rtimes_\alpha G) \\ & \searrow^{(\mathrm{id}\otimes M)\circ\alpha,\, 1\otimes\lambda} & \downarrow \overline{\Lambda} \\ & & M(B \otimes \mathcal{K}) \end{array} \qquad \begin{array}{c} B \rtimes_\alpha G \\ \downarrow \Lambda \\ M(B \otimes \mathcal{K}) \end{array}$$

The reduced crossed product can be identified as $\Lambda(B \rtimes_\alpha G)$, and so

$$B \rtimes_\alpha G / \ker \Lambda \cong B \rtimes_{\alpha, r} G$$

A New Approach to Baum-Connes

Conjecture (*BGW*): For any C^*-dynamical system (B, G, α) with second-countable G, the *maximal assembly map*

$$\mu_{\max}\colon K_*^{top}(G; B) \to K_*(B \rtimes_\alpha G)$$

is an isomorphism.

"There are well-known Property (T) obstructions. . . " (*Higson*)

Another Approach...

"The key idea... is to study crossed products that combine the good properties of the maximal and reduced crossed products." (*BGW*)

An *exotic crossed product* is (provisionally) a quotient of $B \times_\alpha G$ by an ideal I with $\{0\} \subsetneq I \subsetneq \ker \Lambda$. So we have quotient maps

$$B \times_\alpha G \to (B \times_\alpha G)/I \to B \times_{\alpha,r} G$$

Conjecture (*BGW*): For any C^*-dynamical system (B, G, α) with second-countable G, there exists an exotic crossed product $(B \times_\alpha G)/I$ such that the *exotic assembly map*

$$\mu_{\text{exotic}} \colon K_*^{top}(G; B) \to K_*((B \rtimes_\alpha G)/I)$$

is an isomorphism.

Exotic Group C^*-Algebras

If $(B, G, \alpha) = (\mathbb{C}, G, \mathrm{id})$, then

$$B \rtimes_\alpha G = C^*(G)$$

is the *full group C^*-algebra*, and

$$B \rtimes_{\alpha,r} G = C^*_r(G).$$

is the *reduced group C^*-algebra*.

Which intermediate quotients

$$C^*(G) \to C^*(G)/I \to C^*_r(G)$$

behave like group C^*-algebras?

Work of Brown and Guentner...

Let Γ be a countable discrete group, and consider a quotient

$$C^*_D(\Gamma) \stackrel{def}{=} C^*(\Gamma)/J_D,$$

where

D is a two-sided (algebraic) ideal of $\ell^\infty(\Gamma)$

$J_D = \bigcap \{ \ker \pi \mid \pi \text{ is a } D\text{-representation of } D \}$

$\pi \colon \Gamma \to \mathcal{U}(\mathcal{H})$ is a D-representation if
the maps $s \mapsto \langle \pi_s(\xi), \eta \rangle$ are in D
for all ξ,η in some dense subspace of $\mathcal{H}$.

For example:

- $C^*(\Gamma) = C^*_{\ell^\infty}(\Gamma)$
- $C^*_{\ell^p}(\Gamma) = C^*_r(\Gamma)$ for $p \in [1, 2]$
- $C^*(\Gamma) \neq C^*_{\ell^p}(\Gamma)$ for all $p \in [1, \infty)$ if Γ is not amenable

Work of Brown and Guentner...

- For the free group $\mathbb{F}_n$ on $n \geq 2$ generators, there exists $p \in (2, \infty)$ such that

$$C^*(\mathbb{F}_n) \neq C^*_{\ell^p}(\mathbb{F}_n) \neq C^*_r(\mathbb{F}_n)$$

(*Brown-Guentner; Willett*)

- For $p < q$ in $(2, \infty)$, we have

$$C^*_{\ell^q}(\mathbb{F}_2) \neq C^*_{\ell^p}(\mathbb{F}_2)$$

(*Higson-Ozawa; Okayasu*)

- For any infinite Coxeter group Γ, there exists $p \in (2, \infty)$ such that

$$C^*_{\ell^p}(\Gamma) \neq C^*_r(\Gamma)$$

(*Bożejko-Januszkiewicz-Spatzier; Brown-Guentner*)

Work of Brown and Guentner...

Observations:

- Γ is amenable if and only if $C^*(\Gamma) = C^*_{c_c}(\Gamma)$
- Γ has the Haagerup property if and only if $C^*(\Gamma) = C^*_{c_0}(\Gamma)$
- Γ has Property (T) if and only if the *only* (Γ-invariant) ideal D such that $C^*(\Gamma) = C^*_D(\Gamma)$ is ℓ^∞.

So for Brown and Guentner, the intermediate quotients

$$C^*(G) \to C^*(G)/J_D \to C^*_r(G)$$

behave like group C^*-algebras?

The Abelian Case

Recall that for G (locally compact) abelian, the Fourier transform $f \mapsto \hat{f}$ is an isomorphism

$$C^*(G) \cong C_0(\widehat{G})$$

where $\widehat{G}$ is the Pontryagin dual of G.

The group structure of $\widehat{G}$ gives $C_0(\widehat{G})$ extra structure: We have a homomorphism

$$\Delta\colon C_0(\widehat{G}) \to C_b(\widehat{G} \times \widehat{G}) \subseteq M(C_0(\widehat{G}) \otimes C_0(\widehat{G}))$$

defined by

$$\Delta(\hat{f})(\chi, \eta) = \hat{f}(\chi\eta).$$

Moreover, associativity in $\widehat{G}$ means that $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$:

$$\begin{aligned}(\Delta \otimes \mathrm{id}) \circ \Delta(\hat{f})(\xi, \eta, \nu) &= \Delta(\hat{f})(\xi\eta, \nu) = \hat{f}((\xi\eta)\nu) \\ &= \hat{f}(\xi(\eta\nu)) = \Delta(\hat{f})(\xi, \eta\nu) = (\mathrm{id} \otimes \Delta) \circ \Delta(\hat{f})(\xi, \eta, \nu).\end{aligned}$$

Comultiplication on Group C^*-Algebras

For general G, the homomorphism $s \mapsto i_G(s) \otimes i_G(s)$ induces a $*$-homomorphism

$$\delta_G \colon C^*(G) \to M(C^*(G) \otimes C^*(G))$$

Moreover, δ_G is *coassociative* in that $(\delta_G \otimes \mathrm{id}) \circ \delta_G = (\mathrm{id} \otimes \delta_G) \circ \delta_G$:

$$\begin{array}{ccc} C^*(G) & \xrightarrow{\delta_G} & M(C^*(G) \otimes C^*(G)) \\ \downarrow{\scriptstyle \delta_G} & & \downarrow{\scriptstyle \delta_G \otimes \mathrm{id}} \\ M(C^*(G) \otimes C^*(G)) & \xrightarrow[\mathrm{id} \otimes \delta_G]{} & M(C^*(G) \otimes C^*(G) \otimes C^*(G)) \end{array}$$

The map δ_G is called the *comultiplication* on $C^*(G)$.

Comultiplication on Group C^*-Algebras

Similarly, the reduced group C^*-algebra carries a comultiplication

$$\delta_G^r \colon C_r^*(G) \to M(C_r^*(G) \otimes C_r^*(G))$$

The regular representation is *compatible* with δ_G and δ_G^r in the sense that

$$\begin{array}{ccc} C^*(G) & \xrightarrow{\delta_G} & M(C^*(G) \otimes C^*(G)) \\ \Lambda \downarrow & & \downarrow \Lambda\otimes\Lambda \\ C_r^*(G) & \xrightarrow[\delta_G^r]{} & M(C_r^*(G) \otimes C_r^*(G)) \end{array}$$

So, which intermediate quotients

$$C^*(G) \to C^*(G)/I \to C_r^*(G)$$

carry a comultiplication compatible with the quotient maps?

Exotic Group C^*-Algebras

Let G be a locally compact group, and consider a quotient

$$C^*_E(G) \overset{def}{=} C^*(G)/{}^{\perp}E,$$

where

$B(G) = C^*(G)^* \subseteq C_b(G)$ is the Fourier-Stieltjes algebra of G

E is a weak*-closed G-invariant sub*space* of $B(G)$

${}^{\perp}E = \{f \in C^*(G) \mid \langle f, \chi \rangle = 0 \text{ for all } \chi \in E\}$

For example:

$C^*_{B(G)}(G) = C^*(G)$

$C^*_{B_r(G)}(G) = C^*_r(G)$

$C^*_E(G) = C^*_r(G)$ for $E = \overline{span\{L^p(G) \cap P(G)\}}^{\text{wk}^*}$ and $p \in [1, 2]$

Here we view $B_r(G) = C^*_r(G)^* \subseteq B(G)$, and $P(G)$ denotes the positive elements of $C^*(G)^*$.

Exotic Group C^*-Algebras

If E is a weak*-closed G-invariant sub*algebra* of $B(G)$, then $C^*_E(G)$ has a *comultiplication* δ_G^E, and the quotient map is compatible with δ_G and δ_G^E:

$$\begin{array}{ccc} C^*(G) & \xrightarrow{\ \delta_G\ } & M(C^*(G) \otimes C^*(G)) \\ \big\downarrow{\scriptstyle \varrho} & & \big\downarrow{\scriptstyle \varrho\otimes\varrho} \\ C^*_E(G) & \xrightarrow[\ \delta_G^E\ ]{} & M(C^*_E(G) \otimes C^*_E(G)) \end{array}$$

If in addition $C^*_E(G)$ is a *proper* intermediate quotient

$$C^*(G) \xrightarrow{\ \varrho\ } C^*_E(G) \xrightarrow{\ \varsigma\ } C^*_r(G),$$

we call it (after *Kayed-Sołtan*) an *exotic group C^*-algebra*.

Exotic Group C^*-Algebras

If E is a weak*-closed G-invariant *ideal* of $B(G)$, then there is also a *coaction* δ^E of G on $C_E^*(G)$ such that the quotient map is $\delta_G - \delta^E$ equivariant:

$$
\begin{array}{ccc}
C^*(G) & \xrightarrow{\delta_G} & M(C^*(G) \otimes C^*(G)) \\
\downarrow \varrho & & \downarrow \varrho\otimes\mathrm{id} \\
C_E^*(G) & \xrightarrow[\delta^E]{} & M(C_E^*(G) \otimes C^*(G))
\end{array}
$$

More suggestively:

$$
\begin{array}{ccc}
\mathbb{C} \rtimes_{\mathrm{id}} G & \xrightarrow{(i_{\mathbb{C}}\otimes 1)\times(i_G\otimes u)} & M((\mathbb{C} \rtimes_{\mathrm{id}} G) \otimes C^*(G)) \\
\downarrow \varrho & & \downarrow \mathrm{id}\otimes\varrho \\
C_E^*(G) & \xrightarrow[\Sigma\circ\delta^E]{} & M((\mathbb{C} \rtimes_{\mathrm{id}} G) \otimes C_E^*(G))
\end{array}
$$

Mundane Crossed Products

Let (B, G, α) be a C^*-dynamical system.

The full crossed product $B \rtimes_\alpha G$ carries a *dual coaction* $\widehat{\alpha}$ of G, and the reduced crossed product $B \rtimes_{\alpha,r} G$ carries a *dual coaction* $\widehat{\alpha}^r$ of G.

Moreover, the regular representation is $\widehat{\alpha} - \widehat{\alpha}^r$ equivariant:

$$\begin{array}{ccc} B \rtimes_\alpha G & \xrightarrow{\widehat{\alpha}} & M((B \rtimes_\alpha G) \otimes C^*(G)) \\ \Lambda \downarrow & & \downarrow \Lambda\otimes\mathrm{id} \\ B \rtimes_{\alpha,r} G & \xrightarrow[\widehat{\alpha}^r]{} & M((B \rtimes_{\alpha,r} G) \otimes C^*(G)) \end{array}$$

So, which intermediate quotients

$$B \rtimes_\alpha G \to (B \rtimes_\alpha G)/I \to B \rtimes_{\alpha,r} G$$

carry a coaction of G compatible with $\widehat{\alpha}$ and $\widehat{\alpha}^r$?

Exotic Crossed Products

Let (B, G, α) be a C^*-dynamical system, and consider a quotient

$$B \rtimes_{\alpha,E} G \overset{def}{=} (B \rtimes_\alpha G)/I,$$

where

- E is a (nonzero) weak*-closed G-invariant *ideal* of $B(G)$
- $\varrho\colon C^*(G) \to C^*_E(G)$ is the quotient map
- $\widehat{\alpha} = (i_B \otimes 1) \times (i_G \otimes u)$ is the dual coaction of G on $B \rtimes_\alpha G$
- I is the kernel of the map $(\mathrm{id} \otimes \varrho) \circ \widehat{\alpha}$:

$$B \rtimes_\alpha G \xrightarrow{\widehat{\alpha}} M((B \rtimes_\alpha G) \otimes C^*(G)) \xrightarrow{\mathrm{id}\otimes\varrho} M((B \rtimes_\alpha G) \otimes C^*_E(G))$$

For example:

- $B \rtimes_{\alpha,B(G)} G = B \rtimes_\alpha G$
- $B \rtimes_{\alpha,B_r(G)} G = B \rtimes_{\alpha,r} G$

Exotic Crossed Products

Then $B \rtimes_{\alpha,E} G$ carries a *dual coaction* $\widehat{\alpha}_E$ of G, and the quotient map is $\widehat{\alpha} - \widehat{\alpha}_E$ equivariant:

$$\begin{array}{ccc} B \rtimes_\alpha G & \xrightarrow{\widehat{\alpha}} & M((B \rtimes_\alpha G) \otimes C^*(G)) \\ \downarrow{\scriptstyle \mathcal{Q}} & & \downarrow{\scriptstyle \mathcal{Q}\otimes \mathrm{id}} \\ B \rtimes_{\alpha,E} G & \xrightarrow[\widehat{\alpha}_E]{} & M((B \rtimes_{\alpha,E} G) \otimes C^*(G)) \end{array}$$

If $B \rtimes_{\alpha,E} G$ is a *proper* intermediate quotient

$$B \rtimes_\alpha G \xrightarrow{\mathcal{Q}} B \rtimes_{\alpha,E} G \xrightarrow{\mathcal{R}} B \rtimes_{\alpha,r} G,$$

we call it an *exotic crossed-product*.

But the construction shows that exotic crossed products are really about *exotic coactions*.

Mundane Coactions

Let (A, G, δ) be a (*full*) C^**-coaction*:

$\delta\colon A \to M(A \otimes C^*(G))$ is an injective

nondegenerate $*$-homomorphism

such that...

and δ satisfies the *coaction identity*:

$$\begin{CD}
A @>{\delta}>> M(A \otimes C^*(G)) \\
@V{\delta}VV @VV{\delta\otimes\mathrm{id}}V \\
M(A \otimes C^*(G)) @>>{\mathrm{id}\otimes\delta_G}> M(A \otimes C^*(G) \otimes C^*(G))
\end{CD}$$

The *coaction crossed product* $A \rtimes_\delta G$ is universal for covariant representations of $(A, C_0(G))$, and has a *dual action* $\hat{\delta}$ of G. There is a *canonical surjection*

$$\Phi\colon A \rtimes_\delta G \rtimes_{\hat{\delta}} G \to A \otimes \mathcal{K}(L^2(G)).$$

Mundane Coactions

For example:

If (A, G, α) is an *action* and G is abelian, then for each $a \in A$, the rule

$$s \mapsto \alpha_s(a)$$

defines an element $\widehat{\alpha}(a)$ of $C_b(G, A) \subseteq M(C_0(G) \otimes A)$, giving a coaction

$$\widehat{\alpha}\colon A \to M(A \otimes C^*(\widehat{G})) \cong M(C_0(G) \otimes A)$$

of $\widehat{G}$ on A such that $A \rtimes_{\widehat{\alpha}} \widehat{G} \cong A \rtimes_{\alpha} G$.

In the case $A = \mathbb{C}$, the dual coaction $\widehat{\mathrm{id}}$ of G on $\mathbb{C} \rtimes_{\mathrm{id}} G = C^*(G)$ is precisely the comultiplication δ_G.

Observe that here

$$\mathbb{C} \rtimes_{\mathrm{id}} G \rtimes_{\widehat{\mathrm{id}}} G = C^*(G) \rtimes_{\delta_G} G = C_0(G) \rtimes_{\tau} G \cong \mathcal{K}(L^2(G)).$$

Exotic Coactions

Let (A, G, δ) be a C^*-coaction, and consider the quotient

$$A^E \overset{def}{=} A/\ker(\mathrm{id} \otimes q) \circ \delta$$

where

E is a nonzero G-invariant weak*-closed ideal of $B(G)$

$q\colon C^*(G) \to C^*_E(G)$ is the quotient map

$$A \xrightarrow{\delta} M(A \otimes C^*(G)) \xrightarrow{\mathrm{id}\otimes q} M(A \otimes C^*_E(G))$$

Then:

- A^E carries a coaction δ^E of G
- $(B \rtimes_\alpha G)^E = B \rtimes_{\alpha,E} G$, and $\widehat{\alpha}^E = \widehat{\alpha}_E$
- $E = B_r(G)$ gives the *normalization* (A^n, δ^n)
- $E = B(G)$ gives back (A, δ)

Exotic Crossed Product Duality

Let (A, G, δ) be a C^*-coaction, and let E be a nonzero G-invariant weak*-closed ideal of $B(G)$.

(A, G, δ) satisfies *E-crossed-product duality* if the canonical surjection $\Phi\colon A \rtimes_\delta G \rtimes_{\hat{\delta}} G \to A \otimes \mathcal{K}(L^2(G))$ passes to an isomorphism:

$$A \rtimes_\delta G \rtimes_{\hat{\delta},E} G \cong A \otimes \mathcal{K}(L^2(G))$$

- Some coactions do; some don't.
- In general, δ does if and only if... (technical condition).
- δ satisfies $B(G)$-crossed-product duality if and only if δ is maximal.
- δ satisfies $B_r(G)$-crossed-product duality if and only if δ is normal.

Crossed-Product Functors

A *crossed product* is a functor

$$(B, \alpha) \mapsto B \rtimes_{\alpha,\tau} G$$

from G-C^* to C^{**} together with natural transformations

$$B \rtimes_{\alpha} G \to B \rtimes_{\alpha,\tau} G \to B \rtimes_{\alpha,r} G$$

restricting to the identity map on the dense subalgebra(s) $B \rtimes_{\mathrm{alg}} G$.

Each has a τ*-assembly map*

$$\mu_\tau \colon K_*^{top}(G; B) \to K_*(B \rtimes_\alpha G) \to K_*(B \rtimes_{\alpha,\tau} G).$$

Our predilection is to decompose such a crossed-product functor as a composition

$$(B, \alpha) \mapsto (B \rtimes_\alpha G, \widehat{\alpha}) \mapsto (B \rtimes_{\alpha,\tau} G).$$

$^*C^*$-algebras with $*$-homomorphisms

Crossed Product Functors

Crossed product functors are partially ordered by saying $\sigma \leq \tau$ if the natural transformations factor this way:

$$B \rtimes_{\alpha} G \to B \rtimes_{\alpha,\tau} G \to B \rtimes_{\alpha,\sigma} G \to B \rtimes_{\alpha,r} G$$

A crossed product functor τ is *exact* if the sequence

$$0 \to I \rtimes_{\tau} G \to B \rtimes_{\tau} G \to C \rtimes_{\tau} G \to 0$$

is short exact in C^* whenever $0 \to I \to B \to C \to 0$ is short exact in G-C^*.

τ is *Morita compatible* (roughly speaking) if

$$B \rtimes_{\alpha,\tau} G \overset{M}{\sim} C \rtimes_{\gamma,\tau} G$$

whenever $B \overset{M}{\sim} C$ equivariantly .

Both the full and reduced crossed products are Morita compatible.

Back to the Baum-Connes

Conjecture (BGW) For any G-C^*-algebra A, the $\mathcal{E}$-assembly map

$$\mu_{\mathcal{E}}\colon K_*^{top}(G;A) \to K_*(A \rtimes_{\mathcal{E}} G)$$

is an isomorphism, where $\mathcal{E}$ is the *unique minimal exact and Morita compatible crossed product.*

Theorem (BGW, KLQ) For any second countable locally compact group G, there exists a unique minimal exact and Morita compatible crossed product $\mathcal{E}$.

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