THE YANG-MILLS MEASURE IN THE KAUFFMAN BRACKET SKEIN MODULE

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Abstract. For each closed, orientable surface Σ_g, we construct a local, diffeomorphism invariant trace on the Kauffman bracket skein module $K_t(\Sigma_g \times I)$. The trace is defined when $|t|$ is neither 0 nor 1, and at certain roots of unity. At $t = -1$, the trace is integration against the symplectic measure on the $SU(2)$ character variety of the fundamental group of Σ_g.

1. Introduction

Since the introduction of quantum invariants of 3-manifolds [20, 25] the fact that they are only defined at roots of unity has been an obstruction to analyzing their properties. One approach has been to study the perturbative theory of quantum invariants [17]. However, there is ample evidence quantum invariants of three manifolds exist as holomorphic functions on the unit disk, that diverge everywhere on the unit circle but at roots of unity [14]. This paper takes a step towards seeing that this holds in general. The Yang-Mills measure is the path integral on a topological quantization [3] of the $SU(2)$-characters of the fundamental group of a closed surface. The measure displays the same convergence properties as are expected of quantum invariants of 3-manifolds.

The Yang-Mills measure in the Kauffman bracket skein algebra of a cylinder over a closed surface Σ_g is a local, diffeomorphism invariant trace. It quantizes the symplectic measure on the space $\mathcal{M}(\Sigma_g)$ of conjugacy classes of representations of the fundamental group of Σ_g into $SU(2)$. The definition of the symplectic structure and formulas for its computation are in [10, 11]. The volume of $\mathcal{M}(\Sigma_g)$ was computed by Witten in [26] in two ways: via the equivalence of two computations in quantum field theory, and by noting that the symplectic measure is equal to the measure coming from Reidemeister torsion. In Witten's setting the Yang-Mills measure is a path integral in a lattice model of field theory that depends on area. Forman [7] gave a direct proof that Witten's measure converges to the symplectic measure as the area goes to zero.

Alekseev, Grosse and Schomerus [1] conceived of a method of constructing lattice gauge field theory based on a quantum group. This idea was further developed by Buffenoir and Roche [6] who gave a construction of the algebra, its Wilson loops and a trace called the Yang-Mills measure that were completely analogous to Witten's construction. Their theory is topological when the area is set to zero.

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The method of constructing the algebras in [1, 6] is combinatorial and based on generators and relations. We gave a new construction of lattice gauge field theory in [4] that is "coordinate free". The connections form a co-algebra and the product on the gauge fields is a convolution with respect to the co-multiplication of connections. This allows the structure of the observables to be elucidated. We found working over formal power series, basing the theory on quantum sl_2, that the observables are the Kauffman bracket skein algebra of a cylinder over a regular neighborhood of the 1-skeleton. In [5] we recover the same result working over the complex numbers.

These considerations lead one to expect that the Yang-Mills measure exists as a trace on the Kauffman bracket skein algebra of a closed surface. In this paper we affirm this fact, with the only reservation that if the deformation parameter t is a generic point on the unit circle, then the measure does not converge. However, at roots of unity the trace exists and is well known. Furthermore, at $t = -1$ the Yang-Mills measure is the symplectic measure on $\mathcal{M}(\Sigma_g)$.

This paper is organized as follows. Section 2 recalls definitions, associated formulas and the algebra structure of the Kauffman bracket skein module of a cylinder over a surface. In section 3 the Yang-Mills measure is defined for compact surfaces with boundary, and is proved to be a trace. In section 4, working with the parameter t such that $|t| \neq 1$, we obtain estimates for the absolute value of the tetrahedral coefficients and use these to show that the Yang-Mills measure can be defined for closed surfaces. In section 5 we define and investigate the measure when t is a root of unity.

2. Preliminaries

Let M be an orientable 3-manifold. A framed link in M is an embedding of a disjoint union of annuli into M. Framed links are depicted by showing the core of an annulus lying parallel to the plane of the paper (i.e. with blackboard framing). Two framed links in M are equivalent if there is an isotopy of M taking one to the other. Let $\mathcal{L}$ denote the set of equivalence classes of framed links in M, including the empty link. Fix a complex number $t \neq 0$. Consider the vector space $\mathbb{C}\mathcal{L}$ with basis $\mathcal{L}$. Define $S(M)$ to be the smallest subspace of $\mathbb{C}\mathcal{L}$ containing all expressions of the form $\times - t \asymp - t^{-1})($ and $\bigcirc + t^2 + t^{-2}$, where the framed links in each expression are identical outside balls pictured in the diagrams. The Kauffman bracket skein module $K_t(M)$ is the quotient

$$\mathbb{C}\mathcal{L}/S(M).$$

Let F be a compact orientable surface and let $I = [0, 1]$. There is an algebra structure on $K_t(F \times I)$ that comes from laying one link over the other. Suppose that $\alpha, \beta \in K_t(F \times I)$ are skeins represented by links L_α and L_β. After isotopic deformations, to "raise" the first link and "lower" the second, $L_\alpha \subset F \times (\frac{1}{2}, 1]$ and $L_\beta \subset F \times [0, \frac{1}{2})$. The skein $\alpha * \beta$ is represented by $L_\alpha \cup L_\beta$. This product extends to a product on $K_t(F \times I)$. We denote the resulting algebra by $K_t(F)$ to emphasize that it comes from viewing the underlying three manifold as a cylinder over F.

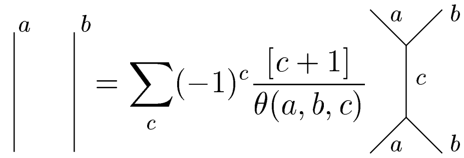
The notation and the formulas in this paper are taken from [13]. However, the variable t replaces A, and we use quantum integers

$$[n] = \frac{t^{2n} - t^{-2n}}{t^2 - t^{-2}}.$$

When $t = \pm 1$, $[n] = n$. Note that Δ_n from [13] is equal to $(-1)^n[n+1]$.

There is a standard convention for modeling a skein in $K_t(M)$ on a framed trivalent graph $\Gamma \subset M$. When Γ is represented by a diagram we assume blackboard framing. An *admissible coloring* of Γ is an assignment of a nonnegative integer to each edge so that the colors at trivalent vertices form admissible triples (defined below). The corresponding skein in $K_t(M)$ is obtained by replacing each edge labeled with the letter m by the m-th Jones–Wenzl idempotent (see [24], or [15], p.136), and replacing trivalent vertices with Kauffman triads (see [15, Fig. 14.7]).

Recall the fusion identity:



where the sum is over all c so that the triples (a, b, c) are admissible, i.e. $a + b + c$ is even, $a \leq b + c$, $b \leq a + c$, and $c \leq a + b$. Value of $\theta(a, b, c)$ is given by equation (4) below. The fusion relation is satisfied in $K_t(M)$ unless t is a root of unity other than ± 1.

3. The Yang-Mills Measure in a Handlebody

Throughout this section we assume that t is not a root of unity. The first result is well known and comes from Przytycki's [18] construction of examples of torsion in skein modules.

Lemma 1 (The Sphere Lemma). *Let s_c be a skein represented by coloring a trivalent framed graph in the manifold M. Suppose further that there is a sphere embedded in M which intersects the underlying graph transversely in a single point in the interior of an edge, and the color of that edge is not zero. Then $s_c = 0$.*

Proof. Using the "light bulb trick" isotope the framed graph s_c so that it is the same graph, but the framing on the edge intersecting the sphere has been changed by adding two kinks. Using the formula for eliminating a kink, notice that s_c is a nontrivial complex multiple of itself. Ergo, s_c represents zero in $K_t(M)$. □

Consider now $K_t(\#_g S^1 \times S^2)$, the Kauffman bracket skein module of the connected sum of g copies of $S^1 \times S^2$.

Proposition 1. *The skein module $K_t(\#_g S^1 \times S^2)$ is canonically isomorphic to $\mathbb{C}$. The isomorphism is given by writing each skein as a complex multiple of the empty skein.*

Proof. This follows easily from theorems of Hoste and Przytycki [12, 18, 19]. In [12] the Kauffman bracket skein module of $S^1 \times S^2$ is computed over $\mathbb{Z}[t, t^{-1}]$. This along with the results in [19] on the Kauffman bracket skein module of a connected sum over rational functions in t, combined with the universal coefficient theorem stated in [18], proves the desired result.

We outline the actual isomorphism with the complex numbers. Choose a system of spheres in $\#_g S^1 \times S^2$ that cut it down to a punctured ball. Given a skein in $\#_g S^1 \times S^2$, represent it as a linear combination of colored, framed, trivalent graphs intersecting the spheres transversely in interior of edges, and so that each graph intersects any sphere at most once. This is done by fusing multiple edges passing through the same sphere. By the sphere lemma, we can assume the graphs miss the spheres. Now take the Kauffman bracket of the skein in the punctured ball to write it as a complex multiple of the empty skein. □

Given a handlebody H of genus g its double is $\#_g S^1 \times S^2$. There is a linear functional $\mathcal{YM} : K_t(H) \to \mathbb{C}$ computed by taking the inclusion of H into $\#_g S^1 \times S^2$ followed by taking the " Kauffman bracket" as above. Let F be a compact, oriented surface with boundary. Since $F \times I$ is a handlebody the linear functional

$$\mathcal{YM} : K_t(F) \to \mathbb{C},$$

is defined. We call this the *Yang-Mills measure.*

Choose a trivalent spine of F. The admissible colorings of that spine form a basis for $K_t(F)$. The skein modules of the disk and annulus are exceptions; the first is spanned by the empty skein and the latter is described in Section 4. In terms of this basis the Yang-Mills measure is just the coefficient of the skein coming from labeling all the edges of the spine with 0.

Proposition 2. *The Yang-Mills measure is a trace, that is*

$$\mathcal{YM}(\alpha * \beta) = \mathcal{YM}(\beta * \alpha).$$

Furthermore, the trace is invariant under the action of the diffeomorphisms of $F \times I$ on $K_t(F)$.

Proof. Let L be the link $\partial F \times \{1/2\}$. The result of removing L from the double of $F \times I$ is homeomorphic to the Cartesian product of the interior of F with a circle. Given any skein in $F \times I$ we can represent it by a linear combination of framed links that miss L. Hence, the Yang-Mills measure factors through the skein module of $F \times S^1$. In $F \times S^1$ the skeins $\alpha * \beta$ and $\beta * \alpha$ are the same.

The group of diffeomorphisms of the handlebody $F \times I$ acts on $K_t(F)$ in the obvious way. If $f : F \times I \to F \times I$ is a diffeomorphism then it can be extended to $Df : \#_g S^1 \times S^2 \to \#_g S^1 \times S^2$. Since the image of the empty skein under a diffeomorphism is the empty skein, the action of Df on $K_t(\#_g S^1 \times S^2)$ is trivial. Therefore, $\mathcal{YM}(f(\alpha)) = \mathcal{YM}(\alpha)$. □

The final commonly used property of the Yang-Mills measure is that it is *local.* Suppose that k is a proper arc in F. Cut F along k to get a surface F'. It is evident that if we write a skein α as a linear combination of admissibly colored graphs, each one intersecting k transversely in at most a single point, then we can throw out any graph such that the edge intersecting k carries a nonzero label. This yields a skein in F', denoted by α_k. Then $\mathcal{YM}(\alpha) = \mathcal{YM}(\alpha_k)$.

4. The Yang-Mills measure on a closed surface

Throughout this section assume that $|t| \neq 1$. In fact, we only work with $0 < t < 1$. However, it is evident that the same proofs are valid when $1 < t$ since the formulas are symmetric in t and t^{-1}. Finally, the arguments extend to the case where t is not real by replacing the estimates for $t \in \mathbb{R}$ by estimates of the absolute value of $t \in \mathbb{C}$.

Recall the Kauffman bracket skein algebra of a cylinder over an annulus A. The central core of the annulus can be seen as a link by giving it the blackboard framing. Let s_i be the skein in the annulus which is the result of plugging the i-th Jones-Wenzl idempotent into the core. The skein module $K_t(A)$ is the vector space with basis $\{s_i\}$, where i runs from zero to infinity. The product with respect to this basis is given by

$$s_i * s_j = \sum_{q \geq |i-j|, \text{by } 2\text{'s}}^{i+j} s_q. \tag{1}$$

Use the Yang-Mills measure on $K_t(A)$ to define a pairing:

$$\langle \alpha, \beta \rangle = \mathcal{YM}(\alpha * \beta). \tag{2}$$

The s_i form an orthonormal basis with respect to (2). This pairing identifies the linear dual of $K_t(A)$ with series of the form $\sum_i \alpha_i s_i$, where the α_i are complex numbers. Note that:

$$\langle \sum_{i=0}^{\infty} \alpha_i s_i, \sum_{j=0}^{n} \beta_j s_j \rangle = \sum_{i=0}^{n} \alpha_i \beta_i.$$

Let $\Sigma_{g,1}$ denote the compact orientable surface of genus g with one boundary component. There is a pairing,

$$K_t(A) \otimes K_t(\Sigma_{g,1}) \to K_t(\Sigma_{g,1})$$

given by representing the skein in $K_t(\Sigma_{g,1})$ by a linear combination of links disjoint from some collar of the boundary, and plugging the skein in $K_t(A)$ into the collar. The Yang-Mills measure can then be applied to give a pairing,

$$K_t(A) \otimes K_t(\Sigma_{g,1}) \to \mathbb{C}. \tag{3}$$

This means there is a well defined map,

$$Y : K_t(\Sigma_{g,1}) \to K_t(A)^*.$$

Topologize $K_t(A)$ by giving it the weak topology from Y. That is a sequence $\sigma_n \in K_t(A)$ is Cauchy if for every skein $\alpha \in K_t(\Sigma_{g,1})$, the sequence of complex numbers $Y(\alpha)(\sigma_n)$ is Cauchy. A linear functional on $K_t(\Sigma_{g,1})$ that comes from an element of this completion via the pairing (3) is called a *distribution*. It is interesting to note that the weak topology from Y on $K_t(A)$ depends on the genus of the surface.

If $g > 1$ there is a distribution on $K_t(\Sigma_{g,1})$ which annihilates all "handle-slides" (*Skeins that are represented by the difference of two links such that one can be obtained from*

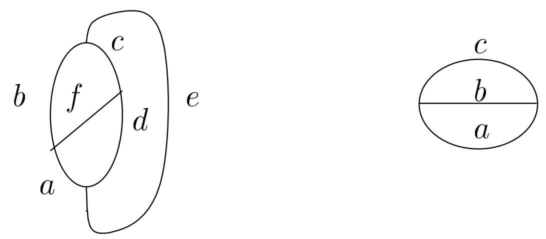


FIGURE 1. Tet and theta

the other by a slide across an imagined disk filling the boundary of $\Sigma_{g,1}$). This linear functional descends to the skein module of the closed surface. Yang-Mills measure on a closed surface is the result of evaluating this distribution followed by a normalization.

Let's think about what a skein in $K_t(A)$ would be like if it annihilated all handle-slides. Begin by writing it as $\sum_i \alpha_i s_i$ and solve for the α_i. A simple computation shows that if α_0 is zero then all α_i are zero. Normalize so that $\alpha_0 = 1$. Notice that if our skein annihilates handle-slides then the skein $s_1 + [2]s_0$ must be annihilated. Using the rules for multiplication (1) we see that the coefficient α_1 is equal to $-[2]$. Continuing on this way we see that this skein has to be

$$\sum_i (-1)^i [i+1] s_i,$$

which is of course not in $K_t(A)$.

The first goal is to show that for $g > 1$ the sequence of partials sums $\sum_{i=0}^n (-1)^i [i+1] s_i$ is Cauchy in the weak topology from Y, and so defines a distribution.

The notation $\mathrm{Tet}\begin{pmatrix} a & b & e \\ c & d & f \end{pmatrix}$ stands for the Kauffman bracket of the skein pictured in Figure 1 on the left. The explicit formula is given in [13]. We also need the quantity $\theta(a,b,c)$ which is the Kauffman bracket of the colored graph on the right in Figure 1. In terms of quantum integers

$$\theta(a,b,c) = (-1)^{\frac{a+b+c}{2}} \frac{[\frac{a+b+c}{2}+1]![\frac{a+b-c}{2}]![\frac{b+c-a}{2}]![\frac{c+a-b}{2}]!}{[a]![b]![c]!}. \tag{4}$$

Another quantity, called a $6j$ symbol, is derived from the tetrahedral evaluation. Specifically,

$$\begin{Bmatrix} a & b & e \\ c & d & f \end{Bmatrix} = \frac{\mathrm{Tet}\begin{pmatrix} a & b & e \\ c & d & f \end{pmatrix} (-1)^e [e+1]}{\theta(a,d,e)\theta(c,b,e)}. \tag{5}$$

The $6j$ symbols can be woven together to give a change of basis matrix for the Whitehead move on graphs. As a consequence they satisfy an orthogonality equation:

$$\sum_e \begin{Bmatrix} a & b & e \\ c & d & f \end{Bmatrix} \begin{Bmatrix} d & a & g \\ b & c & e \end{Bmatrix} = \delta^g_f, \tag{6}$$

where δ^g_f is the Kronecker delta.

The following proposition seems quite weak, but turns out to be a powerful tool for gauging the convergence of series of Kauffman brackets.

Proposition 3.

$$\left| Tet \begin{pmatrix} a & b & e \\ c & d & f \end{pmatrix} \right| \leq \sqrt{\frac{\theta(b,c,e)\theta(a,d,e)\theta(a,b,f)\theta(c,d,f)}{(-1)^{e+f}[e+1][f+1]}}$$

Proof. In order for all the triples at the vertices of a tetrahedron to be admissible , the parity of the sum of the entries in any two columns of

$$\mathrm{Tet} \begin{pmatrix} a & b & e \\ c & d & f \end{pmatrix}$$

has to be the same. Use (5) to expand the formulas for the $6j$ symbols in the orthogonality relation (6), with $g = f$. The tetrahedral evaluations are equal and the signs of the θ's and the $(-1)^{e+f}$ cancel so that each term in the sum is positive. Hence every term in the sum is less than 1. Fixing e and putting everything except for the tetrahedral evaluations on the right hand side, and taking square roots yields the desired result. □

Corollary 1. *There is a real valued function $C(k_1, k_2, k_3)$ so that*

$$\frac{|\,Tet \begin{pmatrix} i & i & i \\ k_1 & k_2 & k_3 \end{pmatrix}|}{\sqrt{|\theta(i,i,k_1)\theta(i,i,k_2)\theta(i,i,k_3)|}} \tag{7}$$

is less than $t^i C(k_1, k_2, k_3)$ whenever the graphs corresponding to the functions in the formula are admissibly labeled.

Proof. Substitute into the inequality from Proposition 3 to get,

$$\left| \mathrm{Tet} \begin{pmatrix} i & i & i \\ k_1 & k_2 & k_3 \end{pmatrix} \right| \leq \sqrt{\frac{\theta(k_1,k_2,k_3)\theta(i,i,k_1)\theta(i,i,k_2)\theta(i,i,k_3)}{(-1)^{i+k_3}[k_3+1][i+1]}}. \tag{8}$$

Shift $\sqrt{\theta(i,i,k_1)\theta(i,i,k_2)\theta(i,i,k_3)}$ to the left hand side. Use the fact that $\frac{1}{[i+1]} \leq t^{2i}$ to make the right hand side bigger. Finally, note that the remaining factor on the right hand side is a function of k_1, k_2 and k_3. □

Theorem 1. *The sequence $\sum_{i=0}^n (-1)^i [i+1] s_i$ defines a distribution for $g > 1$. That is, the limit*

$$\mathcal{YM}_D(\alpha) = \lim_{n \to \infty} \mathcal{YM}(\alpha * \sum_{i=0}^n (-1)^i [i+1] s_i)$$

exists and gives a well defined trace on $K_t(\Sigma_{g,1})$ when $g > 1$.

Proof. Choose a trivalent spine for $\Sigma_{g,1}$ with $4g-2$ vertices and $6g-3$ edges. Basis elements s_c for $K_t(\Sigma_{g,1})$ correspond to labeling the edges admissibly with integers k_j, where j runs from 1 to $6g-3$. Let s_i denote the core of an annulus that runs parallel to the boundary, labeled with the ith Jones-Wenzl idempotent. In order to compute $\mathcal{YM}(s_c * s_i)$ place both skeins in the same diagram. Choose a system of arcs, each intersecting this configuration transversely in three points, that isolate the vertices from one another. The transverse points of intersection are labeled i, k_j, i as you traverse each arc. Fuse along these arcs, until the resulting graphs intersect each arc in at most one point. Discard any term where the label on an edge intersecting an arc is not zero. Given a vertex v, let (k_{v1}, k_{v2}, k_{v3}) be the triple of colors appearing there. The resulting answer is:

$$\mathcal{YM}(s_c * s_i) = \prod_{j=1}^{6g-3} \frac{1}{\theta(i,i,k_j)} \prod_v \mathrm{Tet}\begin{pmatrix} i & i & i \\ k_{v1} & k_{v2} & k_{v3} \end{pmatrix}. \tag{9}$$

Each edge appears at exactly two vertices, so (9) can be written as a product of $4g-2$ factors like (7). By Corollary 1 the absolute value of $\mathcal{YM}(s_c * s_i)$ is less than $C(k_j)t^{i(4g-2)}$, where $C(k_j)$ is a number depending only on the k_j. The nth partial sum for $\mathcal{YM}_D(s_c)$ is

$$\sum_{i=0}^{n}(-1)^i[i+1] \prod_{j=1}^{6g-3} \frac{1}{\theta(i,i,k_j)} \prod_v \mathrm{Tet}\begin{pmatrix} i & i & i \\ k_{v1} & k_{v2} & k_{v3} \end{pmatrix}.$$

Note that $[i+1]$ is less than $(i+1)t^{-2i}$. Hence the i-th summand is less than $(i+1)(-1)^i C(k_j)t^{i(4g-4)}$. The ratio test implies that the sequence of partial sums is absolutely convergent for $0 < t < 1$.

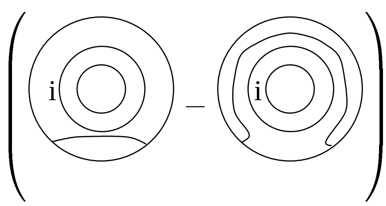
Finally, $\mathcal{YM}_{\mathcal{D}}$ is a trace since the partial sums $\sum_{i=0}^{n}(-1)^i[i+1]s_i$ can be seen as lying in the center of $K_t(\Sigma_{g,1})$. $\square$

Theorem 2. *$\mathcal{YM}_D$ descends to give a well defined trace*

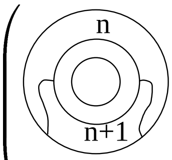
$$\mathcal{YM} : K_t(\Sigma_g) \to \mathbb{C}.$$

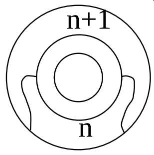
Proof. There is a homomorphism $K_t(\Sigma_{g,1}) \to K_t(\Sigma_g)$ induced by inclusion. The surface $K_t(\Sigma_g)$ is the result of adding a disk to the boundary of surface $K_t(\Sigma_{g,1})$. The kernel of this homomorphism consists of all skeins that can be written as a linear combination of handle-slides. The next step is to show that the linear functional $\mathcal{YM}_D$ annihilates all handle-slides. To this end we analyze the difference of the two skeins in the annulus (relative to a pair of points in the boundary).

(10) $$\sum_{i=0}^{n}(-1)^i[i+1]$$



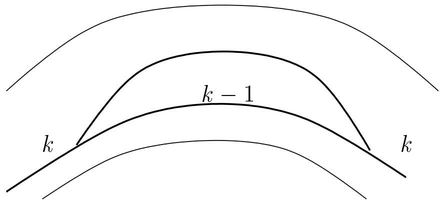
The analysis of the diagram (10) diagram is due to Lickorish, [15]. It is equal to:

(11) $(-1)^n[n+1]\Bigg($ 

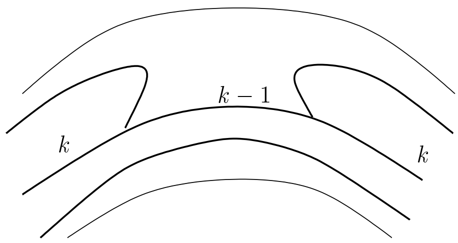
$-$ 

$\Bigg).$

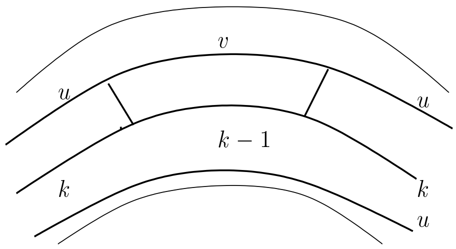
This diagram needs to be set in place. Using standard arguments as in [2] yields that we only need to check handle-slides of the following form. Take a skein corresponding to a colored spine, and separate one strand along an edge.



Now slide the strand over the added disk, locally the diagram looks like:



Multiplying each of the diagrams above by $\sum_{i=0}^{n}(-1)^i[i+1]s_i$, taking their difference, and using the identity (10)=(11), we get a difference of two terms like the one below. In the first one the label $u = n$ and the label $v = n+1$, and in the second one $u = n+1$ and $v = n$.



Fusing to isolate the vertices of this diagram requires two more cross cuts than the diagrams we have been working with up till now. We get the product of

$$(-1)^n[n+1]\frac{1}{\theta(u,k,u)\theta(u,k-1,v)}\mathrm{Tet}\begin{pmatrix} u & u & v \\ 1 & k-1 & k \end{pmatrix}\mathrm{Tet}\begin{pmatrix} u & v & u \\ 1 & k & k-1 \end{pmatrix} \tag{12}$$

with the standard product,

$$\prod_{j=1}^{6g-3}\frac{1}{\theta(u,u,k_j)}\prod_{v}\mathrm{Tet}\begin{pmatrix} u & u & u \\ k_{v1} & k_{v2} & k_{v3} \end{pmatrix}. \tag{13}$$

The product (13) is smaller than a global constant, depending on the k_j, times $t^{n(4g-2)}$. It remains to ascertain that the term (12) is not too large. Using the inequality from Proposition 3 we get that, regardless of whether $u = n$ and $v = n+1$, or $u = n+1$ and $v = n$, the absolute value of (12) is less than $[n+2]$, which is a universal constant times t^{-2n}. As long as the genus of the surface is greater than 1, the full product goes to zero as n goes to infinity. So, in the limit, all handle-slides are annihilated. □

The case of a surface of genus 1 is slightly different. To get a convergent distribution we need to divide the partial sum $\sum_{i=0}^{n}(-1)^i[i+1]s_i$ by n. The sequence is then Cauchy and defines a distribution on $K_t(T^2)$.

The algebra $K_t(T^2)$ is very nice for working examples. If (p,q) is a pair of integers that are relatively prime there is an obvious skein $s_{(p,q)}$ in $K_t(T^2)$ corresponding to the (p,q) curve on the torus . Define a family of skeins based on (p,q) by using the following iterative scheme: $s_{(p,q)_0} = 2s_{(0,0)}$, that is, twice the empty skein, and $s_{(p,q)_1} = s_{(p,q)}$. For $d > 1$ define:

$$s_{(p,q)_d} = s_{(p,q)} * s_{(p,q)_{d-1}} - s_{(p,q)_{d-2}}.$$

Finally, if $d = \gcd\{p,q\}$, let

$$s_{(p,q)} = s_{(p/d,q/d)_d}.$$

Using this notation the product in $K_t(T^2)$ is given by

$$s_{(p,q)} * s_{(u,v)} = t^{\begin{vmatrix} p & q \\ u & v \end{vmatrix}} s_{(p+u,q+v)} + t^{-\begin{vmatrix} p & q \\ u & v \end{vmatrix}} s_{(p-u,q-v)}. \tag{14}$$

The formula (14) is proven in [8].

There is a map

$$\mu : K_t(T^2) \to \mathbb{C}\emptyset \oplus \mathbb{C}H_1(T^2; Z_2)$$

introduced in [16]. Let

$$\mu\left(\sum_{(p,q)} a_{(p,q)} s_{(p,q)}\right) = a_{(0,0)}\emptyset + \sum_{(p,q)\neq(0,0)} a_{(p,q)}[(p,q)],$$

where $[(p,q)]$ is the Z_2–homology class in $H_1(T^2; Z_2)$ corresponding to $d = \gcd\{p,q\}$ copies of a $(p/d, q/d)$ curve on the torus. The map μ has as its kernel the submodule of all commutators. Hence any linear functional on the five dimensional space that is the image of μ is a trace. It is easy to check that there is a three dimensional family of traces that are invariant under diffeomorphism. In this set up

$$\mathcal{YM}\left(\sum_{(p,q)} a_{(p,q)} s_{(p,q)}\right) = a_{(0,0)}.$$

This is the same trace as the one induced from the inclusion of $K_t(T^2)$ into the non-commutative torus [8].

Towards uniqueness of the Yang-Mills measure, it should be normalized, just as the symplectic measure on moduli space needs to be normalized. It should also be invariant under diffeomorphism, and be local. Locality is made up by two rules. One for cutting a surface along an arc and one for removing a point from a closed surface. If we formalize our rules correctly, we get the following:

Theorem 3. *The Yang-Mills measure is the unique, local, diffeomorphism invariant trace on $K_t(\Sigma_g)$ up to normalization.* □

5. Roots of Unity

Fusion no longer holds in $K_t(M)$ when t is a root of unity. However, when $t = e^{\frac{\pi i}{2r}}$ then one can take a quotient, where an appropriate form of the fusion identity is true. This can be done by setting any skein containing the $(r-1)$-st Jones-Wenzl idempotent equal to zero. The quotient is denoted $K_{r,f}(M)$. The *reduced skein* is a central object in the construction of quantum invariants of 3-manifolds [9, 21, 22].

The Yang-Mills measure on a surface with boundary is obtained the same way as for other values of t. Since $[r] = 0$, the iterative procedure for finding a skein in the annulus that annihilates handle-slides terminates, to yield

$$\sum_{i=0}^{r-2}(-1)^i[i+1] \bigcirc^i .$$

There is an induced trace,

$$\mathcal{YM} : K_{r,f}(\Sigma_g) \to \mathbb{C},$$

constructed the same way as for other t except that there is no need to take a limit because the formula is a finite sum.

Notice that Σ_g is the boundary of a handlebody H_g (it doesn't make any difference which one). There is an action of $K_{r,f}(\Sigma_g)$ on $K_{r,f}(H_g)$ given by gluing skeins in $\Sigma_g \times I$ into a collar of the boundary of H_g. The action gives a map

$$\phi : K_{r,f}(\Sigma_g) \to \mathrm{End}(K_{r,f}(H_g)).$$

As we are working at a root of unity, $K_{r,f}(H_g)$ is a finite dimensional vector space. Denote its dimension by d, and let $\omega = \mathcal{YM}(\emptyset) = \sum_{i=0}^{r-2} \frac{1}{[i+1]^{2g-2}}$. The Yang-Mills measure is:

$$\mathcal{YM}(\alpha) = \frac{\omega}{d}\mathrm{tr}(\phi(\alpha)).$$

From [23] the map ϕ is injective and onto. Hence we can identify $K_{r,f}(\Sigma_g)$ with $\mathrm{End}(K_{r,f}(H_g))$. The Yang-Mills measure is zero on commutators. Thus it factors through

$$\mathrm{End}(K_{r,f}(H_g))/[\mathrm{End}(K_{r,f}(H_g)), \mathrm{End}(K_{r,f}(H_g))].$$

This quotient is a 1-dimensional vector space. Hence any two linear functionals that factor through this quotient are equal if they agree on the identity matrix. The trace also vanishes on commutators, thus it factors through the commutator quotient. The normalization in the formula causes the two induced linear functionals to be the same.

Next we address the cases of $t = \pm 1$. Since the formula for the measure of a spine is an even function of t, we only need to consider one value. The value $t = -1$ is more convenient as the correspondence between $K_{-1}(F)$ and the $SU(2)$-characters of $\pi_1(F)$

is simpler. The skein of the disjoint union of curves c_i corresponds to the function that sends the representation ρ to

$$\prod_i -\mathrm{tr}(\rho(c_i)).$$

Theorem 4. *The Yang-Mills measure is well defined on $K_{\pm1}(\Sigma_g)$ for $g > 1$. Let s_c be an admissibly colored trivalent spine of Σ_g. If t_n, with $|t_n| \neq 1$, is a sequence of complex numbers converging to ± 1 then*

$$\lim_{n\to\infty} \mathcal{YM}_{t_n}(s_c) = \mathcal{YM}_{\pm 1}(s_c).$$

Proof. The formulas for working with skeins in $K_{-1}(F)$ are the same as the ones for $|t| \neq 1$ except that quantized integers are replaced by ordinary integers. These formulas are the limits as $t \to -1$ of the values we have been using. Revisiting the fundamental estimate (8), we see that,

$$\frac{|\mathrm{Tet}\begin{pmatrix} i & i & i \\ k_1 & k_2 & k_3 \end{pmatrix}|}{\sqrt{|\theta(i,i,k_1)\theta(i,i,k_2)\theta(i,i,k_3)|}} \leq \sqrt{\frac{\theta(k_1,k_2,k_3)}{(-1)^{i+k_3}(k_3+1)(i+1)}} \tag{15}$$

from which we conclude that the right hand side is less than or equal to

$$\frac{C(k_1,k_2,k_3)}{\sqrt{i+1}}.$$

Considering the series for the Yang-Mills measure of a spine, comparison to the p-series implies that it converges as long as the surface has genus greater than 1. Similarly, the Yang-Mills measure is invariant under handle-slides.

The convergence statement follows from the fact that the series that define the Yang-Mills measure at t_n converge absolutely, and the terms of the series converge to the terms of the series for the Yang-Mills measure at -1. □

For a surface of genus 1 we divide the partial sums, as before, by the number of terms in the sum, and the series then converges.

Theorem 5. *The Yang-Mills measure at $t = -1$ is the symplectic measure on $\mathcal{M}(\Sigma_g)$.*

Proof. Using Weyl orthogonality to compute Witten's Yang-Mills measure for a surface of area ρ yields that its value on the spine s_c is given by the series

$$\sum_{i=0}^{\infty}(-1)^i(i+1)e^{-\rho c_2(i)}\prod_{j=1}^{6g-3}\frac{1}{\theta(i,i,k_j)}\prod_v \mathrm{Tet}\begin{pmatrix} i & i & i \\ k_{v1} & k_{v2} & k_{v3} \end{pmatrix},$$

where the edges of s_c carry colors k_i, and k_{v_i} are the colors of the edges ending at the vertex v, and $c_2(i)$ is the value of the quadratic Casimir operator on the $(i+1)$-dimensional irreducible representation of $SU(2)$. As both Witten's series and our series converge absolutely, and Witten's formula converges term by term to our formula as $\rho \to 0$, the limit of Witten's Yang-Mills measure is equal to our Yang-Mills measure at $t = -1$. Finally, Forman [7] showed that the limit as $\rho \to 0$ of Witten's measure is the symplectic measure on $\mathcal{M}(\Sigma_g)$, normalized as in [7]. □

Suppose now that $|t| = 1$ and t is not a root of unity. Evaluation of the Yang-Mills measure on the empty skein on a surface of genus g yields $\sum_{i=o}^{\infty} \frac{1}{[i+1]^{2g-2}}$. As t is not a root of unity the number $[i+1]^{2g-2}$ gets arbitrarily close to 1 infinitely often, which means that the series does not converge. Therefore the Yang-Mills measure does not exist away from roots of unity on the unit circle.

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