SKEIN MODULES AND THE NONCOMMUTATIVE TORUS

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1. Introduction

This paper introduces a new direction in the study of skein modules. The Kauffman bracket [9] is a knot invariant associated to quantum field theory. The noncommutative torus is an algebra of functions that appears in noncommutative geometry [6]. In this paper we explicate the relationship between the two.

When the variable of the Kauffman bracket is -1, the Kauffman bracket skein algebra of the 2-dimensional torus is isomorphic to the algebra of $SL_2\mathbb{C}$-characters of the fundamental group of the torus. You can think of these as a subalgebra of the algebra of continuous functions on the torus. For an arbitrary value of the variable, the Kauffman bracket skein algebra of the torus can be viewed as a deformation of this particular subalgebra. Similarly, the noncommutative torus is a deformation of the algebra of functions on the torus. The main result of the paper states that the Kauffman bracket skein algebra of the torus is isomorphic to a subalgebra of the noncommutative torus. That is, the two algebras arise from the same deformation.

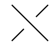
The functions we are working with are in fact trigonometric functions, and hence iterative techniques for dealing with Chebyshev polynomials are a central technique for establishing the results here. Their presence in this context is natural if one thinks of the relation between trigonometric functions and quantum physics.

Although a presentation of the Kauffman bracket skein algebra of the torus appeared before [5], the multiplicative structure of this algebra remained mysterious. Chebyshev polynomials with variables simple closed curves on the torus enable us to give a complete description of the multiplication operation, by the *product-to-sum* formula given below.

Some applications of the approach follow. First, we analize the structure of the Kauffman bracket skein module of the *solid* torus as a module over the Kauffman bracket skein algebra of the torus. Then, we give a short algebraic proof of the result of Hoste and Przytycki describing the Kauffman bracket skein module of a lens space. Finally, we show how to write, in terms of generators, the element of the Kauffman bracket skein algebra of the torus obtained by placing a Jones-Wenzl idempotent on a simple closed curve.

2. Skein Modules

Throughout this paper t will denote a fixed complex number. A *framed link* in an orientable manifold M is a disjoint union of annuli. In the case when the manifold can be written as the product of a surface and an interval, framed links will be identified with curves, using the convention that the annulus is parallel to the surface (i.e. we consider the blackboard framing). Let $\mathcal{L}$ denote the set of equivalence classes of framed links in M modulo isotopy, including the empty link.

Consider the vector space, $\mathbb{C}\mathcal{L}$ with basis $\mathcal{L}$. Define $S(M)$ to be the smallest subspace of $\mathbb{C}\mathcal{L}$ containing all expressions of the form  $-\,t$  $-\,t^{-1}$ $)($ and $\bigcirc + t^2 + t^{-2}$, where the framed links in each expression are identical outside the balls pictured in the diagrams. The *Kauffman bracket skein module* $K_t(M)$ is the quotient

$$\mathbb{C}\mathcal{L}/S(M).$$

Skein modules were introduced by Przytycki [12] as a way to extend the new knot polynomials of the 1980's to knots and links in arbitrary 3-manifolds. They have since become central in the theory of invariants of 3-manifolds. The idea that they could be used to quantize algebras of functions on surfaces is due to Turaev [17]. They were then used as a tool for constructing quantum invariants by Lickorish [11], Kauffman and Lins [10], Blanchet, Habegger, Masbaum, and Vogel [1], Roberts [15] and Gelca [7]. Finally, the connection between skein modules and characters of the fundamental group of the underlying manifold was explained by Bullock [2], Przytycki and Sikora [13] and Sikora [16]. The connection between skein algebras and the algebras of observables arising in lattice gauge field theory has been studied by Bullock, Frohman and Kania-Bartoszyńska [3]. There are also higher skein modules that were introduced in [4].

The Kauffman bracket skein module of the cylinder over a torus has a multiplicative structure, induced by the topological operation of gluing one cylinder on top of the other. The product $\alpha * \beta$ is the result of laying α over β. This multiplication makes $K_t(T^2 \times I)$ into an algebra, which we will call the skein algebra of the torus.

The skein module of a manifold that has a torus boundary has a left $K_t(T^2 \times I)$-module structure induced by gluing the zero end of the cylinder over a torus to that boundary component. In particular, this is true for $K_t(S^1 \times \mathbb{D}^2)$, the Kauffman bracket skein module of the solid torus. Note that $S^1 \times \mathbb{D}^2$ is homeomorphic with the cylinder over an annulus, hence $K_t(S^1 \times \mathbb{D}^2)$ is itself an algebra. However, the algebra structure of the skein module of the solid torus is not related to the algebra structure of the skein module of the cylinder over the torus. In fact $K_t(S^1 \times \mathbb{D}^2)$ is isomorphic to $\mathbb{C}[X]$ under the isomorphism that takes the simple closed curve α, which runs once around the torus, into the variable X. Consequently, a basis of $K_t(S^1 \times \mathbb{D}^2)$ as a $\mathbb{C}$-vector space is given by the elements α^n.

3. The Noncommutative Torus

The *noncommutative torus* is a "virtual" geometric space whose algebra of functions is a certain deformation of the algebra of continuous functions on the classical torus. One usually identifies the noncommutative torus with its associated algebra of functions.

The most natural way in which the noncommutative torus arises is by exponentiating the Heisenberg non-commutation relation $pq - qp = hI$. One then obtains an algebra generated by two unitary operators u and v which satisfy $uv = \lambda vu$ where $\lambda \in \mathbb{C}$ is some constant. The noncommutative torus is the the closure of this algebra in a certain C^*-norm.

As Rieffel [14] pointed out, the noncommutative torus can be obtained as a strict deformation quantization of the algebra of continuous functions on the torus in the following way. Let t be the deformation parameter (denoted this way to be consistent with the rest of the paper). For the space of Laurent polynomials of two variables $\mathbb{C}[l, l^{-1}, m, m^{-1}]$ (here $l = exp(2\pi i x)$ and $m = exp(2\pi i y)$ are the "longitude" and the "meridian" of the torus), one considers the basis over $\mathbb{C}$ given by the vectors $e_{p,q} = t^{-pq} l^p m^q$. Define the multiplication $*$, which depends on the parameter t, by

$$e_{p.q} * e_{r,s} = t^{\begin{vmatrix} p, q \\ t, s \end{vmatrix}} e_{p+r,q+s}.$$

The space of Laurent polynomials becomes a noncommutative algebra which we denote by $\mathbb{C}_t[l, l^{-1}, m, m^{-1}]$. This is the algebra of Laurent polynomials on the noncommutative torus. In order to construct the algebra A_θ of continuous functions on the noncommutative torus, where $t = e^{2\pi i \theta}$, one considers the left regular representation of $\mathbb{C}_t[l, l^{-1}, m, m^{-1}]$ on $L^2(\mathbb{T}^2)$ induced by this product, and takes the norm closure in the operator norm defined by this representation. Let us mention that the above construction corresponds to the deformation of the usual product of functions in the direction of the Poisson bracket associated to the symplectic form $\theta dx \wedge dy$. In the physical setting mentioned at the beginning, the unitary operators are $u = e_{1,0}$ and $v = e_{0,1}$.

There is a large body of literature devoted to the algebra A_θ. In the case when θ is irrational, this algebra is called the irrational rotation algebra, and has appeared in the works of operator theorists. It has been shown that A_θ is the C^*-algebra naturally associated to the Kronecker foliation of the torus $dy = \theta dx$ [6]. Also Weinstein explained how A_θ can be obtained through a geometric quantization procedure applied to the groupoid of this foliation [18].

In the present paper we are interested only in the algebra of Laurent polynomials on the noncommutative torus. Consider the algebra morphism

$$\Theta : \mathbb{C}_t[l, l^{-1}, m, m^{-1}] \to \mathbb{C}_t[l, l^{-1}, m, m^{-1}],\ \Theta(e_{p,q}) = e_{-p,-q}$$

and let $\mathbb{C}_t[l, l^{-1}, m, m^{-1}]^\Theta$ be its invariant part. Note that the algebra $\mathbb{C}_t[l, l^{-1}, m, m^{-1}]^\Theta$ is spanned by the elements $e_{p,q} + e_{-p,-q}$, $p, q \in \mathbb{Z}$. In the next section, we will show that this algebra has a significant role in the study of invariants of knots. As Θ has order two, its only eigenvalues are 1 and -1. The algebra $\mathbb{C}_t[l, l^{-1}, m, m^{-1}]$ then splits into the direct sum of its symmetric part and its antisymmetric part with respect to Θ. The subalgebra $\mathbb{C}_t[l, l^{-1}, m, m^{-1}]^\Theta$ is the symmetric part.

4. The Isomorphism

In this section we will prove that the Kauffman bracket skein algebra of the torus can be embedded in the noncommutative torus. More precisely, we will prove that $K_t(\mathbb{T}^2 \times [0,1])$ is isomorphic to the algebra $\mathbb{C}_t[l, l^{-1}, m, m^{-1}]^\Theta$ defined in the previous section. The proof is based on a multiplication formula, which is the object of Theorem 1, and which is important in its own respect. This formula describes explicitly the multiplication in the Kauffman bracket skein algebra of the torus. Let us point out that a presentation of this algebra was given in [5]. The elements that enable us to clear the picture and obtain a neat, compact formula for the multiplication are *Chebyshev polynomials*.

For two integers m, n we denote by $\gcd(m,n)$ their greatest common divisor, with the convention $\gcd(0,0) = 0$. We denote by T_n the n-th Chebyshev polynomial, defined recursively by $T_0 = 2$, $T_1 = x$ and $T_{n+1} = T_n \cdot T_1 - T_{n-1}$.

For p, q relatively prime and $n \geq 0$, we denote by (p, q) the (p, q)-curve on the torus. For (p, q) not necessarily relatively prime, we define

$$(p,q)_T = T_{\gcd(p,q)}\left(\left(\frac{p}{\gcd(p,q)}, \frac{q}{\gcd(p,q)}\right)\right)$$

which is the element of $K_t(\mathbb{T}^2 \times I)$ obtained by replacing the variable of the Chebyshev polynomial by the curve on the torus.

For $\sum \alpha_i D_i$ and $\sum \beta_j D'_j$ two elements of the Kauffman module of the torus, written as an algebraic combination of diagrams, we define their *intersection number* to be $max_{i,j} D_i \cdot D'_j$, where $D_i \cdot D'_j$ is the geometric intersection number of the diagrams D_i, and D'_j.

Remark 1. *For $m, n > 0$ and $gcd(p, q) = 1$, $gcd(r, s) = 1$, the geometric intersection number of $T_n(p, q)$ and $T_m(r, s)$ is the absolute value of $mn|\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}|$, where $|\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}|$ is the determinant.*

Theorem 1. *(the product-to-sum formula) For any integers p, q, r, s one has*

$$(p,q)_T * (r,s)_T = t^{|\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}|}(p+q, r+s)_T + t^{|\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}|}(p-q, r-s)_T.$$

Proof. The proof will be by induction on the intersection number of $(p, q)_T$ and $(r, s)_T$. If the intersection number is 0 or ± 1, the relation obviously holds, by one application of the skein relation. The case $p = q = 0$ or $r = s = 0$ is also trivial.

Case 1. $\gcd(p, q) = \gcd(r, s) = 1$.

We must show that

$$(p,q) * (r,s) = t^{|\begin{smallmatrix} p & r \\ q & s \end{smallmatrix}|}(p+r, q+s)_T + t^{-|\begin{smallmatrix} p & r \\ q & s \end{smallmatrix}|}(p-r, q-s)_T.$$

By applying a homeomorphism of the torus, this can be transformed in the equivalent identity:

$$(p,q) * (0,1) = t^p (p-1, q)_T + t^{-p}(p+1, q)_T$$

with $0 \leq q < p$.

If $p = 1, 2$, or if $q = 0$, the latter equality is obvious. To prove it for $p \geq 3$ we use the following result.

Lemma 1. *Given $p \geq 3$, $0 < q < p$ with $gcd(p, q) = 1$ there exist integers u, v, w, z satisfying $u+w = p$, $v+z = q$, $|\begin{smallmatrix} u & v \\ w & z \end{smallmatrix}| = \pm 1$, $0 < w < p$, $0 < u < p - 1$, $0 < v, z$.*

Proof. The equation

$$uz + vw = 1$$

can be rewritten as $u(q - v) - v(p - u) = 1$ or $uq - vp = 1$. From the general theory of linear Diophantine equations it follows that there exists a solution (u,v) with $0<u<p$ and $0<v<q$. Let $w=p-u$ and $z=q-v$. If $u=p-1$ exchange u and w, respectively v and z. $\square$

Returning to the proof of the theorem, the relations $u+w=p$, $v+z=q$ and $|{}^{uv}_{wz}|=\pm 1$, together with the skein relation imply

$$(p,q)=t^{-|{}^{uv}_{wz}|}(u,v)*(w,z)-t^{-2|{}^{uv}_{wz}|}(u-w,v-z).$$

Since $|{}^{wz}_{0\,1}|=w<p$ and $|{}^{u\,v}_{wz+1}|=|{}^{uv}_{wz}|+u=u\pm 1<p$ we can apply the induction hypothesis to write

$$(p,q)*(0,1)=t^{-|{}^{uv}_{wz}|}(u,v)*(w,z)*(0,1)-t^{-2|{}^{uv}_{wz}|}(u-w,v-z)*(0,1)=$$

$$t^{-|{}^{uv}_{wz}|}(u,v)*[t^{w}(w,z+1)_T+t^{-w}(w,z-1)_T]-$$

$$t^{-2|{}^{uv}_{wz}|}[t^{u-w}(u-w,v-z+1)_T+t^{w-u}(u-w,v-z-1)_T]=$$

$$t^{w+u}(u+w,v+z+1)_T+t^{-2|{}^{uv}_{wz}|+w-u}(u-w,v-z-1)_T+$$

$$t^{-2|{}^{uv}_{wz}|-w+u}(u-w,v-z+1)_T+t^{-w-u}(u+w,v+z-1)_T-$$

$$t^{-2|{}^{uv}_{wz}|+u-w}(u-w,v-z+1)_T+t^{-2|{}^{uv}_{wz}|-u+w}(u-w,v-z-1)_T=$$

$$t^{p}(p,q+1)_T+t^{-p}(p,q-1)_T.$$

Case 2. One of $\gcd(p,q)$ or $\gcd(r,s)$ is greater then 1.

Assume that $\gcd(p,q)\geq 2$, and let $n=\gcd(p,q)$, $p'=p/n$, $q'=q/n$. Then, an induction on n gives

$$\begin{gathered}
(p,q)_T*(r,s)_T=T_n(p',q')*(r,s)_T=\\
T_{n-1}(p',q')*(p',q')*(r,s)_T-T_{n-2}(p',q')*(r,s)_T=\\
T_{n-1}(p',q')*(t^{|{}^{p'r}_{q's}|}(p'+r,q'+s)_T+t^{-|{}^{p'r}_{q's}|}(p'-r,q'-s)_T)-\\
t^{(n-2)|{}^{p'r}_{q's}|}((n-2)p'+r,(n-2)q'+s)_T+\\
t^{-(n-2)|{}^{p'r}_{q's}|}((n-2)p'-r,(n-2)q'-s)_T=\\
t^{|{}^{p'r}_{q's}|+(n-1)|{}^{p'r}_{q's}|}(np'+r,nq'+s)_T+t^{-|{}^{p'r}_{q's}|-(n-1)|{}^{p'r}_{q's}|}(np'-r,nq'-s)_T=\\
t^{|{}^{pr}_{qs}|}(p+r,q+s)_T+t^{-|{}^{pr}_{qs}|}(p-r,q-s)_T
\end{gathered}$$

and the theorem is proved. $\square$

Theorem 2. *There exists an isomorphism of algebras*

$$\phi:K_t(\mathbb{T}^2\times I)\to\mathbb{C}_t[l,l^{-1},m,m^{-1}]^{\Theta}$$

determined by

$$\phi((p,q)_T) = e_{(p,q)} + e_{(-p,-q)},\ p,q \in \mathbb{Z}.$$

Proof. The fact that the map is a morphism follows from Theorem 1 and the fact that

$$(e_{p,q} + e_{-p,-q}) * (e_{r,s} + e_{-r,-s}) = t^{\left|\begin{smallmatrix} p & r \\ q & s \end{smallmatrix}\right|}(e_{p+r,q+s} + e_{p-r,q-s}) +$$

$$t^{-\left|\begin{smallmatrix} p & r \\ q & s \end{smallmatrix}\right|}(e_{p-r,q-s} + e_{p+r,q+s}).$$

As $\mathbb{C}$-vector spaces the two algebras have the basis $(p,q)_T$, $p \in \mathbb{Z}_+$, $q \in \mathbb{Z}$, respectively $e_{p,q} + e_{-p,-q}$, $p \in \mathbb{Z}_+$, $q \in \mathbb{Z}$, which proves that the map is an isomorphism. □

5. The Solid Torus

In this section we explain how to obtain the skein module of the solid torus from the skein algebra of the torus, and explicate its module structure.

The solid torus is obtained as a quotient of $\mathbb{T}^2 \times [0,1]$ hence $K_t(S^1 \times \mathbb{D}^2)$ is obtained by factoring $K_t(\mathbb{T}^2 \times [0,1])$. As mentioned before $K_t(\mathbb{T}^2 \times [0,1])$ acts on the left on the Kauffman bracket skein module of the solid torus by the gluing map, so the latter is a $K_t(\mathbb{T}^2 \times [0,1])$-module. Hence the the skein module of the solid torus is the quotient of the skein algebra of the torus by a left ideal. The basis for $K_t(S^1 \times \mathbb{D}^2)$ as a $\mathbb{C}$-vector space is given by $\{\alpha^n\}_n$, and these elements are the images of $(0,1)^n$, $n \geq 0$ through the quotient map. However, for a better understanding of the module structure, it is better to work with the basis $\{\alpha_n\}_n$, $\alpha_n = T_n(\alpha)$. We denote by $\cdot$ the left action of $K_t(\mathbb{T}^2 \times [0,1])$ on the skein module of the solid torus.

Let $\mathcal{I}$ be the left ideal that is the kernel of the epimorphism

$$\pi : K_t(\mathbb{T}^2 \times [0,1]) \to K_t(\mathbb{D}^2 \times \mathbb{T}).$$

We want to show that $(0,1) + t^2 + t^{-2}$ and $(1,1) + t^{-3}(1,0)$ form a minimal set of generators for $\mathcal{I}$. For this let $\mathcal{J}$ be the ideal generated by these two elements.

Lemma 2. *Every element in $K_t(\mathbb{T}^2 \times [0,1])$ is of the form $P((0,1)) + u$ where $P \in \mathbb{C}[X]$ and u is in the left ideal $\mathcal{J}$.*

Proof. Since as a vector space $K_t(\mathbb{T}^2 \times [0,1])$ is spanned by $(p,q)_T$, $p,q \in \mathbb{Z}$, it suffices to prove the statement for elements of this form.

If $p = 0$, since $(0,q)_T$ is a polynomial in $(0,1)$,

$$(0,q)_t = a * ((0,1) + t^2 + t^{-2}) + b,\ a \in K_t(\mathbb{T}^2 \times [0,1]), b \in \mathbb{C}.$$

If $p = 1$, then from Theorem 1 we get

$$(1,q)_T = t^{-q+1}(1,1) * (0,q-1)_T + t^{-2q+2}(1,q-2)_T$$

and the previous argument together with an induction on q show that there exist u, u' in $\mathcal{J}$, $c \in \mathbb{C}$ and P a polynomial such that

$$(1,q)_T = u + c(1,1) + u' + P(1,0) =$$

$$u + u' + c((1,1) + t^{-3})(1,0) - t^{-3}(1,0) + P(1,0)$$

which proves this case, too.

For $p \geq 2$ Theorem 1 gives

$$(p,q)_T = t^{-q}(1,0) * (p-1,q)_T + t^{-2q}(p-2,q)_T$$

and an induction on p gives the desired conclusion.

Finally, the case $p < 0$ follows from

$$(p,q)_T = t^{pq}(0,q)_T * (p,0)_T - t^{2pq}(-p,q)_T.$$

□

Theorem 3. $\mathcal{I} = \mathcal{J}$.

Proof. It is easy to see that $(0,1) + t^2 + t^{-2}$ is in $\mathcal{I}$. On the other hand, in the solid torus, $(1,1)$ has framing -1, so $(1,1) = -t^{-3}(1,0)$, from which it follows that the second generator of $\mathcal{J}$ is in $\mathcal{I}$ as well, hence $\mathcal{J} \subset \mathcal{I}$.

Since the restriction of π to the subring of $K_t(\mathbb{T}^2 \times [0,1])$ generated by $(1,0)$, previous lemma shows that $\mathcal{I} \subset \mathcal{J}$, and the theorem is proved. □

Let us now describe the action of $K_t(\mathbb{T}^2 \times [0,1])$ on $K_t(S^1 \times \mathbb{D}^2)$. Define $x_{p,q}$ in $K_t(S^1 \times \mathbb{D}^2)$ by $x_{p,q} = t^{-pq} y_{p,q}$ where $y_{p,q}$ satisfies

$$\begin{aligned} y_{p,q} &= \alpha \cdot y_{p-1,q} - y_{p-2,q}, \\ y_{0,q} &= (-t^2)^q + (-t^{-2})^q, \\ y_{1,q} &= (-t^{-3})^q (1,0). \end{aligned}$$

Lemma 3. *The element $x_{p,q}$ is the image of $(p,q)_T$ in $K_t(S^1 \times \mathbb{D}^2)$.*

Proof. Since

$$(1,0) * (p,q)_T = t^q (p+1,q)_T + t^{-q}(p-1,q)_T$$

it follows that

$$x_{p+1,q} = t^{-q}(1,0) \cdot x_{p,q} - t^{-2q} x_{p,q}$$

from which the desired reccurence is obtained by multiplication by $t^{(p+1)q}$. The initial condition is a consequence of the properties of Chebyshev polynomials. □

Corollary 1. *For any two integers p and q, the element $x_{p,q}$ is a polynomial of $|p|$-th degree in α.*

Theorem 4. *The action of the skein algebra of the torus on the skein module of the solid torus is given by*

$$(p,q)_T \cdot \alpha_n = t^{p-nq} x_{n+p,q} + t^{-p+nq} x_{p-n,q}.$$

Proof. The theorem is a consequence of Lemma 3 and of

$$(p,q)_T * (n,0)_T = t^{p-nq}(n+p,q)_T + t^{-p+nq}(p-n,q)_T.$$

□

6. Lens Spaces

In this section we will give an alternate short proof of the result of Hoste and Przytycki [8] which describes the structure of the Kauffman bracket skein module of a lens space.

Let $L(p,q)$ be a lens space $(p,q \neq 0)$ and let

$$\begin{pmatrix} a & p \\ b & q \end{pmatrix}$$

be the gluing matrix of the two tori. Since the gluing map reverses orientation, the determinant of this matrix is -1. Any link in the lens space can be pushed off the cores of the two tori , thus

$$K_t(L(p,q)) = K_t(S^1 \times \mathbb{D}) \bigotimes_{K_t(\mathbb{T}^2 \times [0,1])} K_t(S^1 \times \mathbb{D}),$$

where the tensor product structure is defined by

$$x_{m,n} \otimes 1 = 1 \otimes x_{am+pn,bm+nq}.$$

Note in particular that $K_t(L(p,q))$ is spanned by the elements $1 \otimes 1$, $1 \otimes \alpha$, $1 \otimes \alpha^2$, $\cdots$. We will prove that in fact $K_t(L(p,q))$ is spanned by $1 \otimes 1$, $1 \otimes \alpha$, $\cdots$, $1 \otimes \alpha^{[\frac{p}{2}]}$. To this end let V be the span of these elements.

We start by noting that Theorem 3 implies that

$$1 \otimes x_{p,q} = ((0,1) \cdot 1) \otimes 1 = (-t^2 - t^{-2}) \otimes 1$$

and

$$1 \otimes x_{p+a,q+b} = ((1,1) \cdot 1) \otimes 1 = -t^{-3}((1,0) \cdot 1) \otimes 1 = -t^{-3} \otimes x_{a,b}.$$

Lemma 4. *For every $k \in \mathbb{Z}$, there exists a constant $c_k \in \mathbb{C}$ such that for all $u \in K_t(S^1 \times \mathbb{D})$ one has the identity*

$$1 \otimes ((a+kp, b+kq) \cdot u) = c_k \otimes ((a,b) \cdot u).$$

Proof. The property is true for $k = 0$ and $k = 1$. Since by Theorem 1

$$\begin{gathered} 1 \otimes ((a+kp, b+kq) \cdot u) = \\ t^{aq-bp} \otimes ((p,q) * ((k-1)p+a, (k-1)q+b) \cdot u) - \\ t^{2(aq-bp)} \otimes (((k-2)p+a, (k-2)q+b) \cdot u) = \\ t^{aq-bp}(-t^2-t^{-2}) \otimes ((k-1)p+a, (k-1)q+b) \cdot u) - \\ t^{2(aq-bp)} \otimes (((k-2)p+a, (k-2)q+b)) \cdot u). \end{gathered}$$

the property follows by induction on k. □

Lemma 5. *For every $m, k \in \mathbb{Z}$, one has $1 \otimes x_{ma+kp,mb+kq} \in V$.*

Proof. We will induct on m. For $m = 0, 1$ the property is true, as a consequence of Lemma 4 and the fact that $1 \otimes x_{kp,kq} = (-t^2-t^{-2})^k \otimes 1$.

Let k_0 be the integer that minimizes the absolute value of $ma + k_0p$. Clearly this minimum is at most $[\frac{p}{2}]$, hence $x_{ma+k_0p,mb+k_0q}$ is in V. On the other hand, by using Theorem 1, we get that, for an arbitrary k,

$$\begin{gathered} x_{ma+kp,mb+kq} = t^{-(m-1)k-mk_0} \otimes ((a+(k-k_0)p, b+(k-k_0)q) * \\ ((m-1)a+k_0p, (m-1)b+k_0q) \cdot 1) - \\ t^{-2(m-1)k-2mk_0} \otimes x_{(m-2)a-(k-2k_0)p,(m-2)b-(k-2k_0)q}. \end{gathered}$$

¿From Lemma 4 and Theorem 1 it follows that

$$\begin{gathered} 1 \otimes (a+(k-k_0)p, b+(k-k_0)q) * \\ ((m-1)a+k_0p, (m-1)b+k_0q) \cdot 1) = \\ c_k \otimes ((a,b) * ((m-1)a+k_0p, (m-1)b+k_0q) = \\ c_k t^{-1} x_{ma+k_0p,mq+k_0q} + c_k t x_{(m-2)q+k_0p,(m-2)b+k_0q}. \end{gathered}$$

So, from the induction hypothesis and the fact that $x_{ma+k_0p,mb+k_0q} \in V$ we get that $x_{ma+kp,mb+kq} \in V$, which completes the induction. □

Theorem 5. *(Theorem 4. in* [8]*) The space $K_t(L(p,q))$ is spanned by $1 \otimes 1$, $1 \otimes \alpha$, $\cdots$, $1 \otimes \alpha^{[\frac{p}{2}]}$.*

Proof. Every natural number n can be written in the form $ma + kp$. From Corollary 1 it follows that

$$1 \otimes x_{ma+kp,mb+kq} = c \otimes \alpha^{ma+kp} + 1 \otimes f(\alpha)$$

where f is a polynomial of degree strictly less than $ma + kp$, and c is a nonzero constant, hence by applying Lemma 5 and inducing on n we deduce that $\alpha^n \in V$ for every n, which proves the theorem. □

7. Jones-Wenzl idempotents

Jones-Wenzl idempotents appeared for the first time in the study of operator algebras, but they are best known to topologists because of their use in the construction of topological quantum field theories ([11, 1, 15, 7]). By placing Jones-Wenzl idempotents on simple closed curves on a torus one obtains certain elements of the skein algebra of the torus. We show below how one can make use of the embedding of this algebra in the noncommutative torus to give a pleasing formula for these skeins.

Fix r an integer greater than 1, and let $t = e^{i\pi/2r}$. For a positive integer n, the n-th Jones-Wenzl idempotent $f^{(n)}$ lives in the Temperley-Lieb algebra TL_n, which, let us remind, is the algebra of diagrams of non-intersecting strands joining $2n$ points on the boundary of a rectangle, with multiplication defined by juxtaposition of rectangles. Jones-Wenzl idempotents are denoted by empty boxes, and are defined inductively as in Fig. 1. Here the convention is that a number k written next to a strand means k parallel strands, and $\Delta_k = (-1)^k[k+1]$, where $[k+1]$ is the quantized integer $\frac{t^{2k+2}-t^{-2k-2}}{t^2-t^{-2}}$. Recall that Jones-Wenzl idempotents are defined only for $n = 0, 1, \cdots, r-2$.

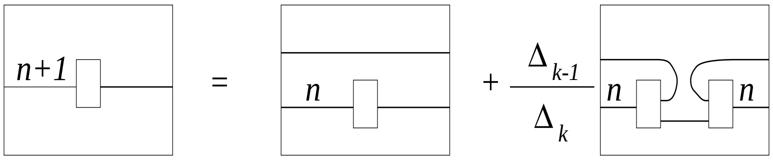


Figure 1.

We will denote by $(p,q)_{JW}$ the element of $K_t(\mathbb{T}^2 \times I)$ obtained by taking $\gcd(p,q)$ parallel copies of the $(\frac{p}{\gcd(p,q)}, (\frac{q}{\gcd(p,q)})$-curve and inserting on them the $\gcd(p,q)$-th Jones-Wenzl idempotent.

Theorem 6. *If p and q are relatively prime and n is a positive integer less than $r-2$, then*

$$\begin{aligned}(np,nq)_{JW} = (np,nq)_T + ((n-2)p,(n-2)q)_T + \cdots \\ +((n-2[\tfrac{n}{2}])p,(n-2[\tfrac{n}{2}])q)_T\end{aligned}$$

Proof. By using the well known identities from Fig. 2 we deduce that the following recurrence relation holds

$$(np,nq)_{JW} = (p,q) * ((n-1)p,(n-1)q)_{JW} - ((n-2)p,(n-2)q)_{JW}.$$

By Theorem 2, the image of the (p,q)-curve in the noncommutative torus is $e_{p,q} + e_{-p,-q}$. In the noncommutative torus the elements

$$e_{mp,mq} + e_{(m-2)p,(m-2)q} + \cdots + e_{-(m-2)p,-(m-2)q} + e_{-mp,-mq}$$

Figure 2.

with $m \geq 0$, satisfy the same recurrence relation as $(np, nq)_{JW}$. Since the image of $f^{(0)}$ is 1, and the image of $(p, q)_{JW}$ is $e_{p,q} + e_{-p,-q}$ (p and q are relatively prime), the conclusion of the theorem follows by induction. $\square$

Note that both $(np, nq)_T$ and $(np, nq)_{JW}$ satisfy the recurrence relation of Chebyshev polynomials. Also the term corresponding to $n = 1$ is the same in both cases. However, the term corresponding to $n = 0$ is 2 in the first case, and 1 in the second, which makes them so much different.

It is a well known fact that the trace of a Jones-Wenzl idempotent is

$$(-1)^n[n+1] = (-1)^n \tfrac{t^{2n+2} - t^{-2n-2}}{t^2 - t^{-2}} =$$
$$(-1)^n(t^{2n} + t^{2n-2} + \cdots + t^{-2n+2} + t^{-2n})$$

As the theorem and its proof show the elements $(np, nq)_{JW}$ are obtained by replacing $(-t^2)^k + (-t^{-2})^k$ in this formula with $(kp, kq)_T$.

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