Traces arising from regular inclusions

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Outline

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(1) an inclusion $B \subset A$ of an abelian C^*-subalgebra which is *non-degenerate* (B contains an approximate identity for A) and *regular* (the multiplicative and self-adjoint set $N(B) = \{n \in A : nBn^* \cup n^*Bn \subset B\}$ spans dense subset of A)

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Note that $n^*n, nn^* \in B$ for any $n \in N(B)$ if B contains an approximate identity for A.

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For which states $\phi \in S(B)$ is the extension $\phi \circ \mathbb{E}$ a *tracial state* on A? If S' is the set of such states, is the map $S' \to T(A)$ given by $\phi \mapsto \phi \circ \mathbb{E}$ a surjection?

Invariant states

Definition

If $\phi \in S(B)$ and $n \in N(B)$, then ϕ is called *n-invariant* if $\phi(nbn^*) = \phi(n^*nb)$ for all $b \in B$. If $N_0 \subset N(B)$, then ϕ is *N_0-invariant* if it is n-invariant for all $n \in N_0$. If ϕ is $N(B)$-invariant we will call ϕ *fully invariant*

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If $\tau \in T(A)$ is a tracial state, then $\phi = \tau|_B$ is a fully invariant state on B.

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Under fairly mild assumptions the converse of the above example is also true.

Normalization of conditional expectations

Definition

Let $E : A \to B$ be a conditional expectation. We say that $\mathbb{E}$ is *normalized* by $n \in N(B)$ if $\mathbb{E}(nan^*) = n\mathbb{E}(a)n^*$ for all $a \in A$. (Similar for $N_0 \subset N(B)$.)

In the cases that we care about, the relevant conditional expectations will be normalized by a set of normalizers that generate A.

Invariant states, part II

Theorem (C., Nagy '15)

Suppose that $B \subset A$ is a regular inclusion and $\mathbb{E} : A \to B$ is a conditional expectation which is normalized by $N_0 \subset N(B)$. Then for any N_0-invariant state $\phi \in S(B)$, the composition $\phi \circ \mathbb{E}$ is a tracial state when restricted to $C^*(B \cup N_0) \subset A$.

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Corollary

Suppose that $\mathbb{E} : A \to B$ is normalized by $N_0 \subset N(B)$ and ϕ is a N_0-invariant state on B, where N_0 generates A as a C^*-algebra. Then $\phi \circ \mathbb{E}$ is a tracial state on A.

Proof

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We show that if $\mathbb{E}$ is normalized by n and $\phi \in S(B)$ is an n-invariant state, then $\phi \circ \mathbb{E}(na) = \phi \circ \mathbb{E}(an)$ for all $a \in A$, because we can then use the fact that the centralizer of a state always forms a C^*-algebra.

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$$\phi(\mathbb{E}((nn^*)^j na)) = \phi(\mathbb{E}(an(n^*n)^j))$$

for any positive integer j, because we have the approximations

$$na = \lim_{k\to\infty} (nn^*)^{1/k} na \qquad an = \lim_{k\to\infty} an(n^*n)^{1/k}$$

and we can find suitable polynomials with zero constant term approximating the k-th root function.

Proof, ctd.

$$\begin{aligned} \phi(\mathbb{E}(an(n^*n)^j)) &= \phi(\mathbb{E}(an(n^*n)^{j-1})n^*n) & (1)\\ &= \phi(n\mathbb{E}(an(n^*n)^{j-1})n^*) & (2)\\ &= \phi(\mathbb{E}(na(nn^*)^j) & (3)\\ &= \phi(\mathbb{E}((nn^*)^j na) & (4) \end{aligned}$$

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Here (1) follows because $\mathbb{E}$ is a conditional expectation, (2) from the n-invariance of ϕ, (3) from the fact that n normalizes $\mathbb{E}$, and (4) follows from conditional expectation and commutativity of B. □

Parametrizing the trace space

The previous result shows that if we have a conditional expectation which is normalized by $N(B)$, then there is a surjective map

$$\mathrm{res} : T(A) \ni \tau \mapsto \tau|_B \in S_{\mathrm{inv}}(B)$$

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Question

When is the restriction map injective? That is, for which inclusions $B \subset A$ is it always the case that any tracial state $\tau \in T(A)$ is fully determined by its restriction to B?

The extension property

Definition

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Proposition (C., Nagy '15)

If $B \subset A$ is a non-degenerate inclusion with the extension property, then the restriction map carrying $T(A)$ to $S_{\text{inv}}(B)$ is injective.

Proof of proposition

Proof.

By a result of Archbold [1], if $B \subset A$ has the extension property, the kernel of the associated conditional expectation $\mathbb{E} : A \to B$ is spanned by the commutators $\{ab - ba : a \in A, b \in B\}$.

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Remark

We do not claim that the tracial state space is non-empty in this case (there are examples of inclusions with the extension property where $T(A) = \emptyset$).

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Remark

We do not claim that the tracial state space is non-empty in this case (there are examples of inclusions with the extension property where $T(A) = \emptyset$). Also, there are cases of inclusions without the extension property for which $\tau \mapsto \tau|_B$ is still injective (for example, $\mathbb{C} \subset C^*_r(\mathbb{F}_2)$).

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Sometimes elements of G are called *morphisms* or *arrows*, as an alternate definition of a groupoid is as a small category with inverses.

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For any element α of G, the compositions $s(\alpha) := \alpha^{-1}\alpha$ and $r(\alpha) = \alpha\alpha^{-1}$ are units referred to as the *source* and *range* of α. The set of all units is denoted by $G^{(0)} \subset G$.

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Étale groupoids turn out to be the appropriate generalization of discrete groups/discrete dynamical systems to the groupoid context.

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where π_u is the representation on $\ell^2(s^{-1}(u))$ given by $\pi_u(f)\delta_\gamma = f * \delta_\gamma$ for $\gamma \in s^{-1}(u)$. The abelian C^*-algebra $C_0(G^{(0)}$ is contained in $C_r^*(G)$ as the completion of $C_c(G^{(0)} \subset C_c(G)$. There is a conditional expectation $\mathbb{E}_{\text{red}} : C_r^*(G) \mapsto C_0(G^{(0)})$ extending restriction $C_c(G) \to C_c(G^{(0)})$.

Bisections and balanced measures

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If μ is a totally balanced measure then the corresponding state ϕ_μ on $C_0(G^{(0)})$ will be n-invariant for every elementary normalizer n.

Balanced measures and tracial states

Let G be étale. If τ is a tracial state on $C_r^*(G)$, then the restriction $\tau|_{C_0(G^{(0)})}$ is a state on $C_0(G^{(0)})$, and the corresponding measure μ_τ on $G^{(0)}$ is balanced.

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Proposition (C., Nagy)

Let G be an étale groupoid, let μ be a probability Radon measure on $G^{(0)}$, and let ϕ_μ be the corresponding state on $C_0(G^{(0)})$. The following conditions are equivalent:

(i) *μ is totally balanced;*

(ii) *ϕ_μ is elementary invariant;*

(iii) *ϕ_μ is fullly invariant;*

(iv) *$\phi_\mu \circ \mathbb{E}_{\mathrm{red}}$ is a tracial state on $C^*_{\mathrm{red}}(G)$.*

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(iv) *$\phi_\mu \circ \mathbb{E}_{\text{red}}$ is a tracial state on $C^*_{\text{red}}(G)$.*

(In particular this shows that $\tau \mapsto \mu_\tau$ is a surjection onto the collection of totally balanced probability measures.)

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Proposition

Let G be a principal étale groupoid. Then the map from $T(C_r^*(G))$ onto the collection of totally balanced probability measures is a bijection. Equivalently, the map $\mu \mapsto \phi_\mu \circ \mathbb{E}_{\text{red}}$ is a surjection.

Proof

Proof.

By a result of Kumjian, if G is principal then the inclusion $C_0(G^{(0)}) \subset C_r^*(G)$ has the extension property. Thus the theorem from the previous section about general regular inclusions ensures that the map from $T(C_r^*(G))$ onto $S_{\text{inv}}(C_0(G^{(0)}))$ is in fact a bijection. □

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Question

What are necessary and sufficient conditions for $\tau \mapsto \mu_\tau$ to be injective? What information needs to be added to μ_τ in order to describe the trace space bijectively?

Graph C^*-algebras

If $E = (E^0, E^1, r, s)$ is a directed graph, then there is a universal C^*-algebra $C^*(E)$ generated by a family $\{s_e, p_v\}_{e\in E^1, v\in E^0}$ such that

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(1) the p_v are mutually orthogonal projections;

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For a directed path $\alpha = e_1 \dots e_n$, we denote the associated partial isometry $s_{e_1} \dots s_{e_n}$ by s_α.Elements of the form $s_\alpha s_\beta^*$, for $\alpha, \beta \in E^*$ (finite path space), span the graph C^*-algebra.

The abelian core

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It is shown in [4] that there is a conditional expectation $\mathbb{E}$ from $C^*(E)$ onto $\mathcal{M}(E)$. It is easy to verify that $\mathcal{M}(E) \subset C^*(E)$ is regular (all the generators of $C^*(E)$ are normalizers of $\mathcal{M}(E)$). The abelian core is a MASA, in fact $\mathcal{M}(E) = \mathcal{D}(E)'$, where $\mathcal{D}(E) = \overline{\text{span}}\{s_\alpha s_\alpha^* : \alpha \in E^*\}$.

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Tomforde in [5] showed that the map $\tau \mapsto g_\tau$ is surjective onto the normalized graph traces, using states on K-theory.

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Question

What additional structure needs to be added to parametrize all the tracial states? When is the map $\tau \mapsto g_\tau$ injective?

The tracial state space, ctd.

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Cyclically tagged graph traces

Definition

The *cyclic support* of a graph trace g is the set $\mathrm{supp}^c\, g$ of vertices v with $g(v) > 0$ that lie on cycles without entry. A *cyclically tagged graph trace* is a pair (g, μ), where g is a normalized graph trace and $\mu : \mathrm{supp}^c\, g \to \mathrm{Prob}(\mathbb{T})$. It is *consistent* if whenever v and w are on the same cycle, then $\mu(v) = \mu(w)$. The space of consistent cyclically tagged graph traces is denoted by $T_1^{\mathrm{CCT}}(E)$.

Example

If τ is a tracial state on $C^*(E)$, we obtain the graph trace g_τ as before, and the cyclic tagging $\mu = \mu_t au$ is defined for $v \in \mathrm{supp}^c\, g$

$$\int_{\mathbb{T}} z^k \, d\mu_v = \frac{\tau(s_\lambda^k)}{\tau(p_v)} \qquad s(\lambda) = r(\lambda) = v \quad |\lambda| \text{ minimal.}$$

Invariant states and cyclically tagged graph traces

Theorem (C., Nagy)

If $(g, \mu) \in T_1^{\mathrm{CCT}}(E)$*, there is a state* $\phi_{(g,\mu)}$ *on* $\mathcal{M}(E)$ *which satisfies* $\phi_{(g,\mu)}(s_\alpha s_\alpha^*) = g(s(\alpha))$ *and* $\phi_{(g,\mu)}(s_\alpha s_\lambda^k s_\alpha^*) = g(s(\alpha)) \int_{\mathbb{T}} z^k d\mu_s(\alpha)$ *(set right-hand side to* 0 *if* $g(s(\alpha)) = 0$*).*

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Idea of proof

Divide the Gelfand spectrum Ω of $\mathcal{M}(E)$ into two parts, and then define the state on $\mathcal{M}(E)$ by choosing a measure on Ω that is suitably invariant. (One part will carry the graph trace and the other will carry the tagging.) $\square$

Parametrizing $T(C^*(E))$

Theorem (C., Nagy)

(1) *for any E, the map*

$$T_1^{\mathrm{CCT}}(E_{\mathrm{tight}}) \ni (g,\mu) \mapsto \tau_{(g,\mu)} \circ \rho_{\mathrm{tight}} \in T(C^*(E))$$

(where $\tau_{(g,\mu)} \in T(C^*(E_{\mathrm{tight}}))$ corresponds to (g,μ)) is an isomorphism.

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(where $\tau_{(g,\mu)} \in T(C^*(E_{\mathrm{tight}}))$ corresponds to (g,μ)) is an isomorphism.

(2) *if E is tight, then $\tau \mapsto (g_\tau, \mu_\tau)$ is an isomorphism from $T(C^*(E))$ onto $T_1^{\mathrm{CCT}}(E)$.*

When is $\tau \mapsto g_\tau$ injective

Tomforde noted that if E satisfies condition (K), then the map $\tau \mapsto g_\tau$ is injective. However this is not necessary.

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Two (finite) paths λ and μ are *incomparable* if neither one contains the other as initial prefix. A vertex v is *essentially left infinite* if there is an infinite set $\{\lambda_k\}$ of finite paths that are pairwise incomparable and such that $s(\lambda_k) = v$ for all k.

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For a directed graph E the following are equivalent:

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Theorem (C., Nagy)

For a directed graph E the following are equivalent:

(i) *the map $\tau \mapsto g_\tau$ is injective;*

(ii) *the source of each cycle in E is essentially left infinite.*

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Proof.

(ii) $\Rightarrow$ (i): If a vertex v is essentially left infinite, then any bounded graph trace g must vanish on v. Thus if the source of each cycle is essentially left infinite, there are no measures to consider (after passing to the tightening) and the map $\tau \mapsto g_\tau$ is injective.

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Proof.

(ii) $\Rightarrow$ (i): If a vertex v is essentially left infinite, then any bounded graph trace g must vanish on v. Thus if the source of each cycle is essentially left infinite, there are no measures to consider (after passing to the tightening) and the map $\tau \mapsto g_\tau$ is injective.
(i) $\Rightarrow$ (ii): If v is the source of a cycle and v is *not* essentially left infinite, then we can define a (non-normalized but bounded) graph trace g on E by $g(w) = |\{\text{paths } v \to w\}|$. Thus there is a normalized graph trace g on E which does not vanish at v, and if we take any non-Lebesgue probability measure for μ_v, the tagging (g, μ_v) is consistent.

□

Directions for future work

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(2) Find necessary and sufficient conditions for the balanced measures to parametrize all of $T(C_r^*(G))$ (should have something to do with non-existence of compact invariant sets or something related, especially for ample groupoids).

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Thank you!