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On the Ergodic Theorem for Affine Actions on Hilbert Space

Ionut Chifan*   Thomas Sinclair†

Abstract

The note establishes a new weak mean ergodic theorem (Theorem A) for 1-cocycles associated to weakly mixing representations of amenable groups.

Introduction

In a groundbreaking paper [15], Shalom discovered deep connections between the representation theory of an amenable group and aspects of its large-scale geometry. One motivation for this work, among others, was the development of a "spectral" approach to Gromov's celebrated theorem on the virtual nilpotency of groups of polynomial growth [10] (see also [1, 7, 12, 13, 16]). More precisely, Shalom established, Theorem 1.11 in [15], that if it could be shown that any group of polynomial growth G possessed property H_{FD} (see Definition 2.9), then this would suffice to establish that G would have a finite-index subgroup with infinite abelianization—the key step in Gromov's proof which involves the use of Hilbert's 5^{th} problem. As a means of establishing property H_{FD}, Shalom conjectured that for a group of polynomial growth, a sequence of almost fixed points for any affine action with weakly mixing linear part could be obtained by averaging the associated 1-cocycle over an appropriate subsequence of n-balls centered at the identity: see section 6.7 in [15]. This conjecture was partly based on his observation that for such groups a subsequence of the n-balls must possess a

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strong, quantitative Følner sequence. While Shalom did manage to show that large classes of amenable groups, including polycyclic groups, do have property H_{FD}, the proofs are non-geometric, relying in the polycyclic case on deep results of Delorme [6] about the structure of connected, solvable Lie groups. In their paper [3], Cornulier, Tessera, and Valette investigated a generalized version of Shalom's notion of a strong, quantitative Følner sequence, which they called a *controlled Følner sequence*. Further, they made a significant contribution to Shalom's program through their investigation of averaging properties of groups over controlled Følner sequences, which has directly influenced the approach taken here.

By the results of Cornulier–Tessera–Valette [3] and Tessera [17, 18, 19] many classes of amenable groups are known to possess controlled Følner sequences (see Proposition 1.10 below), these classes roughly corresponding to the classes of groups known to possess property H_{FD}. This motivates the following question:

Question 0.1. Does every finitely generated group admitting a controlled Følner sequence have property H_{FD} of Shalom?

Statement of results

We prove a weak mean ergodic theorem for affine actions of finitely generated amenable groups on Hilbert space. A sequence (μ_n) of regular Borel probability measures on a countable discrete group G forms a *Reiter sequence* if $\|\mu_n - g * \mu_n\| \to 0$ for all $g \in G$, where $g * \mu_n(h) = \mu_n(g^{-1}h)$. A countable discrete group is said to be *amenable* if it admits a Reiter sequence.

Theorem A (Weak Mean Ergodic Theorem). *Let $\pi : G \to \mathcal{O}(\mathcal{H})$ be an ergodic orthogonal representation of a finitely generated amenable group G, and let $b : G \to \mathcal{H}$ be a 1-cocycle associated to π. Let S be a finite symmetric generating set for G, and let $|\cdot|$ denote the word length in S. If (μ_n) is a Reiter sequence for G, then*

$$\int \frac{1}{|g|}\, b(g)\, d\mu_n(g) \to 0 \tag{0.1}$$

in the weak topology on $\mathcal{H}$. If π is weakly mixing, then

$$\int \frac{1}{|g|}\, |\langle b(g), \xi\rangle|\, d\mu_n(g) \to 0 \tag{0.2}$$

for all $\xi \in \mathcal{H}$.

Note that while $\frac{1}{|e|}$ is technically undefined, by convention it will be understood to denote 0 throughout.

In the weak mixing case, Theorem A states that the 1-cocycle must be "almost weakly sublinear" in the sense that for any $\epsilon > 0$ and $\xi \in \mathcal{H}$, the subset consisting of all elements $g \in G$ such that $|\langle b(g), \xi\rangle| \geq \epsilon|g|$ has measure 0 for all left invariant means on G. We show in Theorem 2.4 below that that for a group admitting a *controlled Følner sequence* (see Definition 1.7), for every "weakly sublinear" 1-cocycle (i.e., one for which for any $\epsilon > 0$ and $\xi \in \mathcal{H}$, the subset consisting of all elements $g \in G$ such that $|\langle b(g), \xi\rangle| \geq \epsilon|g|$ is *finite*) the associated affine action on Hilbert space admits a sequence of almost fixed points. Thus, the obstruction to settling Question 0.1 in the positive is addressing the gap between "measure zero" sets one hand and finite sets on the other. We note, that in the case of a *compact* representation, almost weak sublinearity is equivalent weak sublinearity. It seems plausible that through additional structural analysis on the 1-cocycle or the group, one may be able to derive weak sublinearity in the general case and close the gap.

Specializing Theorem A to the integers, if $b : \mathbb{Z} \to \mathcal{H}$ is a 1-cocycle, then b is completely determined by $\xi \doteq b(1)$, so that for $n \geq 1$ we have $\frac{1}{n}b(n) = A_n(\xi) := \frac{1}{n}\sum_{k=0}^{n-1}\pi(k)\xi$: a similar formula holds for $-n$ via the identity $b(-n) = -\pi(-n)b(n)$. So, in this case the result reduces to the fact that the Cesàro sums $C_n(\xi,\eta) = \frac{1}{n}\sum_{k=1}^{n}\langle A_k(\xi),\eta\rangle$ and $C'_n(\xi,\eta) = \frac{1}{n}\sum_{k=1}^{n}|\langle A_k(\xi),\eta\rangle|$ converge to 0 for all $\xi,\eta \in \mathcal{H}$. In fact, the stronger summation holds for all ergodic representations and is equivalent to the (weak) mean ergodic theorem of von Neumann.

In fact, for the class of abelian groups, the above result gives a new, geometrically flavored proof of the mean ergodic theorem in combination with the following result.

Theorem B. *Let* G *be finitely generated amenable group admitting a* controlled Følner sequence. *Let* $\pi : G \to \mathcal{O}(\mathcal{H})$ *be an orthogonal representation, and let* $b : G \to \mathcal{H}$ *be a 1-cocycle associated to* π. *Suppose that*

$$\int \frac{1}{|g|}\langle b(g^{-1}),\xi\rangle \, d\mu_n(g) \to 0 \tag{0.3}$$

for all $\xi \in \mathcal{H}$ *and all Reiter sequences* (μ_n). *Then the affine action* $G \curvearrowright^T \mathcal{H}$ *associated to* b *admits a sequence of almost fixed points.*

To see how this implies the mean ergodic theorem for $\mathbb{Z}$, we point out that by an observation of Cornulier–Tessera–Valette (Proposition 3.1 in [3]) a consequence of $G \curvearrowright^T \mathcal{H}$ admitting almost fixed points is that

$$\frac{1}{|g|}\|b(g)\| \to 0$$

as $|g| \to \infty$; in other words, the 1-cocycle b has *sublinear growth*. In fact, sublinearity of a 1-cocycle is actually equivalent in general to the mean ergodic theorem, i.e., the statement that

$$\int \frac{1}{|g|}\|b(g)\| \, d\mu_n(g) \to 0$$

for all Reiter sequences (μ_n) (see Proposition 1.16).

The significance of averaging on the right rather than on the left in Theorem B is that it allows one to conclude that the cocycle is weakly sublinear, i.e., $\frac{1}{|g|}b(g) \to 0$ in the weak topology, from which point averaging arguments over a controlled Følner sequence produce the desired sequence of almost fixed points. This key to this argument is the fact that $g \mapsto b(g)$ is a lipschitz function, i.e., $\|b(gs) - b(g)\|$ is uniformly bounded in g. Alas, this is not necessarily the case for $\|b(sg) - b(g)\|$ which again prevents us from settling Question 0.1.

Remarks on the proofs

The paper is an application of the authors' investigations into the "large scale" properties of affine actions of groups on Hilbert space. Though the above results are stated for affine actions, even in this case the proofs rely on a coarsening of the notion of an affine action, the concept of an *array*, formalized by the authors in [2]. The main novelty of this viewpoint is that it allows one to construct the "absolute value" of a 1-cocycle $b : G \to \mathcal{H}$ which lies in the G-invariant positive cone $\mathcal{V} \subset \mathcal{H} \otimes \mathcal{H}$ which allows one to naturally use the weak mixingness to derive the stronger ergodic theorem in that case. Note that such a map cannot lie within a uniformly bounded distance of an (unbounded) 1-cocycle, since the equation $b(g) = -\pi_g b(g^{-1})$ holds for all $g \in G$ for any 1-cocycle b.

The notion of an array is best viewed from a geometric, rather than algebraic, perspective. Indeed, a length function on a discrete group G may be viewed as a positive array associated with the trivial representation. In general, an array can be thought of as a Hilbert-space valued length function on G which is compatible with some orthogonal G-representation π. The presence of an array then becomes a tool through which properties of the representation can be used to impose large scale conditions on the group, and *vice versa*. For example, it is shown in [2], Proposition 1.7.3, that a non-amenable group admitting a proper array into its left-regular representation, e.g., non-elementary Gromov hyperbolic groups, cannot be decomposed as a direct product of infinite groups. Turning to the topic at hand, the presence of a controlled Følner sequence imposes a strong large-scale "finite dimensionality" condition on the group G—for the case of weak polynomial growth, a point already well made in [10]. Viewed in this light, the content of Theorem B is that this forces any geometric realization of the group which is uniformly distributed throughout an infinite-dimensional Hilbert space to be essentially degenerate.

1 Geometry and Representation Theory

In this section we will introduce the main definitions and concepts used in the sequel.

Notation 1.1. Let X be a set and let $f, g : X \to \mathbb{R}_{\geq 0}$ be maps. We write $f \ll g$ if there exists a finite set $F \subset X$ and a constant $C > 0$ such that $f(x) \leq C \cdot g(x)$ for all $x \in X \setminus F$. We will write $f \lesssim g$ if $f \ll g$ for a constant $C \leq 1$.

1.1 Isometric actions on Hilbert space

Definition 1.2. An orthogonal representation $\pi : G \to \mathcal{O}(\mathcal{H})$ is said to be *ergodic* if for any $\xi \in \mathcal{H}$ we have that $\pi_g(\xi) = \xi$ for all $g \in G$ if and only if $\xi = 0$, i.e., π has no non-zero invariant vectors. The representation π is said to be *weakly mixing* if the diagonal representation $\pi \otimes \pi : G \to \mathcal{O}(\mathcal{H} \otimes \mathcal{H})$ is ergodic. In particular weakly mixing representations are ergodic.

If $\pi : G \to \mathcal{O}(\mathcal{H})$ is an orthogonal representation, a map $b : G \to \mathcal{H}$ is said to be a 1-cocycle associated to π if it satisfies the Leibniz identity

$$b(gh) = \pi_g(b(h)) + b(g),$$

for all $g, h \in G$. It is essentially a consequence of the Mazur–Ulam theorem that any isometric action $G \curvearrowright^T \mathcal{H}$ may be written as $T_g(\xi) = \pi_g(\xi) + b(g)$ for some orthogonal representation π and an associated 1-cocycle $b(g)$ and conversely. The representation π is known as the *linear part* of T.

Definition 1.3. An isometric action $G \curvearrowright^T \mathcal{H}$ is said to admit *almost fixed points* if there exists a sequence (ξ_n) of vectors in $\mathcal{H}$ such that

$$\|T_g(\xi_n) - \xi_n\| \to 0$$

for all $g \in G$.

Definition 1.4. We will say that a 1-cocycle b associated to an orthogonal representation $\pi : G \to \mathcal{O}(\mathcal{H})$ is *almost inner* if the associated affine isometric action $G \curvearrowright \mathcal{H}$ admits almost fixed points.

1.2 Geometric group theory

Throughout the paper G will be a countable discrete group, often finitely generated. Recall that a *length function* $|\cdot| : G \to \mathbb{R}_{\geq 0}$ is a map satisfying: (1) $|g| = 0$ if and only if $g = e$ is the identity; (2) $|g^{-1}| = |g|$, for all $g \in G$; and (3) $|gh| \leq |g| + |h|$, for all $g, h \in G$. A length function is proper if the map $g \mapsto |g|$ is proper, i.e., all sets of bounded length are finite. If $|\cdot|$ is a length function, then we denote

$$B(n) = \{g \in G : |g| \leq n\},$$

the *ball of radius* n centered at the identity, and

$$S(n) = \{g \in G : |g| = n\},$$

the *sphere of radius* n centered at the identity. If G is generated by a finite set S, then the function which assigns to each $g \in G$ the least integer k such that g can be written as a product of k elements from $S \cup S^{-1}$ is a proper length function, known as a *word length function*.

Notation 1.5. Let G be a finitely-generated discrete group with a fixed finite, symmetric, generating set S. Let $F \subset G$ be a finite subset. We set

$$\partial F := \bigcup_{g \in S} gF \Delta F,$$

where "Δ" denotes the symmetric difference.

Definition 1.6. A sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of G is said to form a *Følner sequence* if

$$\frac{|gF_n \Delta F_n|}{|F_n|} \to 0$$

for all $g \in G$.

Definition 1.7. Let G be a finitely generated discrete group with a fixed finite, symmetric, generating set S. For a constant $K > 0$, a sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of G is said to be a K-*controlled Følner sequence* if

$$\frac{|\partial F_n|}{|F_n|} \leq \frac{K}{\operatorname{diam} F_n},$$

where $\operatorname{diam} F_n$ is defined to be the least integer m such that $F_n \subset B(m)$. The group admits a controlled Følner sequence if it admits a K-controlled Følner sequence for some K.

Definition 1.8. A finitely generated group G is said to have *polynomial growth* if for some (equivalently, for any) proper word length function we have that

$$\limsup_n \frac{\log|B(n)|}{\log n} < \infty.$$

The group G is said to be of *weak polynomial growth* if

$$\liminf_n \frac{\log|B(n)|}{\log n} < \infty$$

for any proper word length.

The following observation is due to Shalom.

Proposition 1.9 (Shalom, Lemma 6.7.3 in [15]). *If G is a finitely generated group of polynomial growth of degree d, then for any proper word length, there is a subsequence $\mathcal{S} \subset \mathbb{N}$ such that the sequence of balls $(B(n))_{n\in\mathcal{S}}$ form a K-controlled Følner sequence for $K > 10d$.*

In fact, a group G which satisfies a doubling condition $|B(2n)| \leq C \cdot |B(n)|$ for some subsequence admits a controlled Følner sequence by an observation of Tessera, [18], Remark 4.10. Gromov's "Regularity lemma" ([10], section 3) shows that groups of weak polynomial growth have the doubling condition. By the work of Tessera several large classes of groups are known to admit controlled Følner sequences.

Proposition 1.10 (Tessera, Theorem 11 in [18] and Theorem 6 in [19]). *The following classes of groups admit controlled Følner sequences:*

1. *polycyclic groups;*
2. *wreath products $D \wr \mathbb{Z}$ with D finite;*
3. *semi-direct products $\mathbb{Z}[\frac{1}{mn}] \rtimes_{m/n} \mathbb{Z}$, with m, n coprime and $|mn| \geq 2$;*
4. *any closed, undistorted subgroup (e.g., cocompact lattice) of a direct product of a p-adic solvable group with a connected, solvable Lie group.*

By results of Mal'cev and Auslander, it is known that a group G is polycyclic if and only if G is realizable as a solvable subgroup of $\mathrm{GL}(n, \mathbb{Z})$, cf. [5], section III.A.5.

The full extent of the class of amenable groups admitting a controlled Følner sequence is unknown. An interesting problem would be to determine exactly which solvable groups with finite Hirsch number belong to this class or at least have property H_{FD}. (To recall, let G be a solvable group with derived series $G > G^{(1)} > G^{(2)} > \cdots > G^{(n)} > G^{(n+1)} = \{1\}$. The *Hirsch number* is then defined to be the sum of the torsion-free ranks of the abelian groups $G^{(i)}/G^{(i+1)}$, $i = 1, \ldots, n$. See section 6.6 in [15] for a discussion on this problem.) We pose the following, more concrete question:

Question 1.11. If Γ is a solvable subgroup of $\mathrm{GL}(n, \mathbb{Z}[\frac{1}{p}])$, does Γ admit a controlled Følner sequence?

If Γ is an *undistorted* solvable subgroup of $\mathrm{GL}(n, \mathbb{Z}[\frac{1}{p}])$, then the answer is affirmative by item (4) of the previous proposition, so it would be interesting to know whether there are other solvable subgroups of $\mathrm{GL}(n, \mathbb{Z}[\frac{1}{p}])$. We remark that $\mathbb{Z}[\frac{1}{p}]$ cannot be replaced with $\mathbb{Z}[\tau]$ for some non-algebraic number τ, since $\mathrm{GL}(2, \mathbb{Z}[\tau])$ contains a copy of $\mathbb{Z} \wr \mathbb{Z}$ which does not to admit a controlled Følner sequence by an isoperimetric inequality due to Erschler [8].

1.3 Arrays

The definition of an array was formally introduced in [2] as a means for unifying the concepts of length functions and 1-cocycles into orthogonal representations. We now recall the definition.

Definition 1.12. Let $\pi : G \to \mathcal{O}(\mathcal{H})$ be an orthogonal representation of a countable discrete group G. A map $\alpha : G \to \mathcal{H}$ is called an *array* if for every finite subset $F \subset G$ there exists $K \geq 0$ such that

$$\|\pi_g(\alpha(h)) - \alpha(gh)\| \leq K, \tag{1.1}$$

for all $g \in F$, $h \in G$ (i.e., α is *boundedly equivariant*). It is an easy exercise to show that for any array α on a finitely generated group G there exists a proper word length function on G, a scalar multiple of which bounds $\|\alpha(g)\|$ from above.

Lemma 1.13. *Let G be a finitely generated group equipped with some proper word length associated to a finite, symmetric, generating set S. If $\alpha : G \to \mathcal{O}(\mathcal{H})$ is an array into an orthogonal representation π, then $\widetilde{\alpha}(g) := \frac{1}{|g|}\, \alpha(g) \otimes \alpha(g)$, with $\widetilde{\alpha}(e) := 0$, is an array into $\pi \otimes \pi$.*

Proof. The proof is very similar to the proof of Proposition 1.4 of [2]: we include it here only for the sake of completeness. First, for every $g \in G$, we denote by $B_g := \sup_{h \in G} \|\alpha(gh) - \pi_g(\alpha(h))\|$ and from the assumptions we have $B_g < \infty$. Using the triangle inequality together with the bounded equivariance property, for all $k \in G$ we have $\|\alpha(k)\| \leq D|k|$, where $D = \max_{s \in S} B_s$. This further implies that for every $\ell \in G$ we have the following inequality

$$\sup_{k \neq e, \ell^{-1}} \frac{\|\alpha(k)\|}{|\ell k|} = \sup_{k \neq e, \ell^{-1}} \frac{\|\alpha(k)\|}{|k|} \cdot \frac{|k|}{|\ell k|} \leq D(|\ell| + 1). \tag{1.2}$$

To check the bounded equivariance for $\widetilde{\alpha}$, we fix $g, h \in G$ where $h \neq e, g^{-1}$. Applying the triangle inequality and using successively the bounded equivariance property, the basic inequality $||gh| - |h|| \leq |g|$, and the inequality (1.2), we have

$$\begin{aligned}
\|\widetilde{\alpha}(gh) - (\pi \otimes \pi)_g \widetilde{\alpha}(h)\| &\leq \frac{\|(\alpha(gh) - \pi_g \alpha(h)) \otimes \alpha(gh)\|}{|gh|} + \\
&\quad + \frac{\|\pi_g \alpha(h) \otimes (\alpha(gh) - \pi_g \alpha(h))\|}{|gh|} + \\
&\quad + \|\pi_g \alpha(h) \otimes \pi_g \alpha(h)\| \left| \frac{1}{|gh|} - \frac{1}{|h|} \right| \\
&\leq B_g \frac{\|\alpha(gh)\|}{|gh|} + B_g \frac{\|\alpha(h)\|}{|gh|} + ||gh| - |h|| \frac{\|\alpha(h)\|}{|h|} \frac{\|\alpha(h)\|}{|gh|} \\
&\leq B_g D(|g| + 2) + D^2 |g|(|g| + 1).
\end{aligned}$$

This implies that for every $g, h \in G$ we have

$$\|\widetilde{\alpha}(gh) - (\pi \otimes \pi)_g \widetilde{\alpha}(h)\| \leq \max\{B_g D(|g| + 2) + D^2 |g|(|g| + 1), \|\widetilde{\alpha}(g^{-1})\|, \|\widetilde{\alpha}(g)\|\},$$

which concludes our proof as the right hand expression depends only on g. ∎

1.4 Large scale lipschitz maps

Let $\mathcal{V}$ be a normed vector space. We will say a map $f : G \to \mathcal{V}$ is *large scale lipschitz* if there exists a map $C : G \to \mathbb{R}_{\geq 0}$ such that for all $g \in G$, $\|f(g) - f(gs)\| \leq C(s)$. An array can be viewed in some sense as the formal adjoint of some large scale lipschitz map $f : G \to \mathcal{H}$ with respect to the representation π, viz.,

Proposition 1.14. *If $\alpha : G \to \mathcal{H}$ is an array associated to π, then $\alpha^\star(g) := \pi(g)\alpha(g^{-1})$ is large scale lipschitz. Conversely, if $f : G \to \mathcal{H}$ is large scale lipschitz, then $f^\star(g) := \pi(g)f(g^{-1})$ is an array associated to π.*

The proof consists of a straightforward check that the respective identities are satisfied.

Given a finite, symmetric generating set S for G, for any map $f : G \to \mathbb{R}$ we define the variation function $\partial f : G \to \mathbb{R}^S$ by $\partial f(g)(s) := f(g) - f(gs)$.

Definition 1.15. A bounded function $f : G \to \mathbb{R}$ is said to be *slowly oscillating* if $\|\partial f\| \in C_0(G)$, where $\|\cdot\|$ is the euclidean norm on $\mathbb{R}^S$.

Note that if $f : G \to \mathcal{V}$ is a large scale lipschitz map into a normed vector space $\mathcal{V}$, then $g \mapsto \frac{1}{|g|} f(g)$ is slowly oscillating.

We define $\mathcal{H}^\infty(G)$ to be Banach space of all slowly oscillating functions. For all $1 \le p < \infty$, we also define $\mathcal{H}^p(G)$ to be the Banach space of all slowly oscillating functions f such that $\|\partial f\| \in \ell^p(G)$. Note that the definition of $\mathcal{H}^p(G)$ for all $1 \le p \le \infty$ does not depend on the choice of finite generating set.

Our interest in slowly oscillating functions stems from the following "rigidity" phenomenon which can be observed under the assumption of ergodicity.

Proposition 1.16. *If* $f \in \mathcal{H}^\infty(G)$ *is a function such that*

$$\int f(g^{-1})d\mu_n(g) \to 0$$

for all Reiter sequences (μ_n)*, then* $f \in C_0(G)$*.*

Proof. Suppose by contradiction that f does not belong to $C_0(G)$. Without loss of generality, we would have that there would exist $c > 0$ and a sequence (g_n) of elements in G such that $f(g_n) \ge c$ for all $n \in \mathbb{N}$. Since $f \in \mathcal{H}^\infty(G)$, for any finite subset $F \subset G$ there exists $n \in \mathbb{N}$ sufficiently large so that $f(h) \ge c/2$ for all $h \in g_n F$. Hence, passing to a subsequence of (g_n), there is a Følner sequence (F_k) with the property that $f(h) \ge c/2$ for all $h \in g_{n_k} F_k^{-1}$ for all $k \in \mathbb{N}$. Taking μ_k to be the uniform probability measure on the set $F_k g_{n_k}^{-1}$, we would then have constructed a Reiter sequence such that $\liminf_k \int f(g^{-1}) d\mu_k(g) \ge c/2 > 0$, a contradiction. ∎

Definition 1.17. Let G be an amenable group, and let $f : G \to \mathcal{V}$ be a large scale lipschitz map. We say that f has *sublinear growth* if $\limsup_{|g|\ge n} \|f(g)\|/|g| = 0$. We say that f has *almost sublinear growth* if $\int \frac{1}{|g|}\|f(g)\| d\mu_n(g) \to 0$ for all Reiter sequences (μ_n).

Proposition 1.18. *Let* G *be an amenable group. Let* $f : G \to \mathcal{H}$ *be a large scale lipschitz map in to Hilbert space. If* f *is* symmetric*, i.e.,* $\|f(g)\| \equiv \|f(g^{-1})\|$*, then the following statements are equivalent:*

1. f *has sublinear growth;*
2. f *has almost sublinear growth;*
3. $f_\xi(g) := \langle f(g), \xi\rangle$ *has sublinear growth for all* $\xi \in \mathcal{H}$ *and the set* $V := \left\{\frac{1}{|g|} f(g)\right\}_{g \in G}$ *is precompact;*
4. f_ξ *has almost sublinear growth for all* $\xi \in \mathcal{H}$ *and* V *is precompact.*

Proof. The implications (1)⇒(2), (1)⇒(3), and (3)⇒(4) are trivial, while the implication (2)⇒(1) follows directly by Proposition 1.16 applied to the function $\frac{1}{|g|}\|f(g)\|$. Therefore, we only need prove the implication (4)⇒(1).

To this end, note that if V is precompact, then for any $\epsilon > 0$ we can find a set of vectors $\xi_1, \dots, \xi_n \in \mathcal{H}$ so that

$$\int \Big(\frac{1}{|g|}\|f(g)\|\Big)^2 d\mu(g) \le C \sum_{i=1}^{n} \int \frac{1}{|g|} |\langle f(g), \xi_i\rangle| d\mu(g) + \epsilon \tag{1.3}$$

holds for any probability measure μ, where $C := \sup_{g\in G} \frac{\|f(g)\|}{|g|} < \infty$. Thus, by almost sublinear growth of each f_ξ and the Cauchy–Schwarz inequality, we have that $\int \frac{1}{|g|}\|f(g)\|\,d\mu_n(g) \to 0$ along any Reiter sequence. By symmetry, the result then obtains by Proposition 1.16. ∎

2 Main Results

2.1 Arrays and the weak mean ergodic theorem

In this section we present the proof of Theorem A. Though the theorem was stated explicitly for cocycles, the natural context for the theorem is actually the class of arrays. This is essentially due to the fact that there is no well-defined product of cocycles, while such a product exists for the class of arrays. This allows us to exploit the weak mixingness in order to derive the strong form of the theorem in that case.

Theorem 2.1 (Theorem A). *Let $\pi : G \to \mathcal{O}(\mathcal{H})$ be an ergodic orthogonal representation of a finitely generated amenable group G, and let $\alpha : G \to \mathcal{H}$ be an array. Let S be a finite, symmetric, generating set for G, and let $|\cdot|$ denote the word length in S. If $(\mu_n)_{n\in\mathbb{N}}$ is a Reiter sequence for G, then*

$$\int \frac{1}{|g|}\,\alpha(g)\,d\mu_n(g) \to 0 \tag{2.1}$$

in the weak topology. If π is weakly mixing, then

$$\int \frac{1}{|g|}\,|\langle \alpha(g), \xi\rangle|\,d\mu_n(g) \to 0 \tag{2.2}$$

for all $\xi \in \mathcal{H}$.

Before we begin the proof, we pause to introduce some convenient notation to be used here as well as in the sequel.

Notation 2.2. Let $\alpha : G \to \mathcal{H}$ be an array. We set

$$\alpha^\flat(g) = \frac{1}{|g|}\,\alpha(g),$$

where by convention $\alpha^\flat(e) = 0$. $\mathcal{H}\otimes\mathcal{H}$ will be denoted as $\widetilde{\mathcal{H}}$. The representation $\pi\otimes\pi : G \to \mathcal{O}(\widetilde{\mathcal{H}})$ will be denoted as $\widetilde{\pi}$. The array $\widetilde{\alpha} : G \to \widetilde{\mathcal{H}}$ is defined as

$$\widetilde{\alpha}(g) = \frac{1}{|g|}\,\alpha(g)\otimes\alpha(g),$$

where $\widetilde{\alpha}(e) = 0$ by convention.

Proof of Theorem 2.1. The proofs of these formulas are inspired by the standard approach to the (weak) mean ergodic theorem for amenable groups. We begin by proving (2.1). To this end, we fix $\epsilon > 0$, $n \in \mathbb{N}$ and note that there exists a finite subset $F_n \subset G$ such that

$$\|\alpha^\flat(gh) - \pi(g)\alpha^\flat(h)\| \leq \epsilon$$

whenever $g \in B(n)$ and $h \in G \setminus F_n$. Let $\xi \in \mathcal{H}$ be a vector of the form $\xi = (1-\pi(g^{-1}))\eta$ for some $g \in B(n), \eta \in \mathcal{H}$. We then have that

$$\begin{aligned}
&\left|\int \langle \alpha^\flat(h), \xi\rangle \, d\mu_N(h)\right| \\
&= \left|\int \langle \alpha^\flat(h) - \pi(g)\alpha^\flat(h), \eta\rangle \, d\mu_N(h)\right| \\
&\leq \left|\int \langle \alpha^\flat(gh) - \pi(g)\alpha^\flat(h), \eta\rangle \, d\mu_N(h)\right| + \int |\langle \alpha^\flat(k), \eta\rangle| \, d|\mu_N(g^{-1}k) - \mu_N(k)| \\
&\leq \|\eta\| \int \|\alpha^\flat(gh) - \pi(g)\alpha^\flat(h)\| \, d\mu_N(h) + \sup_k \|\alpha^\flat(k)\| \cdot \|\mu_N - g * \mu_N\|_1 \lesssim 2\|\eta\|\epsilon,
\end{aligned} \tag{2.3}$$

since $\lim_N \mu_N(F_n) = 0$ and $\|\alpha^\flat\|$ is bounded. By inspection, the estimate holds for the span $\mathcal{V} := \mathrm{span}\{\xi : \exists g \in G, \eta \in \mathcal{H}(\xi = (1-\pi(g))\eta)\}$, establishing the theorem in that case. Since $\int \|\alpha^\flat(g)\| \, d\upsilon_n(g)$ is uniformly bounded, the result then extends to the closure of $\mathcal{V}$, which by ergodicity is all of $\mathcal{H}$. This concludes the proof of (2.1).

For the proof of the second part, formula (2.2), we note that if $\alpha : G \to \mathcal{H}$ is an array for π, then $\widetilde{\alpha}(g)$ is an array for $\widetilde{\pi}$ by Lemma 1.13. Applying this, we see that

$$\left|\int \langle \widetilde{\alpha}^\flat(h), \xi \otimes \xi\rangle \, d\mu_N(h)\right| = \int |\langle \alpha^\flat(h), \xi\rangle|^2 \, d\mu_N(h) \to 0 \tag{2.4}$$

by the proof of (2.1). By the Cauchy–Schwarz inequality, we have that

$$\int |\langle \alpha^\flat(h), \xi\rangle| \, d\mu_N(h) \leq \left(\int |\langle \alpha^\flat(h), \xi\rangle|^2 \, d\mu_N(h)\right)^{1/2}, \tag{2.5}$$

and we are done. ∎

In the case the 1-cocycle is proper, there is a sharpening of the above result. The proof is identical the the proof of the previous theorem, using Proposition 1.4 from [2] instead of Lemma 1.13.

Proposition 2.3. *Let $\pi : G \to \mathcal{H}$ be a weakly mixing orthogonal representation. If $b : G \to \mathcal{H}$ is a proper 1-cocycle, then*

$$\int \frac{1}{\|b(g)\|} |\langle b(g), \xi\rangle| d\mu_n(g) \to 0 \tag{2.6}$$

for all Reiter sequences (μ_n).

2.2 Theorem B and the mean ergodic theorem

We begin with the main technical theorem in this section, the formulation and proof of which are inspired by Lemma 3.4 in [3].

Theorem 2.4. *Let* G *be a finitely generated discrete group in the class* $\mathcal{CF}$. *Let* $b : G \to \mathcal{H}$ *be a* 1*-cocycle associated to an orthogonal representation* π. *Assume that*

$$\frac{1}{|g|}\langle b(g), \xi\rangle \in C_0(G) \tag{2.7}$$

for all $\xi \in \mathcal{H}$ *(i.e.,* b *is weakly sublinear). Let* $(F_n)_{n\in\mathbb{N}}$ *be a* K*-controlled Følner sequence. Let* υ_n *be the uniform measure on* F_n. *There exists a sequence* (μ_k) *of finitely supported measures which are in the convex hull of* $\{\upsilon_n\}$ *such that* $\xi_k := \int b(g)\, d\mu_k(g)$ *form a sequence of almost fixed points for the affine action* $G \curvearrowright^{T} \mathcal{H}$ *associated to* b.

Proof. Fix a word length $|\cdot|$ coming from some finite, symmetric generating set $S \subset G$. Let $d_n = \operatorname{diam} F_n$. We set $F_n(g) = gF_n \Delta F_n \subset \partial F_n \subset B(d_n+1)$, for each $g \in S$. Let $\eta_n = \int b(g)\, d\upsilon_n(g)$.

For all $n \in \mathbb{N}$ we have the *a priori* estimate

$$\begin{aligned} \|T_g(\eta_n) - \eta_n\| &= \left\|\int b(h)\, d\upsilon_n(g^{-1}h) - \int b(h)\, d\upsilon_n(h)\right\| \\ &\le \frac{1}{|F_n|}\int_{F_n(g)} \|b(h)\|\, dh \\ &\le C(d_n+1)\cdot\frac{|\partial F_n|}{|F_n|} \le 2CK, \end{aligned} \tag{2.8}$$

where $C = \sup_{s\in S}\|b(s)\|$.

Therefore, we need only show that for any $\xi \in \mathcal{H}$ and $g \in S$, we have that

$$\lim_n |\langle T_g(\eta_n) - \eta_n, \xi\rangle| = 0. \tag{2.9}$$

Indeed, the sequence $(T_g(\eta_n) - \eta_n)_{n\in\mathbb{N}}$ would then have 0 as a weak limit point for any $g \in S$. Thus, the sequence $\bigoplus_{g\in S}(T_g(\eta_n) - \eta_n) \subset \bigoplus_{g\in S}\mathcal{H}$ converges weakly to 0, so that by passing to the convex hull, the theorem obtains.

We now fix $\xi \in \mathcal{H}$. By assumption 2.7 for every $\epsilon > 0$ there exists a finite set $E_\epsilon \subset G$ such that

$$|\langle b(g), \xi\rangle| < \epsilon|g| \tag{2.10}$$

for all $g \in G \setminus E_\epsilon$.

Since $\lim_n \upsilon_n(E_\epsilon) = 0$, we have that for any $g \in S$,

$$\begin{aligned} |\langle T_g(\eta_n) - \eta_n, \xi\rangle| &= \left|\int \langle b(h), \xi\rangle\, d\upsilon_n(g^{-1}h) - \int \langle b(h), \xi\rangle\, d\upsilon_n(h)\right| \\ &= \frac{1}{|F_n|}\left|\int_{F_n(g)} \langle b(h), \xi\rangle\, dh\right| \\ &\le \frac{1}{|F_n|}\int_{F_n(g)} |\langle b(h), \xi\rangle|\, dh \\ &\lesssim 2\epsilon(d_n+1)\frac{|\partial F_n|}{|F_n|} \le 4K\epsilon, \end{aligned} \tag{2.11}$$

and we are done. ■

Question 2.5. By Proposition 3.1 in [3] we know that any almost inner 1-cocycle has sublinear growth. For a general amenable group, is it the case that any weakly sublinear 1-cocycle is in fact (strongly) sublinear?

Examining the proof of the previous theorem, we find that the conclusion holds under the following weaker hypothesis.

Proposition 2.6. *Let G be a finitely generated discrete group in the class $\mathcal{CF}$. Let $b : G \to \mathcal{H}$ be a 1-cocycle associated to an orthogonal representation π. For every $c > 0$, $\xi \in \mathcal{H}$ define the set $E_c(\xi) := \{g \in G : |\langle b(g), \xi\rangle| \geq c|g|\}$. Suppose there exists K and $(F_n)_{n\in\mathbb{N}}$ a K-controlled Følner sequence so that for all $c, \xi, \delta > 0$ $|\partial F_n \cap E_c(\xi)| \leq \delta/d_n \cdot |F_n|$ for all n sufficiently large. Let υ_n be the uniform measure on F_n. There exists a sequence (μ_k) of finitely supported measures which are in the convex hull of $\{\upsilon_n\}$ such that $\xi_k := \int b(g)\, d\mu_k(g)$ form a sequence of almost fixed points for the affine action $G \curvearrowright^{T} \mathcal{H}$ associated to b.*

Proof. The proof of Theorem 2.4 carries over nearly identically, except for the last estimate of equation 2.11. Using the same notation and set-up, fixing $\epsilon > 0$ we have instead that for n sufficiently large

$$\begin{aligned}
&\frac{1}{|F_n|}\int_{F_n(g)} |\langle b(h), \xi\rangle|\, dh \\
&= \frac{1}{|F_n|}\int_{F_n(g)\cap E_\epsilon(\xi)} |\langle b(h), \xi\rangle|\, dh + \frac{1}{|F_n|}\int_{F_n(g)\setminus E_\epsilon(\xi)} |\langle b(h), \xi\rangle|\, dh \\
&\lesssim 4\epsilon(d_n+1)\frac{|\partial F_n|}{|F_n|} \leq 8K\epsilon.
\end{aligned}$$

■

Question 2.7. Suppose G admits a controlled Følner sequence and that $E \subset G$ be a set which has zero measure for any left invariant mean on G. Does G also admit a controlled Følner sequence (F_n) so that for every $\delta > 0$ $|\partial F_n \cap E| \leq \delta/d_n \cdot |F_n|$ for all n sufficiently large.

Remark 2.8. Recently, Gournay [9] generalized the argument of Proposition 3.1 in [3] from groups with controlled Følner sequences to the more general class of "transport amenable" groups; see Definition 1.3 in [9]. This class includes, in particular $\mathbb{Z} \wr \mathbb{Z}$. Therefore, it would be highly interesting to know whether Theorem 2.4 likewise holds for all transport amenable groups.

Definition 2.9. A group G has *property H_{FD}* of Shalom if any affine action $G \curvearrowright^{T} \mathcal{H}$ on Hilbert space with weakly mixing linear part admits almost fixed points.

Proposition 2.10. *Suppose that either Question 2.5 or 2.7 has a positive solution for a group G which admits a controlled Følner sequence. Then G has property H_{FD}.*

The proof is an easy consequence of Theorem 2.4 and Proposition 2.6. It follows from an argument given in [15] (Theorem 6.7.2) that a positive solution to either Question 2.5 or 2.7 for all groups of polynomial growth implies Gromov's theorem.

A stated in Proposition 1.10, among the known classes of amenable groups which admit controlled Følner sequences are: groups of (weak) polynomial growth; polycyclic groups, i.e., lattices in solvable Lie groups; wreath products $D \wr \mathbb{Z}$ with D finite; semi-direct products $\mathbb{Z}[\frac{1}{mn}] \rtimes_{m/n} \mathbb{Z}$, with m, n coprime and $|mn| \geq 2$. The latter three classes are the work of Tessera, Theorem 11 in [18]. Each of these classes is known have property H_{FD} by the seminal work of Shalom, Theorems 1.13 and 1.14 in [15], which in the polycyclic case relies in turn on deep work of Delorme [6]from the 1970s. The advantage to the approach suggested here is that it may potentially offer a broad, conceptually unified way of deriving property H_{FD} for large classes of groups.

We also point out that another consequence of a positive solution to Question 2.14 would give an alternate proof of the fact (due to Erchler [8]) that $\mathbb{Z} \wr \mathbb{Z}$, for instance, does not admit a controlled Følner sequences, cf. Theorem 1.15 in [15].

Theorem 2.11 (Theorem B). *Let* G *be finitely generated group in the class* $\mathcal{CF}$. *Let* $\pi : G \to \mathcal{O}(\mathcal{H})$ *be an orthogonal representation, and let* $b : G \to \mathcal{H}$ *be a 1-cocycle associated to* π. *Suppose that*

$$\int \frac{1}{|g|}\langle b(g^{-1}), \xi\rangle d\mu_n(g) \to 0 \tag{2.12}$$

for all $\xi \in \mathcal{H}$ *and all Reiter sequences* (μ_n). *Then the affine action* $G \curvearrowright^{T} \mathcal{H}$ *associated to* b *admits a sequence of almost fixed points.*

Proof. The proof follows directly from Proposition 1.16 combined with Theorem 2.4. ∎

Definition 2.12. Let G be a finitely generated group and let μ be a probability measure on G. A function $u : G \to \mathcal{V}$ into a vector space is said to be μ-*harmonic* if

$$u(g) = \int u(gs)d\mu(s) \tag{2.13}$$

for all $g \in G$.

Let μ be a probability measure with finite second moment, i.e., $\int |g|^2 d\mu(g) < \infty$. We know, by Theorem 6.1 in [14] and Theorem 6.1 in [4], that every group G without property (T) of Kazhdan admits at least one μ-harmonic 1-cocycle for some (irreducible) representation.

Proposition 2.13. *Let* G *be a group in the class* $\mathcal{CF}$, $\pi : G \to \mathcal{O}(\mathcal{H})$ *be an orthogonal representation, and* $b : G \to \mathcal{H}$ *be a* μ*-harmonic* 1*-cocycle with* μ *having finite second moment. Let* π_0 *be the restriction of* π *to the (invariant) subspace* $\mathcal{H}_0$ *spanned by the image of* b. *If* $V := \left\{\frac{1}{|g|}b(g)\right\}$ *is precompact, then* π_0 *is compact.*

Proof. Suppose by contradiction that $\mathcal{H}_0$ contains an non-zero invariant subspace $\mathcal{K}$ on which the restriction of π is weakly mixing. Setting $b' : G \to \mathcal{K}$ defined by $b'(g) := P_{\mathcal{K}}b(g)$, we then would have that b' is a harmonic 1-cocycle into a weakly mixing representation such that $V' := P_{\mathcal{K}}V = \left\{\frac{1}{|g|}b'(g)\right\}$ is precompact. Proposition 1.18 then implies that $b'(g)$ has sublinear growth; hence, by Theorem

2.4 it is almost inner. However, no non-zero harmonic 1-cocycle into an orthogonal representation can be almost inner, cf. Theorem 6.1 in [4]. Therefore, $b' \equiv 0$ which contradicts the fact that the span of V' is dense in $\mathcal{K}$. Thus, we have shown that π_0 contains no non-zero, weakly mixing subrepresentation which implies that π_0 is compact. ∎

Question 2.14. Let G be an amenable group, and let μ be a probability measure with finite second moment and trivial Poisson boundary. If $u : G \to \mathbb{R}$ is a lipschitz μ-harmonic function such that

$$\int \frac{1}{|g|}|u(g)|d\mu_n(g) \to 0$$

for all Reiter sequences (μ_n), does u have sublinear growth?

Notice that if u is harmonic, then $|u|$ is *subharmonic*, i.e., $|u|(g) \leq \frac{1}{|S|}\sum_{s\in S}|u|(gs)$ for all $g \in G$, so the conjecture may be posed in this generality. A positive solution to Question 2.14 also implies that the group G has property H_{FD}. The use of harmonicity as a tool for "regularizing" the cocycle is a key insight in the approach of Kleiner [13].

Remark 2.15. A result of Hebisch and Saloff-Coste, Theorem 6.1 in [11], shows that there a no non-constant real-valued harmonic functions of sublinear growth on a group of polynomial growth. It would be interesting if a variant of this argument could be made to apply to harmonic functions of almost sublinear growth.

2.3 On the space $\mathcal{H}^p(G)$

As a last remark, we develop another line of thought towards establishing the mean ergodic theorem for affine actions of groups of polynomial growth independently of Gromov's theorem.

Theorem 2.16. *Let G be a one-ended group with a finite, symmetric, generating set S. If $f \in \mathcal{H}^1(G)$, then $f \in C_0(G) + \mathbb{C}1$.*

Proof. For every $\epsilon > 0$, choose r sufficiently large so that

$$K_r := \sum_{g\in G\setminus B_r}\sum_{s\in S}|f(g) - f(gs)| < \epsilon.$$

Since G is one-ended $G \setminus B_r$ contains exactly one infinite connected component U_r. For every pair of elements $g, h \in U_r$ there exists a sequence of elements $x_1, \ldots, x_n$ in U_r such that $g = x_1$, $h = x_n$ and $x_{i+1}^{-1}x_i \in S$ for all $i = 1, \ldots, n-1$. Hence it follows by the triangle inequality that

$$|f(g) - f(h)| \leq K_r$$

which proves the claim. ∎

In fact, in the case that f is positive, a slightly weaker condition will suffice:

Theorem 2.17. *For* $f \in \ell^\infty(G)$ *and* $F \in \ell^\infty(G \times S)$*, let* $f \cdot F(g,s) := f(g)F(g,s)$*. Let* G *be a one-ended group with a finite, symmetric, generating set* S*. Suppose that* $f \in \ell^\infty(G)$*,* $f \geq 0$*. If* $\|f \cdot \partial f\| \in \ell^1(G)$*, then* $f \in C_0(G) + \mathbb{C}1$*.*

Note that since $f \geq 0$, we have that $\|f \cdot \partial f\| \leq \|\partial(f^2)\|$; hence, by the boundedness of f and standard estimation techniques it follows that if $f^p \in \mathcal{H}^1(G)$ for any $1 \leq p < \infty$, then it holds that $f \in C_0(G) + \mathbb{C}1$.

Proof. Let $\Gamma = \Gamma(G,S)$ be the Cayley graph of G with respect to the generating set S. We produce a new graph Γ' by subdividing each edge in Γ so the the vertex set of Γ' may be identified with $V(\Gamma) \sqcup E(\Gamma)$ and Γ' is again one-ended. We define a map $f' : V(\Gamma') \to \mathbb{R}$ by $f'(g) := f(g)^2$ for $g \in V(\Gamma)$ and $f'(e) := f(g)f(gs)$ for $e = (g, gs) \in E(\Gamma)$. Now by assumptions we can see that $\|\partial f'\| \in \ell^1(V(\Gamma'))$, so by Theorem 2.16, we can conclude that $f^2 \in C_0(G) + \mathbb{C}1$. By the positivity of f, this suffices to show the result. ∎

Proposition 2.18. *Let* G *be a one-ended group in the class* $\mathcal{CF}$*. If* b *is a* 1*-cocycle associated to an ergodic representation* $\pi : G \to \mathcal{O}(\mathcal{H})$ *such that*

$$\frac{1}{|g|}\langle b(g), \xi\rangle \in \mathcal{H}^1(G) \tag{2.14}$$

for all $\xi \in \mathcal{H}$*, then* b *is almost inner. The same holds assuming that* π *is weakly mixing and*

$$\frac{1}{|g|}|\langle b(g), \xi\rangle| \in \mathcal{H}^1(G). \tag{2.15}$$

Proof. The proof follows directly from Theorem 2.16 and Theorem 2.4. ∎

Proposition 2.19. *If* G *is a group of polynomial growth, then there exists* $1 \leq p < \infty$ *such that for any* 1*-cocycle* $b : G \to \mathcal{H}$ *we have that*

$$\frac{1}{|g|}\langle b(g), \xi\rangle \in \mathcal{H}^p(G) \tag{2.16}$$

for all $\xi \in \mathcal{H}$

Proof. Fixing a finite generating set S, we have that $\sum_{s\in S}\|\frac{1}{|g|}b(g) - \frac{1}{|gs|}b(gs)\| \ll \frac{1}{|g|}$ choosing an integer p such that $R^{p-2} \gg |B(R)|$, we have that

$$\sum_{g\in G}\sum_{s\in S}\left\|\frac{1}{|g|}b(g) - \frac{1}{|gs|}b(gs)\right\|^p \ll \sum_{g\in G}|g|^{-p} \ll \sum_{n\in\mathbb{N}} n^{-2} \tag{2.17}$$

from which the result easily obtains. ∎

Conjecture 2.20. *If* G *is a one-ended group of polynomial growth, then for any* $1 \leq p < \infty$ *any* positive *function* $f \in \mathcal{H}^p(G)$ *belongs to* $C_0(G) + \mathbb{C}1$*.*

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References

[1] E. Breuillard, B. Green, and T. Tao, *The structure of approximate groups*, Publ. Math. IHÉS **116** (2012), 115–221.

[2] I. Chifan and T. Sinclair, *On the structural theory of* II_1 *factors on negatively curved groups*, Ann. Sci. Éc. Norm. Supér. **46** (2013), 1–33.

[3] Y. de Cornulier, R. Tessera, and A. Valette, *Isometric group actions on Hilbert space: growth of cocycles*, Geom. Funct. Anal. **17** (2007), 770–792.

[4] D. Creutz, Ph.D. Thesis, UCLA, June 2011.

[5] P. de la Harpe, *Topics in Geometric Group Theory*, Chicago Lectures in Mathematics, U. Chicago Press, Chicago, IL, 2000, vi + 310 pp.

[6] P. Delorme, *1-cohomologie et des representations unitaires des groupes de Lie semisimples et résolubles. Produits tensoriels continus de représentations*, Bull. Soc. Math. France **105** (1977), 281–336.

[7] L. van den Dries and A. J. Wilkie, *Gromov's theorem on groups of polynomial growth and elementary logic*, J. Algebra **89** (1984), 349–374.

[8] A. Erschler, *On isoperimetric profiles of finitely generated groups*, Geom. Dedicata **100** (2003), 157–171.

[9] A. Gournay, *Vanishing of* ℓ^p*-cohomology and transportation cost*, Bull. London Math. Soc. **46** (2014), 481–490.

[10] M. Gromov, *Groups of polynomial growth and expanding maps*, Publ. Math. I.H.É.S. **53** (1981), 53–78.

[11] W. Hebisch and L. Saloff-Coste, *Gaussian estimates for Markov chains and random walks on groups*, Ann. Prob. **21** (1993), 673–709.

[12] E. Hrushovski, *Stable group theory and approximate subgroups*, J. Amer. Math. Soc. **25** (2012), 189–243.

[13] B. Kleiner, *A new proof of Gromov's theorem on groups of polynomial growth*. J. Amer. Math. Soc. **23** (2010), 815–829.

[14] Y. Shalom, *Rigidity of commensurators and irreducible lattices*, Invent. Math. **141** (2000), 1–54.

[15] Y. Shalom, *Harmonic analysis, cohomology, and the large-scale geometry of amenable groups*, Acta Math. **192** (2004), 119–185.

[16] Y. Shalom and T. Tao, *A finitary version of Gromov's polynomial growth theorem*, Geom. Funct. Anal. **20** (2010), 1502–1547.

[17] R. Tessera: *Vanishing of the first reduced cohomology with values in an* L^p*-representation*, Ann. Inst. Fourier (Grenoble) **59** (2009), no. 2, 851–876.

[18] R. Tessera, *Asymptotic isoperimetry on and uniform embeddings into Banach spaces*, Comment. Math. Helv. **86** (2011), 499–535.

[19] R. Tessera, *Isoperimetric profile and random walks on locally compact solvable groups*, Rev. Mat. Iberoam. **29** (2013), 715–737.

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