A QUANTUM OBSTRUCTION TO EMBEDDING

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Abstract. We use topological quantum field theory to derive an invariant of a three-manifold with boundary. We then show how to use the structure of this invariant as an obstruction to embedding one three-manifold into another.

1. Introduction

In the mid 1980's the Jones polynomial of a link was introduced [5], [6]. It was defined as the normalized trace of an element of a braid group, corresponding to the link, in a certain representation. Although a topological invariant, the extrinsic nature of its computation made the relationship between topological configurations lying in the complement of the knot and the value of the Jones polynomial obscure. In order to elucidate this connection, Witten [21] introduced Topological Quantum Field Theory. TQFT is a cut and paste technique which allows for localized computation of the Jones polynomial. Following his discovery various approaches to topological quantum field theory were introduced ([1], [3], [4], [8], [13], [16], [19], [20] and others). In the work of Witten, it is obvious that the vector spaces of topological quantum field theory are quantizations of the space of connections on the manifold. Initial considerations of TQFT focused on the formal rules of combination of vector spaces and vectors. As more intrinsic developments of TQFT appeared there was a shift towards the use of the Kauffman bracket skein module. The Kauffman bracket skein module of a 3-manifold can be thought of as a quantization of the $SL_2(\mathbb{C})$–characters of the manifold [2]. This approach to TQFT recovers some of the initial feel of Witten's work.

Although the study of 3-manifold invariants is substantial in itself, few applications to classical 3-manifold topology have been found. In

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this paper, we use quantum invariants to develop obstructions to embedding one 3-manifold in another. In order to extract quantum invariants from a 3-manifold a good deal of extra information about its boundary is needed. The big challenge is to sort out what is information about the manifold and what is just data about the parameterization chosen for the boundary. A very similar problem arises in differential geometry. The Riemannian curvature tensor is a classifying invariant of the metric structure of the manifold, but it comes in the form of a tensor that depends intimately on the choice of local coordinates. To sidestep this, invariant functions of the coefficients of the tensor are used in order to obtain numbers that are independent of the choice of local coordinates. Similarly, we take the quantum invariant of a manifold and perform a construction to produce an invariant that is independent of the extra information about how the boundary is parametrized.

In the next section needed definitions are recalled and the TQFT is constructed. The path we take is the elegant approach of Roberts [16], [18]. This is a compact and eminently computable approach to topological quantum field theory. The vector spaces are built out of framed links in the manifold, so that they have a topological feel much like singular homology and fundamental group.

In the following section we prove a theorem that gives an obstruction to embedding one 3-manifold in another, and illustrate it with examples and with an application of quantum obstructions. If a 3-manifold contains a punctured lens space then it cannot be embedded in the 3-sphere. This observation lies at the heart of the technique of Scharlemann cycles - a technique that has been extremely fruitful in resolving questions about Dehn surgery on knots. Our Corollary 1 generalizes this criterion and opens the door to the use of quantum invariants in similar combinatorial arguments. We also give examples which show that the obstruction is not just homological.

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2. Projective TQFT

There is a construction of topological quantum field theory based on the Witten-Reshetikhin-Turaev invariants due to Blanchet, Habegger, Masbaum and Vogel [1]. The vector space associated to a surface is a quotient of the vector space with basis given by all connected oriented 3-manifolds having that surface as boundary. There is a pairing between the vector space of a surface and the vector space associated to the same surface with the opposite orientation. Specifically, if $\partial M = -\partial N$

then $< M, N >= Z_r(M \cup N)$ where Z_r is the Witten-Reshetikhin-Turaev invariant of level r [14], normalized so that the invariant of the 3-sphere is 1. To obtain the vector space of a surface take the quotient of the vector space above by the radical of this form. The invariant of a connected 3-manifold with boundary is just its image in the quotient.

We take a different approach here, with equivalent results, that makes fundamental computations more direct. The full apparatus of topological quantum field theory is not needed, as the invariant we derive is independent of framing. Hence, we set up a projective topological quantum field theory. The approach we take is based on Roberts' work (chapter 6 of [16], and [18]). The construction uses the Kauffman bracket skein module at a root of unity, modulo fusion. The number theoretical properties follow from the computation leading to the main theorem of Masbaum and Roberts [12] which says that the Reshetikhin-Turaev invariants of closed 3-manifolds are algebraic integers.

2.1. **The Kauffman Bracket Skein Module.** Let r be an odd prime. Let u be a primitive 8rth root of unity and $A = u^2$. We will work over $\mathbb{Q}[u]$, the cyclotomic numbers corresponding to u. The quantized integer $[n]$ denotes

$$\frac{A^{2n} - A^{-2n}}{A^2 - A^{-2}},$$

which is a unit in the ring $\mathbb{Z}[u]$ for $n < r$. Let X be the positive square root of $\sum_{c=0}^{r-2}[c+1]^2$. It can be shown that $X \in \mathbb{Z}[u]$. Explicitly, if $A = \exp\left(\frac{a\pi i}{2r}\right)$ with $(a, r) = 1$, then

$$X = \sqrt{\frac{r}{2}} \frac{1}{2 \sin \frac{a\pi}{r}}.$$

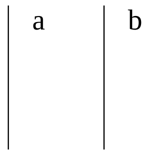
Let M be an orientable 3-manifold. A framed link in M is an embedding of a disjoint union of annuli into M. In our diagrams we draw only the core of an annulus lying parallel to the plane of the paper, i.e. with blackboard framing.

Two framed links in M are equivalent if there is an isotopy of M taking one to the other. Let $\mathcal{L}$ denote the set of equivalence classes of framed links in M, including the empty link. In the vector space $\mathbb{Q}[u]\mathcal{L}$, with basis $\mathcal{L}$, define $S(M)$ to be the smallest subspace of $\mathbb{Q}[u]\mathcal{L}$ containing all expressions of the form $\;- A\;- A^{-1}\;$ and $\bigcirc + A^2 + A^{-2}$, where the framed links in each expression are identical outside the balls pictured in the diagrams. The Kauffman bracket skein module $K_r(M)$ is defined to be the quotient

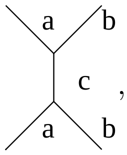
$$\mathbb{Q}[u]\mathcal{L}/S(M).$$

We assume that the reader is familiar with the Jones-Wenzl idempotents and the Kauffman triads as described on pages 136-138 of [9]. We refer the reader to pages 150-152 of [9] for a computation of the number $\theta(a,b,c)$ that is defined whenever (a,b,c) is an admissible triple. Notice that $\theta(a,b,c)$ is a unit in $\mathbb{Z}[u]$. By labeling an edge of a trivalent graph with the letter m we mean that it carries the m-th Jones-Wenzl idempotent.

Fusion (see [9], page 156) is the relation:



$$= \sum_c (-1)^c \frac{[c+1]}{\theta(a,b,c)}$$



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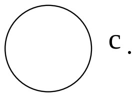
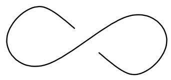
where the sum is over all c so that the triads are admissible. The result of enforcing fusion in the Kauffman bracket skein module $K_r(M)$ is denoted $K_{r,f}(M)$, and will be referred to as the reduced Kauffman bracket skein module of level r. It can be shown (see [18]) that this is equivalent to setting any skein that includes the $r-1$–st Jones-Wenzl idempotent equal to zero. The element Ω_r is the skein in the solid torus,

$$\frac{1}{X}\sum_{c=0}^{r-2}(-1)^c[c+1]$$

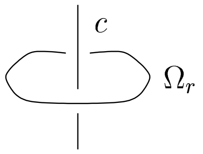


When one plugs Ω_r into a component of a framed link in M, then in $K_{r,f}(M)$ one can alter the framed link by handle slides along that component and not change the element of $K_{r,f}(M)$. The result of plugging in copies of Ω_r into each of the components of the link L is denoted by $\Omega_r(L)$.

Let κ be the Kauffman bracket of the skein in the three sphere obtained by plugging Ω_r into the framed knot shown below.



Notice that the Kauffman bracket of the result of plugging Ω_r into a trivial framed knot in a ball is X. Frequently in this paper we will use the Turaev-Wenzl identity (see [9]), which implies the following. Suppose a diagram contains the figure below, where Ω_r is plugged into the horizontal circle, and the vertical arc carries the cth Jones-Wenzl idempotent.



If $c \neq 0$, the diagram represents zero. If $c = 0$, then we are able to resolve the horizontal component, which removes it from the diagram and multiplies the result by X.

2.2. **preTQFT.** We review a result of Roberts [18] that is used in the construction of the TQFT. Suppose that M and M' are compact oriented 3-manifolds and $f : \partial M \to \partial M'$ is an orientation preserving homeomorphism. We say that framed links $L \subset M$ and $L' \subset M'$ are adapted to f if

1. The result of surgery along L is M', the result of surgery along L' is M.
2. There are regular neighborhoods $N(L)$ and $N(L')$, and an extension of f,

$$f : M - N(L) \to M' - N(L')$$

that is a homeomorphism taking the meridian of L to the framing of L', and the framing of L to the meridian of L'.

Given a pair of links adapted to f and the extension, we can define a map from $K_{r,f}(M)$ to $K_{r,f}(M')$, which we denote

$$K_r(f) : K_{r,f}(M) \to K_{r,f}(M').$$

Given a framed link L_1 in M push it off of L, then map it forward to M' using the extension of f to get $f(L_1)$. The skein in M' induced by the union of $f(L_1)$ and $\Omega_r(L')$ is $K_r(f)(L_1)$. Any two links adapted to f differ by the Kirby-Roberts moves for surgery on a manifold with boundary [15]. From this we see that the map $K_r(f)$ is well defined up to a power of κ.

Theorem 1 ([18])**.** *The map $K_r(f) : K_{r,f}(M) \to K_{r,f}(M')$ is an isomorphism.* □

Let $M\#N$ denote the connected sum of M and N. The Kauffman bracket skein modules of $M\#N$ and of the disjoint union of M and N are not isomorphic. However, there is a natural isomorphism between $K_{r,f}(M\#N)$ and $K_{r,f}(M \sqcup N)$. Let M' and N' be the result of removing a ball from each of M and N. There is a map $\iota : M' \sqcup N' \to M\#N$ coming from inclusion. As the Kauffman bracket skein module of the result of removing a ball from a manifold is isomorphic to the Kauffman bracket skein module of the manifold, this induces,

$$\hat{\imath} : K_{r,f}(M \sqcup N) \to K_{r,f}(M\#N).$$

Proposition 1. *The map $\hat{\imath}$ is an isomorphism.*

Proof. The fact that $\hat{\imath}$ is onto follows immediately from Lemma 1. Since the two vector spaces are finite dimensional the injectivity is implied by the fact that the dimensions are the same. The latter is a consequence of Theorem 1 and the easy computation in the case where M and N are connect sums of handlebodies.

Lemma 1. *If a 3-manifold M contains a family of embedded spheres then any skein in $K_{r,f}(M)$ can be represented by a linear combination of links that miss these spheres.*

Let S^2 be a sphere embedded in M and suppose that T is a trivalent graph which intersects S^2 transversely in one point on an edge colored with the n–th Jones–Wenzl idempotent. One can isotope T so it is the same away from the sphere and has two added positive twists close to the sphere. This implies that $< T >= A^{2n^2+4n} < T >$. If r is odd and $n \leq r-2$ then $A \neq 1$, thus $< L >= 0$.

Take a representative of any skein in $K_{r,f}(M)$. One can assume that it is a linear combination of trivalent graphs admissibly colored with Jones-Wenzl idempotents [10]. Use fusion repeatedly to make sure that the graphs do not have more than one edge intersecting each of the spheres embedded in M. Use the light bulb trick as above to see that the label on that edge has to be zero. □

Hence, we can be fluid about connected sums and disjoint unions so long as we work in the reduced skein module.

2.3. **TQFT.** Recall that a *projective Topological Quantum Field Theory* consists of a functor and a partition function, defined for each level r. The functor V_r maps the category of *marked* surfaces and homeomorphisms to the category of vector spaces over $\mathbb{Q}[u]$, and linear maps defined up to scalar multiplication by a power of κ. The partition function Z_r assigns to every *marked* 3-manifold M a vector $Z_r(M)$ in the vector space corresponding to its boundary, well defined up to scalar multiplication by a power of κ. These have to satisfy the following axioms.

1. **Dimension**: The empty surface is mapped to the vector space $\mathbb{Q}[u]$.

$$\emptyset \mapsto \mathbb{Q}[u]$$

2. **Disjoint union**: Disjoint union of surfaces is mapped to the tensor product of vector spaces.
$$F_1 \sqcup F_2 \mapsto V_r(F_1) \otimes V_r(F_2)$$
3. **Duality**: Oppositely oriented marked surfaces are mapped to naturally dual vector spaces.
$$V_r(-F) = V_r(F)^*$$
This duality determines a pairing:
$$\langle\ ,\ \rangle : V_r(F) \otimes V_r(-F) \to \mathbb{Q}[u]$$
4. **Gluing**: Gluing two 3-manifolds along surfaces in their boundaries corresponds to contracting vectors.

   Let M_1 and M_2 be marked 3-manifolds with $\partial M_1 = F \sqcup F_1$ and $\partial M_2 = -F \sqcup F_2$ where the markings agree on F. Then $Z_r(M_1) \in V_r(F) \otimes V_r(F_1)$ can be written as $\sum v \otimes v_1$, and $Z_r(M_2) \in V_r(F)^* \otimes V_r(F_2)$ can be written as $\sum \phi \otimes v_2$. If $M = M_1 \cup_F M_2$ then
$$Z_r(M) = \sum \phi(v) v_1 \otimes v_2.$$
5. **Mapping cylinder**: The value of the partition function on the mapping cylinder of a homeomorphism corresponds to a linear map given by that homeomorphism.

   Let $f : F_1 \to F_2$ and let $M = F_1 \times I \cup_f F_2$, where $(x, 1) \equiv f(x)$, so that $\partial M = -F_1 \sqcup F_2$. Then
$$Z_r(M) = V_r(f) \in V_r(F_1)^* \otimes V_r(F_2)$$

We will now construct a projective TQFT. Fix a level r.

Vector Spaces. Let F be a closed oriented surface. A *marking* of F is a choice of a handlebody H with $\partial H = F$, up to homeomorphisms that are the identity on F. We assign to the marked surface (F, H) the vector space $V_r(F, H) = K_{r,f}(H)$.

Linear Maps. Let $f : F \to F'$ be an orientation preserving homeomorphism of the marked surfaces (F, H) and (F', H'). Given links $L \subset H$ and $L' \subset H'$ adapted to f, we let $V_r(f) = K_r(f)$. Our morphism is defined up to multiplication by a power of κ.

Suppose now that a surface is not connected. In this case, a marking for the surface is a choice of a handlebody H_i for each component. The vector space associated with the surface is $\otimes_i K_{r,f}(H_i)$. We can realize this tensor product by taking any combination of connected sums and disjoint unions of the H_i, since $K_{r,f}(\#_i H_i)$ is canonically isomorphic to $\otimes_i K_{r,f}(H_i)$.

Duality. Let (F, H) be a marked surface. The marked surface with the orientations of F and H reversed is denoted by $-(F, H)$. We can form the double $H \cup_F -H$, which is homeomorphic to $\#_g S^1 \times S^2$, where g is the genus of F. Skeins $x \in K_{r,f}(H)$ and $y \in K_{r,f}(-H)$ determine a skein in $K_{r,f}(\#_g S^1 \times S^2)$ obtained by taking their disjoint union. In order to pair x and y, choose a family of spheres that cut $\#_g S^1 \times S^2$ down to punctured balls, and represent the union of x and y by a linear combination of links missing the spheres. Since the corresponding links live in punctured balls we can take their Kauffman bracket to get a complex number times the empty link, finally multiply the coefficient by X^g. The final multiplication makes the answer coincide with a number computed in the 3-sphere. To be more precise, choose an isomorphism between $K_{r,f}(\#_g S^1 \times S^2)$ and $K_{r,f}(S^3)$ coming from a surgery diagram, and the answer will differ from the one we have given by multiplication by a power of κ. We have defined a pairing,

$$< \, , \, >: V_r(F, H) \otimes V_r(-F, -H) \to \mathbb{Q}[u].$$

This is a duality pairing. Since $K_{r,f}(H) = K_{r,f}(-H)$, we can see the pairing as defined on $V_r(F, H) \otimes V_r(F, H)$.

Choose a trivalent spine for the handlebody H. A coloring a of the spine is a choice of an integer between 0 and $r-2$ for each edge. When the numbers at each vertex are admissible, we can form a skein by running that number of arcs between the vertices and putting the appropriate Kauffman triad at each vertex. We denote that skein by x_a. The skeins x_a where the coloring gives an admissible triad at each vertex form a basis for $K_{r,f}(H)$ [10]. This can easily be verified by computing $< x_a, x_b >$. If $a \neq b$ the answer is 0, and $< x_a, x_a >= X^g u(a)$, where $u(a)$ is a unit in $\mathbb{Z}[u]$. The dual basis x^a are the skeins with $< x_b, x^a >= \delta_a^b$ where δ_a^b is Kronecker's delta. It is easy to see that $x^a = \frac{1}{u(a)X^g} x_a$.

Suppose now that $C = \# H_i$. There is a standard embedding of C in $\#_g S^1 \times S^2$, where g is the sum of the genera of the components of ∂C. In the case that C is a handlebody, the double of C is homeomorphic to $\#_g S^1 \times S^2$, and this is the standard embedding of C. If C is a connected sum of handlebodies, take the connected sum of the doubles of each of the handlebodies. Choose the balls along which the sum is taken to lie inside the handlebodies H_i. It is clear that this embedding is unique up to homeomorphism and the complement of C is a disjoint union of handlebodies. Choose a trivalent spine for each H_i. A basis for $K_{r,f}(C)$ is given by all admissible colorings of the disjoint union of the spines, which we denote by $\otimes x_{a_i}$ or x_a, where a_i is the restriction of the coloring a to the ith spine.

We can see our pairing quite nicely in the three-sphere. To this end we say that C is embedded in S^3 in the standard way if its complement is a disjoint union of handlebodies, so that each handlebody is unknotted and there is a family of disjoint balls in S^3, each containing exactly one of the complementary handlebodies. Such an embedding is unique up to topological equivalence. To get from a standard embedding of C in S^3 to a standard embedding in $\#_g S^1 \times S^2$ we can do surgery on a link that lies in the complementary handlebodies. Specifically, choose an unknotted circle with zero framing, linking each 1-handle of C lying in the complementary handlebodies so that the aggregate of such circles forms the unlink which we denote U. These are the same as the circles used in the constructions of [12], [16]. Surgery along this link yields $\#_g S^1 \times S^2$ with C embedded in a standard way.

In the three-sphere picture we can see the skeins $\otimes x_{a_i}$ lying in C. To see a skein on the right hand side of the pairing, push it through the boundary of C and take its disjoint union with $\Omega_r(U)$. Notice that this is just the map of skein modules associated to the identity on the boundary of the complementary handlebodies in $\#_g S^1 \times S^2$ to the boundary of the complementary handlebodies in S^3.

To pair $\otimes x_{a_i}$ with $\otimes x_{b_i}$, take the Kauffman bracket of the union $x_a \cup x_b \cup \Omega_r(U)$. The result of the pairing is $\delta_a^b X^g u(a)$, where $u(a)$ is a unit in $\mathbb{Z}[u]$ depending on the coloring a.

2.4. **Partition Function.** Let M be a compact, connected, oriented three-manifold. A marking for M is a choice of handlebodies, one for each connected component of ∂M. We denote the connected sum of these handlebodies by C. If ∂M is empty then the marking is S^3. Since there is only one choice we suppress it from the notation. The *preinvariant* of (M, C) is the pair consisting of $K_{r,f}(M)$ and the empty skein. Let $L \subset M$ and $L' \subset C$ be framed links adapted to the identity map on ∂M. Let $K_r(Id) : K_{r,f}(M) \to K_{r,f}(C)$ be the associated isomorphism. We define

$$Z_r(M, C) = K_r(Id)(\emptyset).$$

Once again, from the Kirby calculus for manifolds with boundary, this is well defined up to multiplication by a power of κ. Notice that the orientation on ∂M is always the orientation inherited from M. Using the main result from [12] we see that if M is any manifold then $Z_r(M, C)$ can be written as

$$\frac{1}{X^g} \sum_a I_a \otimes x_{a_i},$$

where the I_a are algebraic integers. The dual basis allows us to rewrite this answer without any fractions:

$$Z_r(M,C) = \sum_a J_a \otimes x^{a_i},$$

where the J_a are just the I_a times a unit depending on a.

If $\partial M = \emptyset$ then $C = S^3$ and the basis consists of the empty skein. In this case, by Masbaum-Roberts theorem [12], $Z_r(M) \in \mathbb{Z}[u]$ and, up to multiplication by a power of κ, it is equal to the Witten-Reshetikhin-Turaev invariant $\tau_r(M)$ [7], [14].

2.5. **Gluing.** Suppose that (M_1, C_1) and (M_2, C_2) are marked 3-manifolds, and that there is a surface F with $F \subset \partial M_1$ and $-F \subset \partial M_2$. We will assume that the genus of F is g. Suppose further that the handlebodies in C_1 and C_2 corresponding to F and $-F$ are a handlebody H for F and $-H$ for $-F$. The invariants of the manifolds are

$$\frac{1}{X^g}\sum_a I_a x_{a_1} \otimes \left(\otimes_{i>1} x^{a_i}\right),$$

and

$$\sum_b J_b x^{b_1} \otimes \left(\otimes_{j>1} x^{b_j}\right),$$

where the first factor in each tensor corresponds to F or $-F$. The invariant of the result of gluing (M_1, C_1) and (M_2, C_2) with respect to the marking C, coming from deleting the handlebodies H and $-H$ and taking the disjoint union of the remaining handlebodies, is

$$\frac{1}{X^g}\sum_{a,b} I_a J_b < x_{a_1}, x^{b_1} > \left(\otimes_{i>1} x^{a_i}\right) \otimes \left(\otimes_{j>1} x^{b_j}\right).$$

To see this, glue C_1 to C_2 and then choose the obvious surgery diagram in $-H$ to take the union to a connect sum of handlebodies corresponding to the deletion of H and $-H$.

Using ideas of Roberts [16], we can establish the equivalence of this projective TQFT and the one derived from the constructions in [1]. There is a natural equivalence of the functors from the compact marked oriented surfaces to vector spaces, that takes every manifold M to a surgery diagram for M in the handlebody that is its marking. It is easy to check that this equivalence preserves the invariants of the three-manifolds (projectively).

3. Ideals and Obstructions

In this section we introduce three ideals in $\mathbb{Z}[u]$ derived from quantum invariants. The first two are invariants of the manifold, and can be used as obstructions to embedding one manifold into another. The third ideal is not a manifold invariant but it can be easily computed and is used to estimate one of the first two ideals. If an ideal is all of $\mathbb{Z}[u]$ we say it is trivial. It is clear that an ideal is trivial if it contains a unit [11].

Let M be a compact, oriented 3-manifold with boundary.

Definition 1.

$$I_r(M) = \langle Z_r(M \cup N) \rangle_{\partial M = -\partial N} \leq \mathbb{Z}[u]$$

That is, $I_r(M)$ is the ideal of $\mathbb{Z}[u]$ generated by the Witten-Reshetikhin-Turaev invariants of all closed manifolds containing M.

Similarly,

Definition 2. *Let*

$$I_r^{TV}(M) = \langle Z_r\left((M \cup N) \# (-M \cup -N)\right) \rangle_{\partial M = -\partial N} \leq \mathbb{Z}[u]$$

be the ideal generated by the Turaev-Viro invariants of all closed manifolds containing M.

Recall [17], [20] that the Turaev-Viro invariant of a closed manifold P can be computed as $TV(P) = Z_r(P \# -P)$.

Proposition 2. *The ideal $I_r(M)$ is an invariant of oriented homeomorphism. The ideal $I_r^{TV}(M)$ is an invariant of unoriented homeomorphism.* □

These are related in the following way.

Proposition 3.

$$I_r^{TV}(M) \subset I_r(M) \tag{1}$$

Proof. Recall [7] that, for closed 3-manifolds M_1, M_2, $Z_r(-M_1) = \overline{Z_r(M_1)}$ and $Z_r(M_1 \# M_2) = Z_r(M_1) Z_r(M_2)$. Any element of $I_r^{TV}(M)$ can be written as a sum, taken over all the manifolds N with $\partial N = -\partial M$,

$$\sum_N \alpha_N Z_r(M \cup N) \overline{Z_r(M \cup N)}.$$

Since for every such N, the element $Z_r(M \cup N)$ is in the ideal $I_r(M)$, thus also $Z_r(M \cup N)\overline{Z_r(M \cup N)} \in I_r(M)$, and so is the linear combination of such elements. □

In particular, if $I_r(M)$ is nontrivial then $I_r^{TV}(M)$ is also nontrivial.

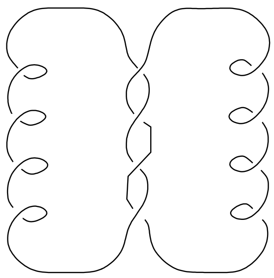


FIGURE 1. A manifold that demonstrates that the ideal is not determined by homology.

Proposition 4. *If the compact, oriented 3-manifold M embeds in the compact, oriented 3-manifold N, then $I_r(N) \subset I_r(M)$ and $I_r^{TV}(N) \subset I_r^{TV}(M)$.*

Proof.

$$I_r(N) = \langle Z_r(N \cup N')\rangle_{\partial N' = -\partial N} = \langle Z_r\left((M \cup \mathrm{Cl}(N-M)) \cup N'\right)\rangle_{\partial N' = -\partial N}$$

$$\subset \langle Z_r(M \cup N'')\rangle_{\partial N'' = -\partial M}$$

The argument for I_r^{TV} is analogous. □

Corollary 1. *Suppose that N is a compact oriented 3-manifold which contains a submanifold M such that $I_r(M)$ or $I_r^{TV}(M)$ is nontrivial, then N does not embed in S^3.* □

Corollary 2. *If M is a compact oriented 3-manifold which contains a punctured lens space $L(p,q)$ then M cannot be embedded into S^3.*

Proof. It is an easy consequence of a Mayer-Vietoris sequence. For $p \neq 2^n$ with $n > 1$ we can recover this using I_r^{TV} ideals. It follows form Theorem 2 of Yamada [22], where he computes $Z_r(L(p,q))$. In particular for any lens space there exists r for which $Z_r(L(p,q))$ is not a unit. If $p = 2$ then for any odd prime r the ideal is zero. Otherwise take r to be an odd prime dividing p, and the ideal will be either 0 or nontrivial. For $n > 1$, the quantum invariant of $L(2^n, q)$ is unfortunately always a unit when r is an odd prime. □

Remark 1. *The obstruction given by proposition 4 is not merely homological.*

Consider the 3-manifold M given by the surgery diagram in Figure 1.

Notice that $H_1(M) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$. However, $I_3(M) =< 1 >$, so M cannot contain a punctured $RP(3)$, since $I_3(RP(3)) = 0$, even though first homology groups do not rule it out.

Second family of examples is given by lens spaces, $L(7,1)$ and $L(7,2)$, which are homotopic but not homeomorphic. Using Theorem 2 of Yamada [22] we can see that $I_7^{TV}(L(7,2)) = \langle 0 \rangle$ while $I_7^{TV}(L(7,1)$ is a nonzero ideal. This leads to a multitude of examples where the quantum obstruction given by Proposition 4 rules out one manifold being embedded into another, while homotopy theory is useless. In particular, $L(7,2)$ is not a summand of any manifold whose ideal is nonzero (e.g. of $L(7,1)\#L(7,1)$) yet such a manifold could contain $L(7,1)$.

Definition 3. *Let (M,C) be a marked 3-manifold, denote by g the sum of the genera of the handlebodies making up C, and suppose that*

$$Z_r(M,C) = \frac{1}{X^g} \sum_a i(a) x_a,$$

with respect to a basis for C made up of admissibly colored trivalent spines. Define the ideal $J_r(M,C)$ to be generated by all coefficients $i(a)$.

$$J_r(M,C) = \langle i(a) \rangle \leq \mathbb{Z}[u]. \tag{2}$$

Any two trivalent spines of a handlebody can be related by isotopy and Whitehead moves. The coefficients of the corresponding change of basis matrices are units. Thus the ideal is independent of the choice of spine. However, the ideal $J_r(M,C)$ is only an invariant of the *marked* three-manifold. For instance, if we mark the solid torus $S^1 \times D^2$ with itself, then the surgery diagram is empty and the ideal is generated by X. If we mark the solid torus so that its longitude corresponds to the meridian of the marking solid torus and its meridian corresponds to the longitude of the marking torus, then the surgery diagram consists of the core of the marking torus with Ω_r plugged in, hence the ideal is 1. We are less interested in $J_r(M,C)$ as an invariant than we are as a tool to estimate $I_r^{TV}(M)$.

We restrict our attention now to the case when ∂M is connected. The ideal $I_r^{TV}(M)$ can be estimated from the ideal $J_r(M\natural - M, C)$ for a particular marking C. Here $M\natural - M$ denotes the boundary–connected sum of M and $-M$ along a disk in ∂M. The construction is an extension for analyzing the invariant of a 3-manifold with boundary of Roberts' "chain mail" [16].

Proposition 5. *Suppose that $\partial M = F$ is a connected surface of genus g, let $D \subset F$ be a disk and let $F' = Cl(F - D)$. Then*

$$I_r^{TV}(M) \subset \frac{1}{X^g} J_r(M\natural - M, C),$$

where $C = F' \times I$.

Remark 2. *It is not hard to show that*

$$I_r^{TV}(M) \subset \frac{1}{X^{2g}} J_r(M\natural - M, C). \tag{3}$$

However, we will show that $2g$ in the equation 3 can be replaced by g, giving a stronger bound.

Proof. As in the statement of the proposition, let $F = \partial M$ be a surface of genus g. Let $D \subset F$ be a disk and let $F' = \mathrm{Cl}(F - D)$, and denote the double of F' by $DF = F' \cup_{\partial F'} -F'$. The boundary of $M\natural - M$ can be naturally identified with DF. We will use the handlebody $F' \times I$ for the marking C of $M\natural - M$, obviously $\partial(F' \times I) = DF$.

We will show that for any N with $\partial M = -\partial N$, the Turaev-Viro invariant of the manifold $M \cup_{\partial M = -\partial N} N$ can be expressed as an integral linear combination of the coefficients $i(a)$ where,

$$Z_r(M\natural - M, C) = \frac{1}{X^g} \sum_a i(a) x_a.$$

Notice that g is *one half* of the genus of C. We achieve this by constructing an efficacious surgery description in the three-sphere. First we find a surgery diagram in S^3 for the double of $F' \times I$. Next, we locate a framed link in $F' \times I$ that is adapted to the marking of $\partial(M\natural - M)$ by DF. Finally we combine these diagrams to effect the computation.

The double of $F' \times I$ can be constructed in two steps. First identify $F' \times \{1\}$ with $F' \times \{0\}$. Second, fill the boundary so that the curves $\{pt\} \times I$, where $pt \in \partial F'$, bound meridian disks in the filling torus. Alternatively, one can do 0-surgery along a fiber of $F \times S^1$. A surgery diagram for $F \times S^1$ is drawn below for genus $g = 2$. In general, the number of clasped pairs woven through the middle circle is equal to g as in Figure 2.

Surgery along a fiber amounts to adding a zero framed unknotted circle that links the middle circle as in Figure 3.

The circumcision move says that if the small unknotted circle and the middle circle are removed, then surgery along the resulting diagram, pictured in Figure 4, yields the same result as in Figure 3.

Let $N(\partial D)$ be a regular neighborhood of the boundary of the disk D in F. Notice that $N(\partial D) = S^1 \times [0, 1]$. Let $f : F \to \mathbb{R}$ be a smooth

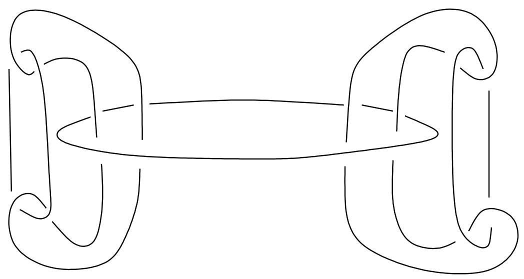


FIGURE 2. A surgery diagram that yields a cylinder over a surface of genus 2.

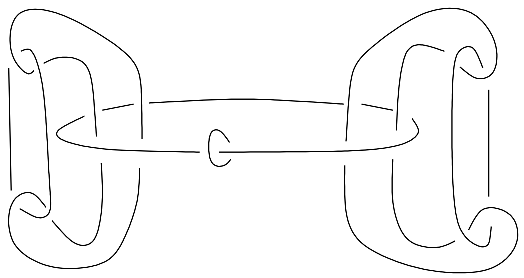


FIGURE 3. Zero framed surgery along a fiber.

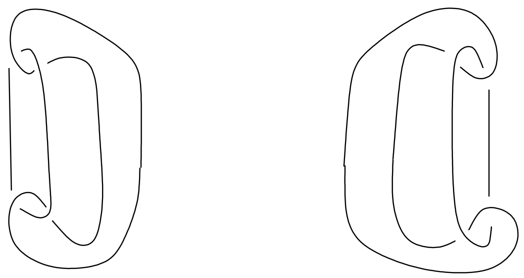


FIGURE 4. The result of applying the circumcision move to Figure 3.

function, equal to 1 on the disk $D - N(\partial D)$, and 0 outside of $D \cup N(\partial D)$. Suppose that on $N(\partial D)$ the function f has the circles $S^1 \times \{pt\}$ as level sets, and that it decreases from 1 to 0 as you move from the boundary component in D to the boundary component outside of D, without any critical points. Extend f to a smooth function on M. Perturb f so that it is Morse and then move all index 1 critical points below all of the index 2 critical points, and cancel all local minima and maxima. Let l be a level between the index 1 and index 2 critical points. Notice that $f^{-1}([0,l])$ is the result of adding 1 handles to $F' \times I$ along $F' \times \{1\}$. We get M from $f^{-1}([0,l])$ by adding 2-handles along some curves γ_i' in

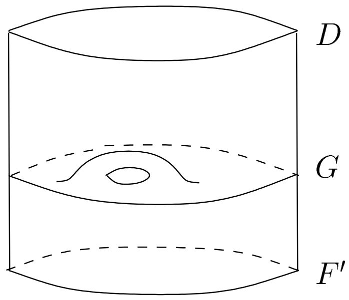


FIGURE 5. A schematic picture indicating the surface G in the cylinder $F \times I$.

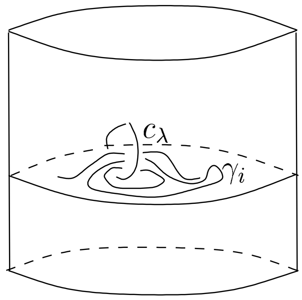


FIGURE 6. The curves c_λ, and γ_i.

$f^{-1}(l)$. Using this picture we can build a surgery diagram for $M\natural - M$. Let G be the result of adding h trivial 1-handles from above to $F' \times \{\frac{1}{2}\}$, so that it has the same genus as $f^{-1}(l)$. We indicate the surface G in Figure 5.

Choose a homeomorphism from $f^{-1}([0,l])$ to the part of $F' \times I$ between $F' \times \{0\}$ and G, so that $f|_{F' \times \{0\}}$ is the identity. Let γ_i be the image of the curves along which you add the 2-handles. The surgery diagram will consist of $g + h$ curves γ_i, along with h 0-framed curves c_λ above G linking each added trivial 1-handle in $F' \times I$. We indicate the curves c_λ, and γ_i in Figure 6

Notice that there is an orientation reversing map of order 2 on the result of surgery along the curves c_λ, that is the identity on G. This establishes that surgery on this diagram yields $M\natural - M$.

From this diagram and the marking C we can compute

$$Z_r(M\natural - M, C) = \frac{1}{X^g} \sum_a i(a) x_a, \tag{4}$$

where the x_a are standard basis elements coming from an admissibly colored trivalent spine for F' and g is the genus of F'. The Turaev-Wenzl identity is applied h times to cancel out the curves c_λ. This

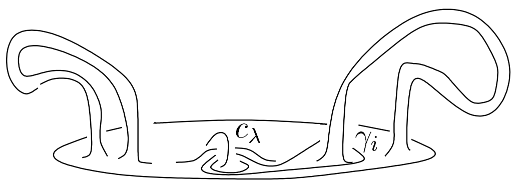


FIGURE 7. The surface G in the surgery diagram in S^3.

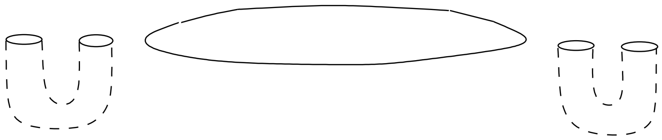


FIGURE 8. An example of a G'.

introduces X^h in the numerator. Since genus of G is $h+g$ the resulting power of X is $\dfrac{1}{X^g}$.

To see G in the surgery diagram for the double of $F' \times I$, take a disk whose boundary is where the middle curve of our initial surgery diagram was, and let it swallow the arcs of the clasped curves going through it upwards. Finally add enough trivial handles above to obtain a surface homeomorphic to G. Visualize the surgery diagram for $M \natural - M$ by seeing the curves γ_i lying on G and the curves c_λ linking the trivial handles. We show the surface G lying in the surgery diagram in S^3 in Figure 7

Since a regular neighborhood of the version of G lying in S^3, along with balls containing the trivial one handles and the linking curves, is homeomorphic to $F' \times I$, we can expand the surgery curves γ_i and the trivial linking handles c_λ to get the same formal sum $\frac{1}{X^g}\sum_a i(a)x_a$ as above, where the x_a now lie in S^3.

We can perform an analogous construction based on a disk outside the middle curve, only swallowing the clasping curves downwards, and add trivial handles downwards. If N is any 3-manifold with $\partial N = -F$ we can find a surgery diagram for the disk sum of N with $-N$ from a Heegaard surface, and construct a surface G' analogous to G. We can place this surgery diagram on the surface outside that we just constructed (with linking curves below the trivial handles). In the picture we just show the tubes that swallow the clasping curves below. Trivial handles below can be added as needed. We show an example of the surface G' in Figure 8

In this way we can build a surgery diagram for any

$$(M\natural - M) \cup_{-\partial M = \partial N} (N\natural - N).$$

Expand the surgery curves in terms of a standard basis x'_a associated to $F' \times I$, with x'_a lying in the copy of F' outside the middle curve. Similar to equation (4),

$$Z_r(N\natural - N, C) = \frac{1}{X^g} \sum_a j(a) x'_a. \tag{5}$$

The $2g$ clasped circles are trivial and unlinked. Furthermore, the disks these circles bound cut the Heegaard surfaces down to planar surfaces. Hence, the Turaev-Wenzl identity can be applied enough times to get rid of all the fractional part of the invariant. Therefore,

$$Z_r\left((M\natural - M) \cup (N\natural - N)\right)$$

is an integral linear combination of $i(a)j(b)$. This implies that

$$I_r^{TV}(M) \subset \frac{1}{X^g} J_r(M\# - M, C)$$

where C is the marking we constructed here. □

Remark 3. *As an immediate consequence of Propositions 5 and 4 we see that if the ideal* $\frac{1}{X^g} J_r(M\natural - M, C)$ *is nontrivial then* M *does not embed in the* 3*-sphere.*

This leads to the following example.

Let M be the result of gluing two solid tori along annuli in their boundaries. Choose the annuli so that their cores are $(2, 1)$ curves. The disk sum of M with $-M$ can be realized as surgery on a diagram in the cylinder over a punctured torus. The ideal $\frac{1}{X} J_3(M\natural - M, C)$ is the principal ideal generated by the integer 2 in the ring $Z[u]$. As this ideal is nontrivial, M cannot be embedded in S^3.

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