Regular and Positive noncommutative rational functions

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Joint work with Igor Klep and Jurij Volčič

Positive numbers: the start of real algebraic geometry

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Via developments in logic in the early 20th century, Tarski noted that the above observation implies the systematic study of real inequalities could be made algebraic.

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We note that the above theorem is usually stated for trigonometric polynomials and was very important in the classical study of orthogonal polynomials.

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No. (Hilbert, although explicit examples were found much later.)

Motzkin polynomial

Theorem (Motzkin 1967)

The polynomial

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Motzkin polynomial as a rational function

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Hilbert's 17th problem

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Hilbert's seventeenth problem asks whether any positive polynomial in several variables can be written as a sum of squares of rational functions.

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In fact, q_i can be chosen such that they are well defined for all real inputs. That is, their denominators can be chosen so that they never vanish on real inputs. (Rational functions with such nonvanishing denominators are sometimes called *regular*.) The proof goes by a clever application of the Tarski principle.

The Artin theorem over rational functions

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The techniques involved will also shift from logic-algebra to functional analysis.

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For example, the free polynomial

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is positive, since it can be written as

$$p(x_1, x_2) = x_1 x_2 (x_1 x_2)^*.$$

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Note the difference from the commutative case: in the noncommutative case a free polynomial can be written as a sum of squares of free polynomials. (There is no mention of rational functions.)

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Now

$$L(p) = \langle p(X)v, v\rangle < 0$$

which witnesses a tuple of self-adjoint operators where the p is not positive semidefinite.

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We note that the first two are equal. (ie $1 = x_1 x_1^{-1}$)

Regular free rational functions

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Follows from minimal realization theory.

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Here the situation is not as simple as clearing denominators as in the commutative case. Additionally, we note that q_i can be taken to be in the subring of noncommutative rational functions generated by subexpressions of a regular formula for r.

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- $p + q \in \mathcal{S} \Rightarrow p, q \in \mathcal{S}$
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Regularity allowed us to conclude that what the GNS produced would be in the domain of our rational function.

Regular functions and Realizations

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A realization is a formula of the form

$$r(X) = c^*(A_0 + \sum A_i X_i)^{-1} b.$$

(Here we have suppressed tensors.)

Stably bounded functions

- A regular rational function *r* is said to be *stably bounded* if there is an $\epsilon > 0$ such that for all inputs with imaginary part having norm less than ϵ, the function *r* is bounded.

Stably bounded functions

- A regular rational function r is said to be *stably bounded* if there is an $\epsilon > 0$ such that for all inputs with imaginary part having norm less than ϵ, the function r is bounded.
- We showed that r is stably bounded if and only if for its minimal realization there exists a D such that DA_0 has positive real part and each DA_i is skew-self-adjoint for $i > 0$. We called such realizations *stably privileged.*

Privileged realizations

Let $d \geq e$ and $L = A_0 + \sum_j A_j x_j$ with $A_j \in M_{d,e}(\mathbb{R})$.

- We recursively define L to be **privileged** if
  1. it is stably privileged; or
  2. there exists $D \in Me, d(\mathbb{R})$ such that
     $0 \neq \text{Re}\,(DA_0) \geq 0, \qquad \text{Re}\,(DA_j) = 0 \quad$ for $j > 0$ and LV is privileged, where columns of V form a basis for $\ker \text{Re}\,(DA_0)$.

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Theorem (Klep, P., Volčič)

A rational function is regular if and only if it has a privileged realization.

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