SIMPLE C*-INCLUSIONS VIA STATE EXTENSIONS

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I. UNIQUENESS

A prototypical uniqueness statement:

> If a $*$-representation $\pi : A \to B(H)$ "acts nicely" on some subset $\mathcal{S} \subset A$, then π is injective.

Example

- A = Toeplitz algebra = $C^*(\{T\})$ ↖ non-unitary isometry
- $\mathcal{S} = \{1 - TT^*\}$
- π "nice on $\mathcal{S}$" means $\pi(1-TT^*) \neq 0$

Uniqueness Examples in Graph Algebras

Preliminaries

- All graphs are directed & countable.
- Path spaces:

For any graph E and any integer $n \geq 0$, we denote by E^n the set of paths of length n in E, so

- $E^0 = \{\text{vertices in } E\}$
- $E^1 = \{\text{edges in } E\}$
- $E^n = \left\{ \lambda = (e_1, \dots, e_n) \in (E^1)^n : \begin{array}{l} s(e_k) = r(e_{k+1}) \\ \forall k = 1, \dots, n-1 \end{array} \right\}$

$$E^* = \bigcup_{n \geq 0} E^n \quad \text{(finite path space)}$$

Source & range maps denoted by $s, r : E^* \to E^0$.

- Path concatenation:

If $\lambda = (e_1, \dots, e_m) \in E^m$

$\mu = (f_1, \dots, f_n) \in E^n$

are such that $s(\lambda) = r(\mu)$, then

we can form $\lambda\mu = (e_1, \dots, e_m, f_1, \dots, f_n) \in E^{m+n}$,

which has $s(\lambda\mu) = s(\mu)$, $r(\lambda\mu) = r(\lambda)$

- Path comparison:

For $\lambda, \mu \in E^*$ we write $\lambda \succ \mu$ (or $\mu \prec \lambda$)

if there exists $\sigma \in E^*$ such that $\lambda = \mu\sigma$.

In this case, the path σ is denoted by $\lambda \ominus \mu$

(If $\lambda = \mu$, we let $\lambda \ominus \mu = s(\lambda) \in E^0$.)

- A **Cuntz-Krieger E-family** in A (a C^*-algebra) is a system $\left((P_v)_{v \in E^0}, (T_e)_{e \in E^1} \right) \subset A$ satisfying:

  (CK1) All $(P_v)_{v \in V}$ are mutually orthogonal projections

  (CK2) $\forall e \in E^1$: $T_e^* T_e = P_{s(e)}$

  (CK3) $\forall v \in E^0$, $\forall F \subset E^1 \cap r^{-1}(v)$ finite:
  $$\sum_{e \in F} T_e T_e^* \leq P_v$$

  (CK3') If $v \in E^0$ has $E^1 \cap r^{-1}(v)$ non-empty & finite,

  call such v **regular**

  then
  $$\sum_{e \in E^1 \cap r^{-1}(v)} T_e T_e^* = P_v$$

- Cuntz-Krieger families viewed as path reps.

Given a family $\left((P_v)_{v\in E^0}, (T_e)_{e\in E}\right) \subset A$ as above we can define, for each path $\lambda = (e_1, \dots, e_n) \in E^n$ the element

$$T_\lambda = T_{e_1} T_{e_2} \cdots T_{e_n} \in A$$

(If $\lambda = v \in E^0$, let $T_\lambda = P_v$.)

The system $(T_\lambda)_{\lambda \in E^*} \subset A$ constitutes a path representation, in the sense that

$$T_\lambda T_\mu = \begin{cases} T_{\lambda\mu}, & \text{if } s(\lambda) = r(\mu) \\ 0, & \text{otherwise} \end{cases}$$

In particular $T_\lambda = P_{r(\lambda)} T_\lambda P_{s(\lambda)}, \forall \lambda \in E^*$

- Orthogonality Relations

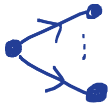
$$T_\lambda^* T_\mu = \begin{cases} T^*_{\lambda \ominus \mu} & , \text{if } \lambda \geq \mu \\ T_{\mu \ominus \lambda} & , \text{if } \mu \geq \lambda \\ 0 & , \text{otherwise} \end{cases}$$

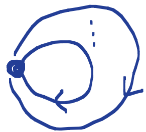
- The Cuntz-Krieger C^*-algebra $C^*(E)$ is the universal C^*-algebra generated by symbols $(p_v)_{v \in E^0} \cup (t_e)_{e \in E^1}$ subject to (CK1)–(CK3')

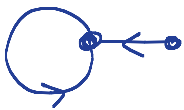
- Spanning monomials

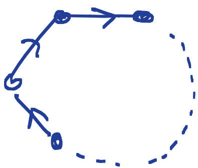
$$C^*(E) = \overline{\operatorname{span}} \underbrace{\{t_\lambda t_\mu^* : \lambda, \mu \in E^*, s(\lambda) = s(\mu)\}}_{\text{denote this set by } \mathcal{G}(E)}$$

- Examples

(a) E :  (n edges) $\Rightarrow C^*(E) \simeq M_{n+1}(\mathbb{C})$

(b) E :  (n loops) $\Rightarrow C^*(E) \simeq \begin{cases} C(\mathbb{T}), & n=1 \\ \mathcal{O}_n, & n>1 \end{cases}$

(c) E :  $\Rightarrow C^*(E) \simeq \mathcal{T}$

(d) E :  $\Rightarrow C^*(E) \simeq C(\mathbb{T}) \otimes M_n$

(n-gon)

Uniqueness Example 1 (Kumjian - Pask - Raeburn - Fowler - ... and others).

$A = C^*(E)$ - with E satisfying condition (L)

$\mathcal{S} = \{ p_v : v \in E^0 \}$

see below

π "nice" on $\mathcal{S}$ means $\pi(p_v) \neq 0, \forall v \in E^0$

Recall that:

- a cycle is a path λ of positive length, with $s(\lambda) = r(\lambda)$
- given a cycle $\lambda = (e_1, \dots, e_n)$, an entry into λ is an edge $f \in E^1$, such that there is some k with $r(f) = r(e_k)$ but $e_k \neq f$.
- Condition (L): every cycle has an entry

Uniqueness Example 2 (Szymański 2001; see also N-Rosnikoff 2010)

$A = C^*(E)$ with E arbitrary graph

$\mathcal{G} = \{p_v\}_{v \in E^0} \cup \{t_\lambda\}_{\lambda \text{ entry-less cycle}}$

π "nice" on $\mathcal{G}$ means:

- $\pi(p_v) \neq 0, \ \forall v \in E^0$
- $\pi(t_\lambda)$ has full spectrum $(\mathbb{T})$ for every entry-less cycle λ

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II SIMPLE INCLUSIONS

When interested in generalizing / abstracting the preceding uniqueness results to other settings, it might be helpful to move away from the actual combinatorial framework, and take a different look at our uniqueness prototype.

Instead of cooking up a (small) set $\mathcal{S}$, and an ad-hoc definition of "acts nicely on $\mathcal{S}$", we may want to enlarge $\mathcal{S}$, say, to a C^*-subalgebra $M \subset A$, which should satisfy the following.

Definition A C^*-algebra inclusion $M \subset A$ is said to be a simple inclusion, if whenever a $*$-rep $\pi : A \to B(H)$ is injective on M, it is also injective on A.

Equivalently, whenever $J \subset A$ is a closed 2-sided ideal such that $J \cap M = \{0\}$, it follows that $J = \{0\}$.

Preferably, we want M to be <u>abelian</u>.

Graph Examples revisited

$A = C^*(E)$

For any $\lambda \in E^*$ let $P_\lambda = t_\lambda t_\lambda^*$

The diagonal in $C^*(E)$:

$$\mathcal{D}(E) = \overline{\text{span}}\,\{P_\lambda : \lambda \in E^*\}$$

$$\boxed{\text{1. If } E \text{ satisfies condition (L), then } \mathcal{D}(E) \subset C^*(E) \text{ is a simple inclusion}}$$

In the general case, consider

$\mathcal{M}(E) = \mathcal{D}(E)'$ (commutant of $\mathcal{D}(E)$ in $C^*(E)$)

$$\boxed{\text{② } (E \text{ arbitrary}) : \mathcal{M}(E) \subset C^*(E) \text{ is a simple inclusion}}$$

Additional Features (N - Reznikoff)

- $\mathcal{M}(E)$ is always a MASA
- there exists a unique conditional expectation $\Phi : C^*(E) \to \mathcal{M}(E)$
- E satisfies (L) $\iff$ $\mathcal{D}(E) = \mathcal{M}(E)$ (i.e. $\mathcal{D}(E)$ is MASA)
- $\mathcal{M}(E) = \overline{\text{span}} \underbrace{\{ x \in \mathcal{G}(E) : x \text{ normal} \}}_{\mathcal{G}_{\mathcal{M}}(E)}$

- Given $\lambda, \mu \in E^*$ with $s(\lambda) = s(\mu)$, the spanning monomial $x = t_\lambda t_\mu^*$ is normal, if and only if one of the following holds:
  - $\lambda = \mu$
  - $\lambda > \mu$ & $\lambda \ominus \mu$ is an entry-less cycle
  - $\mu > \lambda$ & $\mu \ominus \lambda$ is an entry-less cycle.
- the unique conditional expectation Φ acts on the spanning monomial set $\mathcal{G}(E)$ as:

$$\Phi(x) = \begin{cases} x, & \text{if } x \text{ is normal} \\ 0, & \text{otherwise} \end{cases}$$

- Φ is faithful : $\Phi(a^* a) = 0 \Rightarrow a = 0$.

Higher rank graphs

Def: Given a positive integer k, a **k-graph** is a system $(\Lambda, d, s, r, \cdot)$, where:

- Λ = countable set
- $d : \Lambda \to \mathbb{N}^k$ degree map

  Notation: $d^{-1}(\underline{n})$ is denoted by $\Lambda^{\underline{n}}$
- $s, r : \Lambda \to \underbrace{\Lambda^{\underline{0}}}_{\text{vertex set}}$ source & range maps
- concatenation rule: $\lambda\mu \in \Lambda$ is defined, whenever $\lambda, \mu \in \Lambda$ have $s(\lambda) = r(\mu)$.

subject to ...

- Grading Rule: $d(\lambda\mu) = d(\lambda) + d(\mu)$

- Unique Factorization Rule: For any $\lambda \in \Lambda$ and any pair $\underline{m}, \underline{n} \in \mathbb{N}^k$ with $\underline{m} + \underline{n} = d(\lambda)$, there exists a unique factorization $\lambda = \mu\nu$, such that $d(\mu) = \underline{m}$ & $d(\nu) = \underline{n}$.

Remark 1-graphs are precisely path spaces E^* of ordinary graphs.

For technical reasons, we only consider row-finite k-graphs without sources, which are those s.t. for every vertex v & any degree $\underline{n} \in \mathbb{N}^k$, the set $v\Lambda^{\underline{n}} = \Lambda^{\underline{n}} \cap r^{-1}(v)$ is finite.

<u>Cuntz-Krieger families</u>: $(T_\lambda)_{\lambda \in \Lambda} \subset A$.

(CK1) $(T_\lambda)_{\lambda \in \Lambda^0}$ form an orthogonal family of projections

(CK2) $\forall \lambda \in \Lambda$: $T_\lambda^* T_\lambda = T_{s(\lambda)}$

(CK3) $\forall v \in \Lambda^0, \underline{n} \in \mathbb{N}^k$: $\displaystyle\sum_{\lambda \in v\Lambda^{\underline{n}}} T_\lambda T_\lambda^* = T_v$

(CK4) $s(\lambda) = r(\mu) \Rightarrow T_\lambda T_\mu = T_{\lambda\mu}$

As was the case of ordinary graphs, the Cuntz-Krieger C^*-algebra $C^*(\Lambda)$ associated to a k-graph is defined as the universal C^*-algebra generated by symbols $(t_\lambda)_{\lambda\in\Lambda}$ subject to (CK1)-(CK4).

Spanning monomials: As for ordinary graphs, we have

$$C^*(\Lambda) = \overline{\operatorname{span}}\{t_\lambda t_\mu^* : \lambda,\mu\in\Lambda, s(\lambda)=s(\mu)\}$$

Question: Are there an analogues of 1 & 2 for k-graphs?

Answer: Yes

Let $\mathcal{D}(\Lambda) = \overline{\operatorname{span}}\{p_\lambda : \lambda \in \Lambda\}$

$\boxed{\text{k-1}}$ (Kumjian-Pask, 2000) If Λ is aperiodic, then $\mathcal{D}(\Lambda) \subset C^*(\Lambda)$ is a simple inclusion.

For a version of 2, we define cycline pairs to be those pairs $(\lambda, \mu) \in \Lambda \times \Lambda$ with $s(\lambda) = s(\mu)$ that satisfy one of the following equivalent conditions

(a) $\forall\, \gamma \in r^{-1}(s(\lambda)) : p_{\lambda\gamma} = p_{\mu\gamma}$

(b) $t_\lambda t_\mu^*$ is normal and commutes with $\mathcal{D}(\Lambda)$

The cycline subalgebra:

$$\mathcal{M}(\Lambda) = \overline{\operatorname{span}}\{t_\lambda t_\mu^* : (\lambda,\mu) \text{ cycline pair}\}$$

$\boxed{k=2}$ (Brown - N - Reznikoff, 2014)

$\mathcal{M}(\Lambda) \subset C^*(\Lambda)$ is a simple inclusion

Thm (Brown-N-Reznikoff-Sims-Williams, Yang)

$\mathcal{M}(\Lambda)$ is a MASA in $C^*(\Lambda)$

However, a conditional expectation onto $\mathcal{M}(\Lambda)$ may fail to exist in general. (See BNRSW)

Simple Inclusions for Groupoid C^*-algebras

Start with a locally compact Hausdorff second countable étale groupoid G and consider the isotropy subgroupoid: $\mathrm{Iso}(G) = \{\gamma \in G : s(\gamma) = r(\gamma)\}$.

The interior $\mathrm{Iso}(G)^\circ$ is an open subgroupoid, which with the help of the inclusion

$C_c(\mathrm{Iso}(G)^\circ) \subset C_c(G)$ gives rise to a C^*-algebra embedding

$$C^*_{red}(\mathrm{Iso}(G)^\circ) \subset C^*_{red}(G)$$

<u>Thm</u> (BNRSW) The above is a simple inclusion.

<u>Note</u> $\boxed{\text{K-2}}$ is a special case of ↑.

III STATE EXTENSIONS

As it turns out, all the above examples of simple inclusions (and many more!) are derived from certain "abstract uniqueness theorems," all of which relie on state extension analysis.

Notation: Assume $M \subset A$ is a non-degenerate C^*-algebra inclusion.

M contains an approx. unit for A.

$$P_1(M \uparrow A) = \{ \varphi \in \overbrace{P(M)}^{\text{pure states}} : \varphi \text{ extends to a unique state } \tilde{\varphi} \text{ on } A \}$$

(Note: $\varphi \in P_1(M \uparrow A) \Rightarrow \tilde{\varphi} \in P(A)$.)

Special Properties based on state extensions

- (EP) $\mathcal{M}$ abelian & $P_1(\mathcal{M} \uparrow \mathcal{A}) = P(\mathcal{M})$

  Notes: (EP) implies:

  - unique cond. expectation $\Phi : \mathcal{A} \to \mathcal{M}$
  - $\mathcal{M}$ = MASA

- (EP^+) : (EP) & Φ is faithful.

  non-standard

- (AEP) $\mathcal{M}$ abelian & $P_1(\mathcal{M} \uparrow \mathcal{A})$ is dense in $P(\mathcal{M})$

  Notes: • Neither existence of expectations, nor masa are implied.

  • $\mathcal{M}(\Lambda) \subset C^*(\Lambda)$ is a masa with (AEP)

- (CAEP) : (AEP) & existence of expectation

  Note : implies uniqueness of expectation.

- Pseudo-diagonal : (CAEP) & expectation is faithful.

Note (results of N-Reznikoff) implies

- M masa
- $M \subset A$ simple inclusion
- $\mathcal{M}(E) \subset C^*(E)$ is a pseudo-diagonal
- $\mathcal{M}(\Lambda) \subset C^*(\Lambda)$ may fail to be pseudo-diagonal (no expectation)

- Cartan (Renault) : M masa, non-degenerate, existence of faithful conditional expectation, regular

$\boxed{\begin{array}{l} N(M) = \{n \in A : nMn^* \cup n^*Mn \subset M\} \\ \text{generates } A \end{array}}$

<u>Notes</u>

- implies pseudo-diagonal (BNRSW)
- $\mathcal{M}(E)$ is Cartan in $C^*(E)$ (NR)

- C^*-diagonals (Kumjian) : Cartan & (EP)

- (♡) (Un-named, but used in BNRSW) :

$$\bigoplus_{\varphi \in \Sigma} \pi_{\tilde{\varphi}} \text{ is faithful, for some } \Sigma \subset P_1(M \uparrow A).$$

GNS rep. of A associated with $\tilde{\varphi}$.

__Notes:__

- implied by pseudo-diagonal
- $C^*_{red}(\mathrm{Iso}(G)^\circ) \subset C^*_{red}(G)$ satisfies $(\heartsuit)$ (BNRSW)

$$\boxed{\begin{array}{l}\text{"\underline{Abstract Uniqueness" Thm} (BNRSW).}\\ (\heartsuit) \Rightarrow M \subset A \text{ is a simple inclusion.}\end{array}}$$

- M is not required to be abelian
- M may fail to have conditional expectations onto it
- Even when a conditional expectation exists, it may fail to be unique or faithful.

Irregular Inclusions

All examples given so far are regular inclusions.

N(M) generates A

Here are a few irregular examples

A $\quad C(\mathbb{T}) \simeq C^*(\{u\}) \subset C^*_{red}(\mathbb{F}_2)$

free group with generators u, v

Irregular with property (EP^+). No big deal about simplicity of the inclusion, because $C^*_{red}(\mathbb{F}_2)$ is known to be simple

(B) $\ell^\infty \subset B(\ell^2)$ [Kadison-Singer]

Irregular with (EP^+)

uses proof of

(C) $\ell^\infty/c_0 \subset Q(\ell^2)$ Calkin

Irregular with (EP) but not (EP^+)

Hints for (B), (C)

- Regular & (CAEP) implies $L_\Phi = \{a \in A : \phi(a^*a) = 0\}$ is 2-sided ideal.
- regularity & (EP) pass to quotients
- Failure of (EP^+) & simplicity of Calkin imply irregularity of (C).

(D) Assume $\Gamma \curvearrowright X$ is a free action

discrete grp. — compact — $\gamma \neq e \Rightarrow \gamma x \neq x, \forall x \in X$

Then both inclusions: $\begin{cases} \text{(i) } C(X) \subset C(X) \rtimes \Gamma; \\ \text{(ii) } C(X) \subset C(X) \rtimes_{red} \Gamma \end{cases}$

satisfy (EP). In fact, (ii) satisfies (EP$^+$), but in general (i) does not, for instance when the ideal

$$J = \mathrm{Ker}\left[C(X) \rtimes \Gamma \longrightarrow C(X) \rtimes_{red} \Gamma\right]$$

is non-trivial.

Caution! Both (i) and (ii) are regular, but (assuming $J \neq \{0\}$) when we look at $A = J + C(X)$ $(\subset C(X) \rtimes \Gamma)$, the inclusion $C(X) \subset A$ has (EP), but is irregular.

Note that in fact the (non-faithful) expectation $\Phi : A \to C(X)$ is actually a $*$-homomorphism (!)

Select References

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THANK YOU!