Sampling in de Branges Spaces of Entire Functions

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Paley-Wiener Space

PW_π consists of f which are:

1. entire;
2. square integrable
$$\int_{\mathbb{R}} |f(t)|^2 dt < \infty;$$
3. exponential type π, i.e. for all $\epsilon > 0$,
$$|f(z)| \leq C_\epsilon e^{(\pi+\epsilon)|z|}.$$

Paley-Wiener Theorem

Theorem (Paley-Wiener, ∼1930)

If $f \in PW_\pi$, then there exists a $g \in L^2[-1/2, 1/2]$ such that

$$f(z) = \int_{-1/2}^{1/2} g(t) e^{-2\pi i t z} dt.$$

Colloquially,

$$PW_\pi = L^2\widehat{[-\frac{1}{2}, \frac{1}{2}]}.$$

Shannon-Whitaker-Kotelnikov Sampling

Theorem (Whitaker 1929, Shannon 1949, Kotelnikov 1933)

If $f \in PW_\pi$, then for all $x \in \mathbb{R}$,

$$f(x) = \sum_{n \in \mathbb{Z}} f(n) \frac{\sin(\pi(x-n))}{\pi(x-n)}.$$

The convergence takes place both uniformly as well as in the mean.

Note:

$$\begin{aligned} f(n) &= \int_{-1/2}^{1/2} g(t) e^{-2\pi int} dt \\ &= \langle g(t), e^{2\pi int} \rangle \\ &= \left\langle f(x), \frac{\sin(\pi(x-n))}{\pi(x-n)} \right\rangle. \end{aligned}$$

The Sampling Problem

Definition

A sequence $\{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ is a sampling sequence for PW_π if there exist $A, B > 0$ such that for all $f \in PW_\pi$,

$$A\|f\|^2 \leq \sum_n |f(\lambda_n)|^2 \leq B\|f\|^2.$$

Question: which sequences are sampling sequences?

Reconstruction: if $\{\lambda_n\}$ is a sampling sequence, then there exists $\{h_n\} \subset PW_\pi$ such that

$$f(x) = \sum_n f(\lambda_n) h_n(x).$$

The Interpolation Problem

Definition

A sequence $\{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ is an interpolating sequence for PW_π if for every $(c_n) \in \ell^2(\mathbb{Z})$ there exists an $f \in PW_\pi$ such that $f(\lambda_n) = c_n$, and $\|f\| \simeq \|(c_n)\|$.

Frames

Duffin and Schaeffer, *A Class of Non-Harmonic Fourier Series*, 1952

Definition

For a Hilbert space H, a sequence $\{v_n\} \subset H$ is a frame if there exist $A, B > 0$ such that for all $v \in H$,

$$A\|v\|^2 \leq \sum_n |\langle v, v_n \rangle|^2 \leq B\|v\|^2.$$

For PW_π, $\{\lambda_n\}_n$ is a sampling sequence if and only if

$$\left\{ \frac{sin(\pi(x - \lambda_n))}{\pi(x - \lambda_n)} \right\}_n$$

is a frame.

Beurling Density

Definition

For a sequence $\{\lambda_n\} \subset \mathbb{R}$, the lower and upper Beurling density are given by:

$$D_-(\{\lambda_n\}) = \liminf_{r \to \infty} \inf_{x \in \mathbb{R}} \frac{\#(\{\lambda_n\} \cap (x-r, x+r))}{2r},$$

$$D_+(\{\lambda_n\}) = \limsup_{r \to \infty} \sup_{x \in \mathbb{R}} \frac{\#(\{\lambda_n\} \cap (x-r, x+r))}{2r}.$$

Landau Inequalities

Theorem (Landau, 1967)

1. *If* $\{\lambda_n\}$ *is a sampling sequence for* PW_π, *then* $1 \leq D_-(\{\lambda_n\}) \leq D_+(\{\lambda_n\}) < \infty$.
2. *If* $1 < D_-(\{\lambda_n\}) \leq D_+(\{\lambda_n\}) < \infty$, *then* $\{\lambda_n\}$ *is a sampling sequence for* PW_π.
3. *If* $\{\lambda_n\}$ *is an interpolating sequence for* PW_π, *then* $D_+(\{\lambda_n\}) \leq 1$.
4. *If* $D_+(\{\lambda_n\}) < 1$, *then* $\{\lambda_n\}$ *is an interpolating sequence for* PW_π.

- Interpolating sequences can also be characterized by the Carleson criterion.
- Complete interpolating sequences are characterized by the Hruschév-Nikolskii-Pavlov theorem.

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Solution to the Sampling Problem

Theorem (Ortega-Cerdá and Seip 2002)

A sequence $\{\lambda_n\}$ is a sampling sequence for PW_π if and only if there exist entire functions E, F such that

1. *for all $z \in UHP$, $|E(\overline{z})| < |E(z)|$ and $|F(\overline{z})| < |F(z)|$;*
2. $\mathcal{H}(E) \simeq PW_\pi$;
3. $\{\lambda_n\}$ *is the zero sequence of* $EF + E^* F^*$.

1. E and F are *Hermite-Biehler* class, $\mathcal{HB}$;
2. $\mathcal{H}(E)$ is the de Branges space generated by E;
3. $E^*(z) = \overline{E(\overline{z})}$.

de Branges Spaces

For $E \in \mathcal{HB}$, define

$$K_E(w, z) = \frac{\overline{E(w)}E(z) - E(\overline{w})E^*(z)}{2\pi i(\overline{w} - z)}.$$

This is a positive matrix (Moore-Aronszajn) and so generates a RKHS: $\mathcal{H}(E)$.

$\mathcal{H}(E)$ consists of all entire functions f that satisfy:

1. $$\|f\|_E^2 := \int_{\mathbb{R}} \frac{|f(t)|^2}{|E(t)|^2} dt < \infty,$$
2. for all $z \in \mathbb{C}$,
$$|f(z)| \leq K_E(z, z)\|f\|_E.$$

Paley-Wiener, again

Example: $E(z) = e^{-i\pi z}$:

$$K_E(w, z) = \frac{\sin \pi(z - \bar{w})}{\pi(z - \bar{w})}.$$

Thus, $\mathcal{H}(e^{-i\pi z}) = PW_\pi$, both as sets, and as Hilbert spaces.

Phase Function

For E, we define $\varphi : \mathbb{R} \to \mathbb{R}$ such that $x \in \mathbb{R}$,

$$|E(x)| = e^{i\varphi(x)}E(x).$$

The function φ is C^1, unique up to additive constant, and

$$\varphi'(x) = \frac{\pi K(x,x)}{|E(x)|^2} \geq 0$$

so is increasing.
Example: for $E(z) = e^{-i\pi z}$, $\varphi(x) = \pi x$.

Zeros of the Kernel

Sequences that satisfy the condition

$$\varphi(\lambda_n) = \pi n + \alpha$$

for some α correspond to zeros of K_E.

Example: $E(z) = e^{-i\pi z}$, then $\{\lambda_n\} = \{n + \alpha\}$ for some $\alpha \in [0, 1]$.

These are the zeros of the translations of the sinc function, and also correspond to the frequencies of orthogonal exponentials on $[-1/2, 1/2]$.

Shannon's Sampling Theorem in de Branges Spaces

Under suitable conditions, for $f \in \mathcal{H}(E)$, $f(z) = \sum_{n\in\mathbb{Z}} f(\gamma_n) \frac{K(\gamma_n,z)}{\|K(\gamma_n,.)\|_E^2}$.

Theorem (de Branges, 1960)

Let $\mathcal{H}(E)$ be a de Branges space with phase function $\varphi(x)$, and let $\alpha \in \mathbb{R}$. If $\Gamma = \{\gamma_n\}_{n\in\mathbb{Z}}$ is a sequence of real numbers, such that $\varphi(\gamma_n) = \alpha + \pi n$, $n \in \mathbb{Z}$, then the functions $\{K(\gamma_n, z)\}_{n\in\mathbb{Z}}$ form an orthogonal set in $\mathcal{H}(E)$.
If, in addition, $e^{i\alpha}E(z) - e^{-i\alpha}E^*(z) \notin \mathcal{H}(E)$, then $\left\{\frac{K(\gamma_n,z)}{\|K(\gamma_n,.)\|}\right\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $\mathcal{H}(E)$. Moreover, for every $f(z) \in \mathcal{H}(E)$,

$$f(z) = \sum_{n\in\mathbb{Z}} f(\gamma_n) \frac{K(\gamma_n, z)}{\|K(\gamma_n, .)\|_E^2}. \tag{1}$$

Homogeneous Approximation Property

1. Ramanathan-Steger (1994)
2. Gröchenig-Razafinjatovo (1998)
3. Heil-Kutyniok (2002)

Theorem (al-Sa'di and W)

Let $\mathcal{H}(E)$ be a de Branges space such that the phase function of $E(z)$ satisfies $0 < \delta \leq \varphi'(x)$ for all $x \in \mathbb{R}$. Let $\{\mu_n\}_{n \in \mathbb{Z}} \subset \mathbb{R}$ be a separated sequence such that $\{k_{\mu_n}(z)\}_{n \in \mathbb{Z}}$ is a frame in $\mathcal{H}(E)$. Then given $\epsilon > 0$ there exists $R = R(\epsilon) > 0$ such that for all $y \in \mathbb{R}$ and all $r > 0$

$$\sup_{|x-y| \leq r} \left\| k_x(.) - Q_{y,r+R} k_x(.) \right\| < \epsilon, \tag{2}$$

where $k_x(z) = \frac{K(x,z)}{\|K(x,.)\|}$, and the supremum is taken over $x \in \mathbb{R}$.

$Q_{y,r+R}$ is the projection onto the span of $\{k_{\mu_n} : |\mu_n - y| \leq r + R\}$.

Comparison Theorem

Theorem (al-Sa'di and W)

Let $\mathcal{H}(E)$ be a de Branges space, and the corresponding phase function of E satisfies $0 < \delta \leq \varphi'(x)$ for all $x \in \mathbb{R}$. Suppose that $\mathcal{M} = \{\mu_n\}, \Gamma = \{\gamma_n\} \subseteq \mathbb{R}$ are two separated sequences, such that $\{k_{\mu_n}(z)\}_{n \in \mathbb{Z}}$ is a frame in $\mathcal{H}(E)$, and $\{k_{\gamma_n}(z)\}_{n \in \mathbb{Z}}$ is a Riesz basis for a closed subspace of $\mathcal{H}(E)$. Then for every $\epsilon > 0$, there exists $R = R(\epsilon) > 0$, such that for all $r > 0$ and $y \in \mathbb{R}$, we have

$$(1 - \epsilon) \sharp\big(\Gamma \cap [y - r, y + r)\big) \leq \sharp\big(\mathcal{M} \cap [y - r - R, y + r + R)\big).$$

Therefore,

$$D^-(\Gamma) \leq D^-(\mathcal{M}), \quad \text{and} \quad D^+(\Gamma) \leq D^+(\mathcal{M})$$

Necessary Densities of Sampling and Interpolating Sets

Theorem

Let $E \in \mathcal{HB}$, with phase function satisfying $0 < \delta \leq \varphi'(x)$, for all $x \in \mathbb{R}$. If $\mathcal{M} = \{\mu_n\}_{n \in \mathbb{Z}}$ is a uniformly separated sampling sequence in $\mathcal{H}(E)$, then $D^-(\mathcal{M}) \geq \frac{\delta}{\pi}$.

Theorem

Let $E \in \mathcal{HB}$, with phase function satisfying $0 < \delta \leq \varphi'(x) \leq M < \infty$, for all $x \in \mathbb{R}$. If $\Gamma = \{\gamma_n\}_{n \in \mathbb{Z}}$ is a uniformly separated interpolating sequence in $\mathcal{H}(E)$, then $D^+(\Gamma) \leq \frac{M}{\pi}$.

We recover the Landau inequalities on PW_π.

Toward Necessary and Sufficient Conditions

In general, density criteria are not valid in de Branges spaces (Lyubarskii and Seip, 2002).

Theorem (al-Sa'di and W; Baranov)

Let $E_0 \in \mathcal{HB}$. If $\{\lambda_n\}$ is a separated sampling sequence for $\mathcal{H}(E_o)$, then there exists two functions E, F such that

1. $E, F \in \mathcal{HB}$,
2. $\mathcal{H}(E_o) \simeq \mathcal{H}(E)$,
3. $\{\lambda_n\}$ *constitutes the zero sequence of* $EF + E^* F^*$.

Note: still only necessary condition.

Naimark's Dilation Theorem

Theorem (Naimark ∼1930)

Let $\mathcal{E}$ be regular, positive, $B(H)$-valued measure on Ω. Then there exists a Hilbert space K, a bounded linear operator $V : H \to K$, and a regular, self-adjoint, spectral, (i.e. PVM) $B(K)$-valued measure $\mathcal{F}$ on Ω such that for all measurable sets S

$$\mathcal{E}(S) = V^{*}\mathcal{F}(S)V.$$

Theorem (Han and Larson 2000)

If $\{v_n\} \subset H$ is a frame, then there exists a Hilbert space K and a frame $\{w_n\} \subset K$ such that $\{v_n \oplus w_n\} \subset H \oplus K$ is a Riesz basis.

The converse was observed in Aldroubi (1994).

Embedding de Branges Spaces

We define $\mathcal{I} : \mathcal{H}(E) \to \mathcal{H}(EF) : f \mapsto fF$; $\mathcal{I}$ is a linear isometry.

Lemma

The mapping $\mathcal{J} : \mathcal{H}(F) \to \mathcal{H}(EF)$ defined by $g \mapsto gE^*$ is a linear isometry. Consequently, for every $g_1, g_2 \in \mathcal{H}(F)$,

$$\langle g_1 E^*, g_2 E^* \rangle_{EF} = \langle g_1, g_2 \rangle_F. \tag{3}$$

Dilation of de Branges Spaces

Lemma

The images of $\mathcal{I}$ and $\mathcal{J}$ are orthogonal in $\mathcal{H}(EF)$. Consequently,

$$\mathcal{H}(EF) = F\mathcal{H}(E) \oplus E^*\mathcal{H}(F).$$

Lemma

The following equation holds for the kernel K_{EF}:

$$K_{EF}(w,z) = \overline{F(w)}[\mathcal{I}(K_E(w,\cdot))](z) + E(\overline{w})[\mathcal{J}(K_F(w,\cdot))](z). \tag{4}$$

Naimark Dilation of Kernel Functions

Recall: if $\{\lambda_n\}$ is a sampling sequence in $\mathcal{H}(E_0)$, then there exists $E, F \in \mathcal{HB}$ satisfying conditions 1-3.

Theorem (al-Sa'di and Weber)

Suppose that $\{\lambda_n\}$ is a sampling sequence for $\mathcal{H}(E_0)$. Suppose $E, F \in \mathcal{HB}$ is given by the Necessary Condition Theorem. Then $\mathcal{H}(E_0)$ can be embedded into $\mathcal{H}(EF)$ such that the frame $\{K_{E_0}(\lambda_n, \cdot)\}$ is embedded into the Riesz basis

$$\left\{ \frac{\overline{F(\lambda_n)}[\mathcal{I}(K_{E_0}(\lambda_n, \cdot))](z)}{\sqrt{K_{EF}(\lambda_n, \lambda_n)}} \oplus \frac{E(\lambda_n)[\mathcal{J}(K_F(\lambda_n, \cdot))](z)}{\sqrt{K_{EF}(\lambda_n, \lambda_n)}} \right\}_n$$

Sufficient Conditions for Sampling

Theorem (al-Sa'di and Weber)

Suppose that $E_0, E, F \in \mathcal{HB}$ have no real roots such that $\mathcal{H}(E_0) \simeq \mathcal{H}(E)$, and $\varphi'_F \lesssim \varphi'_E$. Suppose $\{\lambda_n\}$ satisfies the equation $\varphi_{EF}(\lambda_n) = n\pi + \alpha$ for some $\alpha \in [0, \pi)$. Then the sequence $\{\lambda_n\}$ is a normalized sampling sequence for $\mathcal{H}(E_0)$.

Idea: the kernel functions $\{K_{EF}(\lambda_n, \cdot)\}$ is a Riesz basis in the big space, so the projection onto $\mathcal{H}(E_0)$ is a frame, hence corresponds to a sampling sequence (though we need to normalize).

Sufficient Conditions for Sampling (cont'd)

Corollary

Assume the conditions of the previous theorem, if $\{\lambda_n\}$ *is the zero set of* $EF + E^*F^*$*, then* $\{\lambda_n\}$ *is a normalized sampling set for* $\mathcal{H}(E_0)$*.*

Corollary

Assume the conditions of the previous theorem; assume also that $K_{E_0}(x,x) \simeq 1$*. Then the zero set of* $EF + E^*F^*$ *is a (non-normalized) sampling sequence for* $\mathcal{H}(E_0)$*.*

Multiplexing of Sampled Signals

Corollary

Suppose E, F and $\{\lambda_n\}$ satisfy the hypotheses of the previous theorem, with $f \in \mathcal{H}(E)$ and $g \in \mathcal{H}(F)$. Given the samples $\{f(\lambda_n)\}$ and $\{g(\lambda_n)\}$, f and g can be reconstructed from the multiplexed samples as follows:

$$f(z) = \sum_n \left(f(\lambda_n)F(\lambda_n) + g(\lambda_n)E^*(\lambda_n)\right) \frac{\overline{F(\lambda_n)}K_E(\lambda_n, z)}{K_{EF}(\lambda_n, \lambda_n)} \tag{5}$$

$$g(z) = \sum_n \left(f(\lambda_n)F(\lambda_n) + g(\lambda_n)E^*(\lambda_n)\right) \frac{E(\lambda_n)K_F(\lambda_n, z)}{K_{EF}(\lambda_n, \lambda_n)}. \tag{6}$$

The End

Thank you!