# SKEIN QUANTIZATION AND LATTICE GAUGE FIELD THEORY

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## 1. Introduction

The Kauffman bracket skein module is a deceptively simple construction, occurring naturally in several fields of mathematics and physics. This paper surveys various ways in which it is a quantization of a classical object.

Przytycki [38] and Turaev [46] introduced skein modules. Shortly thereafter, Turaev [47] discovered that they formed quantizations of loop algebras, and Hoste and Przytycki pursued related quantizations in [25]. We will look at some of the heuristic reasons for treating skein modules as deformations, and then realize the Kauffman bracket module as a precise quantization in two different ways.

One type of quantization consists of the construction of a non-commutative algebra that deforms a commutative algebra in a manner coherent with a Poisson structure. It is well known that the $SL_2(\mathbb{C})$-characters of a surface group form a Poisson algebra [6, 22]. The Kauffman bracket skein module is the appropriate deformation; a fact which emerged following a surprising discovery relating the skein module and $SL_2(\mathbb{C})$ invariant theory.

There is an alternative approach to the quantization of surface group characters, via lattice gauge field theory, due to Fock and Rosly [21]. It was developed by Alekseev, Grosse and Schomerus [2] and by Buffenoir and Roche [9, 8]. We tie the approaches together by showing that the skein module coincides with the lattice quantization.

## 2. The Kauffman Bracket Skein Module

Quantum topology began with the discovery of several new link polynomials, the first and most well known being the Jones polynomial [29], [30]. Many subsequent invariants arose from alternative proofs of its existence. The state sum approach [32] yielded an invariant known as the Kauffman bracket, on which we will focus. The Kauffman bracket is a function on the set of framed links in $\mathbb{R}^3$. Since we will take a combinatorial view throughout, one may as well think of a link as represented by a diagram in $\mathbb{R}^2$ (see Figure 1).

A link is an embedding of circles, of which a diagram is a particularly convenient picture. Two diagrams represent the same link if one can be deformed into the other.[1] In a framed link each circle is actually the centerline of an embedded annulus. Since we always work with diagrams, it makes sense to assume the annulus lies flat in $\mathbb{R}^2$ as illustrated in Figure 2.

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[1] These deformations may include some Reidemeister moves.

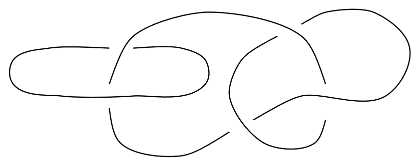

FIGURE 1. Diagram of a two-component link.

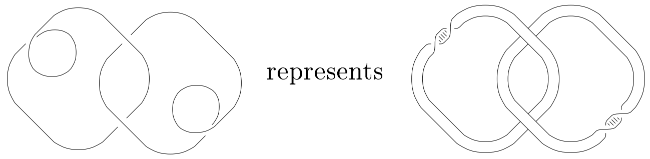


FIGURE 2. Diagram of a framed Hopf link

The Kauffman bracket, $\langle\ \rangle$, takes values in the ring $\mathbb{Z}[A^{\pm 1}]$ and is uniquely determined by the rules:[2]

1. $\langle \emptyset \rangle = 1$,
2. $\left\langle \times \right\rangle = A \left\langle \asymp \right\rangle + A^{-1} \left\langle )( \right\rangle$, and
3. $\langle \bigcirc \rangle = (-A^2 - A^{-2})\langle \emptyset \rangle$.

The arguments of $\langle\ \rangle$ in (2) and (3) represent diagrams which are identical except in a neighborhood where they differ as shown in the formulas. One evaluates the function on a diagram by first applying (2) until no crossings remain, and then reducing each diagram to a polynomial via (3) and (1).

**Theorem 1** (Kauffman)**.** *The function $\langle\ \rangle$ is well defined. If $D_1$ and $D_2$ represent the same framed link, then $\langle D_1 \rangle = \langle D_2 \rangle$.*

Kauffman's construction is an example of a link invariant defined by skein relations on the set of all diagrams. Since skein relations are defined only in small neighborhoods, the idea generalizes naturally to spaces locally modeled on $\mathbb{R}^3$.

The notion of a skein module of a 3-manifold[3] was introduced independently by Przytycki in [38] and Turaev in [46]. Roughly speaking, the construction consists of dividing the linear space of all links by an appropriate set of skein relations, usually the same as those known to define a polynomial invariant in $\mathbb{R}^3$. We will give the explicit definition for the Kauffman bracket skein module.

Let $\mathcal{L}_M$ be the set of framed links (including $\emptyset$) in a 3-manifold $M$. Denote by $\mathbb{C}\mathcal{L}_M$ the vector space consisting of all linear combinations of framed links. Take

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[2]Actually, Kauffman did not include the empty diagram $\emptyset$, and his normalization was $\langle \text{unknot} \rangle = 1$. Later, the quantum group approach [33, 41] to knot polynomials—and to some extent skein modules—indicated that normalization at $\emptyset$ was preferable.

[3]A suggestion appeared in Conway's treatment of the Alexander polynomial [18].

$\mathbb{C}\mathcal{L}_M[[h]]$ to be formal power series with coefficients in $\mathbb{C}\mathcal{L}_M$, and give it the $h$-adic topology.[4] This is an example of a topological module (see [31] for a nice introduction), however, one may think of $\mathbb{C}\mathcal{L}_M[[h]]$ as just the completion of a vector space with basis $\mathcal{L}_M$ and scalars $\mathbb{C}[[h]]$.

Let $t$ denote the formal series $e^{h/4}$ in $\mathbb{C}[[h]]$. We define the module of skein relations, $S(M)$, to be the the smallest subset of $\mathbb{C}\mathcal{L}_M[[h]]$ that is closed under addition, scalar multiplication and the $h$-adic topology, and which contains all expressions of the form

1.  $+\, t$  $+\, t^{-1}\, )($, and
2. $\bigcirc + t^2 + t^{-2}$.

As before, (1) and (2) indicate relations that hold among links which can be isotoped in $M$ so that they are identical except in the neighborhood shown. The Kauffman bracket skein module is the quotient

$$K(M) = \mathbb{C}\mathcal{L}_M[[h]]/S(M).$$

This process can be mimicked for any choice of basis (oriented links, links up to homotopy, etc.), any choice of scalars, any set of skein relations, and with or without requiring topological completion. The resulting quotient is generically called a skein module. For instance, an older version of the Kauffman bracket skein module is

$$K_A(M) = \mathbb{Z}[A^{\pm 1}]\mathcal{L}_M/S(M),$$

with $t = -A$ in the skein relations, and without topology. If $M = \mathbb{R}^3$ (or $B^3$ or $S^3$) the new version is just an outrageous way of expanding the Kauffman bracket into a power series.

**Theorem 2** (Kauffman–Przytycki–B–F–K)**.** *$K(\mathbb{R}^3) \cong \mathbb{C}[[h]]$ via $L \mapsto \langle L\rangle_{A=-t}$.*

On one level, $K(M)$ is a generalization of the Kauffman bracket polynomial. If $K(M)$ is topologically free (i.e. isomorphic to $V[[h]]$ for some vector space $V$) then the isomorphism gives a power series link invariant for each vector in a basis of $V$. The coefficients behave like finite type link invariants [15], generalizing a well known property of the Jones polynomial expanded as a power series [5]. In order to utilize the module in this fashion, one would like to know that it is free, and what the basis is; information that is decidedly difficult to come by.

The survey article [27] contains a nearly complete list of those manifolds for which the explicit computation has been done, the exception being [10]. These computations predate the topological version of the module, but whenever $K_A(M)$ is free, $K(M)$ is just its completion after substituting $A = -t$.

There is, however, a deeper understanding of skein modules, $K(M)$ in particular. Przytycki often refers to skein theory as "algebraic topology based on knots," alluding strongly to skein modules as a sort of non-commutative alternative to homology. This is also reflected in the principle that loops up to homotopy carry classical information, whereas knots up to isotopy carry quantum information. The notion that a skein module is a quantization or a deformation of some kind can be made very explicit for $M = F \times I$, $F$ being a compact oriented surface.

In this case, $K(F \times I)$ is an algebra. Multiplication of links in $\mathbb{C}\mathcal{L}_{F\times I}$ is by stacking one atop the other. It extends obviously to $\mathbb{C}\mathcal{L}_{F\times I}[[h]]$, and it is a simple

[4]Skein modules were originally less technical [36, 38, 46]. Power series first appeared in [47]. Topological considerations were first addressed in [15].

matter to check that $S(F\times I)$ is an ideal. Crossings form a barrier to commutativity in $\mathbb{C}\mathcal{L}_{F\times I}$, and, for most surfaces, the obstruction survives in the quotient.[5]

It is possible for non-homeomorphic surfaces $F_1$ and $F_2$ to have homeomorphic cylinders, $F_1 \times I$ and $F_2 \times I$. The homeomorphism does not preserve the algebra structure.[6] For this reason, it makes sense to compress the notation into $K(F) = \mathbb{C}\mathcal{L}_F[[h]]/S(F)$.

For an example of $K(F)$ as a "deformation", recall that the commutative algebra of polynomials in three variables (over your favorite scalars) is presented by

$$\langle x, y, z \mid xy - yx = 0, yz - zy = 0, zx - xz = 0\rangle.$$

**Theorem 3** (B–Przytycki). *If $F$ is a once punctured torus, then $K_A(F)$ is presented by*[7]

$$\begin{aligned}\langle x, y, z \mid Axy - A^{-1}yx &= (A^2 - A^{-2})z \\ Ayz - A^{-1}zy &= (A^2 - A^{-2})x \\ Azx - A^{-1}xz &= (A^2 - A^{-2})y\rangle.\end{aligned}$$

Theorem 3 can be thought of as a 1-parameter family of presentations,[8] which reduces to the commutative polynomials if $A = \pm 1$. Other examples can be found in [17].

Yet another way to see the module as a deformation is to let the parameter $h$ go to 0 (or $t$ to 1, or $A$ to $-1$). Formally, this is achieved by passing to the quotient $K_0(F) = K(F)/hK(F)$. In this case the skein relations would become

1.  +  + , and
2.  $+ 2$.

Taken together these allow crossings to be changed at will and make framing irrelevant. Hence, the "undeformed" module is a commutative algebra spanned by free homotopy classes of collections of loops.

This is all quite heuristic for we lack a precise definition of quantization or deformation. The next section will address this. However, the realization that a non-commutative algebra deforms some specialization of itself is not especially deep. Discovering a deformation of an apparently unrelated object is much more challenging.

## 3. Group Characters

Suppose that $G$ is a finitely presented group with generators $\{a_i\}_{i=1}^m$ and relations $\{r_j\}_{j=1}^n$. The space of $SL_2(\mathbb{C})$-representations of $G$ is a closed affine algebraic set.[9] You can view the representations as lying in $\prod_{i=1}^m SL_2(\mathbb{C}) \subset \mathbb{C}^{4m}$. Each of the relations $r_j$ induces four equations from the entries of $r_j(A_1, \ldots, A_m) = I$. The zero set of these polynomials restricted to the variety $\prod_{i=1}^m SL_2(\mathbb{C})$ is the representation space.

We might naively try to construct the coordinate ring of the representations as follows. Let $\mathcal{I}$ be the ideal generated by the equations $r_j(A_1, \ldots, A_m) = I$ in the

[5] The exceptions are planar surfaces with $\chi(F) \geq -1$.

[6] $F \times I$ in Theorem 3 is homeomorphic to  $\times I$, whose skein algebra is commutative.

[7] The variables $x$, $y$ and $z$ are a meridian, a longitude and a slope one curve on $F$.

[8] Barrett [4] showed that a spin structure on $M$ induces an isomorphism $K_A(M) \cong K_{-A}(M)$.

[9] Shafarevich [43] is a good reference for the algebraic geometry.

coordinate ring $C[\prod_{i=1}^m SL_2(\mathbb{C})]$. Let $R(G) = C[\prod_{i=1}^m SL_2(\mathbb{C})]/\mathcal{I}$. The problem is that $R(G)$ might have nilpotents (equivalently, $\sqrt{\mathcal{I}} \neq \mathcal{I}$). However, it was proved in [37] that $R(G)$ is an isomorphism invariant of $G$. There is an action of $SL_2(\mathbb{C})$ on $R(G)$ induced by conjugation in the factors of $\prod_{i=1}^m SL_2(\mathbb{C})$. The part of the ring fixed by this action is the affine $SL_2(\mathbb{C})$-characters of $G$, denoted $R(G)^{SL_2(\mathbb{C})}$. It, too, is an isomorphism invariant of $G$ [7]. For our purposes, it suffices to define the ring of $SL_2(\mathbb{C})$-characters of $G$ to be[10]

$$\Xi(G) = R(G)^{SL_2(\mathbb{C})}/\sqrt{0}.$$

If $X$ is a manifold, we will write $\Xi(X)$ rather than $\Xi(\pi_1(X))$.

The connection with 3-manifolds is quite simple (see [12, 13, 14, 39] for details). Suppose that $\rho : \pi_1(M) \to SL_2(\mathbb{C})$ is a representation and $\chi_\rho$ is its character. Let $K$ be a loop, thought of as a conjugacy class in $\pi_1(M)$. Since the trace of a matrix in $SL_2(\mathbb{C})$ is invariant under inversion and conjugation, it makes sense to speak of $\chi_\rho(K)$ regardless of the choice of a starting point or orientation. The loop $K$ determines an element of $R(\pi_1(M))^{SL_2(\mathbb{C})}$ by $K(\rho) = -\chi_\rho(K)$. The function extends to

$$\Phi : K_0(M) \to R(\pi_1(M))^{SL_2(\mathbb{C})}$$

by requiring it to be a map of algebras. It is well defined because the skein relations for $K_0(M)$ are sent to the fundamental $SL_2(\mathbb{C})$ trace identities:

1. $\mathrm{tr}(AB) + \mathrm{tr}(AB^{-1}) = \mathrm{tr}(A)\mathrm{tr}(B)$, and
2. $\mathrm{tr}(I) = 2$.

It is shown in [14] that the image of $\Phi$ is a particular presentation of the affine characters [23]. Sikora [44] has achieved this by directly identifying a version of $K_0(M)$ with $R(\pi_1(M))^{SL_2(\mathbb{C})}$ as defined by Brumfiel and Hilden. Przytycki and Sikora [39, 40] have computed $K_0(M)$ for a large number of manifolds, including $M = F \times I$, for which they can prove it has no nilpotents.[11] Summarizing, we have a good idea of what $K_0(M)$ is in general, and we know exactly what $K_0(F)$ is.

**Theorem 4** (B-Przytycki-Sikora). *$\Phi : K_0(F) \to \Xi(F)$ is an isomorphism.*

## 4. Poisson Quantization of Surface Group Characters

In the previous section we saw how a non-commutative algebra shrinks to a commutative specialization for some particular value of a deformation parameter. The formal definition of quantization reverses this process. Beginning with a commutative algebra, one introduces a parameter $h$, and a "direction" of deformation. The direction is a Poisson bracket.

To make this precise, a commutative algebra $A$ is called a Poisson algebra if it is equipped with a bilinear, antisymmetric map $\{\ ,\ \} : A \otimes A \to A$ which satisfies the Jacobi identity:

$$\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0,$$

[10] It is a deep result of Culler and Shalen [19] that the set of characters of $SL_2(\mathbb{C})$ representations of $G$ is a closed affine algebraic set. Its coordinate ring is $\Xi(G)$.

[11] They have also worked with various scalars. Certainly if the scalar ring has nilpotents then $K_0(M)$ does as well, and they have even located a nilpotent with scalar field $Z_2$. However, no nilpotents have ever been found with scalar ring $\mathbb{C}[[h]]$.

and is a derivation:

$$\{ab, c\} = a\{b, c\} + b\{a, c\},$$

for any $a, b, c \in A$.

A quantization of a complex Poisson algebra $A$ is a $\mathbb{C}[[h]]$-algebra, $A_h$, together with a $\mathbb{C}$-algebra isomorphism, $\Phi : A_h/hA_h \to A$, satisfying the following properties:

- as a $\mathbb{C}[[h]]$-module $A_h$ is topologically free (i.e. $A_h \cong V[[h]]$);
- if $a, b \in A$ and $a', b'$ are any elements of $A_h$ with $\Phi(a') = a$ and $\Phi(b') = b$, then

$$\Phi\left(\frac{a'b' - b'a'}{h}\right) = \{a, b\}.$$

Hoste and Przytycki [25] and Turaev [47] knew that certain skein modules gave Poisson quantizations of various algebras based on loops in a surface.[12] Since $K(F)$ is topologically free ([38], [15]), one can easily see it as a Poisson quantization of $K_0(F)$ with the obvious bracket:

$$\{a, b\} = \text{lead coefficient of } a'b' - b'a' \text{ in } K(F).$$

As noted in Section 2, however, understanding the Poisson algebra $K_0(F)$ as a formal quotient of $K(F)$ yields no new insight. This is where character theory reenters.

Since $\Xi(F)$ is the complexification of the $SU(2)$-characters of $\pi_1(F)$, it has a Poisson structure given by complexifying the standard one on $SU(2)$-characters [22, 6]. Recall (Theorem 4) that the algebra $\Xi(F)$ is generated by the functions corresponding to loops. The Poisson bracket is given by an intersection pairing on oriented loops, and extended to all of $\Xi(F)$. In [15] this is reformulated as a state sum using unoriented loops, proving

**Theorem 5** (B-F-K)**.** *$K(F)$ and the map $\Phi : K_0(F \times I) \to \Xi(F)$ form a Poisson quantization of the standard Poisson algebra $\Xi(F)$.*

## 5. Lattice Gauge Field Theory

In this section we restrict our attention to surfaces with nonempty boundary. Lattice gauge field theory gives an alternative quantization of $\Xi(F)$. To see this, we first sketch how an $SU(2)$ gauge theory on $F$ recovers the $SU(2)$-characters of $\pi_1(F)$. We then pass to a lattice model of the theory, in which a Lie group may be replaced with its universal enveloping algebra. Finally, the enveloping algebra may be deformed to a quantum group. Along the way we will complexify to return to the $SL_2(\mathbb{C})$ setting.

An $SU(2)$ gauge theory over $F$ consists of connections, gauge transformations (also called the gauge group) and gauge fields. These objects have technical definitions involving the geometry of an $SU(2)$-bundle over $F$, but for our purposes only a few consequences are relevant.

First of all, a connection determines a notion of parallel transport along a path, $\gamma$, which assigns to it an element $hol(\gamma)$ of $SU(2)$. This element is called the holonomy

[12] Their modules have a slightly different flavor than the one defined here, both because topology is not considered and because the scalars are not necessarily power series. Their definitions of Poisson quantization are analogously distinct. It is also interesting to note that their work predates the appearance of quantum groups in low-dimensional topology.

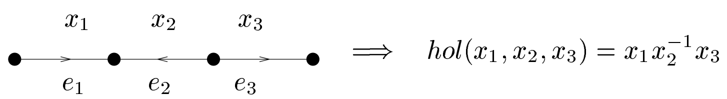



FIGURE 3. Example of holonomy in a lattice.

of the connection along $\gamma$. If you traverse the path in the opposite direction then the holonomy is the inverse. A connection is flat if holonomy only depends on the homotopy class of a path relative to its endpoints.

Second, the gauge group acts on connections. A gauge transformation can be thought of as an element of $SU(2)$ assigned to each point of $F$. Its effect on a connection is irrelevant; its effect on holonomy is $hol(\gamma) \mapsto g\ hol(\gamma)\ h^{-1}$, where $g$ and $h$ correspond to the beginning and end points of $\gamma$.

Finally, gauge fields are (real analytic) functions on connections. There is an adjoint action of the gauge group on gauge fields.[13] Invariant gauge fields are called observables.

Flat connections give rise to representations of $\pi_1(F)$ into $SL_2(\mathbb{C})$ via holonomy of loops. There are more flat connections than representations. However, two connections are gauge equivalent if and only if their holonomy representations are conjugate.

The observables, restricted to flat connections, are a space of (real analytic) functions on $SU(2)$-representations that are invariant under conjugation. The "polynomials" in this space—a dense set—are the $SU(2)$-characters of $F$.

Much of the technical detail glossed over in the last few paragraphs vanishes if we pass to a lattice model; a combinatorial setting in which geometry is disposed of and the behavior of holonomy is axiomatized. As a bonus, one need not base the theory on a compact Lie group. What follows works for any affine algebraic group, but we will stick to $SL_2(\mathbb{C})$ for continuity.

Suppose that $F$ is triangulated. The 1-skeleton of the triangulation of $F$ is a graph. Let $V$ denote the set of vertices and $E$ the set of edges, each with an orientation. The objects of a lattice gauge field theory over $F$ are:

1. the connections, $\mathbb{A} = \prod_{e \in E} SL_2(\mathbb{C})$,
2. the gauge group, $\mathcal{G} = \prod_{v \in V} SL_2(\mathbb{C})$, and
3. the gauge fields, $C[\mathbb{A}] = \bigotimes_{e \in E} C[SL_2(\mathbb{C})]$.

In the formula above $C[SL_2(\mathbb{C})]$ is the coordinate ring of $SL_2(\mathbb{C})$. One thinks of a connection as assigning an element of $SL_2(\mathbb{C})$ to each edge. A path is a string of edges. Holonomy of $(x_1, x_2, x_3)$ along the path $\{e_1, e_2, e_3\}$ is depicted in Figure 3. Note that holonomy is clearly inverted if the path is reversed.

One thinks of a gauge transformation as an element of $SL_2(\mathbb{C})$ at each vertex. The action of the gauge group on a connection is illustrated, for one vertex, in Figure 4. Note that the action is by $y^{-1}$ on the right if an edge points in and by $y$ on the left if it points out, a convention we adhere to through this and the next two sections.

[13] For a gauge transformation $g$, a gauge field $f$ and a connection $x$, $(g \bullet f)(x) = f(g \bullet x)$.

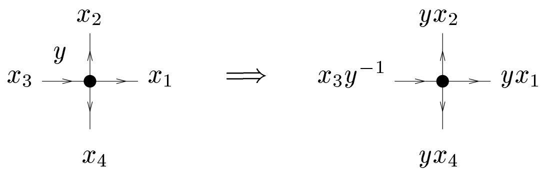



FIGURE 4. Gauge group action at a vertex.

The gauge fields can be evaluated on connections in the obvious way. By taking adjoints we get an action of the gauge group on the gauge fields. The fixed subring of this action is the ring of $SL_2(\mathbb{C})$-characters of the one skeleton. If $G$ is the fundamental group of the 1-skeleton this ring is isomorphic to $\Xi(G)$.

Flatness should amount to holonomy being independent of path, but in a lattice model we prefer the following equivalent definition. A connection is flat on a face of the triangulation if it is gauge equivalent to one which has 1 on each edge of the face. A flat connection is flat on each face. Invariant gauge fields evaluated on flat connections form a ring of observables which, regardless of the choice of triangulation, is isomorphic to $\Xi(F)$.[14]

This is an easily manipulated model of a gauge theory, but groups do not deform; algebras do. So, replace $SL_2(\mathbb{C})$ with the universal enveloping algebra $U(sl_2)$. This is a cocommutative Hopf algebra. The interested reader may find a full explanation in [1] for example, but we can get by with less. There is an involution $S : U(sl_2) \to U(sl_2)$ that corresponds to inversion in the group, a counit $\epsilon : U(sl_2) \to \mathbb{C}$, and a comultiplication $\Delta : U(sl_2) \to U(sl_2) \otimes U(sl_2)$. One may regard $\Delta^n$ as an operation that breaks an element of $U(sl_2)$ into states residing in $U(sl_2)^{\otimes(n+1)}$. The notation for this is due to Sweedler [45]. For example,

$$\Delta^3(y) = \sum_{(y)} y^{(1)} \otimes y^{(2)} \otimes y^{(3)} \otimes y^{(4)}.$$

Since $C[SL_2(\mathbb{C})]$ lies in the dual of $U(sl_2)$, we can almost repeat the entire process with

1. connections $\mathbb{A} = \bigotimes_{e \in E} U(sl_2)$,
2. gauge algebra $\mathcal{G} = \bigotimes_{v \in V} U(sl_2)$, and
3. gauge fields $C[\mathbb{A}] = \bigotimes_{e \in E} C[SL_2(\mathbb{C})]$.

The catch is the gauge action. In order to make sense of it, we need to assign an ordering to the edges at each vertex. This is done by marking the vertex with a cilium (see Figure 5) after which the orientation on $F$ gives a counter-clockwise ordering of the edges.

It is best to think of connections and gauge transformations as pure tensors, remembering always to extend linearly. We thus view a connection as an assignment of an element of $U(sl_2)$ to each edge of the triangulation. Holonomy is apparent; for the path in Figure 3 it would be $x_1 S(x_2) x_3$, where $S$ is the antipode of $U(sl_2)$.

[14] Technically, divide the gauge field algebra by the annihilator of all flat connections and then restrict to the gauge invariant part of the quotient.

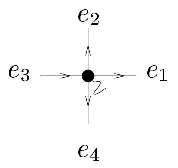



FIGURE 5. Ciliated vertex with edges ordered $e_1 < e_2 < e_3 < e_4$.

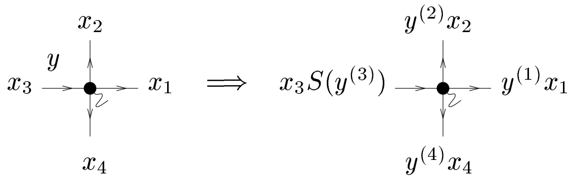



FIGURE 6. Gauge algebra action at a vertex.

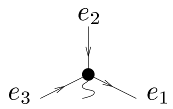



FIGURE 7. Trivalent ciliated vertex.

We continue to think of a gauge transformation as an element of $U(sl_2)$ at each vertex, with the action at a vertex illustrated in Figure 6.

A further problem with the action is that gauge "equivalence" is not an equivalence relation anymore, necessitating a technical modification of flatness which we will not address here. Also, the word "invariant" means $y \bullet x = \epsilon(y)x$. However, the passage to gauge fields on flat connections modulo the gauge algebra proceeds as before, giving exactly the same ring.

Finally we pass to $U_h(sl_2)$. This is a quasi-triangular ribbon Hopf algebra [31]. It is non-cocommutative in a fashion constrained by an element of $U_h(sl_2) \otimes U_h(sl_2)$ called the universal $R$-matrix. The antipode $S$ is no longer an involution; rather $S^2$ acts as conjugation by the so-called charmed element, $k$. The definition of flat connection is further altered, preserving independence of path but deforming the holonomy of a trivial loop to $k^{\pm 1}$. The dual of $U_h(sl_2)$ contains a deformation, ${}_qSL_2$, of $C[SL_2(\mathbb{C})]$. Thus one hopes to obtain a quantized ring of observables by replacing each object with its quantum analogue.

There is one small problem. The natural multiplication on $C[\mathbb{A}] = \otimes_{e \in E\,q}SL_2$ (i.e. the one dual to the natural comultiplication on $\mathbb{A} = \otimes_{e \in E} U_h(sl_2)$) is not gauge invariant. This is a major obstruction, and the solution is notable enough to occupy the next section. However, once it has been addressed, we will have

**Theorem 6.** *Quantum observables exist. They form a ring, $\Xi_h(F)$, independent of triangulation and ciliation, which quantizes $\Xi(F)$.*

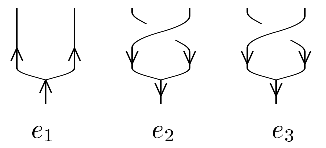



FIGURE 8. Coupons for each edge.

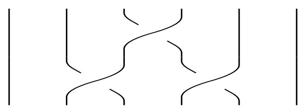


FIGURE 9. Six-braid encoding the permutation (1)(2453)(6).

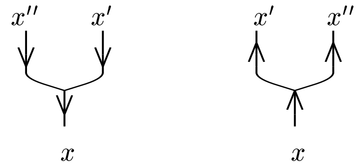



FIGURE 10. Comultiplication acting at a branch.

## 6. NABLA

The natural comultiplication on the coalgebra of quantum connections is a tensor power of $\Delta$ composed with a permutation. For instance, it would send

$$x_1 \otimes x_2 \mapsto x_1^{(1)} \otimes x_2^{(1)} \otimes x_1^{(2)} \otimes x_2^{(2)}.$$

Expanding on a theme of quantum topology, we denote this morphism by a tangle built from branches for each application of $\Delta$ and a braid corresponding to the permutation. We then obtain a quantized comultiplication

$$\nabla : \mathbb{A} \to \mathbb{A} \otimes \mathbb{A}$$

by allowing crossings to encode actions of the $R$-matrix.

There is a fundamental tangle associated to any vertex—the one in Figure 7, for example—whose construction proceeds in stages. First, assign a coupon to each edge as in Figure 8. There are two types of these, depending on whether the edge points in or out, and they must be ordered left to right matching the cilial order of the edges. Next, we construct a $2n$-braid ($n$ = valence of the vertex) by dragging odd numbered strands left and even numbered strands right.[15] Odds lie over evens. Our example is the the 6-braid in Figure 9. The fundamental tangle is formed by stacking the braid atop the coupons. Orientation of the coupons carries over to the strands of the braid.

Now imagine $x_1 \otimes x_2 \otimes x_3$ entering the tangle from the bottom and traveling upward. Each branch indicates comultiplication with the output ordered as in Figure 10. Note that we are suppressing the summation symbols.

[15]The inherent ambiguity evaporates when we construct the morphism because the $R$-matrix solves the Yang-Baxter equation [31]. It is an elegant feature of quantum topology that isotopies of tangles correspond to identities in a quantum group.

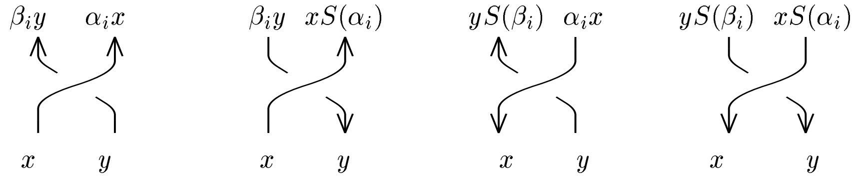



FIGURE 11. Actions of the $R$-matrix at crossings.

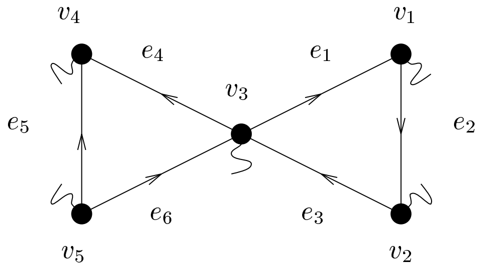



FIGURE 12. Oriented ciliated graph $\Gamma$.

Each crossing corresponds to an action of the $R$-matrix, which we write as $R = \sum_i \alpha_i \otimes \beta_i$. The four possibilities are shown in Figure 11, again suppressing summation. Note that, as usual, left and right multiplication correspond respectively to outward and inward pointing edges, and that right multiplication is preceded by an application of $S$.

Sweeping out, we get a morphism

$$\mathbb{F}_v : \bigotimes_{\text{edges at } v} U_h(sl_2) \to \left( \bigotimes_{\text{edges at } v} U_h(sl_2) \right)^{\otimes 2}.$$

In our example,

$$x_1 \otimes x_2 \otimes x_3 \mapsto$$
$$x_1' \otimes x_2' S(\beta_1) S(\beta_3) \otimes x_3' S(\beta_2) S(\beta_4) S(\beta_5) \otimes \alpha_5 \alpha_3 x_1'' \otimes x_2'' S(\alpha_2) S(\alpha_4) \otimes x_3'' S(\alpha_2).$$

Eight summations are suppressed, and the subscripts on $\alpha_j$ and $\beta_j$ are shorthand for summation over the $j$-th application of the $R$-matrix.

The morphism $\mathbb{F}_v$ is coassociative in the sense that $(Id \otimes \mathbb{F}_v) \circ \mathbb{F}_v = (\mathbb{F}_v \otimes Id) \circ \mathbb{F}_v$. Furthermore, its effect in a given factor of $(\bigotimes_{\text{edges at } v} U_h(sl_2))^{\otimes 2}$ is either entirely by right multiplication or entirely by left multiplication. This allows us to combine the effects of $\{\mathbb{F}_v \mid v \in V\}$ into a single morphism

$$\nabla : \mathbb{A} \to \mathbb{A} \otimes \mathbb{A}.$$

**Theorem 7.** $\nabla$ *is coassociative and gauge invariant.*

## 7. QUANTUM OBSERVABLES FOR $SL_2(\mathbb{C})$

At the end of Section 2 we saw how loops became functions generating $\Xi(F)$. In this section we will describe the quantum analogue of that fact. All definitions are given in terms of a running example, so assume that $\Gamma$ is the oriented, ciliated graph in Figure 12.

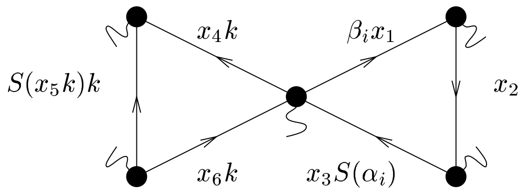



FIGURE 13. Action of the $q$-loop $l$ on $x$.

Following Section 6 we have

1. the connection coalgebra, $\mathbb{A} = U_h(sl_2)^{\otimes 6}$,
2. the gauge algebra, $\mathcal{G} = U_h(sl_2)^{\otimes 5}$, and
3. the algebra of gauge fields, $C[\mathbb{A}] = ({}_qSL_2)^{\otimes 6}$.

Bowing to technicalities, a loop in $\Gamma$ will be allowed to meet each edge at most once, and each vertex at most twice. In accordance with the theme of quantization by crossings, we say a $q$-loop is a loop with a choice of under- or over-crossing whenever it intersects itself transversely. We express this as a sequence of edges with $+$ and $-$ signs interspersed. For example,

$$l = \{e_1, e_2, e_3, +, e_4, e_5, e_6, -\}$$

is a $q$-loop with a single crossing. The $+$ and $-$ indicate that the arc along $e_3, e_4$ passes over the arc along $x_6, x_1$. The loop defines an element of $C[\mathbb{A}]$ via the following graphical recipe.

1. Choose a pure tensor $x = x_1 \otimes \cdots \otimes x_6 \in \mathbb{A}$. Draw a picture of $\Gamma$ with $x_i$ labeling each corresponding edge.
2. Apply $\epsilon$ to any edge not appearing in the loop. (No effect on this example.)
3. At the crossing, act by an $R$-matrix, either $\sum \alpha_i \otimes \beta_i$ or $\sum \beta_i \otimes S(\alpha_i)$. The action will take place on the first two edges in the ciliation. The $R$-matrix is chosen so that $\beta_i$ acts on the bottom strand ($\sum \alpha_i \otimes \beta_i$ in this example), and the action follows left/right rules as in Section 5. Write this on the appropriate edges, suppressing summations.
4. If an edge is oriented against the direction of the loop, multiply on the right by $k$, and then apply $S$.
5. Each time the loop passes through a vertex check to see if the incoming edge goes before the outgoing edge in the ciliation. If not, then right multiply by $k$ on the incoming edge. The picture should now look like Figure 13.
6. Multiply everything together as you traverse the loop. Take the image of this "quantum holonomy" in the fundamental representation (see [33] or [41]) of $U_h(sl_2)$. Finally, take the trace to get a scalar:
$$x \mapsto \sum_i \mathrm{tr}(\beta_i x_1 x_2 x_3 S(\alpha_i) x_4 k S(x_5 k) k x_6 k).$$
7. Extend linearly over all of $\mathbb{A}$ to obtain a function, $W_l$.

The function we have defined is usually called a Wilson loop in the literature. It should be clear that a $q$-loop is just a knot diagram with a base point and an orientation. Our goal is to assign a quantum observable to each equivalence class of link diagrams.

The first step is to note that the rules we gave for acting on edges prior to computing holonomy are local. One could just as well apply them to a link of loops, compute holonomy along each, and take the product of the resulting traces. Clearly the individual traces are independent of base points. Reversing orientation is less trivial, as it involves $S$ rather than inversion. But $\mathrm{tr}(S(x)) = \mathrm{tr}(x)$ in the fundamental representation, so orientations don't matter. Gauge invariance is easily checked at individual vertices. Finally, suppose that $\Gamma$ is the 1-skeleton of a triangulated surface. Let $l$ and $l'$ be equivalent link diagrams and $x$ a flat connection. Since flatness implies independence of path, $W_l = W_{l'}$.

At this point we see that a link $L$ determines an observable, $W_L$, provided one has a fine enough triangulation of $F$. Since $\Xi_h(F)$ is independent of triangulation, we can make the assignment

$$L \mapsto (-1)^{|L|} W_L,$$

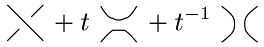
where $|L|$ is the number of components of $L$. Linearity and continuity extend it uniquely to a map

$$\Phi_h : \mathbb{C}\mathcal{L}_{F\times I}[[h]] \to \Xi_h(F).$$

This is the quantum analogue of $\Phi$ from Section 3, taking loops to the character ring. That map carried skein relations in $K_0$ to the fundamental $SL_2(\mathbb{C})$ trace identity and to $\mathrm{tr}(I) = 2$. The corresponding quantized identities in $U_h(sl_2)$ are

$$t\ \mathrm{tr}(ZW) + t^{-1}\ \mathrm{tr}(S(Z)W) = \sum_i \mathrm{tr}(\alpha_i Z)\mathrm{tr}(\beta_i W), \quad \text{and}$$
$$\mathrm{tr}(k^{\pm 1}) = t^2 + t^{-2}.$$

Flatness implies that the skein relation $\bigcirc + (t^2 + t^{-2})$ maps to $\mathrm{tr}(k) = t^2 + t^{-2}$, However,



is more complicated because it has less symmetry than the quantum trace identity. In some cases the skein relation maps directly to the identity given above, while others require more manipulation.

**Theorem 8.** $\Phi_h(S(F)) = 0$. *Furthermore, the quotient map*

$$\widetilde{\Phi}_h : K(F) \to \Xi_h(F)$$

*is an isomorphism.*

## 8. The Future

The results described in this survey place skein theory at the confluence of ideas from topology, representation theory, non-commutative algebra and mathematical physics. Standard techniques from skein theory [27, 28] extend our lattice construction of $K(F)$ to a description of the Kauffman bracket skein module of an arbitrary compact 3-manifold. Consequently, $K(M)$ has an intensive definition in terms of links and skein relations, and an extensive definition in terms of quantized invariant theory. This nexus suggests some avenues for further research.

It is proved in [15] that the affine $SL_2(\mathbb{C})$-characters induce topological generators of $K(M)$. In particular, the Kauffman bracket skein module of a small 3-manifold (i.e. containing no incompressible surface) is finitely generated, and thus can be used as a classification tool.

If $K(M)$ is topologically free then there is a pairing between it and the set of equivalence classes of $SL_2(\mathbb{C})$-representations of $\pi_1(M)$. For nilpotent free $K_0(M)$

this is a duality pairing. In the case of $F \times I$ the pairing has an especially easy form because the basis is a canonical set of links [15]. The Yang-Mills measure on the algebra of observables can be computed along the same lines [8]. This holds out the promise of producing integral formulas for the Witten-Reshetikhin-Turaev invariants of a 3-manifold that will admit to asymptotic analysis.

The focus in this paper has been $SL_2(\mathbb{C})$, but the lattice gauge field theory works for any algebraic group [16]. There should be skein modules corresponding to the other groups, just as $K(M)$ corresponds to $SL_2(\mathbb{C})$. We will need two kinds of skein relations: fundamental relations in the Hecke algebra associated to the group, and the quantized Cayley-Hamilton identity. There has been some study of these ideas due to Kuperberg [34] and Anderson, Mattes and Reshetikhin [3].

In another direction, it should be possible to commence the study of the syzygies of skein modules. A syzygy is a relationship between relationships. For instance, we can define a homology theory for the Kauffman bracket skein module. The 0-chains are spanned by all links; the 1-chains by all "Kauffman bracket skein triples"; the 2-chains by all "triples of triples", etc. The 0-th homology of this complex is $K(M)$, and we have examples to show that the theory is not always trivial. Notice that the $n$-th homology is measuring relations among relations.

The opacity of the structure of $K(M)$ poses many questions. Is it possible for $K(M)$ of a compact manifold to have torsion and still be topologically finitely generated? What is the relationship between $K_A(M)$ and $K(M)$? Przytycki has an example of a non-compact manifold where $K_A(M)$ is infinitely generated yet $K(M)$ is trivial. There is a grading of $K(M)$ by cables. The top term in the grading is everything; after that you take the span of all 2-fold cables, then 3-fold cables, etc. How is torsion in $K(M)$ reflected in this grading? Are there nilpotents in any $K_0(M)$, and if so, how do they affect the geometry of the representation space? Finally, in the interest of computability, what is a relative skein module and is there a gluing theorem?

The Kauffman bracket skein module is organic to many fields. We hope it, and other skein modules, will act as catalysts for the synergistic mixing of ideas from these fields.

## References


[1] E. Abe, *Hopf Algebras*, Cambridge University Press, New York (1977).

[2] A. Y. Alekseev, H. Grosse, V. Schomerus, *Combinatorial quantization of the Hamiltonian Chern-Simons theory I-II*, hep-th/9403066 and hep-th/9408097.

[3] J. E. Anderson, J. Mattes and N. Yu. Reshetikhin, *The Poisson structure on the moduli space of flat connections and chord diagrams*, preprint.

[4] J. W. Barrett, *Skein spaces and spin structures,* preprint.

[5] J. Birman and X. S. Lin, *Knot polynomials and Vassiliev's invariants*, Invent. Math. **111** (1993) 225-270.

[6] I. Biswas and K. Guruprasad, *Principle bundles on open surfaces and invariant functions on Lie groups*, Int. J. Math. **4** (1993) 535–544.

[7] G. Brumfiel and H. M. Hilden, *$SL(2)$ representations of finitely presented groups*, Contemporary Mathematics **187** (1995).

[8] E. Buffenoir, *Chern-Simons theory on a lattice and a new description of 3-manifold invariants*, q-alg/9509020.

[9] E. Buffenoir and Ph. Roche, *Two dimensional lattice gauge theory based on a quantum group*, Comm. Math. Phys. **170** (1995).

[10] D. Bullock, *The $(2,\infty)$-skein module of the complement of a $(2,2p+1)$ torus knot*, J. Knot Theory Ramifications **4** no. 4 (1995) 619–632.

[11] D. Bullock, *A finite set of generators for the Kauffman bracket skein algebra,* preprint.

[12] D. Bullock, *Estimating a skein module with $SL_2(\mathbb{C})$ characters,* Proc. Amer. Math. Soc., to appear.

[13] D. Bullock, *Estimating the states of the Kauffman bracket skein module*, Proc. Banach Center Mini-Semseter on Knot Theory, to appear.

[14] D. Bullock, *Rings of $SL_2(\mathbb{C})$-characters and the Kauffman bracket skein module*, q-alg/9604014.

[15] D. Bullock, C. Frohman and J. Kania-Bartoszyńska, *Understanding the Kauffman bracket skein module*, q-alg/9604013.

[16] D. Bullock, C. Frohman and J. Kania-Bartoszyńska, in preparation.

[17] D. Bullock and J. H. Przytycki, *Kauffman bracket skein module quantization of symmetric algebra and $so(3)$*, in preparation.

[18] J. H. Conway, *An enumeration of knots and links, and some of their algebraic properties*, Computational Problems in Abstract Algebra, Pergamon Press, New York (1970) 329-358.

[19] M. Culler and P. Shalen, *Varieties of group representations and splittings of 3-manifolds*, Ann. Math. **117** (1983) 109–146.

[20] V. G. Drinfeld, *Quantum groups*, Proc. Int. Congress Math., Berkeley 1986, American Mathematical Society (1987) 798–820.

[21] V. V. Fock and A. A. Rosly, *Flat connections and polyubles*, Teoreticheskaya i Matematicheskaya Fizika **95** no. 2, 228-238.

[22] W. Goldman, *Invariant functions on Lie groups and Hamiltonian flows of surface group representations*, Invent. Math. **85** (1986) 263–302.

[23] F. González-Acuña and J. M. Montesinos, *On the character variety of group representations in $SL(2,\mathbb{C})$ and $PSL(2,\mathbb{C})$*, Math. Z. **214** (1993) 627–652.

[24] J. Hoste and J. H. Przytycki, *An invariant of dichromatic links*, Proc. Amer. Math. Soc. **105** no. 4 (1989) 1003–1007.

[25] J. Hoste and J. H. Przytycki, *Homotopy skein modules of orientable 3-manifolds*, Math. Proc. Camb. Phil. Soc. **108** (1990) 475-488.

[26] J. Hoste and J. Przytycki, *A survey of skein modules of 3-manifolds*, Knots 90, de Gruyter (1992) 363–379.

[27] J. Hoste and J. H. Przytycki, *The $(2,\infty)$-skein module of lens spaces; a generalization of the Jones polynomial*, J. Knot Theory Ramifications **2** no. 3 (1993) 321–333.

[28] J. Hoste and J. H. Przytycki, *The Kauffman bracket skein module of $S^1 \times S^2$*, Math Z. **220** (1995) 65–73.

[29] V. F. R. Jones, *A polynomial invariant for knots via von Neumann algebras*, Bull. Amer. Math. Soc. **12** (1985) 103–111.

[30] V. F. R. Jones, *Hecke algebra representations of braid groups and link polynomials*, Ann. Math. **126** (1987) 335–388.

[31] C. Kassel, *Quantum Goups*, Springer-Verlag (1995).

[32] L. Kauffman, *State models and the Jones polynomial*, Topology **26** no. 3 (1987) 395–401.

[33] R. Kirby and P. Melvin, *The 3-manifold invariants of Witten and Reshetikhin-Turaev for $sl(2,\mathbf{C})$*, Invent. Math. **105** (1991) 473–545.

[34] G. Kuperberg, *Spiders for rank 2 Lie algebras*, preprint.

[35] W. B. R. Lickorish, *Polynomials for links,* Bull. London Math. Soc. **20** (1988) 558–588.

[36] W. B. R. Lickorish, *Three-manifolds and the Temperly-Lieb algebra,* Math. Ann. **290** (1991) 657–670.

[37] A. Lubotzky and A. Magid, *Varieties of representations of fintely presented groups*, Mem. Amer. Math. Soc. **336** (1985).

[38] J. H. Przytycki, *Skein modules of 3-manifolds*, Bull. Pol. Acad. Sci. **39(1-2)** (1991) 91–100.

[39] J. H. Przytycki and A. Sikora, *Skein algebra of a group*, Proc. Banach Center Mini-Semseter on Knot Theory, to appear.

[40] J. H. Przytycki and A. Sikora, personal communication.

[41] N. Yu. Reshetikhin and V. G. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Comm. Math. Phys. **127** (1990) 1–26.

[42] N. Yu. Reshetikhin and V. G. Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, Invent. Math. **103** (1991) 547–597.

[43] I. R. Shafarevich, *Basic Algebraic Geometry*, Springer Verlag, New York (1977).

[44] A. Sikora, personal communication.

[45] M. Sweedler, *Hopf Algebras*, W.A. Benjamin, New York (1969).

[46] V. G. Turaev, *The Conway and Kauffman modules of the solid torus,* Zap. Nauchn. Sem. LOMI; English trans. in J. Soviet Math. **167** (1988) 79-89.
[47] V. G. Turaev, *Algebras of loops on surfaces, algebras of knots, and quantization*, Adv. Ser. in Math. Physics **9** (1989), ed. C. N. Yang, M. L. Ge, 59–95.
[48] E. W. Witten, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. **121** ( 1989) 351–399.

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