

PigeonHole-Probl.txt

1. Show that, in every group of n people, there are two who know the same number (at least one) of people in the group.
2. Each of 500 boxes contains at most 240 apples, and none is empty. Show that at least three boxes contain the same number of apples.
3. There are 17 participants in an international conference. Each participant knows not more than three languages and every two participants have a language in common. Show that at least three participants know the same language. (Here it is not clear whether the three languages are all the same or different, but the result seems obvious either way; with the same three languages, 7 will work instead of 17)
4. Several circles, with total length 10 are inside a unit square. Show that there exists a line passing through at least four circles.
5. Five points are inside an equilateral triangle with side 1. Show that two of those points are less than $1/2$ unit apart.
6. There are 25 points in the plane, and any two points out of every three are less than 1 unit apart. Show that there exists a unit disk containing at least 13 of those points.
7. m^3+1 points are inside the unit cube. Show that there are two points less than $\sqrt[3]{3}/m$ units apart.
8. Show that every convex $2n$ -gon contains a diagonal that is not parallel to any of the sides.
9. Show that every nine-gon has two diagonals with the angle between them less than 7 degrees.
10. Show that every set of $2^{n+1}-1$ integers contains a subset of size 2^n such that the sum of its elements is divisible by 2^n .
11. 30 teams are playing in a soccer tournament. Show that, at every moment, there are two teams with the same number of games played. (here, have to assume that the same teams can play either at most once - then it is easy, or at most twice - then it is harder; with unlimited number of games between the same teams, this is probably not true)
12. Let n_1, n_2, n_3 be three prime numbers, all bigger than 3. Show that either the sum or the difference of two of them is divisible by 12.

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1. The purpose of this document is to provide a comprehensive overview of the project's objectives and scope.

2. The project is designed to address the current challenges faced by the organization in the market.

3. The primary goal is to increase operational efficiency and reduce costs. This will be achieved through the implementation of new technologies and processes.

4. The project will be managed using a structured approach, ensuring that all tasks are completed on time and within budget.

5. The project team consists of experienced professionals from various departments, ensuring a multidisciplinary approach.

6. The project is expected to be completed by the end of the fiscal year, with regular progress reports provided to the management.

7. The project is subject to change based on the evolving needs of the organization and the market.

8. The project is a high-priority initiative and will receive the necessary resources and support from the organization.

9. The project is a key strategic initiative that will significantly impact the organization's long-term success.

10. The project is a complex undertaking that requires close collaboration and communication among all stakeholders.

11. The project is a critical component of the organization's overall strategy and will be closely monitored throughout its duration.

12. The project is a significant investment in the organization's future and will be a key driver of growth.

Problems on ordered sets and extremal elements.

- (1) Show that each of the following numbers is never integer: (a) $\sum_{k=0}^n \frac{1}{2k+1}$; (b) $\sum_{k=1}^n \frac{1}{k}$.
- (2) Show that equation $x^2 + y^2 = 3(u^2 + z^2)$ has no positive integer solutions.
- (3) Show that equation $x^2 + y^2 + z^2 = 2^{1999}$ has no positive integer solutions. What is special about 2^{1999} ?
- (4) What is the largest area of a triangle with each side at most 1?
- (5) Show that a convex polygon of area 1 fits inside a rectangle of area 2.
- (6) The entries of an $n \times n$ matrix are either 0 or 1. The matrix has the property that if an entry is 0, then the sum of all entries in the corresponding row and column is at least n . What is the minimal possible number of non-zero entries in the matrix?
- (7) In some country, there are 100 airports, and all pair-wise distances between them are different. One plane starts from each airport and flies to the closest airport. What is the maximal number of planes that can end up at the same airport?
- (8) Show that some three diagonals of a convex pentagon can be put together to form a triangle.
- (9) Given n points in the plane, no three of which are collinear, show that there exists a closed polygonal line with no self-intersections having these points as vertices.
- (10) In some country all roads between cities are one-way and such that once you leave a city you cannot return to it again. Prove that there exists a city into which all roads enter and a city from which all roads exit.
- (11) Given finitely many squares whose areas add up to 1, show that they can be arranged without overlaps inside a square of area 2.
- (12) Prove that it is impossible to dissect a cube into finitely many cubes, no two of which are the same size.
- (13) Consider a finite set of spherical planets, all of the same radius and no two intersecting. On the surface of each planet consider the set of points not visible from any other planet. Prove that the total area of these sets is equal to the surface area of one planet.
- (14) Show that if the positive integer numbers a and b are relatively prime, then there are infinitely many prime numbers of the form $ak + b$, $k = 1, 2, 3, \dots$. For an exercise, take specific values of a and b , such as $a = 4$ and $b = 3$.

THE UNIVERSITY OF CHICAGO
DIVISION OF THE PHYSICAL SCIENCES
DEPARTMENT OF CHEMISTRY

REPORT OF THE
COMMISSION ON THE
FUTURE OF THE
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PHYSICAL SCIENCES
DEPARTMENT OF CHEMISTRY

CHICAGO, ILLINOIS
1964

THE UNIVERSITY OF CHICAGO
DIVISION OF THE PHYSICAL SCIENCES
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① Suppose we have a necklace of n beads. Each bead is labeled with an integer and the sum of all these labels is $n-1$. Prove that we can cut the necklace to form a string whose consecutive labels x_1, x_2, \dots, x_n satisfy

$$\sum_{i=1}^k x_i \leq k-1 \quad \text{for } k=1, 2, \dots, n.$$

② For a partition π of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $\pi(x)$ be the number of elements in the part containing x . Prove that for any two partitions π and π' , there are two distinct numbers x and y in $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that $\pi(x) = \pi'(y)$ and $\pi'(x) = \pi(y)$.
[A partition of a set S is a collection of disjoint subsets (parts) whose union is S .]

③ A game starts with four heaps of beans, containing 3, 4, 5 and 6 beans. The two players move alternately. A move consists of taking either
(a) one bean from a heap, provided at least two beans are left behind in that heap, or
(b) a complete heap of two or three beans.

The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy.

④ Suppose that each of 20 students has made a choice of anywhere from 0 to 6 courses from a total of 6 courses offered. Prove or disprove: there are 5 students and 2 courses such that all 5 have chosen both courses or all 5 have chosen neither course.

⑤ Let S be the set of ordered triples (a, b, c) of distinct elements of a finite set A . Suppose that

1. $(a, b, c) \in S$ if and only if $(b, c, a) \in S$

2. $(a, b, c) \in S$ if and only if $(c, b, a) \notin S$

3. (a, b, c) and (c, d, a) are both in S if and only if (b, c, d) and (d, a, b) are both in S .

Prove that there exists a one-to-one function from A to \mathbb{R} such that $g(a) < g(b) < g(c)$ implies $(a, b, c) \in S$. Note: \mathbb{R} is the set of real numbers.

⑥ Define a selfish set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \dots, n\}$ which are minimal selfish sets, that is, selfish sets none of whose ^{proper} subsets is selfish.

⑦ Given a finite string S of symbols X and O , we write $\Delta(S)$ for the number of X 's in S minus the number of O 's. For example, $\Delta(XOOXOOX) = -1$. We call a string S balanced if every substring T of (consecutive symbols of) S has $-2 \leq \Delta(T) \leq 2$. Thus, $XOOXOOX$ is not balanced, since it contains the substring $OOXOO$. Find, with proof, the number of balanced strings of length n .

⑧ Players $1, 2, 3, \dots, n$ are seated around a table, and each has a single penny. Player 1 passes a penny to player 2, who then passes two pennies to player 3. Player 3 then passes one penny to player 4, who passes two pennies to player 5, and so on, players alternately passing one penny or two to the next player who still has some pennies. A player who runs out of pennies drops out of the game and leaves the table. Find an infinite set of numbers n for which some player ends up with all n pennies.

⑨ Find necessary and sufficient conditions on positive integers m and n so that

$$\sum_{i=0}^{m-1} (-1)^i \lfloor Li/m \rfloor + \lfloor Li/m \rfloor = 0$$

(10) Let S be a finite set of integers each greater than 1. Suppose that for each integer n there is some $s \in S$ such that $\gcd(s, n) = 1$ or $\gcd(s, n) = s$. Show that there exist $s, t \in S$ such that $\gcd(s, t)$ is prime.

(11) Let a_j, b_j, c_j be integers for $1 \leq j \leq N$. Assume for each j at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $\lfloor N/7 \rfloor$ values of j , $1 \leq j \leq N$.

(12) Let S_0 be a finite set of positive integers. We define finite sets S_1, S_2, \dots of positive integers as follows: the integer a is in S_{n+1} if and only if exactly one of $a-1$ or a is in S_n . Show that there exist infinitely many integers N for which $S_N = S_0 \cup \{N+a : a \in S_0\}$.

(13) Let B be a set of more than $2^{n+1}/m$ distinct points with coordinates of the form $(\pm 1, \pm 1, \dots, \pm 1)$ in the n -dimensional space with $m \geq 3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle.

(14) You have coins C_1, C_2, \dots, C_m . For each k , C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the m coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of m .

(15) Let m be an even number. Write the numbers $1, 2, \dots, m^2$ in the squares of an $m \times m$ grid so that the k -th row, from left to right, is

$$(k-1)m+1, (k-1)m+2, \dots, (k-1)m+m.$$

Color the squares of the grid so that half of the squares in each row and each column are red and the other half are black (a checkerboard is one possibility). Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

(16) Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

(17) Let $m \geq 2$ be an integer and T_m be the number of non-empty subsets S of $\{1, 2, 3, \dots, m\}$ with the property that the average of the elements of S is an integer. Prove that $T_m - m$ is always even.

(18) In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

(19) Shamille O'Neal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?

(20) An integer n , unknown to you, has been randomly chosen in the interval $[1, 2002]$ with uniform probability. Your objective is to select n in an odd number of guesses. After each incorrect guess, you are informed whether n is higher or lower, and you must guess an integer on your next turn among the numbers that are still feasibly correct. Show that you have a strategy so that the chance of winning is greater than $2/3$.

(21) Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \dots + a_k$, with k an arbitrary positive integer and $a_1 \leq a_2 \leq \dots \leq a_k \leq a_1 + 1$? For example, with $n=4$ there are four ways: 4 , $2+2$, $1+1+2$, $1+1+1+1$.

22 Let n be a positive integer. Starting with the sequence $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}$ form a new sequence of $n-1$ entries $\frac{3}{4}, \frac{5}{12}, \dots, \frac{2n-1}{2n(n-1)}$ by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbours on the second sequence to obtain a third sequence of $n-2$ entries, and continue until the final sequence produced consists of a single number x_n . Show that $x_n < \frac{2}{n}$.

23 Basketball star Shaquille O'Neal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made on her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N , but by the end of the season, $S(N)$ was more than 80% of N . Was there necessarily a moment ~~between~~ in between when $S(N)$ was exactly 80% of N ?

24 An $m \times m$ checkerboard is colored randomly: each square is independently assigned red or black with probability $1/2$. We say that two squares, p and q , are in the same connected monochromatic component if there is a sequence of squares, all of the same color, starting at p and ending at q , in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than $\frac{m \cdot m}{8}$.

(25) Let $S = \{(a,b) \mid a = \{1, 2, \dots, m\}, b = \{1, 2, 3\}\}$. A rook tour of S is a polygonal path made up of line segments connecting points p_1, p_2, \dots, p_{3m} in sequence such that

(i) $p_i \in S$

(ii) p_i and p_{i+1} are a unit distance apart, for $1 \leq i < 3m$

(iii) for each $p \in S$ there is a unique i such that $p_i = p$. How many rook tours are there that begin at $(1,1)$ and end at $(m,1)$?

(26) Let H be an $m \times m$ matrix all of whose entries are ± 1 and whose rows are mutually orthogonal. Suppose H has a $a \times b$ submatrix whose entries are all 1. Show that $ab \leq m$.

(27) Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

(28) Prove that for every set $X = \{x_1, x_2, \dots, x_m\}$ of m real numbers, there exists a non-empty subset S of X and an integer n such that

$$\left| n + \sum_{s \in S} s \right| \leq \frac{1}{m+1}$$

- (29) Let Z denote the set of points in \mathbb{R}^m whose coordinates are 0 or 1. (Thus, Z has 2^m elements, which are the vertices of a unit hypercube in \mathbb{R}^m). Given a vector subspace V of \mathbb{R}^m , let $Z(V)$ denote the number of members of Z that lie in V . Let k be given, $0 \leq k \leq m$. Find the maximum over all vector subspaces $V \subseteq \mathbb{R}^m$ of $Z(V)$.
- (30) Let k be a positive integer. Suppose that the integers $1, 2, 3, \dots, 3k+1$ are written down in random order. What is the probability that at no time during this process, the sum of the integers that have been written up to that time is a positive integer divisible by 3? Your answer should be in closed form, but may include factorials.
- (31) Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is non-zero; Barbara wins if it is zero. Which player has a winning strategy?

(32) Start with a finite sequence a_1, a_2, \dots, a_n of positive integers. If possible, choose two ~~indices~~ indices $j < k$ such that a_j does not divide a_k , and replace a_j and a_k by $\gcd(a_j, a_k)$ and $\text{lcm}(a_j, a_k)$, respectively. Prove that if this process is repeated, it must eventually stop and the final sequence does not depend on the choices made. (Note: \gcd means the greatest common divisor and lcm means the least common multiple.)

hypothesis, $A_{i_1} \cap \cdots \cap A_{i_{k-1}} \in S$, and also $A_{i_k} \in S$, so $(A_{i_1} \cap \cdots \cap A_{i_{k-1}}) \cap A_{i_k}$ is in S . This completes the induction. For $k = 2^{n-1}$, we obtain that the intersection of all sets in S is nontrivial. \square

We found the following problem in the *Mathematics Magazine for High Schools (Budapest)*.

Example. Let A be a nonempty set and let $f : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ be an increasing function on the set of subsets of A , meaning that

$$f(X) \subset f(Y) \quad \text{if } X \subset Y.$$

Prove that there exists T , a subset of A , such that $f(T) = T$.

Solution. Consider the family of sets

$$\mathcal{F} = \{K \in \mathcal{P}(A) \mid f(K) \subset K\}.$$

Because $A \in \mathcal{F}$, the family \mathcal{F} is not empty. Let T be the intersection of all sets in \mathcal{F} . We will show that $f(T) = T$.

If $K \in \mathcal{F}$, then $f(T) \subset f(K) \subset K$, and by taking the intersection over all $K \in \mathcal{F}$, we obtain that $f(T) \subset T$. Hence $T \in \mathcal{F}$.

Because f is increasing it follows that $f(f(T)) \subset f(T)$, and hence $f(T) \in \mathcal{F}$. Since T is included in every element of \mathcal{F} , we have $T \subset f(T)$. The double inclusion proves that $f(T) = T$, as desired. \square

Since it will be needed below, let us recall that a graph consists of a set of vertices connected by edges. Unless otherwise specified, our graphs have finitely many edges, there is at most one edge connecting two vertices, and the endpoints of each edge are distinct.

821. Let A and B be two sets. Find all sets X with the property that

$$\begin{aligned} A \cap X &= B \cap X = A \cap B, \\ A \cup B \cup X &= A \cup B. \end{aligned}$$

822. Prove that every graph has two vertices that are endpoints of the same number of edges.

823. Prove that a list can be made of all the subsets of a finite set such that

- (i) the empty set is the first set;
- (ii) each subset occurs once;
- (iii) each subset is obtained from the preceding by adding or deleting an element.

- 824.** Let M be a subset of $\{1, 2, 3, \dots, 15\}$ such that the product of any three distinct elements of M is not a square. Determine the maximum number of elements in M .
- 825.** Let S be a nonempty set and \mathcal{F} a family of $m \geq 2$ subsets of S . Show that among the sets of the form $A \Delta B$ with $A, B \in \mathcal{F}$ there are at least m that are distinct. (Here $A \Delta B = (A \setminus B) \cup (B \setminus A)$.)
- 826.** Consider the sequence of functions and sets

$$\dots \rightarrow A_n \xrightarrow{f_{n-1}} A_{n-1} \xrightarrow{f_{n-2}} A_{n-2} \xrightarrow{f_{n-3}} \dots \xrightarrow{f_3} A_3 \xrightarrow{f_2} A_2 \xrightarrow{f_1} A_1.$$

Prove that if the sets A_n are nonempty and finite for all n , then there exists a sequence of elements $x_n \in A_n$, $n = 1, 2, 3, \dots$, with the property that $f_n(x_{n+1}) = x_n$ for all $n \geq 1$.

- 827.** In a society of n people, any two persons who do not know each other have exactly two common acquaintances, and any two persons who know each other don't have other common acquaintances. Prove that in this society every person has the same number of acquaintances.
- 828.** Let A be a finite set and let $f : A \rightarrow A$ be a function. Prove that there exist the pairwise disjoint sets A_0, A_1, A_2, A_3 such that $A = A_0 \cup A_1 \cup A_2 \cup A_3$, $f(x) = x$ for any $x \in A_0$ and $f(A_i) \cap A_i = \emptyset$, $i = 1, 2, 3$. What if the set A is infinite?

6.1.2 Permutations

A permutation of a set S is a bijection $\sigma : S \rightarrow S$. Composition induces a group structure on the set of all permutations. We are concerned only with the finite case $S = \{1, 2, \dots, n\}$. The standard notation for a permutation is

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ a_1 & a_2 & a_3 & \dots & a_n \end{pmatrix},$$

with $a_i = \sigma(i)$, $i = 1, 2, \dots, n$.

A permutation is a cycle $(i_1 i_2 \dots i_n)$ if $\sigma(i_1) = i_2$, $\sigma(i_2) = i_3$, \dots , $\sigma(i_n) = i_1$, and $\sigma(j) = j$ for $j \neq i_1, i_2, \dots, i_n$. Any permutation is a product of disjoint cycles. A cycle of length two $(i_1 i_2)$ is called a transposition. Any permutation is a product of transpositions. For a given permutation σ , the parity of the number of transpositions in this product is always the same; the signature of σ , denoted by $\text{sign}(\sigma)$, is 1 if this number is even and -1 if this number is odd. An inversion is a pair (i, j) with $i < j$ and $\sigma(i) > \sigma(j)$.

Let us look at a problem from the 1979 Romanian Mathematical Olympiad, proposed by I. Raşa.

an angle of $\frac{2\pi}{n}$ around its center. Such a rotation can be written as the composition of two reflections that map the n -gon to itself, namely the reflection with respect to the perpendicular bisector of A_1A_3 and the reflection with respect to the perpendicular bisector of A_2A_3 (see Figure 37). These reflections define the permutations σ_1 and σ_2 . \square

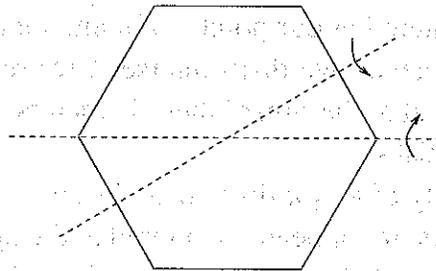


Figure 37

The following problems are left to the reader.

829. For each permutation a_1, a_2, \dots, a_{10} of the integers $1, 2, 3, \dots, 10$, form the sum

$$|a_1 - a_2| + |a_3 - a_4| + |a_5 - a_6| + |a_7 - a_8| + |a_9 - a_{10}|.$$

Find the average value of all such sums.

830. Find the number of permutations $a_1, a_2, a_3, a_4, a_5, a_6$ of the numbers $1, 2, 3, 4, 5, 6$ that can be transformed into $1, 2, 3, 4, 5, 6$ through exactly four transpositions (and not fewer).

831. Let $f(n)$ be the number of permutations a_1, a_2, \dots, a_n of the integers $1, 2, \dots, n$ such that (i) $a_1 = 1$ and (ii) $|a_i - a_{i+1}| \leq 2, i = 1, 2, \dots, n-1$. Determine whether $f(1996)$ is divisible by 3.

832. Consider the sequences of real numbers $x_1 > x_2 > \dots > x_n$ and $y_1 > y_2 > \dots > y_n$, and let σ be a nontrivial permutation of the set $\{1, 2, \dots, n\}$. Prove that

$$\sum_{i=1}^n (x_i - y_i)^2 < \sum_{i=1}^n (x_i - y_{\sigma(i)})^2.$$

833. Let a_1, a_2, \dots, a_n be a permutation of the numbers $1, 2, \dots, n$. We call a_i a *large* integer if $a_i > a_j$ for all $i < j < n$. Find the average number of large integers over all permutations of the first n positive integers.

834. Given some positive real numbers $a_1 < a_2 < \dots < a_n$ find all permutations σ with the property that

$$a_1 a_{\sigma(1)} < a_2 a_{\sigma(2)} < \dots < a_n a_{\sigma(n)}.$$

Example. Prove that any n points in the plane can be covered by finitely many disks with the sum of the diameters less than n and the distance between any two disks greater than 1.

Solution. First, note that if two disks of diameters d_1 and d_2 intersect, then they can be included in a disk of diameter $d_1 + d_2$.

Let us place n disks centered at our points, of some radius $a > 1$ the size of which will be specified later. Whenever two disks intersect, we replace them with a disk that covers them, of diameter equal to the sum of their diameters. We continue this procedure until we have only disjoint disks.

We thus obtained a family of $k \leq n$ disks with the sum of diameters equal to na and such that they cover the disks of diameter a centered at the points. Now let us shrink the diameters of the disks by b , with $1 < b < a$. Then the new disks cover our points, the sum of their diameters is $na - kb \leq na - b$, and the distances between disks are at least b . Choosing a and b such that $1 < b < a$ and $na - b \leq n$ would then lead to a family of circles with the sum of diameters less than n and at distance greater than 1 from each other. For example, we can let $a = 1 + \frac{1}{n}$ and $b = 1 + \frac{1}{2n}$. \square

- 837.** In how many regions do n great circles, any three nonintersecting, divide the surface of a sphere?
- 838.** In how many regions do n spheres divide the three-dimensional space if any two intersect along a circle, no three intersect along a circle, and no four intersect at one point?
- 839.** Given $n > 4$ points in the plane such that no three are collinear, prove that there are at least $\binom{n-3}{2}$ convex quadrilaterals whose vertices are four of the given points.
- 840.** An equilateral triangle of side length n is drawn with sides along a triangular grid of side length 1. What is the maximum number of grid segments on or inside the triangle that can be marked so that no three marked segments form a triangle?
- 841.** 1981 points lie inside a cube of side length 9. Prove that there are two points within a distance less than 1.
- 842.** What is the largest number of internal right angles that an n -gon (convex or not, with non-self-intersecting boundary) can have?
- 843.** A circle of radius 1 rolls without slipping on the outside of a circle of radius $\sqrt{2}$. The contact point of the circles in the initial position is colored. Any time a point of one circle touches a colored point of the other, it becomes itself colored. How many colored points will the moving circle have after 100 revolutions?
- 844.** Several chords are constructed in a circle of radius 1. Prove that if every diameter intersects at most k chords, then the sum of the lengths of the chords is less than $k\pi$.

- 845.** Inside a square of side 38 lie 100 convex polygons, each with an area at most π and the perimeter at most 2π . Prove that there exists a circle of radius 1 inside the square that does not intersect any of the polygons.
- 846.** Given a set M of $n \geq 3$ points in the plane such that any three points in M can be covered by a disk of radius 1, prove that the entire set M can be covered by a disk of radius 1.
- 847.** Prove that if a convex polyhedron has the property that every vertex belongs to an even number of edges, then any section determined by a plane that does not pass through a vertex is a polygon with an even number of sides.

6.1.4 Euler's Formula for Planar Graphs

This section is about a graph-theoretical result with geometric flavor, the famous Euler's formula. Recall that a graph is a collection of points, called vertices, some of which are joined by arcs, called edges. A planar graph is a graph embedded in the plane in such a way that edges do not cross. The connected components of the complement of a planar graph are called faces. For example, the graph in Figure 39 has four faces (this includes the infinite face). Unless otherwise specified, all our graphs are assumed to be connected.

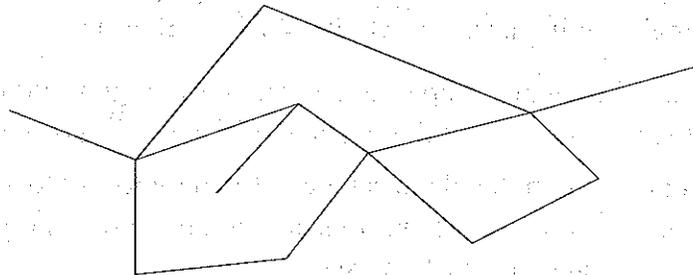


Figure 39

Euler's theorem. *Given a connected planar graph, denote by V the number of vertices, by E the number of edges, and by F the number of faces (including the infinite face). Then*

$$V - E + F = 2.$$

Proof. The proof is an easy induction on F . If $F = 1$ the graph is a tree, and the number of vertices exceeds that of edges by 1. The formula is thus verified in this case.

Let us now consider some $F > 1$ and assume that the formula holds for all graphs with at most $F - 1$ faces. Since there are at least two faces, the graph is not a tree. Therefore, it must contain cycles. Remove one edge from a cycle. The new graph is still

4. $m = 4, n = 3$, in which case $E = 12, V = 6, F = 8$; this is the regular octahedron.
5. $m = 5, n = 3$, in which case $E = 30, V = 12, F = 20$; this is the regular icosahedron.

We have proved the well-known fact that there are five Platonic solids. \square

- 848.** In the plane are given $n > 2$ points joined by segments, such that the interiors of any two segments are disjoint. Find the maximum possible number of such segments as a function of n .
- 849.** Three conflicting neighbors have three common wells. Can one draw nine paths connecting each of the neighbors to each of the wells such that no two paths intersect?
- 850.** Consider a polyhedron with at least five faces such that exactly three edges emerge from each vertex. Two players play the following game: the players sign their names alternately on precisely one face that has not been previously signed. The winner is the player who succeeds in signing the name on three faces that share a common vertex. Assuming optimal play, prove that the player who starts the game always wins.
- 851.** Denote by V the number of vertices of a convex polyhedron, and by Σ the sum of the (planar) angles of its faces. Prove that $2\pi V - \Sigma = 4\pi$.
- 852.** (a) Given a connected planar graph whose faces are polygons with at least three sides (no loops or bigons), prove that there is a vertex that belongs to at most five edges.
 (b) Prove that any map in the plane can be colored by five colors such that adjacent regions have different colors (the regions are assumed to be polygons, two regions are adjacent if they share at least one side).
- 853.** Consider a convex polyhedron whose faces are triangles and whose edges are oriented. A *singularity* is a face whose edges form a cycle, a vertex that belongs only to incoming edges, or a vertex that belongs only to outgoing edges. Show that the polyhedron has at least two singularities.

6.1.5 Ramsey Theory

Ramsey theory is a difficult branch of combinatorics, which gathers results that show that when a sufficiently large set is partitioned into a fixed number of subsets, one of the subsets has a certain property. Finding sharp bounds on how large the set should be is a truly challenging question, unanswered in most cases.

The origins of this field lie in Ramsey's theorem, which states that for every pair of positive integers (p, q) there is a smallest integer $R(p, q)$, nowadays called the Ramsey

such that for any partition of $\{1, 2, \dots, S(n)\}$ into n sets one of the sets will contain a Schur triple. No general formula for $S(n)$ exists although upper and lower bounds have been found. Our problem proves that $S(4) > 40$. In fact, $S(4) = 45$.

- 854.** What is the largest number of vertices that a complete graph can have so that its edges can be colored by two colors in such a way that no monochromatic triangle is formed?
- 855.** For the Ramsey numbers defined above, prove that $R(p, q) \leq R(p-1, q) + R(p, q-1)$. Conclude that for $p, q \geq 2$,

$$R(p, q) \leq \binom{p+q-2}{p-1}.$$

- 856.** The edges of a complete graph with $[k!e] + 1$ edges are colored by k colors. Prove that there is a triangle whose edges are colored by the same color.
- 857.** An international society has members from six different countries. The list of members contains 1978 names, numbered $1, 2, \dots, 1978$. Prove that there exists at least one member whose number is the sum of the numbers of two members from his/her own country, or twice as large as the number of one member from his/her country.
- 858.** Let n be a positive integer satisfying the following property: If n dominoes are placed on a 6×6 chessboard with each domino covering exactly two unit squares, then one can always place one more domino on the board without moving any other dominoes. Determine the maximum value of n .

6.2 Binomial Coefficients and Counting Methods

6.2.1 Combinatorial Identities

The binomial coefficient $\binom{n}{k}$ counts the number of ways one can choose k objects from given n . Binomial coefficients show up in Newton's binomial expansion

$$(x+1)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1} + \dots + \binom{n}{n-1}x + \binom{n}{n}.$$

Explicitly,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\dots(n-k+1)}{k!} \quad \text{if } 0 \leq k \leq n.$$

The recurrence relation

$$\binom{n}{m}_q = \binom{n-1}{m}_q + q^{n-m} \binom{n-1}{m-1}_q,$$

gives rise to what is called the q -Pascal triangle.

859. Prove that

$$\binom{2k}{k} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} (2 \sin \theta)^{2k} d\theta.$$

860. Consider the triangular $n \times n$ matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Compute the matrix A^k , $k \geq 1$.

861. Let $(F_n)_n$ be the Fibonacci sequence, $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$. Prove that for any positive integer n ,

$$F_1 \binom{n}{1} + F_2 \binom{n}{2} + \cdots + F_n \binom{n}{n} = F_{2n}.$$

862. For an arithmetic sequence $a_1, a_2, \dots, a_n, \dots$, let $S_n = a_1 + a_2 + \cdots + a_n$, $n \geq 1$. Prove that

$$\sum_{k=0}^n \binom{n}{k} a_{k+1} = \frac{2^n}{n+1} S_{n+1}.$$

863. Show that for any positive integer n , the number

$$S_n = \binom{2n+1}{0} \cdot 2^{2n} + \binom{2n+1}{2} \cdot 2^{2n-2} \cdot 3 + \cdots + \binom{2n+1}{2n} \cdot 3^n$$

is the sum of two consecutive perfect squares.

864. For a positive integer n define the integers a_n , b_n , and c_n by

$$a_n + b_n \sqrt[3]{2} + c_n \sqrt[3]{4} = (1 + \sqrt[3]{2} + \sqrt[3]{4})^n.$$

Prove that

$$2^{-\frac{n}{3}} \sum_{k=0}^n \binom{n}{k} a_k = \begin{cases} a_n & \text{if } n \equiv 0 \pmod{3}, \\ b_n \sqrt[3]{2} & \text{if } n \equiv 2 \pmod{3}, \\ c_n \sqrt[3]{4} & \text{if } n \equiv 1 \pmod{3}. \end{cases}$$

865. Prove the analogue of Newton's binomial formula

$$[x + y]_n = \sum_{k=0}^n \binom{n}{k} [x]_k [y]_{n-k},$$

where $[x]_n = x(x-1)\cdots(x-n+1)$.

866. Prove that the quantum binomial coefficients $\binom{n}{k}_q$ previously defined satisfy

$$\sum_{k=0}^n (-1)^k q^{\frac{k(k-1)}{2}} \binom{n}{k}_q = 0.$$

6.2.2 Generating Functions

The terms of a sequence $(a_n)_{n \geq 0}$ can be combined into a function

$$G(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots,$$

called the generating function of the sequence. Sometimes this function can be written in closed form and carries useful information about the sequence. For example, if the sequence satisfies a second-order linear recurrence, say $a_{n+1} + ua_n + va_{n-1} = 0$, then the generating function satisfies the functional equation

$$G(x) - a_0 - a_1x + ux(G(x) - a_0) + vx^2G(x) = 0.$$

This equation can be solved easily, giving

$$G(x) = \frac{a_0 + (ua_0 + a_1)x}{1 + ux + vx^2}.$$

If r_1 and r_2 are the roots of the characteristic equation $\lambda^2 + u\lambda + v = 0$, then by using the partial fraction decomposition, we obtain

$$G(x) = \frac{a_0 + (ua_0 + a_1)x}{(1 - r_1x)(1 - r_2x)} = \frac{\alpha}{1 - r_1x} + \frac{\beta}{1 - r_2x} = \sum_{n=0}^{\infty} (\alpha r_1^n + \beta r_2^n)x^n.$$

And we recover the general-term formula $a_n = \alpha r_1^n + \beta r_2^n$, $n \geq 0$, where α and β depend on the initial condition.

It is useful to notice the analogy with the method of the Laplace transform used for solving linear ordinary differential equations. Recall that the Laplace transform of a function $y(t)$ is defined as

$$\mathcal{L}y(s) = \int_0^{\infty} y(t)e^{ts} dt.$$

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869. (a) Prove the identity

$$\binom{m+n}{k} = \sum_{j=0}^k \binom{m}{j} \binom{n}{k-j}.$$

(b) Prove that the quantum binomial coefficients defined in the previous section satisfy the identity

$$\binom{m+n}{k}_q = \sum_{j=0}^k q^{(m-j)(k-j)} \binom{m}{j}_q \binom{n}{k-j}_q.$$

870. Compute the sum

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^m \binom{n}{m}.$$

871. Write in short form the sum

$$\binom{n}{k} + \binom{n+1}{k} + \binom{n+2}{k} + \cdots + \binom{n+m}{k}.$$

872. Prove that the Fibonacci numbers satisfy

$$F_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots.$$

873. Denote by $P(n)$ the number of partitions of the positive integer n , i.e., the number of ways of writing n as a sum of positive integers. Prove that the generating function of $P(n)$, $n \geq 1$, is given by

$$\sum_{n=0}^{\infty} P(n)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots}$$

with the convention $P(0) = 1$.

874. Prove that the number of ways of writing n as a sum of distinct positive integers is equal to the number of ways of writing n as a sum of odd positive integers.

875. Let p be an odd prime number. Find the number of subsets of $\{1, 2, \dots, p\}$ with the sum of elements divisible by p .

876. For a positive integer n , denote by $S(n)$ the number of choices of the signs “+” or “-” such that $\pm 1 \pm 2 \pm \cdots \pm n = 0$. Prove that

$$S(n) = \frac{2^{n-1}}{\pi} \int_0^{2\pi} \cos t \cos 2t \cdots \cos nt dt.$$

- 877.** The distinct positive integers $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$, with $n \geq 2$, have the property that the $\binom{n}{2}$ sums $a_i + a_j$ are the same as the $\binom{n}{2}$ sums $b_i + b_j$ (in some order). Prove that n is a power of 2.
- 878.** Let $A_1, A_2, \dots, A_n, \dots$ and $B_1, B_2, \dots, B_n, \dots$ be sequences of sets defined by $A_1 = \emptyset, B_1 = \{0\}, A_{n+1} = \{x + 1 \mid x \in B_n\}, B_{n+1} = (A_n \cup B_n) \setminus (A_n \cap B_n)$. Determine all positive integers n for which $B_n = \{0\}$.

6.2.3 Counting Strategies

We illustrate how some identities can be proved by counting the number of elements of a set in two different ways. For example, we give a counting argument to the well-known reciprocity law, which we have already encountered in Section 5.1.3, of the greatest integer function.

Example. Given p and q coprime positive integers, prove that

$$\left\lfloor \frac{p}{q} \right\rfloor + \left\lfloor \frac{2p}{q} \right\rfloor + \dots + \left\lfloor \frac{(q-1)p}{q} \right\rfloor = \left\lfloor \frac{q}{p} \right\rfloor + \left\lfloor \frac{2q}{p} \right\rfloor + \dots + \left\lfloor \frac{(p-1)q}{p} \right\rfloor.$$

Solution. Let us look at the points of integer coordinates that lie inside the rectangle with vertices $O(0, 0), A(q, 0), B(q, p), C(0, p)$ (see Figure 41). There are $(p-1)(q-1)$ such points. None of them lies on the diagonal OB because p and q are coprime. Half of them lie above the diagonal and half below.

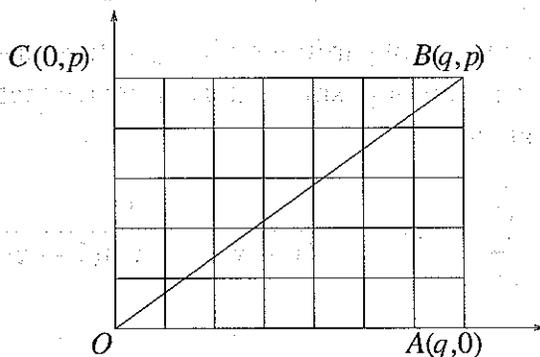


Figure 41

Now let us count by a different method the points underneath the line OB . The equation of this line is $y = \frac{p}{q}x$. For each $0 < k < q$ on the vertical segment $x = k$ there are $\lfloor kp/q \rfloor$ points below OB . Summing up, we obtain

$$\left\lfloor \frac{p}{q} \right\rfloor + \left\lfloor \frac{2p}{q} \right\rfloor + \dots + \left\lfloor \frac{(q-1)p}{q} \right\rfloor = \frac{(p-1)(q-1)}{2}.$$

The expression on the right remains unchanged if we switch p and q , which proves the identity. \square

880. Prove the combinatorial identity

$$\sum_{k=1}^n k \binom{n}{k}^2 = n \binom{2n-1}{n-1}.$$

881. Prove the identity

$$\sum_{k=0}^m \binom{m}{k} \binom{n+k}{m} = \sum_{k=0}^m \binom{m}{k} \binom{n}{k} 2^k.$$

882. For integers $0 \leq k \leq n$, $1 \leq m \leq n$, prove the identity

$$\sum_{j=0}^m \binom{m}{i} \binom{n-i}{k} = \sum_{i=0}^m \binom{m}{i} \binom{n-m}{k-i} 2^{m-i}.$$

883. Show that for any positive integers p and q ,

$$\sum_{k=0}^q \frac{1}{2^{p+k}} \binom{p+k}{k} + \sum_{k=0}^p \frac{1}{2^{q+k}} \binom{q+k}{k} = 2.$$

884. Let $c_n = \binom{n}{\lfloor n/2 \rfloor}$. Prove that

$$\sum_{k=0}^n \binom{n}{k} c_k c_{n-k} = c_n c_{n+1}.$$

885. Let p and q be odd, coprime positive integers. Set $p' = \frac{p-1}{2}$ and $q' = \frac{q-1}{2}$. Prove the identity

$$\left(\left\lfloor \frac{q}{p} \right\rfloor + \left\lfloor \frac{2q}{p} \right\rfloor + \cdots + \left\lfloor \frac{p'q}{p} \right\rfloor \right) + \left(\left\lfloor \frac{p}{q} \right\rfloor + \left\lfloor \frac{2p}{q} \right\rfloor + \cdots + \left\lfloor \frac{q'p}{q} \right\rfloor \right) = p'q'.$$

Now we turn to more diverse counting arguments.

Example. What is the number of ways of writing the positive integer n as an ordered sum of m positive integers?

Solution. This is a way of saying that we have to count the number of m -tuples of positive integers (x_1, x_2, \dots, x_m) satisfying the equation $x_1 + x_2 + \cdots + x_m = n$. These m -tuples are in one-to-one correspondence with the strictly increasing sequences $0 < y_1 < y_2 < \cdots < y_m = n$ of positive integers, with the correspondence given by $y_1 = x_1$, $y_2 = x_1 + x_2, \dots, y_m = x_1 + x_2 + \cdots + x_m$. The numbers y_1, y_2, \dots, y_{m-1} can be chosen in $\binom{n-1}{m-1}$ ways from $1, 2, \dots, n-1$. Hence the answer to the question is $\binom{n-1}{m-1}$.

$i \neq j$ the societies to which C_i belongs are all different from the societies to which C_j belongs. Moreover, condition (ii) guarantees that any society will contain one of the clubs C_i . Therefore, $m_1 + m_2 + \cdots + m_r = k$.

From condition (i) we see that any two clubs C_i and C_j have in common exactly the student x . Therefore, in C_1, C_2, \dots, C_r there are altogether $2(m_1 + m_2 + \cdots + m_r) + 1$ students. But these are all the students, because by condition (i) any other student is in some club with x . We obtain

$$2(m_1 + m_2 + \cdots + m_r) + 1 = 2k + 1 = n.$$

Hence $k = \frac{n-1}{2}$ is the only possibility. And this situation can be achieved when all students belong to one club, which then belongs to $\frac{n-1}{2}$ societies. \square

Here is a third example.

Example. On an 8×8 chessboard whose squares are colored black and white in an arbitrary way we are allowed to simultaneously switch the colors of all squares in any 3×3 and 4×4 region. Can we transform any coloring of the board into one where all the squares are black?

Solution. We claim that the answer is no. It is a matter of counting into how many regions can an all-black board be transformed by applying the two moves several times. The total number of 3×3 regions is $(8 - 2) \times (8 - 2) = 36$, which is the same as the number of moves in which the colors in a 3×3 region are switched. As for the 4×4 regions, there are $(8 - 3) \times (8 - 3) = 25$ of them. Hence the total number of colorings that can be obtained from an all-black coloring by applying the specified operations does not exceed

$$2^{36} \times 2^{25} = 2^{61}.$$

This number is less than the total number of colorings, which is 2^{64} . Hence there are colorings that cannot be achieved. Since the operations are reversible, this actually proves our claim. \square

And now the problems.

886. Two hundred students took part in a mathematics contest. They had 6 problems to solve. It is known that each problem was correctly solved by at least 120 participants. Prove that there exist two participants such that every problem was solved by at least one of them.

887. Prove that the number of nonnegative integer solutions to the equation

$$x_1 + x_2 + \cdots + x_m = n$$

is equal to $\binom{m+n-1}{m-1}$.

- 888.** A number n of tennis players take part in a tournament in which each of them plays exactly one game with each of the others. If x_i and y_i denote the number of victories, respectively, losses, of the i th player, $i = 1, 2, \dots, n$, show that

$$x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2 + y_2^2 + \cdots + y_n^2.$$

- 889.** Let A be a finite set and f and g two functions on A . Let m be the number of pairs $(x, y) \in A \times A$ for which $f(x) = g(y)$, n the number of pairs for which $f(x) = f(y)$, and k the number of pairs for which $g(x) = g(y)$. Prove that

$$2m \leq n + k.$$

- 890.** A set S containing four positive integers is called *connected* if for every $x \in S$ at least one of the numbers $x - 1$ and $x + 1$ belongs to S . Let C_n be the number of connected subsets of the set $\{1, 2, \dots, n\}$.

(a) Evaluate C_7 .

(b) Find a general formula for C_n .

- 891.** Prove that the set of numbers $\{1, 2, \dots, 2005\}$ can be colored with two colors such that any of its 18-term arithmetic sequences contains both colors.

- 892.** For $A = \{1, 2, \dots, 100\}$ let A_1, A_2, \dots, A_m be subsets of A with four elements with the property that any two have at most two elements in common. Prove that if $m \geq 40425$ then among these subsets there exist 49 whose union is equal to A but with the union of any 48 of them not equal to A .

- 893.** Let S be a finite set of points in the plane. A linear partition of S is an unordered pair $\{A, B\}$ of subsets of S such that $A \cup B = S$, $A \cap B = \emptyset$, and A and B lie on opposite sides of some straight line disjoint from S (A or B may be empty). Let L_S be the number of linear partitions of S . For each positive integer n , find the maximum of L_S over all sets S of n points.

- 894.** Let A be a 101-element subset of the set $S = \{1, 2, \dots, 1000000\}$. Prove that there exist numbers t_1, t_2, \dots, t_{100} in S such that the sets

$$A_j = \{x + t_j \mid x \in A\}, \quad j = 1, 2, \dots, 100,$$

are pairwise disjoint.

- 895.** Given a set A with n^2 elements, $n \geq 2$, and \mathcal{F} a family of subsets of A each of which has n elements, suppose that any two sets of \mathcal{F} have at most one element in common.

(a) Prove that there are at most $n^2 + n$ sets in \mathcal{F} .

(b) In the case $n = 3$, show with an example that this bound can be reached.

896. A sheet of paper in the shape of a square is cut by a line into two pieces. One of the pieces is cut again by a line, and so on. What is the minimum number of cuts one should perform such that among the pieces one can find one hundred polygons with twenty sides.

897. Twenty-one girls and twenty-one boys took part in a mathematics competition. It turned out that

- (i) each contestant solved at most six problems, and
- (ii) for each pair of a girl and a boy, there was at least one problem that was solved by both the girl and the boy.

Show that there is a problem that was solved by at least three girls and at least three boys.

6.2.4 The Inclusion–Exclusion Principle

A particular counting method that we emphasize is the inclusion–exclusion principle, also known as the Boole–Sylvester formula. It concerns the counting of the elements in a union of sets $A_1 \cup A_2 \cup \dots \cup A_n$, and works as follows. If we simply wrote

$$|A_1 \cup A_2 \cup \dots \cup A_n| = |A_1| + |A_2| + \dots + |A_n|,$$

we would overcount the elements in the intersections $A_i \cap A_j$. Thus we have to subtract $|A_1 \cap A_2| + |A_1 \cap A_3| + \dots + |A_{n-1} \cap A_n|$. But then the elements in the triple intersections $A_i \cap A_j \cap A_k$ were both added and subtracted. We have to put them back. Therefore, we must add $|A_1 \cap A_2 \cap A_3| + \dots + |A_{n-2} \cap A_{n-1} \cap A_n|$. And so on. The final formula is

$$|A_1 \cup A_2 \cup \dots \cup A_n| = \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \dots + (-1)^{n-1} |A_1 \cap A_2 \cap \dots \cap A_n|.$$

Example. How many integers less than 1000 are not divisible by 2, 3, or 5?

Solution. To answer the question, we will count instead how many integers between 1 and 1000 are divisible by 2, 3, or 5. Denote by A_2 , A_3 , and A_5 be the sets of integers divisible by 2, 3, respectively, 5. The Boole–Sylvester formula counts $|A_2 \cup A_3 \cup A_5|$ as

$$\begin{aligned} & |A_2| + |A_3| + |A_5| - |A_2 \cap A_3| - |A_2 \cap A_5| - |A_3 \cap A_5| + |A_2 \cap A_3 \cap A_5| \\ &= \left\lfloor \frac{1000}{2} \right\rfloor + \left\lfloor \frac{1000}{3} \right\rfloor + \left\lfloor \frac{1000}{5} \right\rfloor - \left\lfloor \frac{1000}{6} \right\rfloor - \left\lfloor \frac{1000}{10} \right\rfloor - \left\lfloor \frac{1000}{15} \right\rfloor + \left\lfloor \frac{1000}{30} \right\rfloor \\ &= 500 + 333 + 200 - 166 - 100 - 66 + 33 = 734. \end{aligned}$$

It follows that there are $1000 - 734 = 266$ integers less than 1000 that are not divisible by 2, 3, or 5. \square

The second example comes from I. Tomescu's book *Problems in Combinatorics* (Wiley, 1985).

Example. An alphabet consists of the letters a_1, a_2, \dots, a_n . Prove that the number of all words that contain each of these letters twice, but with no consecutive identical letters, is equal to

$$\frac{1}{2^n} \left[(2n)! - \binom{n}{1} 2(2n-1)! + \binom{n}{2} 2^2(2n-2)! - \dots + (-1)^n 2^n n! \right].$$

Solution. The number of such words without imposing the restriction about consecutive letters is

$$\frac{(2n)!}{(2!)^n} = \frac{(2n)!}{2^n}.$$

This is so because the identical letters can be permuted.

Denote by A_i the number of words formed with the n letters, each occurring twice, for which the two letters a_i appear next to each other. The answer to the problem is then

$$\frac{(2n)!}{2^n} - |A_1 \cup A_2 \cup \dots \cup A_n|.$$

We evaluate $|A_1 \cup A_2 \cup \dots \cup A_n|$ using the inclusion-exclusion principle. To this end, let us compute $|A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}|$ for some indices $i_1, i_2, \dots, i_k, k \leq n$. Collapse the consecutive letters $a_{i_j}, j = 1, 2, \dots, k$. As such, we are, in fact, computing the number of words made of the letters a_1, a_2, \dots, a_n in which $a_{i_1}, a_{i_2}, \dots, a_{i_k}$ appear once and all other letters appear twice. This number is clearly equal to

$$\frac{(2n-k)!}{2^{n-k}},$$

since such a word has $2n-k$ letters, and identical letters can be permuted. There are $\binom{n}{k}$ k -tuples (i_1, i_2, \dots, i_k) . We thus have

$$\begin{aligned} |A_1 \cup A_2 \cup \dots \cup A_n| &= \sum_k \sum_{i_1, \dots, i_k} (-1)^{k-1} |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}| \\ &= \sum_k (-1)^{k-1} \binom{n}{k} \frac{(2n-k)!}{2^{n-k}}, \end{aligned}$$

and the formula is proved. \square

898. Let m, n, p, q, r, s be positive integers such that $p < r < m$ and $q < s < n$. In how many ways can one travel on a rectangular grid from $(0, 0)$ to (m, n) such that at each step one of the coordinates increases by one unit and such that the path avoids the points (p, q) and (r, s) ?

- 899.** Let E be a set with n elements and F a set with p elements, $p \leq n$. How many surjective (i.e., onto) functions $f : E \rightarrow F$ are there?
- 900.** A permutation σ of a set S is called a derangement if it does not have fixed points, i.e., if $\sigma(x) \neq x$ for all $x \in S$. Find the number of derangements of the set $\{1, 2, \dots, n\}$.
- 901.** Given a graph with n vertices, prove that either it contains a triangle, or there exists a vertex that is the endpoint of at most $\lfloor \frac{n}{2} \rfloor$ edges.
- 902.** Let $m \geq 5$ and n be given positive integers, and suppose that \mathcal{P} is a regular $(2n+1)$ -gon. Find the number of convex m -gons having at least one acute angle and having vertices exclusive among the vertices of \mathcal{P} .
- 903.** Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. For all functions $f : S^1 \rightarrow S^1$ set $f^1 = f$ and $f^{n+1} = f \circ f^n$, $n \geq 1$. Call $w \in S^1$ a periodic point of f of period n if $f^i(w) \neq w$ for $i = 1, \dots, n-1$ and $f^n(w) = w$. If $f(z) = z^m$, m a positive integer, find the number of periodic points of f of period 1989.
- 904.** For positive integers x_1, x_2, \dots, x_n denote by $[x_1, x_2, \dots, x_n]$ their least common multiple and by (x_1, x_2, \dots, x_n) their greatest common divisor. Prove that for positive integers a, b, c ,

$$\frac{[a, b, c]^2}{[a, b][b, c][c, a]} = \frac{(a, b, c)^2}{(a, b)(b, c)(c, a)}$$

- 905.** A $150 \times 324 \times 375$ rectangular solid is made by gluing together $1 \times 1 \times 1$ cubes. An internal diagonal of this solid passes through the interiors of how many of the $1 \times 1 \times 1$ cubes?

6.3 Probability

6.3.1 Equally Likely Cases

In this section we consider experiments with finitely many outcomes each of which can occur with equal probability. In this case the probability of an event A is given by

$$P(A) = \frac{\text{number of favorable outcomes}}{\text{total number of possible outcomes}}$$

The computation of the probability is purely combinatorial; it reduces to a counting problem.

We start with the example that gave birth to probability theory.

906. Let v and w be distinct, randomly chosen roots of the equation $z^{1997} - 1 = 0$. Find the probability that $\sqrt{2 + \sqrt{3}} \leq |v + w|$.
907. Find the probability that in a group of n people there are two with the same birthday. Ignore leap years.
908. A solitaire game is played as follows. Six distinct pairs of matched tiles are placed in a bag. The player randomly draws tiles one at a time from the bag and retains them, except that matching tiles are put aside as soon as they appear in the player's hand. The game ends if the player ever holds three tiles, no two of which match; otherwise, the drawing continues until the bag is empty. Find the probability that the bag will be emptied.
909. An urn contains n balls numbered $1, 2, \dots, n$. A person is told to choose a ball and then extract m balls among which is the chosen one. Suppose he makes two independent extractions, where in each case he chooses the remaining $m - 1$ balls at random. What is the probability that the chosen ball can be determined?
910. A bag contains 1993 red balls and 1993 black balls. We remove two balls at a time repeatedly and
- discard them if they are of the same color,
 - discard the black ball and return to the bag the red ball if they are of different colors.
- What is the probability that this process will terminate with one red ball in the bag?
911. The numbers 1, 2, 3, 4, 5, 6, 7, and 8 are written on the faces of a regular octahedron so that each face contains a different number. Find the probability that no two consecutive numbers are written on faces that share an edge, where 8 and 1 are considered consecutive.
912. What is the probability that a permutation of the first n positive integers has the numbers 1 and 2 within the same cycle.
913. An unbiased coin is tossed n times. Find a formula, in closed form, for the expected value of $|H - T|$, where H is the number of heads, and T is the number of tails.
914. Prove the identities

$$\sum_{k=1}^n \frac{1}{(k-1)!} \sum_{i=0}^{n-k} \frac{(-1)^i}{i!} = 1,$$

$$\sum_{k=1}^n \frac{k}{(k-1)!} \sum_{i=0}^{n-k} \frac{(-1)^i}{i!} = 2.$$

- 915.** An exam consists of 3 problems selected randomly from a list of $2n$ problems, where n is an integer greater than 1. For a student to pass, he needs to solve correctly at least two of the three problems. Knowing that a certain student knows how to solve exactly half of the $2n$ problems, find the probability that the student will pass the exam.
- 916.** The probability that a woman has breast cancer is 1%. If a woman has breast cancer, the probability is 60% that she will have a positive mammogram. However, if a woman does not have breast cancer, the mammogram might still come out positive, with a probability of 7%. What is the probability for a woman with positive mammogram to actually have cancer?
- 917.** Find the probability that in the process of repeatedly flipping a coin, one will encounter a run of 5 heads before one encounters a run of 2 tails.
- 918.** The temperatures in Chicago and Detroit are x° and y° , respectively. These temperatures are not assumed to be independent; namely, we are given the following:
- (i) $P(x^\circ = 70^\circ) = a$, the probability that the temperature in Chicago is 70° ,
 - (ii) $P(y^\circ = 70^\circ) = b$, and
 - (iii) $P(\max(x^\circ, y^\circ) = 70^\circ) = c$.
- Determine $P(\min(x^\circ, y^\circ) = 70^\circ)$ in terms of a , b , and c .
- 919.** An urn contains both black and white marbles. Each time you pick a marble you return it to the urn. Let p be the probability of drawing a white marble and $q = 1 - p$ the probability of drawing a black marble. Marbles are drawn until n black marbles have been drawn. If $n + x$ is the total number of draws, find the probability that $x = m$.
- 920.** Three independent students took an exam. The random variable X , representing the students who passed, has the distribution

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{2}{5} & \frac{13}{30} & \frac{3}{20} & \frac{1}{60} \end{pmatrix}.$$

Find each student's probability of passing the exam.

- 921.** Given the independent events A_1, A_2, \dots, A_n with probabilities p_1, p_2, \dots, p_n , find the probability that an odd number of these events occurs.
- 922.** Out of every batch of 100 products of a factory, 5 are quality checked. If one sample does not pass the quality check, then the whole batch of one hundred will be rejected. What is the probability that a batch is rejected if it contains 5% faulty products.
- 923.** There are two jet planes and a propeller plane at the small regional airport of Gauss City. A plane departs from Gauss City and arrives in Eulerville, where there were