Graph Algorithms

Graph Algorithms

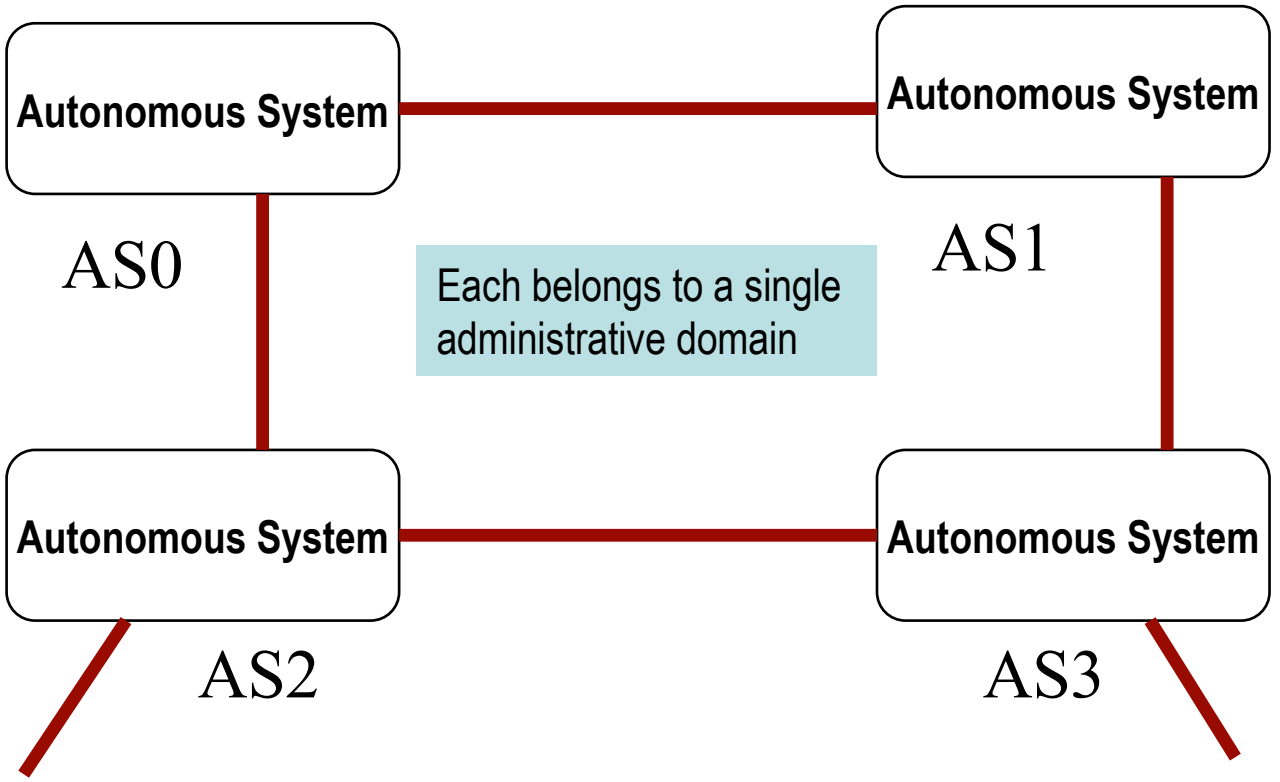
Many problems in networks can be modeled as graph problems.

- The topology of a distributed system is a graph.
- Routing table computation uses the shortest path algorithm
- Efficient broadcasting uses a spanning tree of a graph
- maxflow algorithm determines the maximum flow between a pair of nodes in a graph, etc etc.
- graph coloring, maximal independent set etc have many applications

Routing

- Shortest path routing
- Distance vector routing
- Link state routing
- Routing in sensor networks
- Routing in peer-to-peer networks

Internet routing

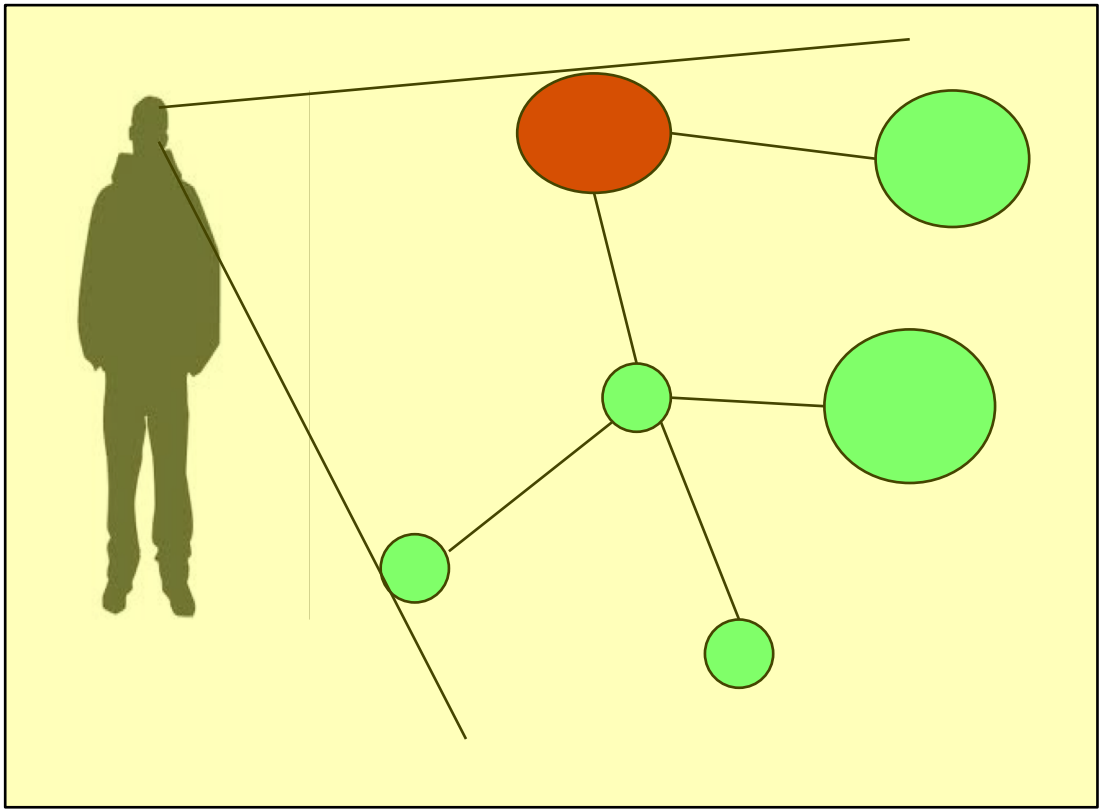


Intra-AS vs. Inter-AS routing

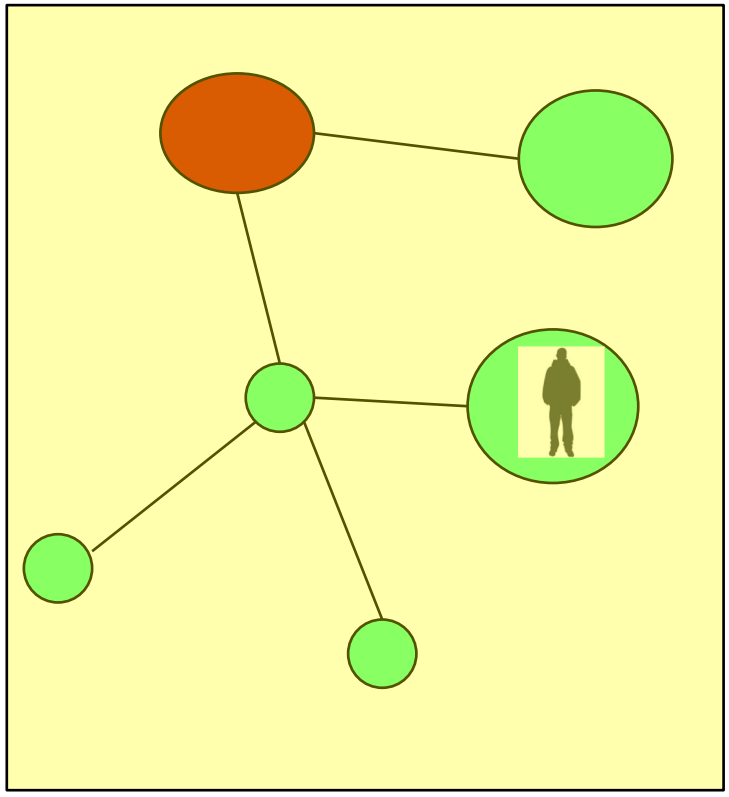
Open Shortest Path First (OSPF) is an adaptive routing protocol for Internet Protocol (IP) network

Routing: Shortest Path

Classical algorithms like Bellman-Ford, Dijkstra's shortest-path algorithm etc are found in most algorithm books.



In an (**ordinary**) graph algorithm
the entire graph is visible to the process

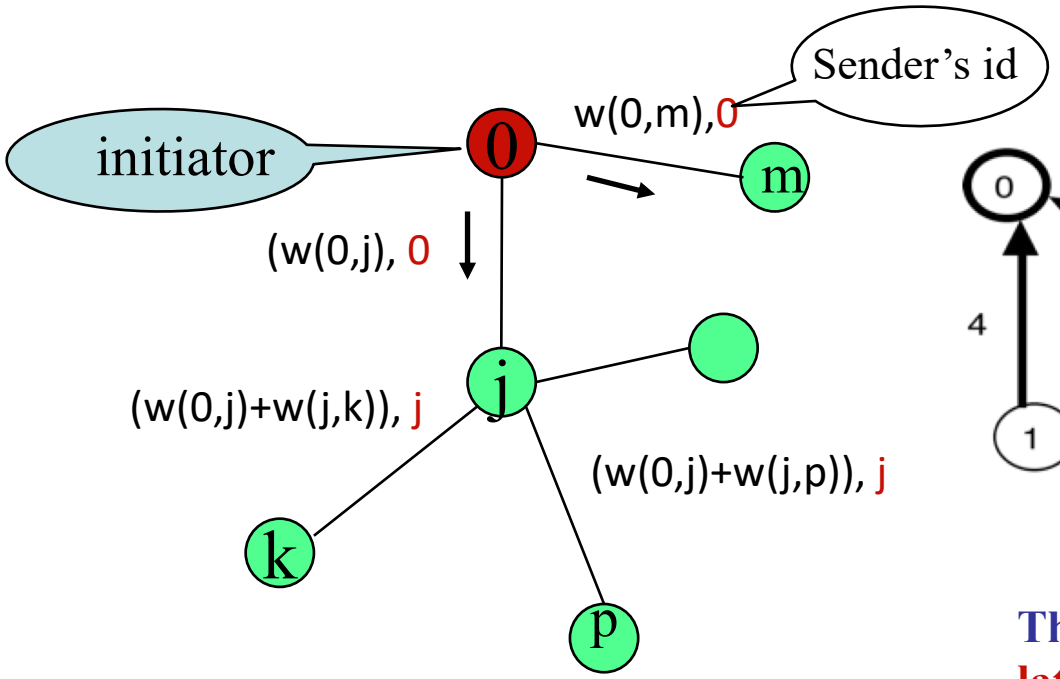


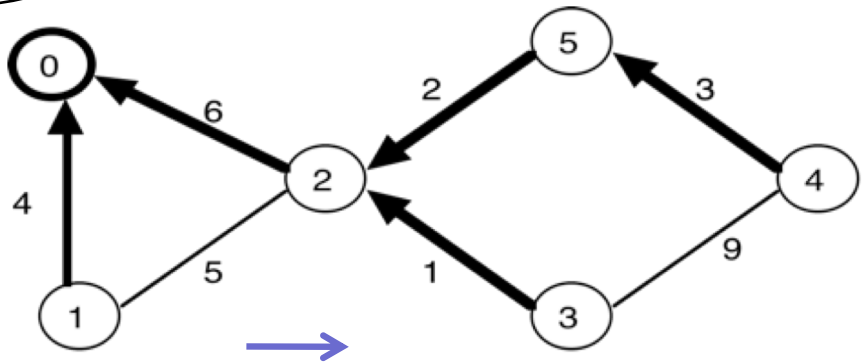
In a ***distributed graph algorithm*** only one node and its neighbors are visible to a process

Routing: Shortest Path

Most distributed algorithms for shortest path are *adaptations* of **Bellman-Ford algorithm**. It computes single-source shortest paths in a weighted graphs. Designed for directed graphs. Computes shortest path if there are no cycle of negative weight.

Let **D(j)** = **shortest distance of node j from initiator 0**. **D(0)** = 0. Initially, ∀i ≠ 0, D(i) =∞. Let **w(i, j)** = weight of the edge from node i to node j





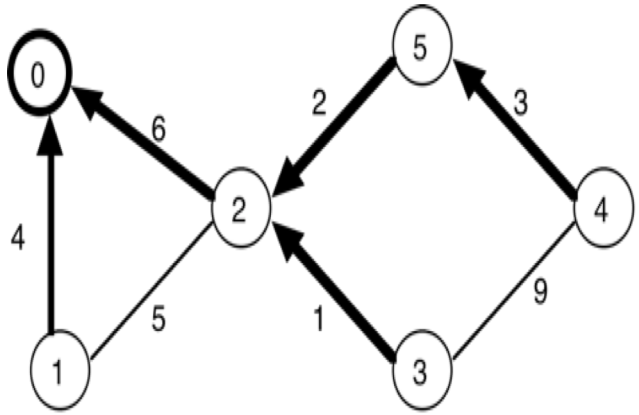
Parent relation

The edge weights can represent latency or distance or some other appropriate parameter.

Shortest path

Distributed Bellman Ford

Consider a static topology



{Process 0 sends w(0,i) to each neighbor i}

{Program for process i}

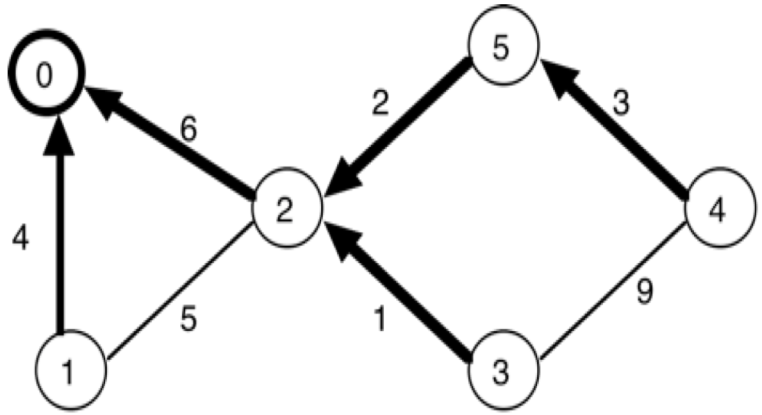
do message = (S,k) $\wedge$ S < D(i) $\rightarrow$
  if $parent \neq k \rightarrow parent := k$ **fi;**
  D(i):= S;
  send (D(i) + w(i,j), i) to each neighbor j $\neq parent$;
[] message (S, k) $\wedge$ S $\geq$ D(i) $\rightarrow$ skip
od

The parent links help the packets reach the initiator

Computes the shortest distance from all nodes to the initiator node

Shortest path

Synchronous or asynchronous?
The time and message complexities depend on the model.
The goal is to lower the complexity



[Synchronous version] In each round every process i sends out D(i) + w(i,j),j to each neighbor j

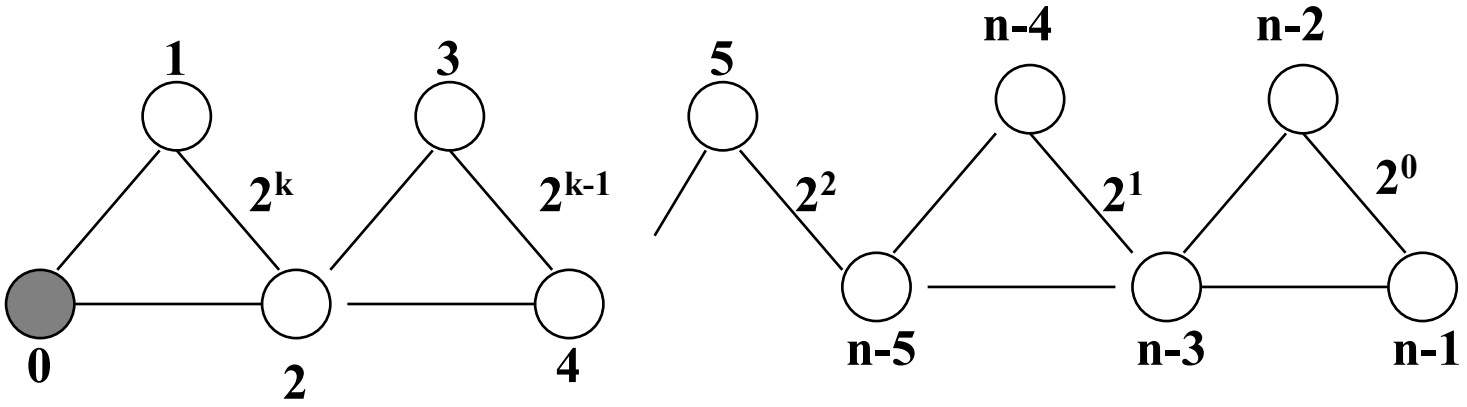
Observation: for a node i, once D(parent(i)) becomes stable, it takes one more round for D(i,0) to be stable

Message complexity = O(|V|)(|E|)

Complexity of Bellman-Ford

Theorem. The message complexity of *asynchronous* Bellman-Ford algorithm is exponential.

Proof outline. Consider a topology with an odd number of nodes 0 through n-1 (the unmarked edges have weight 0)



An adversary can regulate the speed of the messages D(n-1) reduces from $(2^{k+1}- 1)$ to 0 in steps of 1. Since k = (n-3)/2, it will need $2^{(n-1)/2}$-1 messages to reach the goal. So, the message complexity is exponential.

Shortest path

Chandy & Misra's algorithm : basic idea

(includes termination detection)

Process 0 sends w(0,i),0 to each neighbor i

{for process i > 0}

do message = (S ,k) ∧ S < D →

if parent ≠ k → send ack to parent **fi;**

parent := k; D := S;

send (D + w(i,j), i) to each neighbor j ≠ parent;

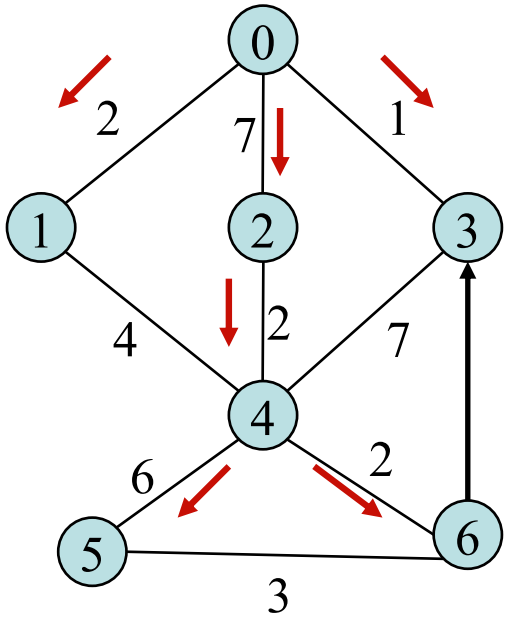
deficit := deficit + |N(i)| -1

[] message (S,k) ∧ S ≥ D → send ack to sender

[] ack → deficit := deficit – 1

[] deficit = 0 ∧ parent ≠ i → send ack to parent

od



Combines shortest path computation with termination detection. Termination is detected when the initiator receives ack from each neighbor

Shortest path

An important issue is: how well do such algorithms perform when the topology changes?

No real network is static!

Let us examine ***distance vector routing*** that is adaptation of the shortest path algorithm

Distance Vector Routing

Distance Vector D for each node **i** contains **N** elements **D[i,0], D[i,1],** … **D[i, N-1]**.

Here, D[i,j] denotes the distance from node i to node j.

- Initially $\forall i$, D[i,j] =0 when j=i

D(i,j) = 1 when $j \in N(i)$. and

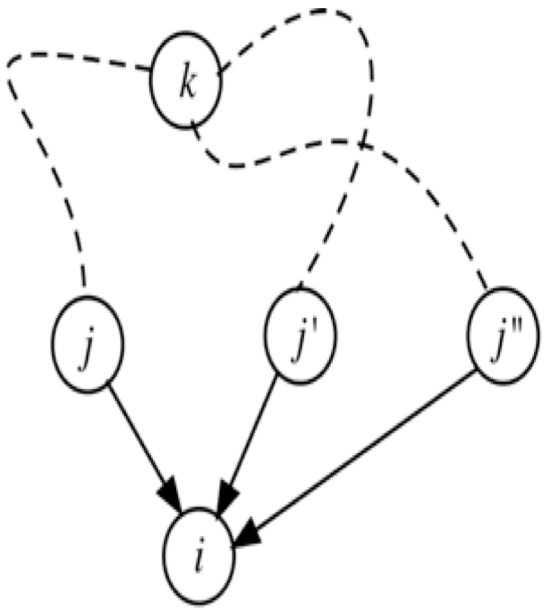
D[i,j] = ∞ when $j \notin N(i) \cup \{i\}$

- Each node **j** periodically sends its distance vector to its immediate neighbors.

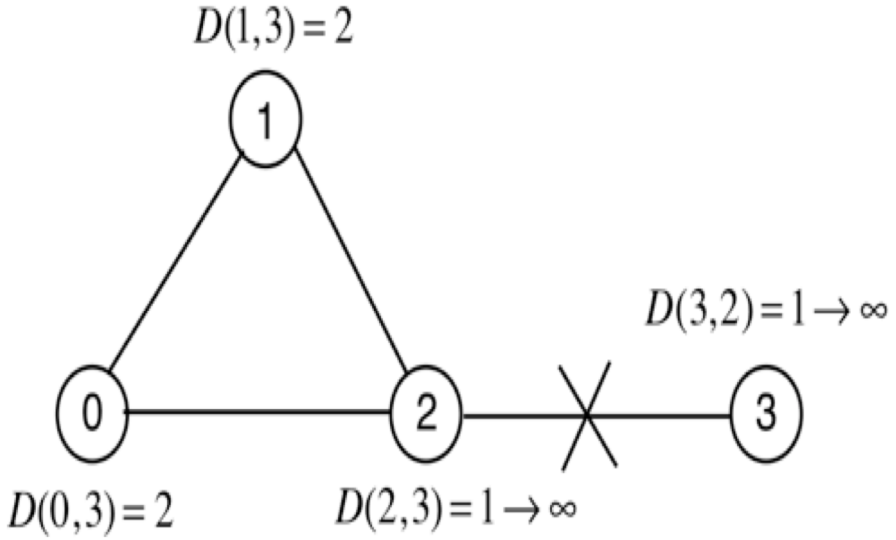
- Every neighbor **i** of **j**, after receiving the broadcasts from its neighbors, updates its distance vector as follows: $\forall k \neq i$: $D[i,k] = \min_j(w[i,j] + D[j,k])$

Used in RIP, IGRP etc

Distance Vector Routing



(a)



(b)

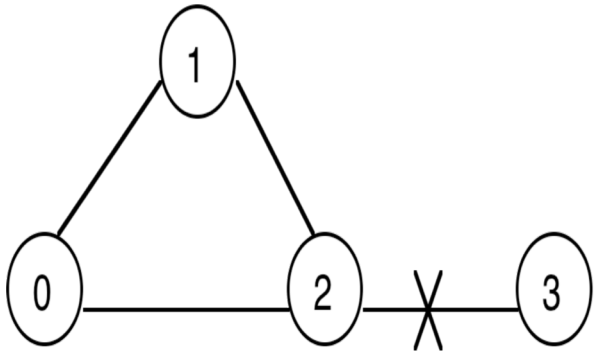
What if the topology changes?

Assume that each edge has weight = 1. Currently,

Node 1: d(1,0) = 1, d(1, 2) = 1, d(1,3) = 2

Node 2: d(2,0) = 1, d(2,1) =1, d(2,3) = 1

Node 3: d(3,0) = 2, d(3,1) = 2, d(3,2) = 1



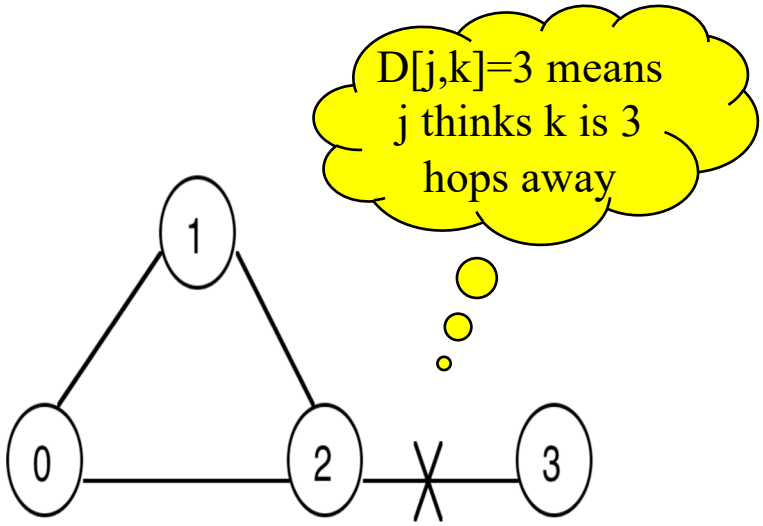
Counting to infinity

Observe what can happen when the link (2,3) fails.

Node 1 thinks d(1,3) = 2 (old value)

Node 2 thinks d(2,3) = d(1,3) +1 = 3

Node 1 thinks d(1,3) = d(2,3) +1 = 4



and so on. So it will take forever for the distances to stabilize. A partial remedy is the **split horizon** method that prevents node 1 from sending the advertisement about d(1,3) to 2 since its first hop (to 3) is node 2

$\forall k \neq i: D[i,k] = \min_j(w[i,j] + D[j,k])$

Suitable for smaller networks. Larger volume of data is disseminated, but to its immediate neighbors only. Poor convergence property.

Link State Routing

Each node **i** periodically broadcasts the weights of all edges **(i,j)** incident on it (this is the *link state*) to all its neighbors. Each link state packet (LSP) has a sequence number *seq*. The mechanism for dissemination is flooding.

This helps each node eventually compute the topology of the network, and independently determine the shortest path to any destination node using some standard graph algorithm like Dijkstra's.

Smaller volume data disseminated over the entire network
Used in OSPF of IP

Link State Routing: the challenges

(Termination of the reliable flooding)

How to guarantee that LSPs don't circulate forever?

A node forwards a given LSP at most once. It remembers the last LSP that it forwarded for each node.

(Dealing with node crash)

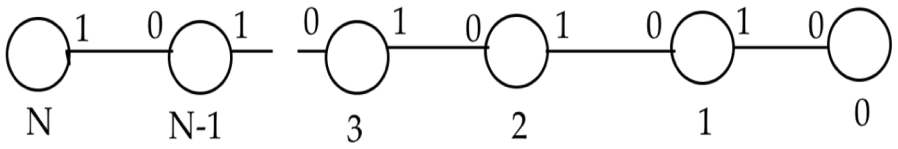
When a node crashes, all packets stored in it may be lost. After it is repaired, new packets are sent with **seq = 0**. So these new packets may be discarded in favor of the old packets! Problem resolved using TTL

See: http://www.ciscopress.com/articles/article.asp?p=24090&seqNum=4

Interval Routing

(**Santoro and Khatib**)

Conventional routing tables have a space complexity O(n).



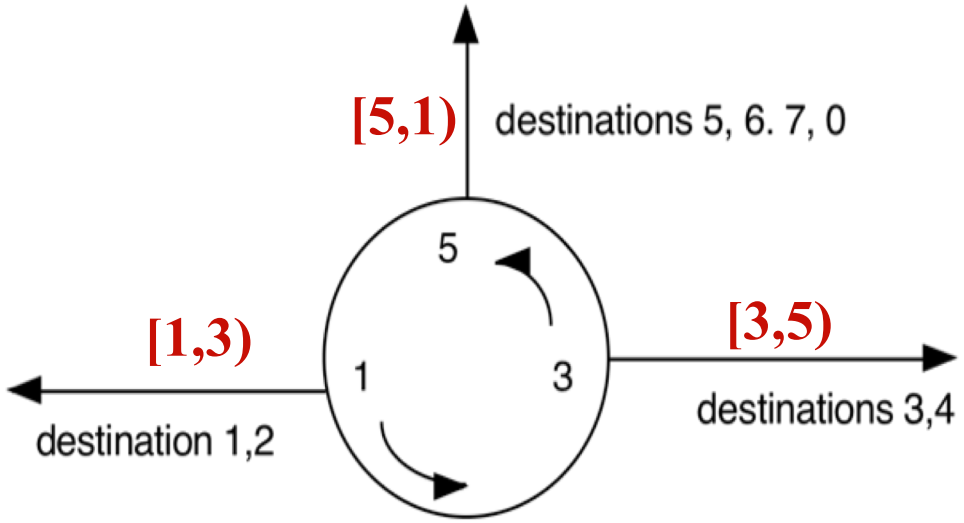
Can we route using a “smaller” routing table? This is relevant since the network sizes are constantly growing. One solution interval routing.

condition	**port number**
Destination > id	0
destination < id	1
destination = id	(local delivery)

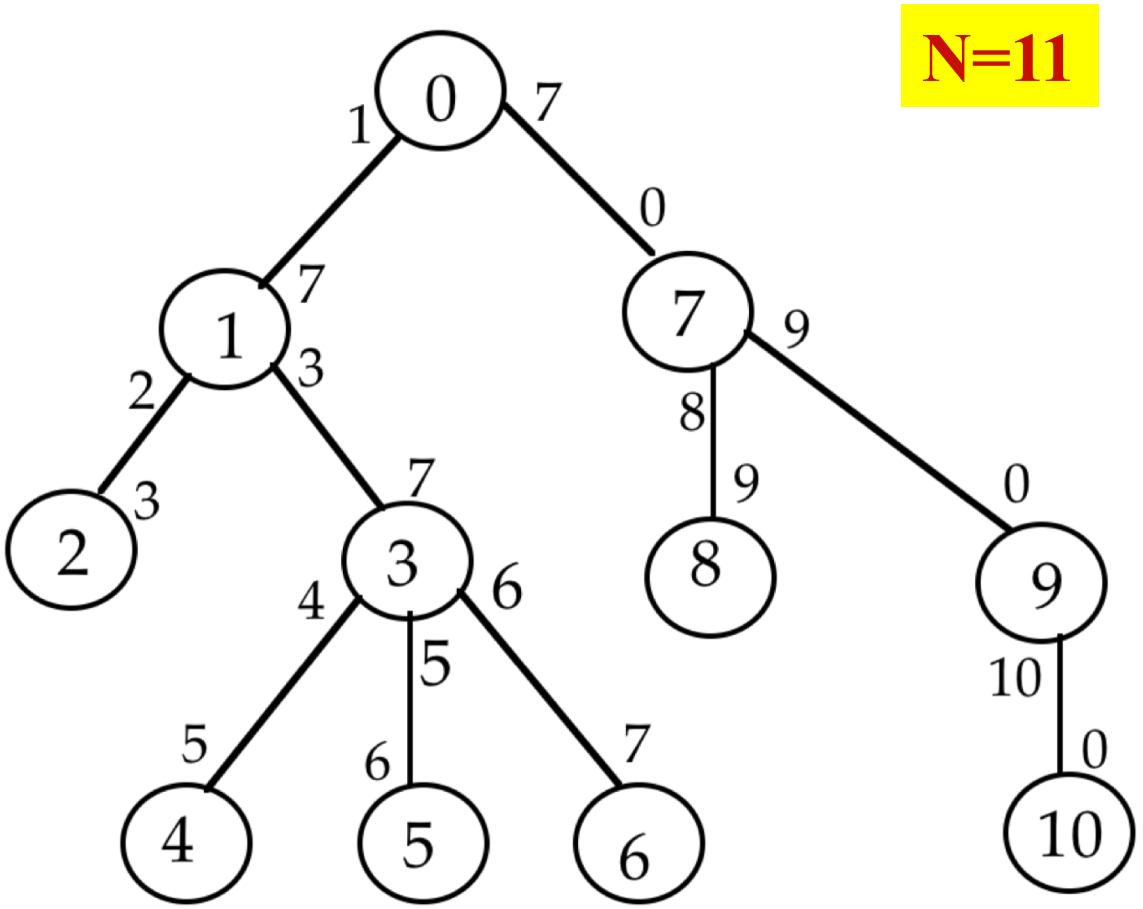
Interval Routing: Main idea

- Determine the **interval** to which the destination belongs.
- For a set of **N** nodes **0 . . N-1**, the interval **[p,q)** between **p** and **q** (**p, q < N**) is defined as follows:

- if **p < q** then **[p,q) = p, p+1, p+2, ... q-2, q-1**
- if **p ≥ q** then **[p,q) = p, p+1, p+2, ..., N-1, N, 0, 1, ..., q-2, q-1**



Example of Interval Routing



Labeling is the crucial part

Labeling algorithm

Label the **root** as 0.

Do a ***pre-order traversal*** of the tree. Label successive nodes as 1, 2, 3

For each node, label the port towards a child by the node number of the child.

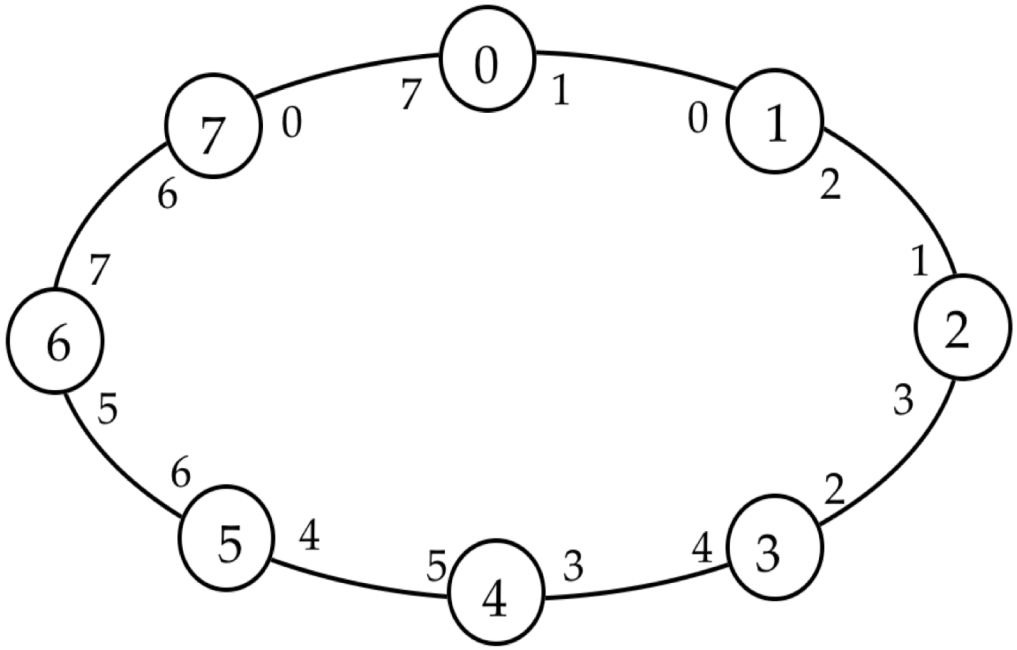
Then label the port towards the parent by **L(i) + T(i) + 1 mod N**, where

- **L(i)** is the label of the node i,
- **T(i) =** # of nodes in the subtree under node i (excluding i),

Question 1. Why does it work?

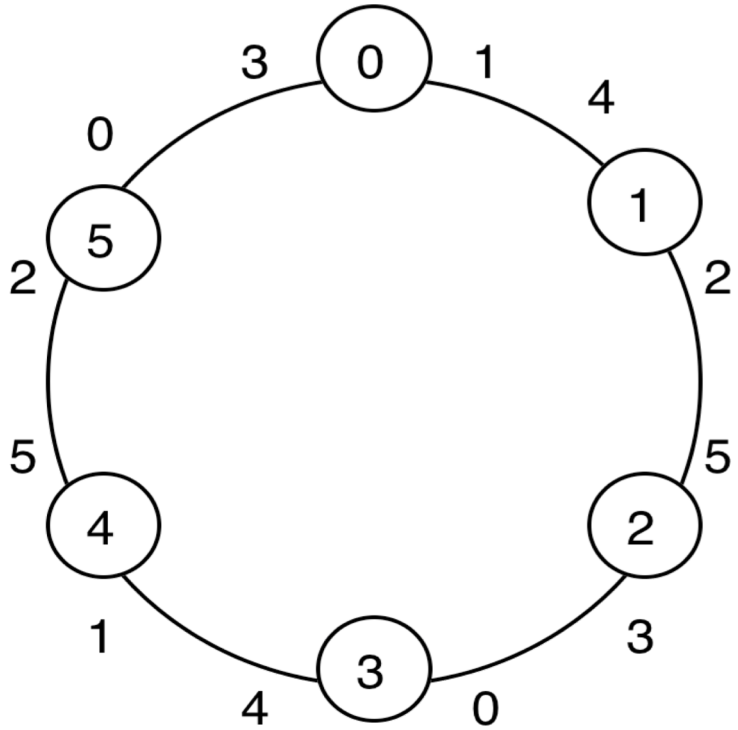
Question 2. Does it work for non-tree topologies too? YES, but the construction is a bit more complex.

Another example



Interval routing on a ring. The routes are not optimal. To make it optimal, label the ports of node i with i+1 mod 8 and i+4 mod 8.

Example of optimal routing



Optimal interval routing scheme on a ring of six nodes

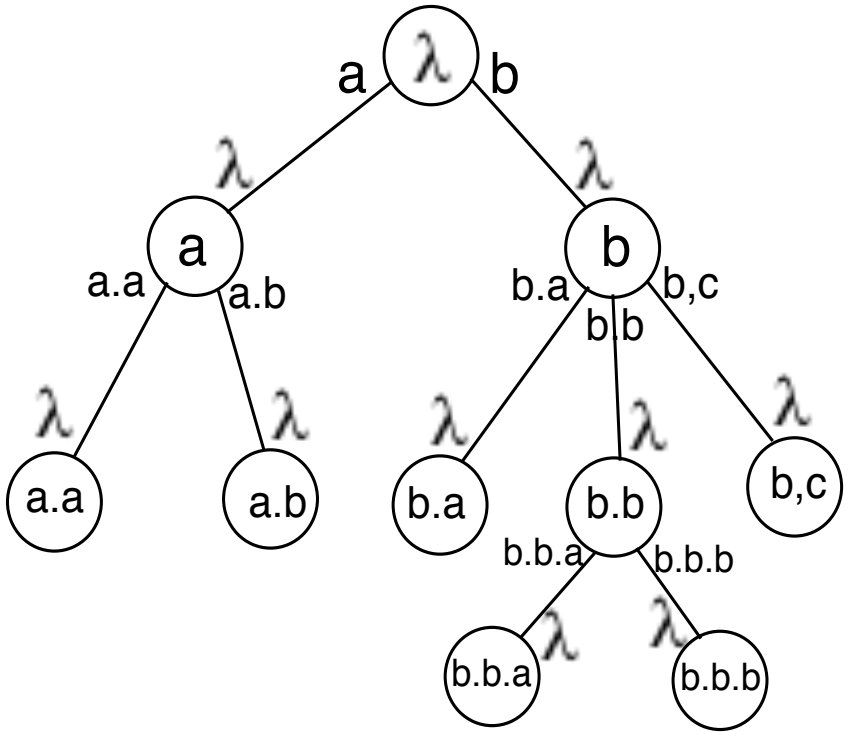
So, what is the problem?

Works for static topologies. Difficult to adapt to changes in topologies.

Some recent work on compact routing addresses dynamic topologies (Amos Korman, ICDCN 2009)

Prefix routing

Easily adapts to changes in topology, and uses small routing tables, so it is scalable. Attractive for large networks, like P2P networks.

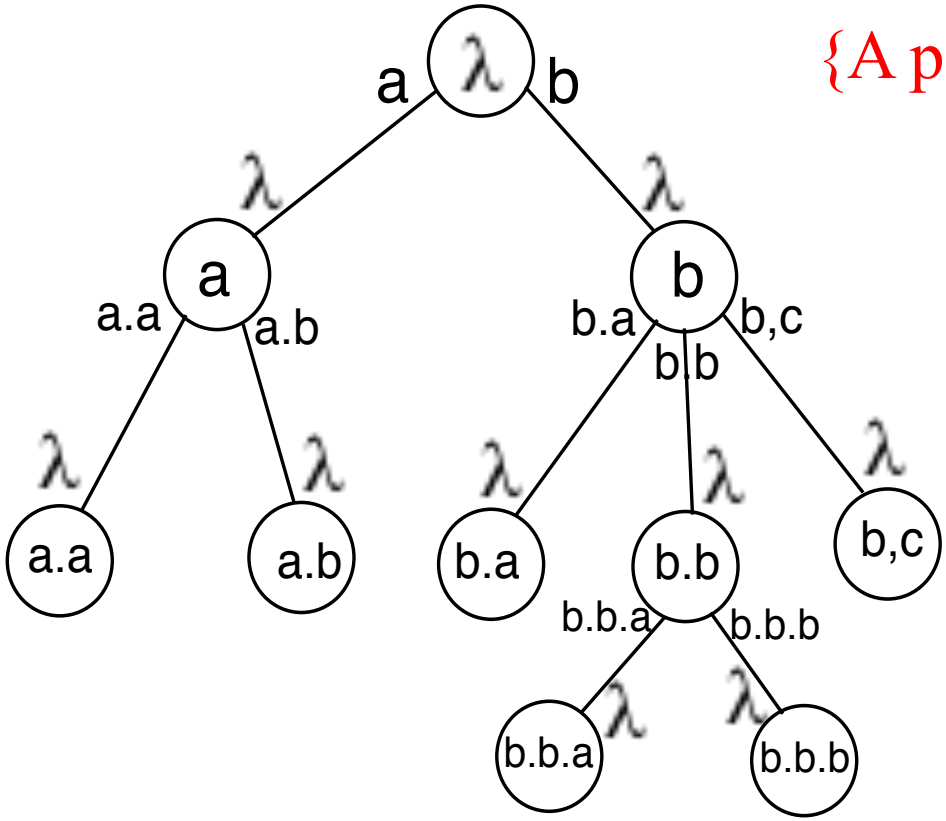


λ = the empty string

Label the root by λ, the empty string
Label each child of node with label L
by L.x (x is a unique for each child.
Label the port to connecting to a child
by the label of the child.
Label the port to the parent by λ

When new nodes are added or existing nodes are deleted, changes are only local.

Prefix routing



λ = the empty string

{A packet arrives at the current node}

{Let X = destination, and Y = current node}

if X=Y → local delivery

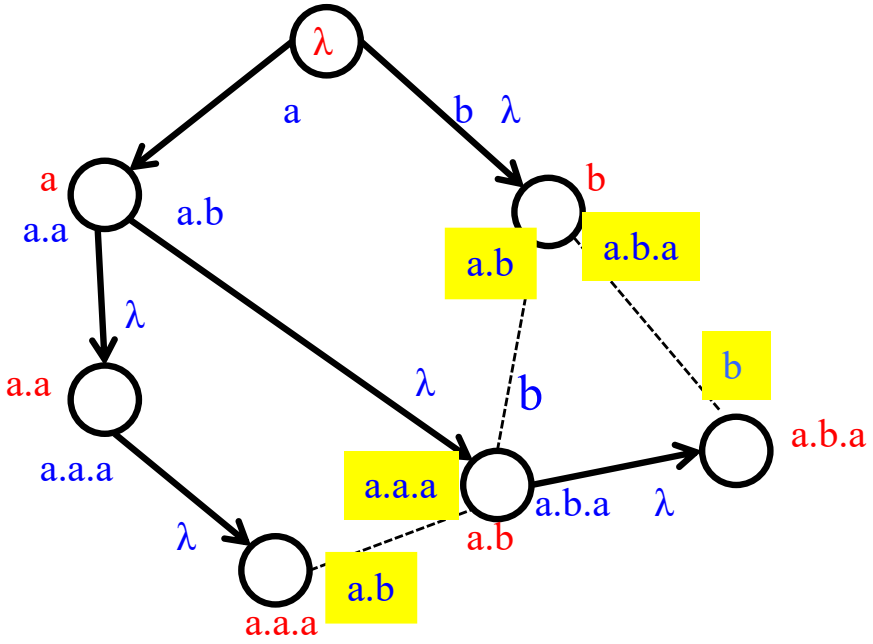
[] X ≠ Y → Find a port p labeled with the longest prefix of X

Forward the message to p

fi

Prefix routing for non-tree topology

Does it work on non-tree topologies too? Yes. Start with a spanning tree of the graph.



If (u,v) is a non-tree edge, then

Label the edge from u to v by the label of node v. If v is the root, then label the port from u to its parent p by the label of p and label the port from u towards the root by λ.

Routing in P2P networks: Example of Chord

- Small routing tables: **log n**
- Small routing delay: **log n** hops
- Load balancing via Consistent Hashing
- Fast join/leave protocol (polylog time)

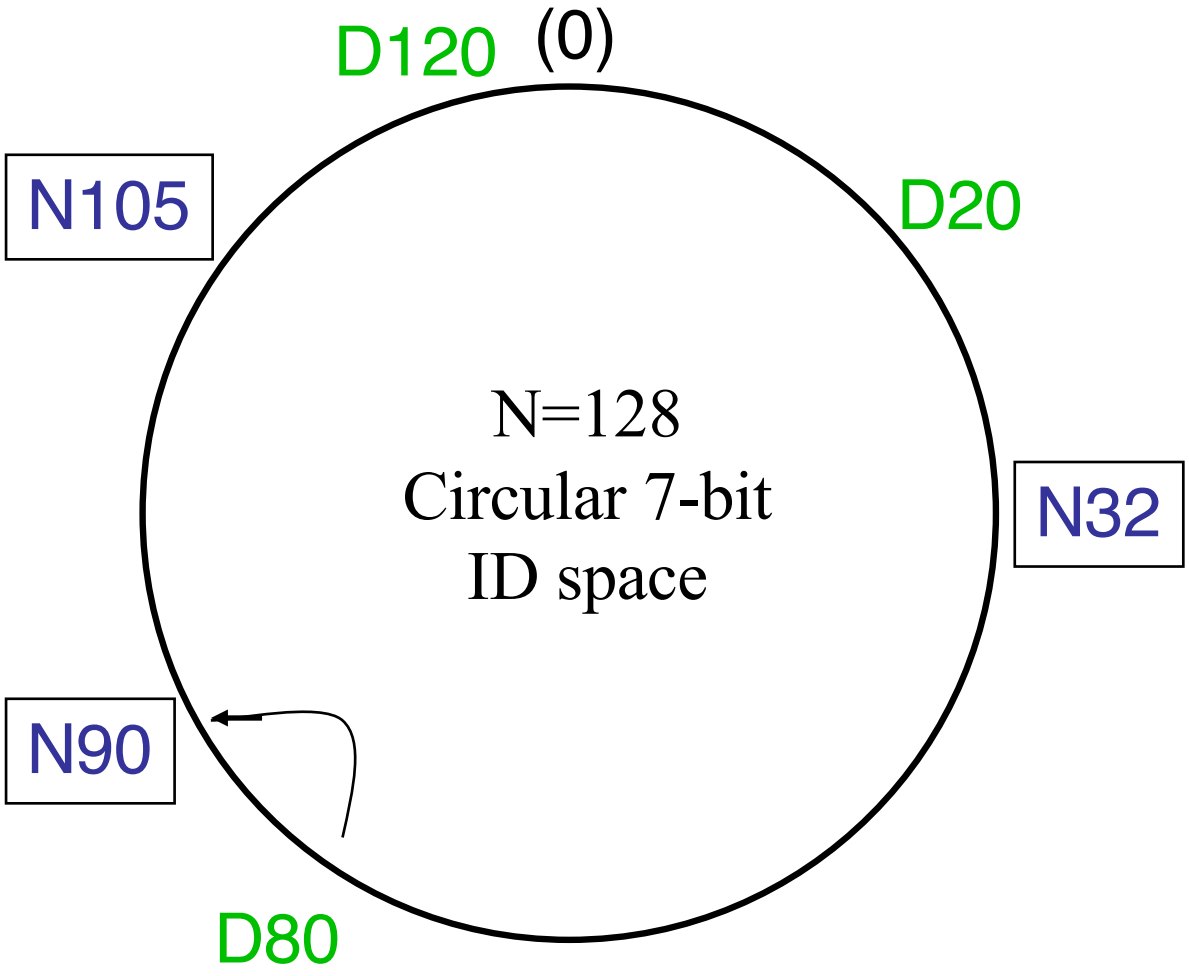
Consistent Hashing

Assigns an m-bit key to both nodes and objects from.

Order these nodes around an identifier circle (what does a circle mean here?) according to the order of their keys (0 .. 2^m-1). This ring is known as the Chord Ring.

Object with key k is assigned to the ***first node*** whose key is $\geq$ k (called the successor node of key k)

Consistent Hashing



Example: Node 90 is the "successor" of document 80.

Consistent Hashing [Karger 97]

Property 1

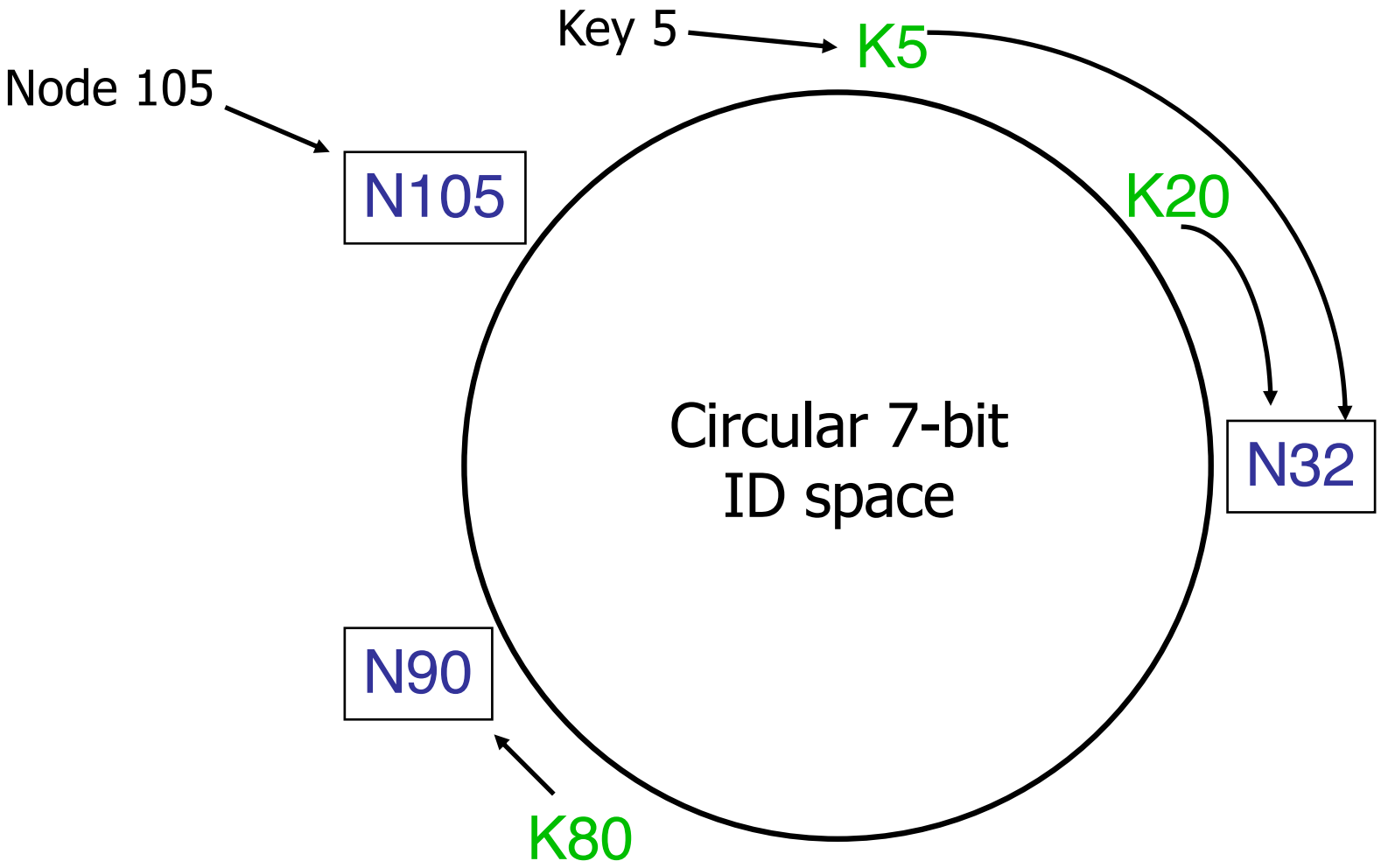
If there are N nodes and K keys, then *with high probability*, each node is responsible for (1+∈)K/N keys.

Property 2

When a node joins or leaves the network, the responsibility of at most O(K/N) keys changes hand (only to or from the node that is joining or leaving.

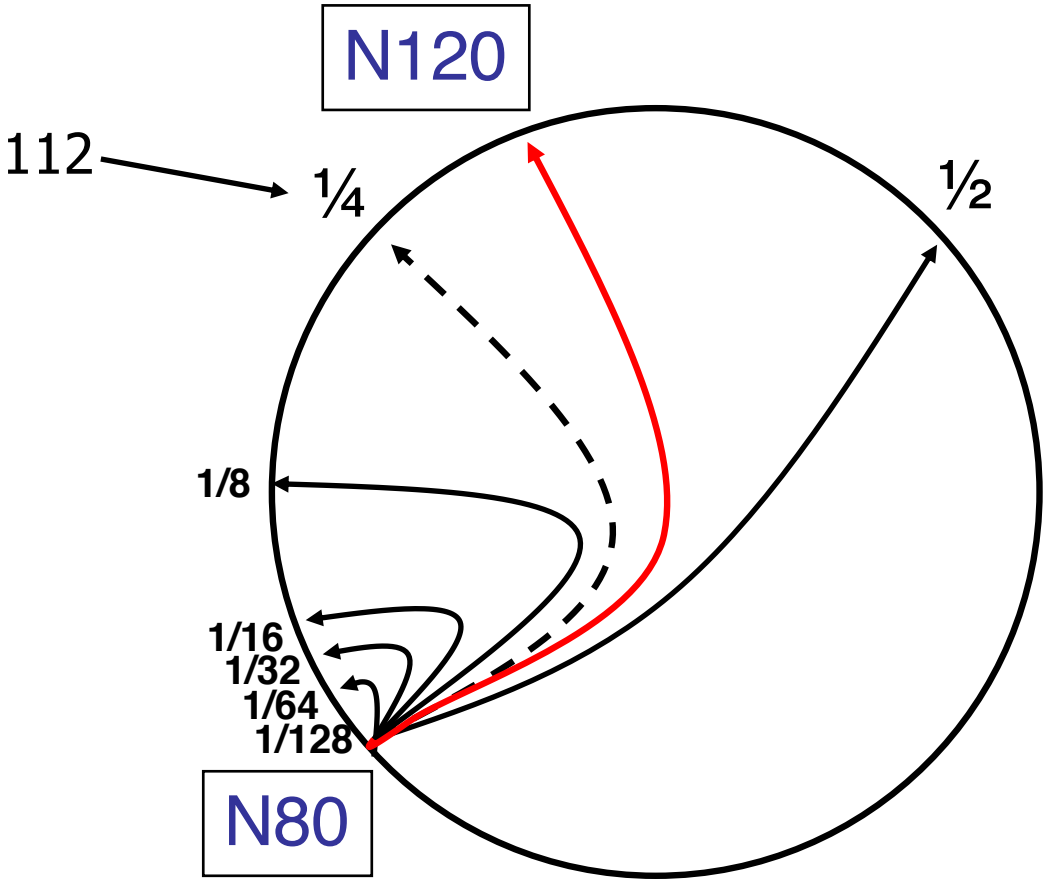
When K is large, the impact is quite small.

Consistent hashing



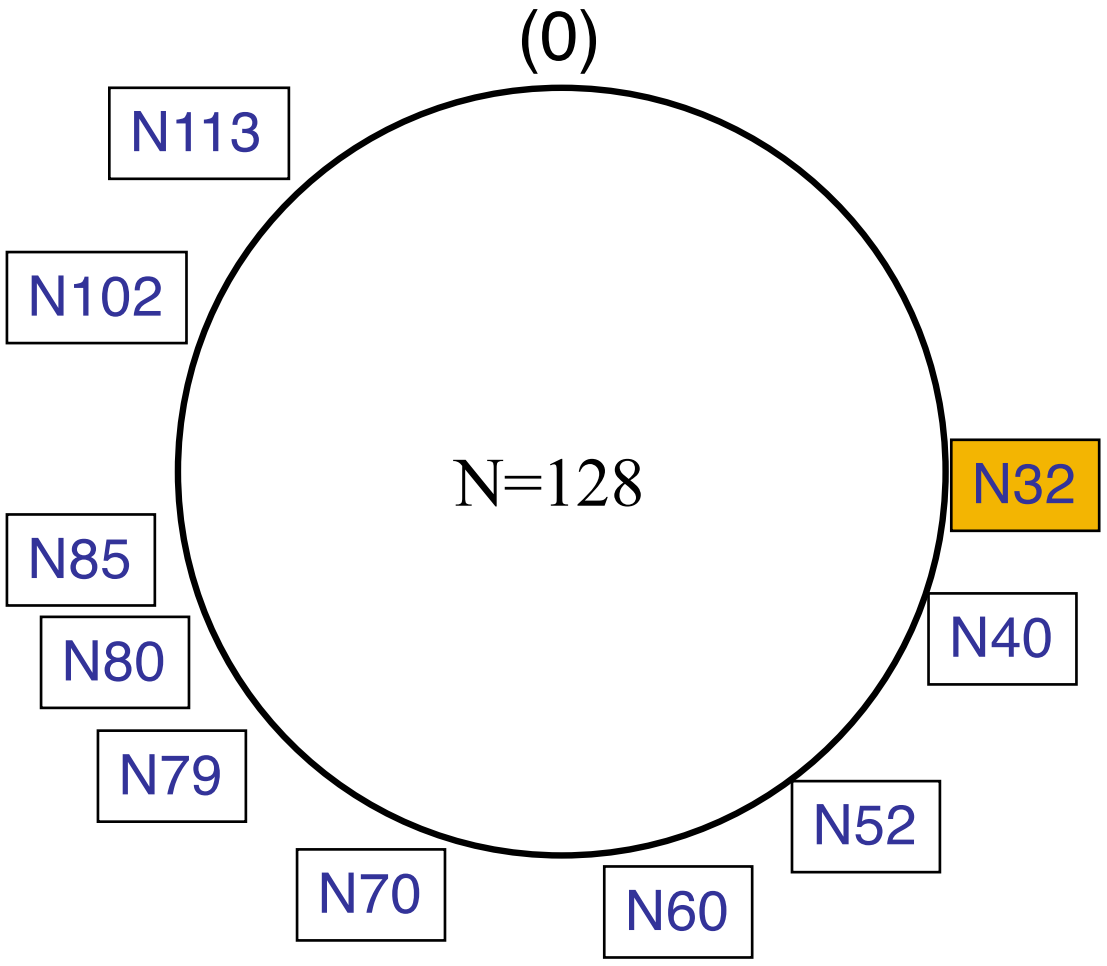
A key **k** is stored at its successor (node with key ≥ k)

The log N *Fingers*



Finger i points to successor of $n+2^i$

Chord Finger Table



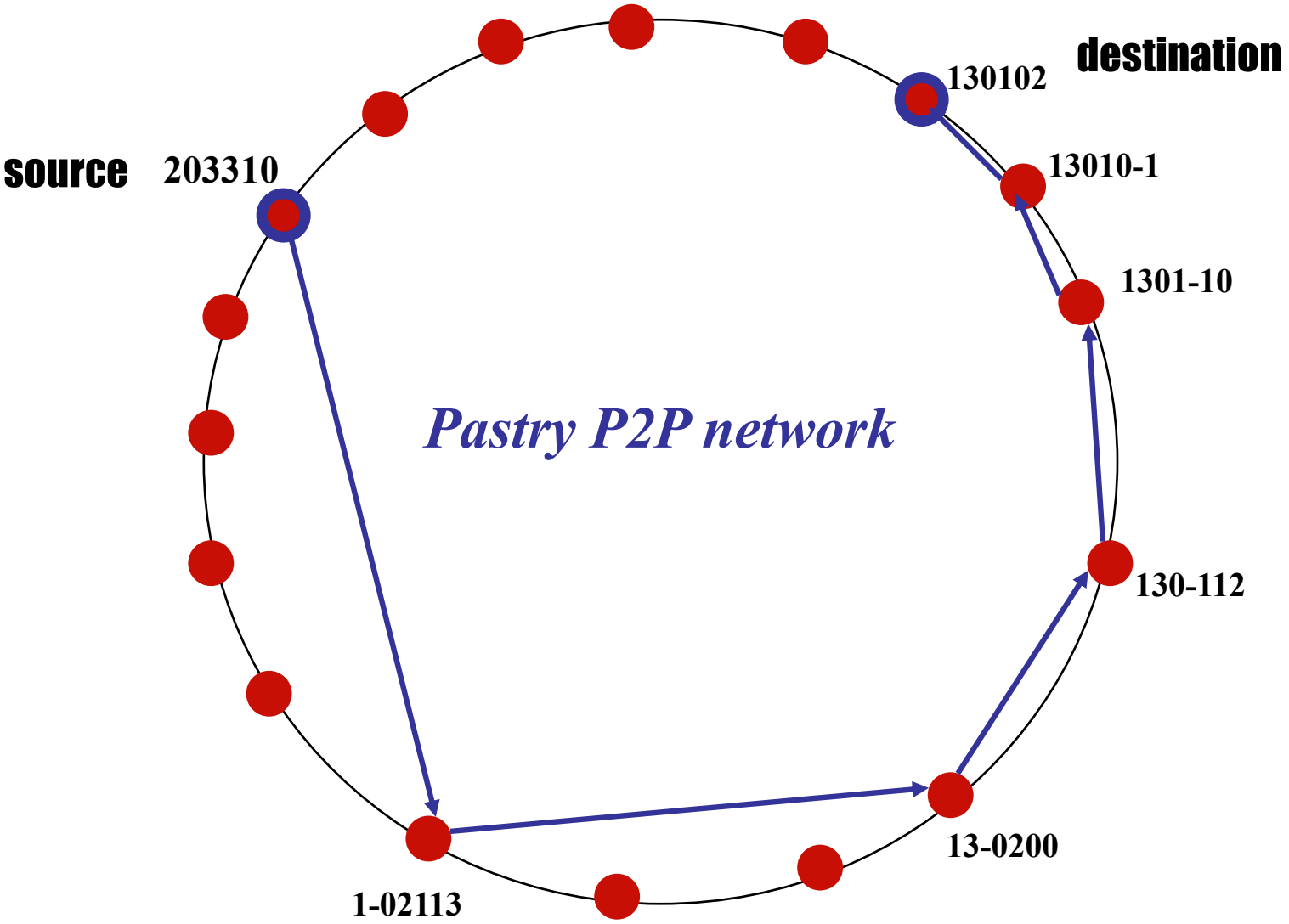
N32's Finger Table

33..33	N40
34..35	N40
36..39	N40
40..47	N40
48..63	N52
64..95	N70
96..31	N102

Finger table actually contains ID and IP address

Node n's i-th entry: first node with id $\geq n + 2^{i-1}$

Routing in Peer-to-peer networks

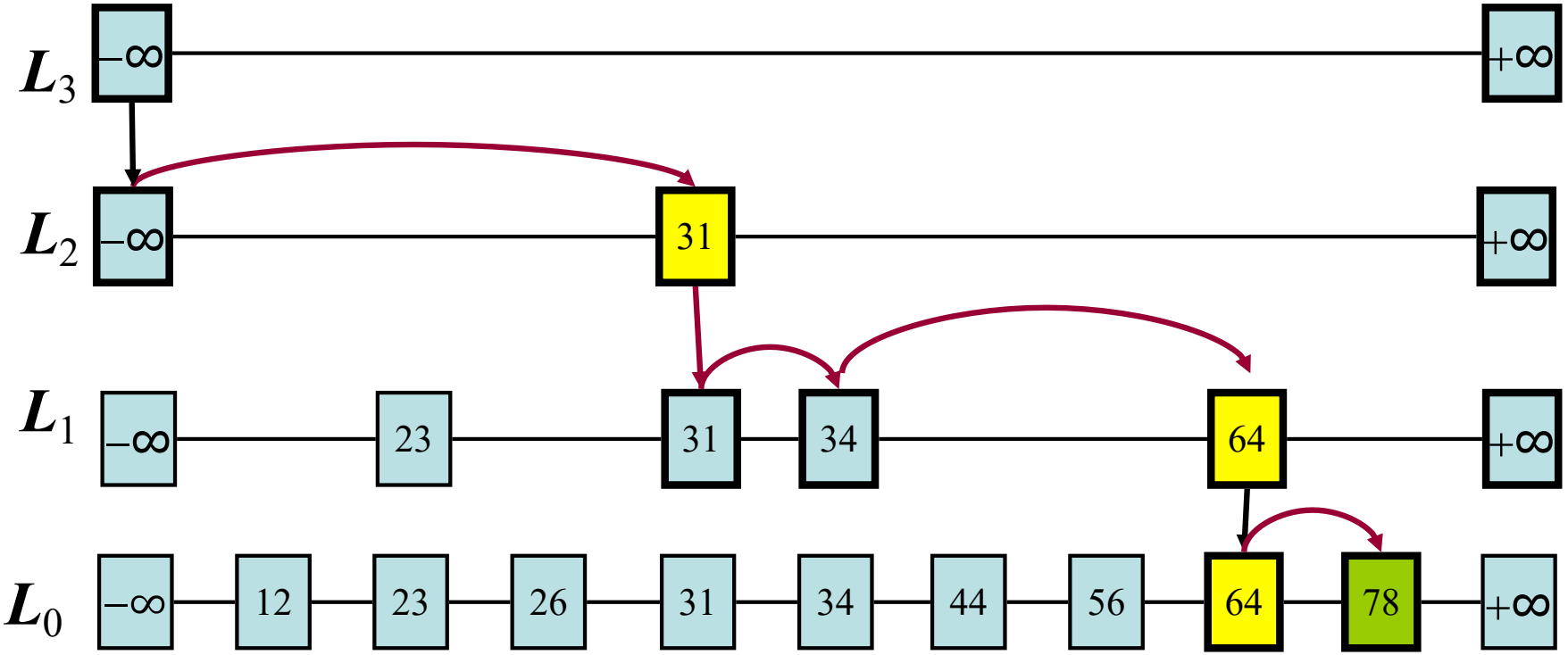


Skip lists and Skip graphs

- Start with a sorted list of nodes.
- Each node has a random sequence number
  of sufficiently large length, called its *membership vector*
- There is a route of length O(log N) that can be discovered using a greedy approach.

Skip List

(Think of train stations)

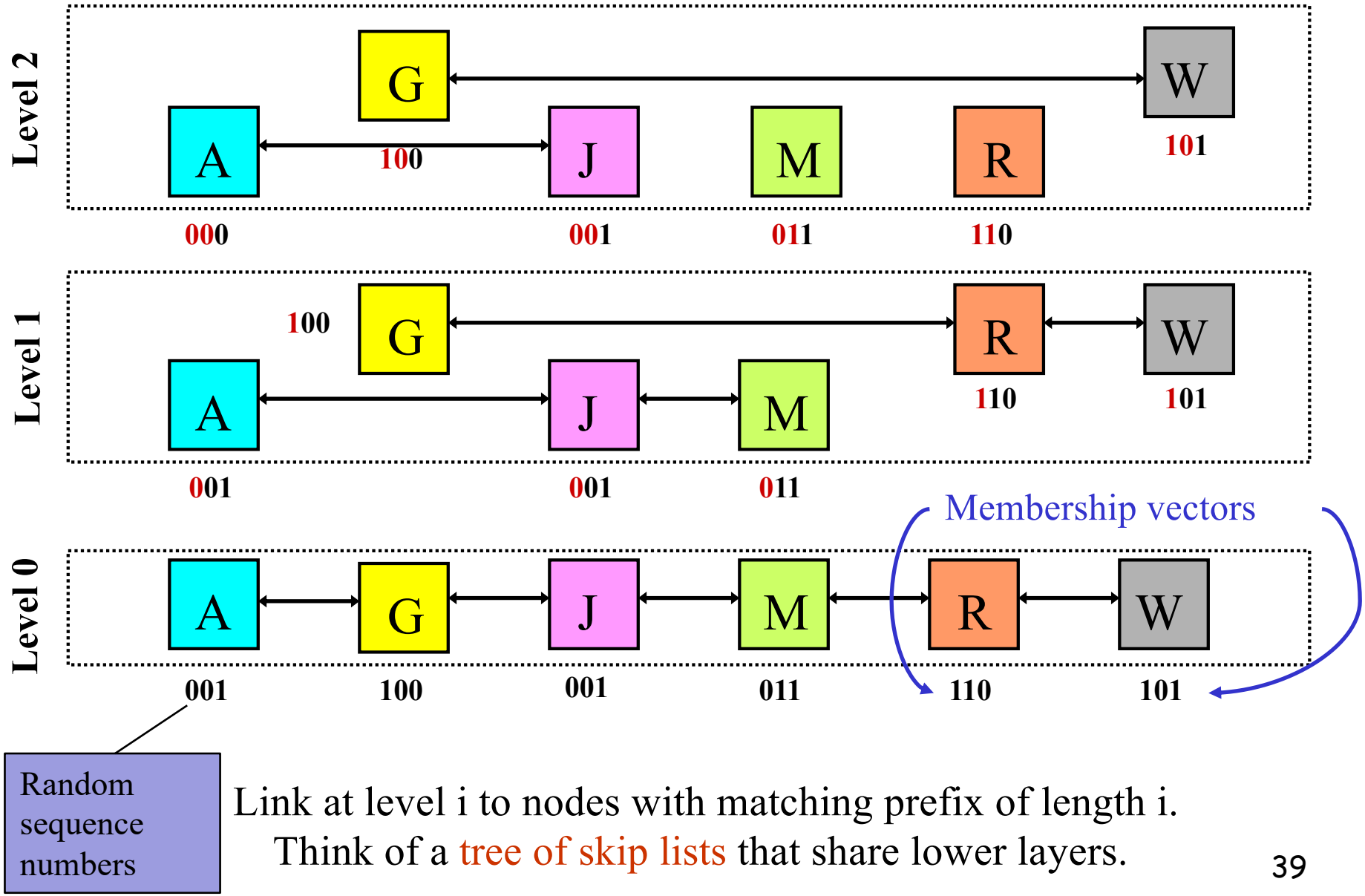


Example of routing to (or searching for) node 78. At L2, you can only reach up to 31.. At L1 go up to 64, As +∞ is bigger than 78, we drop down At L0, reach 78, so the search / routing is over.

Properties of skip graphs

1. Skip graph is a generalization of skip list.
2. Efficient Searching.
3. Efficient node insertions & deletions.
4. Locality and range queries.

Routing in Skip Graphs



Link at level i to nodes with matching prefix of length i.

Think of a tree of skip lists that share lower layers.

Properties of skip graphs

1. Efficient routing in O(log N) hops w.h.p.
2. Efficient node insertions & deletions.
3. Independence from system size.
4. Locality and range queries.

Spanning tree construction

Chang's algorithm {The root is known}

{Uses probes and echoes, and mimics the approach

in Dijkstra-Scholten's termination detection algorithm} {initially ∀i, parent (i) = i}

{***program of the initiator***}

Send probe to each neighbor;

do *number of echoes ≠ number of probes* →

echo received → echo := echo +1

probe received → send echo to the sender

od

{***program for node j, after receiving a probe*** }

first probe --> parent: = sender; forward probe to non-parent neighbors;

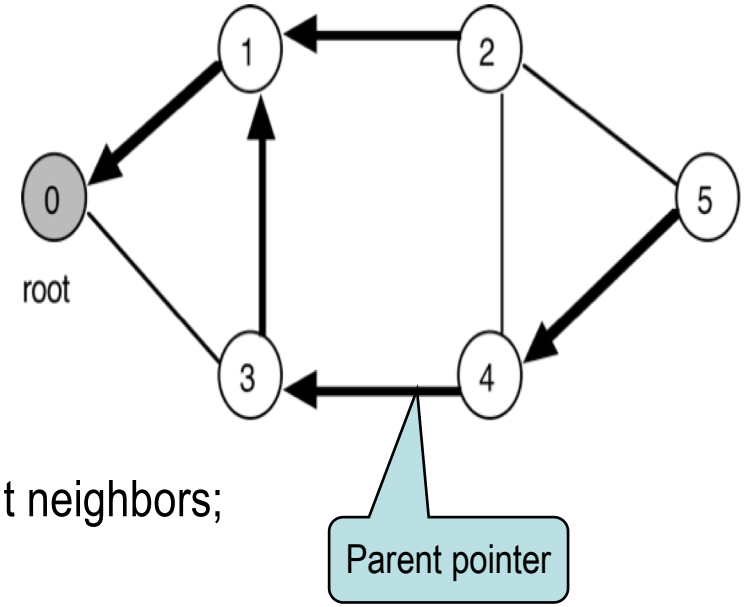
do *number of echoes ≠ number of probes* →

echo received → echo := echo +1

probe received → send echo to the sender

od

Send echo to parent; parent(i):= i



Question: ***What if the root is not designated?***

Graph traversal

Think about **web-crawlers**, exploration of **social networks**, planning of graph layouts for visualization or drawing etc.

Many applications of exploring an unknown graph by a visitor (a **token** or **mobile agent** or a **robot**). The goal of traversal is to visit every node at least once, and return to the starting point.

Main issues

- How efficiently can this be done?
- What is the guarantee that all nodes will be visited?
- What is the guarantee that the algorithm will terminate?

Graph traversal

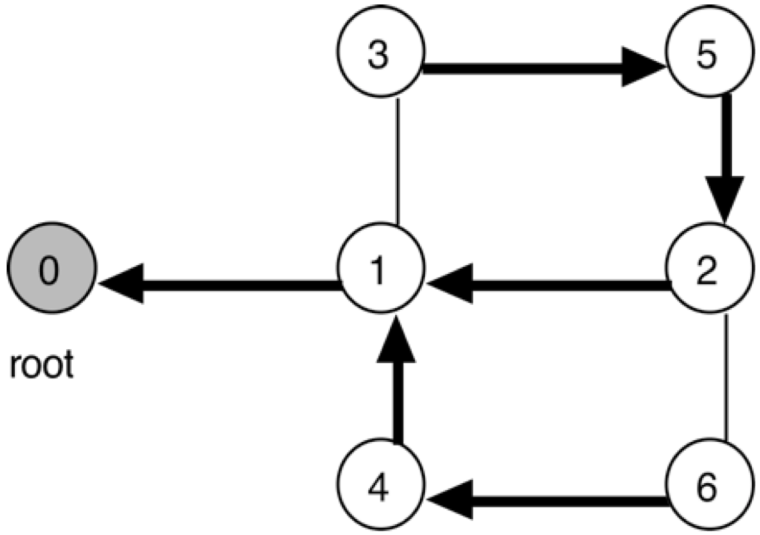
Review DFS (or BFS) traversals. These are well known, so we will not discuss them. There are a few papers that improve the complexity of these traversals

Graph traversal

Tarry's algorithm is one of the oldest (1895)

Rule 1. Send the token towards each neighbor exactly once.

Rule 2. If rule 1 is not applicable, then send the token to the *parent.*



A possible route is: **0 1 2 5 3 1 4 6 2 6 4 1 3 5 2 1 0**

The ***parent*** relation induces a spanning tree.