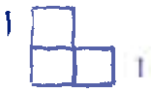
1 Practice Problems (Algebra, Mathematical Induction, Pigeonhole principle)

1. Prove that $\sqrt{2}+\sqrt{3}+\sqrt{5} \in \mathbb{R} \setminus \mathbb{Q}$.
2. Prove that $3^n \geq n^3$ for all positive integers.
3. (hw) Let $n \geq 6$ be an integer. Show that
$$\left(\frac{n}{3}\right)^n < n! < \left(\frac{n}{2}\right)^n.$$
4. Let p be a prime number and n be a positive integer. Prove that $n^p - n$ is divisible by p.
5. Prove that for any $n \geq 1$ a $2^n \times 2^n$ checkerboard with 1×1 corner square removed can be tiled by pieces of the form described below:



6. (hw) Let any numbers $a_1, a_2, \ldots, a_n > 1$. Show the following inequality:
$$\sum_{k=1}^{n} \frac{1}{1+a_k} \geq \frac{n}{1+\sqrt[n]{a_1 a_2 \cdots a_n}}$$
7. For positive integers n let $c(n)$ be a sequence defined by the rules $c(1) = 1$, $c(2n) = c(n)$, and $c(2n+1) = (-1)^n c(n)$ for all $n \geq 1$. Compute
$$\sum_{k=1}^{2013} c(k)c(k+2).$$
8. Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.