Convergence Properties and Upper-Triangular Forms in Finite von Neumann Algebras

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Motivation

Theorem (Schur)

For every matrix $T \in M_n(\mathbb{C})$ there exists a unitary matrix $U \in M_n(\mathbb{C})$ such that $U^{-1}TU$ is an upper-triangular matrix.

Note that the diagonal entries of $U^{-1}TU$ are the eigenvalues of T. Hence the theorem gives a decomposition $T = N + Q$, where N is normal and Q is nilpotent.

Quasinilpotents

An operator T is called **quasinilpotent** if $\|T^n\|^{1/n} \to 0$ as $n \to \infty$.

Equivalently, T is quasinilpotent if $((T^*)^n T^n)^{1/2n}$ converges to 0 in norm as $n \to \infty$.

T is called **sot-quasinilpotent** if $((T^*)^n T^n)^{1/2n}$ converges to 0 in s.o.t. as $n \to \infty$.
Example: the left shift operator on $\ell^2(\mathbb{N})$.

Brown measure

Theorem (Brown 1983)

Let $\mathcal{M}$ be a finite von Neumann algebra with trace τ and let $T \in \mathcal{M}$. There exists a unique probability measure ν_T supported on a compact subset of the spectrum of T such that for any $\lambda \in \mathbb{C}$,

$$\tau(\log(|T-\lambda|)) = \int_{\mathbb{C}} \log(|z-\lambda|) d\nu_T(z).$$

The measure ν_T is called the **Brown measure** of T.
The Brown measure of a normal operator is the trace composed with the projection valued spectral decomposition measure.

Theorem (Haagerup,Schultz 2009)

An operator T in a finite von Neumann algebra is sot-quasinilpotent if and only if $\nu_T(\{0\}) = 1$.

Haagerup-Schultz projections

Theorem (Haagerup, Schultz 2009)

Let $\mathcal{M}$ be a finite von Neumann algebra with trace τ and let $T \in \mathcal{M}$. For any Borel set $B \subset \mathbb{C}$, there exists a unique projection $HS(T, B) \in \mathcal{M}$ such that

1. $\tau(HS(T, B)) = \nu_T(B)$, *where ν_T is the Brown measure of T,*
2. $THS(T, B) = HS(T, B)THS(T, B)$,
3. *if $HS(T, B) \neq 0$, then the Brown measure of $THS(T, B)$, considered as an element of $HS(T, B)\mathcal{M}HS(T, B)$, is concentrated in B,*
4. *if $HS(T, B) \neq 1$, then the Brown measure of $(1 - HS(T, B))T$, considered as an element of $(1 - HS(T, B))\mathcal{M}(1 - HS(T, B))$, is concentrated in $\mathbb{C} \setminus B$.*

In fact, $HS(T,B)$ is T-hyperinvariant, meaning that if $S \in B(\mathcal{H})$ commutes with T, then $HS(T,B)$ is S-invariant.
If $B_1 \subset B_2 \subset \mathbb{C}$, then $HS(T,B_1) \leq HS(T,B_2)$

The projection $HS(T,B)$ is called the Haagerup-Schultz projection of T associated to the set B.
The Haagerup-Schultz projections of a normal operator are the spectral projections.

Haagerup-Schultz projections

The Haagerup-Schultz projection of the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

associated with the set $\{0\}$ is the projection onto the kernel of the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$

which is

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Haagerup-Schultz projections

The Haagerup-Schultz projection of the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

associated with the set $\{1\}$ is the projection onto the kernel of the matrix

$$\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix},$$

which is

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

Theorem (Haagerup,Schultz 2009)

If T is an element of a finite von Neumann algebra, then the sequence $((T^*)^n T^n)^{1/2n}$ converges in the strong operator topology to a positive operator A. The spectral projection of A associated with the set $[0, r]$ is $HS(T, \overline{r\mathbb{D}})$.

DSZ decomposition theorem

Theorem (Dykema, Sukochev, Zanin 2015)

Let $\mathcal{M}$ be a finite von Neumann algebra with trace τ and let $T \in \mathcal{M}$. Then there exist $N, Q \in \mathcal{M}$ such that

1. $T = N + Q$
2. *the operator N is normal and the Brown measure of N equals that of T*
3. *the operator Q is sot-quasinilpotent.*

DSZ decomposition theorem

Dykema, Sukochev and Zanin use a continuous surjection $\phi : [0,1] \to \|T\|\overline{\mathbb{D}}$. The normal operator N is the conditional expectation of T onto the algebra generated by the Haagerup-Schultz projections $HS(T, \phi([0,t]))$ for $t \in [0,1]$. Since all of these projections are T-invariant, this is a generalization of the Schur decomposition.

For the remainder of the talk, the term upper-triangular decomposition will refer to a decomposition $T = N + Q$ obtained by the construction of Dykema, Sukochev and Zanin.

Questions

1. (Dykema) Is it possible for an operator T to have two upper-triangular decompositions $T = N_1 + Q_1$ and $T = N_2 + Q_2$ such that Q_1 is quasinilpotent but Q_2 is not?

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2. When does the sequence $((T^*)^n T^n)^{1/2n}$ converge in norm?

If $supp(\nu_T)$ is a finite set, then Question 1 has a negative answer.

Operators with finitely supported Brown measure

Observation (Brown 1983)

If T is an operator in a von Neumann algebra with a trace, then every connected component of the spectrum of T intersects the support of ν_T.

Lemma (DNZ)

If T is an operator in a von Neumann algebra with a trace, and there exists an upper-triangular decomposition $T = N + Q$ of T with Q quasinilpotent, then $\sigma(T) = supp(\nu_T)$.

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Proposition (DNZ)

If T is an operator with finitely supported Brown measure, then there exists an upper-triangular decomposition $T = N + Q$ with Q quasinilpotent if and only if for any upper-triangular decomposition $T = N + Q$, Q is quasinilpotent.

Initial conjecture

Definition (DNZ)

An operator T has the **norm convergence property** if the sequence $((T^*)^n T^n)^{1/2n}$ converges in norm.

Conjecture (N)

T has the norm-convergence property if and only if there exists an upper-triangular decomposition $T = N + Q$ with Q quasinilpotent.

Why it was wrong

Proposition (DNZ)

If Q is sot-quasinilpotent, then $1 + Q$ has the norm-convergence property if and only if $\sigma(Q) \subset \mathbb{T} - 1$.

Theorem (DNZ)

There exists an sot-quasinilpotent operator Q with $\sigma(Q) = \mathbb{T} - 1$.

New definition

Definition (DNZ)

An operator T in a finite von Neumann algebra is said to have the **shifted norm-convergence property** if $T - \lambda I$ has the norm convergence property for all $\lambda \in \mathbb{C}$.

Norm convergence property and spectra

Lemma (DNZ)

Let T be an element of a tracial von Neumann algebra $\mathcal{M}$ and for $r \geq 0$, let P_r denote the Haagerup-Schultz projection of T associated with the closed disc of radius r. Then T has the norm convergence property if and only if for any $s \geq 0$, the spectrum of $P_s T P_s$ is contained in the closed disc of radius s and when $P_s \neq 1$, the spectrum of $(1 - P_s)T(1 - P_s)$ is contained in the complement of the open disc of radius s.

Proposition (DNZ)

Let $\mathcal{M}$ be a finite von Neumann algebra with a trace τ and $T \in \mathcal{M}$. Then T and T^* both have the shifted norm convergence property if and only if

1. *For any Borel set B, the spectrum of $HS(T,B)THS(T,B)$ is a subset of $\overline{B}$ when $HS(T,B) \neq 0$ and*
2. *For any Borel set B, the spectrum of $(1 - HS(T,B))T(1 - HS(T,B))$ is a subset of $\overline{\mathbb{C} \setminus B}$ when $HS(T,B) \neq 1$.*

Convergence properties and upper-triangular forms

Theorem (DNZ)

T and T^* both have the norm convergence property if and only if for every upper-triangular decomposition $T = N + Q$, Q is quasinilpotent.

Finitely supported Brown measure

Theorem (DNZ)

If T is an operator in a finite von Neumann algebra and ν_T has finite support, then the following are equivalent:

1. *There exists an upper-triangular decomposition $T = N + Q$ such that Q is quasinilpotent*
2. *For any upper-triangular decomposition $T = N + Q$, Q is quasinilpotent*
3. *T has the shifted norm-convergence property.*

Guess

If T is an operator in a von Neumann algebra with a trace τ, then the following are equivalent:

1. *There exists an upper-triangular decomposition $T = N + Q$ such that Q is quasinilpotent*
2. *For any upper-triangular decomposition $T = N + Q$, Q is quasinilpotent*
3. *T has the shifted norm-convergence property.*

Thank you.