TOPOLOGICAL INTERPRETATIONS OF LATTICE GAUGE FIELD THEORY

DOUG BULLOCK, CHARLES FROHMAN, AND JOANNA KANIA-BARTOSZYŃSKA

ABSTRACT. We construct lattice gauge field theory based on a quantum group on a lattice of dimension one. Innovations include a coalgebra structure on the connections and an investigation of connections that are not distinguishable by observables. We prove that when the quantum group is a deformation of a connected algebraic group G (over the complex numbers), then the algebra of observables forms a deformation quantization of the ring of G-characters of the fundamental group of the lattice. Finally, we investigate lattice gauge field theory based on quantum $SL_2\mathbb{C}$, and conclude that the algebra of observables is the Kauffman bracket skein module of a cylinder over a surface associated to the lattice.

Introduction

Lattice gauge field theory based on an algebraic group G is a finite element approximation of a smooth gauge field theory with G as its structure group. Infinitesimally varying connections and gauge transformations on a principal bundle are discretized via a lattice embedded in the base manifold. To each edge in the lattice a connection imparts an element of G encoding the holonomy along that edge. Gauge fields (i.e., functions on connections) are represented by a copy of the coordinate ring of G associated to each edge. The action of the gauge group is then concentrated at vertices. All computations become merely algebraic with analytic and geometric considerations swept aside. The end result is an algebra of observables (the gauge invariant gauge fields) that can be understood as the character theory for representations of the fundamental group of the lattice into G.

Lattice gauge field theory based on a quantum group yields a deformation of this theory. Technically, the result is an algebra of observables which, with respect to the standard Poisson structure [2, 11], gives a deformation quantization of the ring of G-characters. Here one must think of the fundamental group of the lattice as the fundamental group of a surface with boundary.

In this paper we develop, from an elementary and computational viewpoint, the basic objects of lattice gauge field theory based on a quasitriangular ribbon Hopf algebra (e.g., a quantum group). We then use this foundation to begin the study of the structure of algebras of observables (paying particular attention to quantum groups) and to recognize the observables as algebras that have already been studied in a topological framework.

The first author is partially supported by an Idaho SBOE Specific Research Grant; the second and third by NSF-DMS-9204489 and NSF-DMS-9626818.

The genesis of our approach can be found in the papers [1, 3, 4, 5, 10]. Fock and Rosly were the first to derive the Poisson structure on G-characters from a lattice gauge field theory. Their formula is written in terms of a solution of the modified classical Yang-Baxter equation [9]. Also, recall that the characters of a surface group are only a homotopy invariant while the Poisson structure is a topological invariant. For this reason, Fock and Rosly endow the lattice with extra information, called a ciliation, so that it determines a surface.

Passing to quantum groups, Alekseev, Grosse and Schomerus defined an exchange algebra over a ciliated lattice so that basic elements of the algebra of gauge fields commute according to a solution of the quantum Yang-Baxter equation. This algebra is related to a quantization of the characters with respect to the usual Poisson structure. It is based on a full solution of the Clebsch-Gordan problem for the quantum group being used, and has both gauge transformations and gauge fields in the same place.

Buffenoir and Roche [4] took this approach farther. First they isolated the gauge fields from the gauge transformations. Their gauge algebra is dual to that of [1], hence they have a coaction of the gauge algebra on the gauge fields. The coinvariant part of the gauge fields is the algebra of observables, which is a deformation of the classical ring of characters. They proceed to define Wilson loops and the Yang-Mills measure and to derive 3-manifold invariants from this setting [3, 5].

We found ourselves unable to compute examples in the exchange algebra formulation. We instead define our gauge fields as functions on the space of connections. This makes the structure of the algebra of observables more clear. Working from the point of view of low-dimensional topology, we assume a familiarity with the basics of knot theory. Otherwise, one can read most of this paper knowing only the definition of a ribbon Hopf algebra and the basics of its representation theory. Kassel [13] and Sweedler [18] are sufficient references.

Part 1 is devoted to our translation of the basic objects of a lattice gauge field theory and to our devices for computing in the reformulated version. We do not merely alter the language of [4]. Three significant innovations provide added computing power. The first is to realize that gauge fields come from the restricted dual of the Hopf algebra on which the theory is based. This leads to a coordinate free formulation. Next, we do not multiply gauge fields as abstract variables modulo exchange relations. Rather we comultiply connections in a way that implies the usual exchange relations for fields while preserving their evaluabililty. Finally, we are able to mimic the classical phenomenon of pushing the support of a gauge field around. Our new foundations allow us to compute Wilson loops and many other operators using a simple extension of tangle functors.

The second part is devoted to analyzing the structure of the algebra of observables. Our viewpoint is that the observables corresponding to quantum groups generalize the rings studied by Procesi [15]. He arrived at these rings as the invariants of n-tuples of matrices under conjugation. The connection with lattice gauge field theory is that each n-tuple of matrices corresponds to a connection on a lattice with one vertex and n-edges, with the gauge fields based on a classical group.

In passing from Procesi's work to ours, we find that the algebra of observables corresponding to a quantum group is a more subtle object. Instead of depending solely on the fundamental group of the lattice, the observables are classified by the topological type of a surface specified by a ciliated lattice. The construction given in this paper leads to an algebra of "characters" of a surface group with respect to any quasitriangular ribbon Hopf algebra. The algebras are interesting from many points of view: They generalize objects studied in invariant theory; they should provide tools for investigating the structure of the mapping class groups of surfaces; and they should give a way of understanding quantum invariants of 3-manifolds.

In the case that the data correspond to a connected affine algebraic group G, it is possible to make explicit parallels with the existing theory. The algebra of observables based on $U(\mathfrak{g})$ is shown to be the ring of G-characters of the fundamental group of the associated surface. Then, the original motivating problem is solved: Given the ring of G-characters of a surface group, show that the observables based on the corresponding Drinfeld-Jimbo algebra form a quantization with respect to the usual Poisson structure. We also prove for the classical groups that the algebra of observables is generated by Wilson loops. Finally, invoking a quantized Cayley-Hamilton identity, we obtain a new proof, independent of [7], that the $U_h(sl_2)$-characters of a surface are exactly the Kauffman bracket skein module of a cylinder over that surface.

Many further avenues of research present themselves. Working with quantum groups defined over local fields side steps several interesting and subtle structural questions. What happens when one uses a quantum group at a root of unity? How about lattice gauge field theory based on a quasitriangular quasi-Hopf algebra?

There is a graphical calculus of the characters of the fundamental group of any manifold with respect to any algebraic group, for example, see [6], [16] or [17]. It derives from the fact that Wilson loops are a pictorial description of characters of the fundamental group of a manifold, after which the tools of classical invariant theory express all functional relations between characters in a diagrammatic fashion. The graphical models have only been worked out for special linear groups.

The power of lattice gauge field theory is that it places the representation theory of the underlying manifold and the quantum invariants in the same setting. Ultimately the asymptotic analysis of the quantum invariants of a 3-manifold in terms of the representations of its fundamental group should flow out of this setting. The identification of the representation theory of a quantum group with that of a compact Lie group leads to rigorous integral formulas for quantum invariants of 3-manifolds. This should in turn lead to a simple explication of the relationship between quantum invariants and more classical invariants of 3-manifolds.

Finally, there should be a similarly clean treatment of lattice gauge field theory for lattices of higher dimension. Although, in dimension greater than two, the answers will no longer be topological in nature, the constructions and objects should be of interest to geometers, algebraists and analysts.

This work was done at the Banach Center, the University of Iowa, Boise State University, The George Washington University, the University of Missouri, MSRI, and the Free University, Berlin. The authors thank all their hosts for their hospitality. We also thank Daniel Altschuler, Jorgen Anderson, Georgia Benkhart, Vic Camillo, Fred Goodman, Joseph Mattes, Michael Polyak, Florin Radulescu, Arun Ram, Justin Roberts, Don Schack and Bob Sulanke for helpful conversations.

Part 1. **Lattice Gauge Field Theory**

Herein we develop, from a self contained and axiomatic approach, the machinery of gauge field theory on an abstract, oriented, ciliated graph. For basic background on Hopf algebras we rely on Sweedler [18] and Kassel [13]. The discussion here is restricted to the definitions and basic results confirming that the theory is consistent and computationally viable. For origins of the ideas we refer the reader to [1, 3, 4, 5, 8, 10].

1. Objects

The elementary objects of a lattice gauge field theory are: a ribbon Hopf algebra; an abstract graph which is oriented and ciliated; discretized connections, gauge transformations, and gauge fields.

1.1. Let H be a ribbon Hopf algebra defined over a field $\mathfrak{k}$ or its power series ring $\mathfrak{k}[[h]]$. In the latter case all objects carry the h-adic topology (see [13]), and all morphisms are continuous. In discussions germane to both settings we will refer to the base over which the algebra is defined as $\mathfrak{b}$. Following Kassel we let μ, η, Δ, ϵ and S denote the multiplication, unit, comultiplication, counit and antipode of H. The universal R-matrix is $R = \sum_i s_i \otimes t_i$, which we usually write as $s \otimes t$ with summation understood. The ribbon element is θ, and we add a **charmed element**, $k = \theta^{-1}S(t)s$. The charmed element is **grouplike**, meaning $\Delta(k) = k \otimes k$. It also satisfies $k^{-1} = S(k) = \theta t S^2(s)$ and $S^2(x) = kxk^{-1}$ for all $x \in H$.

The Hopf algebra dual is well documented in [18] provided $\mathfrak{b} = \mathfrak{k}$. The topological case, however, has been neglected. If H is a Hopf algebra over $\mathfrak{k}[[h]]$ then the sets $U_n = \{L \in H^* \mid L(H) \subset h^n\mathfrak{k}[[h]]\}$ form a neighborhood basis of the origin. An ideal $J \leq H$ is cofinite if H/J is topologically free and modeled on a finite dimensional vector space; $L \in H^*$ is cofinite if $\ker(L)$ contains a cofinite ideal. The restricted dual, H^o, is the completion of the cofinite functionals.

It is not hard to check that H^o is topologically free. In the case that H is a Drinfeld-Jimbo deformation of a simple Lie algebra [13], it is clear that H^o is modeled on $(H/hH)^o$. To see that it is a Hopf algebra one must check that μ^*, η^*, Δ^*, ϵ^* and S^* restricted to H^o or $H^o \otimes H^o$ take values in the appropriate spaces. The only point that needs any discussion is why $\mu^*(L) \in H^o \otimes H^o$. Suppose that L is the limit of cofinite linear functionals $\{L_i\}$. It follows that $\{L_i\}$ is a Cauchy sequence in H^o. The classical discussion of μ^* in [18] provides $L_i \circ \mu \in H^o \otimes H^o$, which is complete by definition [13]. It is easy to see that $L_i \circ \mu$ is also Cauchy, so continuity of μ^* gives $\mu^*(L) \in H^o \otimes H^o$.

The Hopf algebra H acts on its restricted dual in two obvious ways: $x \cdot \phi(y) = \phi(xy)$ and $x \cdot \phi(y) = \phi(yx)$. A subalgebra of H° is stable if it is invariant under both actions. For the remainder of this section we fix a stable subalgebra B of H°. (Stability implies that B is actually a Hopf subalgebra.)

The adjoint action $\mathrm{ad} : H \otimes H \to H$ is given, in Sweedler notation, by

$$\mathrm{ad}(x \otimes y) = \sum_{(x)} x'' y S(x'),$$

which is further compressed to $\mathrm{ad}(x)y = x'' y S(x')$. (Readers unfamiliar with this notation for comultiplication are referred to [13].) By taking duals we get the adjoint (right) action of H on B: if $x, y \in H$, and $\phi \in B$ then

$$(\phi \mathrm{ad}(x))(y) = \phi(\mathrm{ad}(x)y) = \phi(x'' y S(x')).$$

An element $\phi \in B$ is **invariant** if for every $x \in H$, $\phi \mathrm{ad}(x) = \epsilon(x)\phi$. Our definition of the adjoint action is chosen so that any function ϕ with the property that $\phi(xy) = \phi(yx)$ will be invariant. The invariant elements of B form a subalgebra denoted B^H.

1.2. A graph consists of a set E called edges, a fixed point free involution $- : E \to E$, and a partition V of E into subsets called vertices. Let $\mathfrak{i} : E \to V$ be the map sending e to the vertex v containing it. Let $\mathfrak{t} = \mathfrak{i} \circ -$. We call $\mathfrak{t}(e)$ the terminal vertex of e and $\mathfrak{i}(e)$ the initial vertex. An **orientation** is a choice O of one edge from each orbit of the involution. An oriented graph is denoted by the data $(E, -, V, O)$.

There is a one-dimensional CW-complex associated to $(E, -, V, O)$, which is called its **geometric realization**. The 0-cells are in one to one correspondence with V, the 1-cells are in one to one correspondence with O, and the characteristic maps are determined by $\mathfrak{t}$ and $\mathfrak{i}$.

A ciliation C of a graph is a linear ordering of each vertex. The additional data is denoted V^c, although we will continue to use V for the partition of E. A **lattice** is an oriented, ciliated graph. The geometric realization of an oriented graph is insufficient to support a ciliation so for a lattice we construct an oriented surface called its **envelope**. Each vertex becomes an oriented disk and each edge in O becomes an oriented band. The orientation of a fattened vertex induces an orientation on its boundary, one point of which is marked with a cilium. Attach the band corresponding to each e to the disks (or disk) at its initial and terminal ends. The attaching points along the oriented boundary of each disk must be arranged in the order given by the ciliation of the vertex. Further annotate the resulting surface by orienting the core of each band from $\mathfrak{i}(e)$ to $\mathfrak{t}(e)$.

For example, consider

$$\begin{aligned} E &= \{\pm e_1, \pm e_2, \pm e_3, \pm e_4, \pm e_5, \pm e_6\}, \\ V^c &= \{\{-e_1, e_2\}, \{-e_2, e_3\}, \{-e_3, e_1, e_4, -e_6\}, \{-e_5, -e_4\}, \{e_6, e_5\}\}, \quad \text{and} \\ O &= \{e_1, e_2, e_3, e_4, e_5, e_6\}. \end{aligned}$$

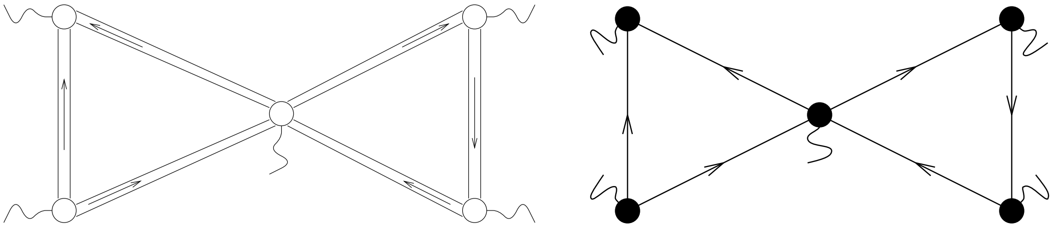


FIGURE 1. Envelope of an oriented, ciliated graph.

Ciliation is given by the order in which the elements of each vertex are written above. The envelope of $(E, -, V^c, O)$ is shown in Figure 1, alongside a streamlined schematic version.

An envelope determines its lattice. The edge set consists of a pair $\pm e$ for each band, with e assigned to the orientation. Label the initial and terminal ends of a band core by e and $-e$ respectively. Each disk forms a ciliated vertex by reading off these labels, beginning at the cilium and traveling along the induced orientation.

1.3. For this subsection we fix $(E, -, V^c, O)$. A gauge field theory on this lattice is defined by the interactions of three algebraic objects:

- A set of **connections**, $\mathbb{A} = \bigotimes_{e\in O} H$.
- A **gauge algebra**, $\mathcal{G} = \bigotimes_{v\in V} H$.
- And a set of **gauge fields**, $C[\mathbb{A}] = \bigotimes_{e\in O} B$.

The gauge algebra is a Hopf algebra in the natural sense of a tensor power of Hopf algebras, whereas $\mathbb{A}$ and $C[\mathbb{A}]$ inherit only the vector space structure of H. However, we will shortly endow the connections with a $\mathcal{G}$-action and a comultiplication, which induce dual structures on $C[\mathbb{A}]$ via the evaluation pairing.

For each $v \in V^c$ there is a function $\mathrm{ord}_v : v \to \mathbb{N}$ that assigns to $e \in v$ the ordinal number corresponding to its position in the ciliation. Connections become a left $\mathcal{G}$-module under the action

$$\otimes_{v\in V} y_v \cdot \otimes_{e\in O} x_e = \otimes_{e\in O} y_{\mathfrak{i}(e)}^{(\mathrm{ord}_{\mathfrak{i}(e)}(e))} x_e\, S\left(y_{\mathfrak{t}(e)}^{(\mathrm{ord}_{\mathfrak{t}(e)}(e))}\right).$$

This is a busy formula, even with the summations of Sweedler notation suppressed. For a graphical description of the action (and the usual method of computing it) see [8]. The gauge algebra acts adjointly on gauge fields via $(f \cdot y)(x) = f(y \cdot x)$, so $C[\mathbb{A}]$ is a right $\mathcal{G}$-module. A gauge field f is called an **observable** if for all $y \in \mathcal{G}$ we have $f \cdot y = \epsilon(y)f$. Observables form a submodule $\mathfrak{O}$ of $C[\mathbb{A}]$.

1.4. There is a construction of gauge fields which uses a direct interpretation of the restricted dual of H. Suppose that W is a finite dimensional left H-module. We use W^* to denote the dual with H acting on the left by $(x \cdot f)(v) = f(S(x) \cdot v)$. The action $(f \cdot x)(v) = f(x \cdot v)$ makes the dual into a right H-module, denoted W'.

Comultiplication supplies a left action on $W^* \otimes W$, namely $x \cdot (f \otimes v) = (x' \cdot f) \otimes (x'' \cdot v)$. Forcing the natural identification of $W^* \otimes W$ with $\mathrm{Hom}(W, W)$ to be an intertwiner makes the later into a left module as well. In Sweedler notation the action is $(x \cdot f)(v) = x'' \cdot f(S(x') \cdot v)$. Now suppose that $\rho : H \to \mathrm{Hom}(W, W)$ is the original representation. I.e., $x \cdot v = \rho(x)(v)$. The reader may check that for any $x, y \in H$, we have $y \cdot \rho(x) = \rho(\mathrm{ad}(y)x)$.

A finite dimensional representation $\rho : H \to \mathrm{Hom}(V, V)$ is said to be **adapted to** B if

$$\{h \circ \rho \mid h \in (\mathrm{Hom}(V, V))'\} \subset B.$$

A **coloring** of a lattice is a labeling of each $e \in O$ by a representation adapted to B. Let W_e denote the representation associated to e, and let $W_{-e} = W_e^*$. A coloring naturally associates the left H-module $W_v = \bigotimes_{e \in v} W_e$ to each vertex.

Given a coloring, there is a map of right $\mathcal{G}$-modules,

$$\bigotimes_{v \in V} W_v' \to \bigotimes_{e \in O} (W_{-e} \otimes W_e)' \to \bigotimes_{e \in O} (\mathrm{Hom}(W_e, W_e))' \to C[\mathbb{A}]$$

defined as follows. The first stage is just reordering of the factors with the natural distribution of primes over tensor products. The next is the canonical identification. The last is composition with $\otimes_{e \in O} \rho_e$, where the maps $\rho_e : H \to \mathrm{Hom}(V_e, V_e)$ are the actual representations of the coloring.

Theorem 1. *The images of these maps, taken over all colorings, add up to* $C[\mathbb{A}]$.

Proof. This is evident after establishing the following claim: For each $f \in B$ there is a finite dimensional H-module W so that

$$f \in \{h \circ \rho \mid h \in (\mathrm{Hom}(W, W))'\} \subset B.$$

Fix a nonzero $f \in B$. Choose I to be maximal among ideals of H contained in $\ker f$. Let $\rho : H \to \mathrm{Hom}(W, W)$ be the representation induced by left multiplication of H on $W = H/I$. Define T to be the linear span of the functionals $\{y \mapsto f(xyz) \mid x, z \in H\}$.

Since I is an ideal and it lies in the kernel of f, T may be thought of as a subspace of W^*. Choose $\{x_1, \ldots, x_n\} \subset H$ so that $x_1 = 1_H$ and so that, in the quotient, this is a basis for W. By evaluation, each x_i is a functional on T. Suppose that, as a functional on T, $X = \sum a_i x_i = 0$. For any $y, z \in H$ we have $f(yXz) = 0$. Maximality of I then implies linear independence of $\{x_i\}$ on T, which means T is all of W^*.

Choose a basis $\{f_1, \ldots, f_n\}$ for W^* that is dual to $\{x_i\}$. Let $\rho(x_j)$ be the matrix M^j in the basis $\{x_i\}$. By duality of bases, we know that every element $y \in H$ can be written as

$$y = z + \sum_i f_i(y) x_i$$

where $z \in I$. Hence

$$\rho(y) = \sum_i f_i(y) M^i.$$

The j-k entry of the matrix $\rho(y)$ is

$$\sum_i f_i(y) M^i_{jk},$$

which proves the assertion

$$\{h \circ \rho \mid h \in (\mathrm{Hom}(W, W))'\} \subset B.$$

Finally, let h_j be the function

$$y \mapsto \sum_i f_i(y) M^i_{j1}.$$

Note that $h_j(x_i) = M^i_{j1}$. The first column of M^i expresses $x_i x_1$ in the chosen basis. However, $x_1 = 1_H$, so we have $M^i_{j1} = \delta_{ji}$. Since $h_j(I) = 0$, we have shown that it agrees with f_j on all of H. This proves that $\{h \circ \rho \mid h \in (\mathrm{Hom}(W, W))'\}$ contains a spanning set for T. In particular, it contains f. □

If H is semisimple there is a way of getting an isomorphism out of the construction above. Restrict the colors to lie in an exhaustive list of irreducible representations adapted to B, so that no representation appears twice in the list. Once this has been done then the map in Theorem 1 becomes an isomorphism. This is the definition of gauge fields used in [1, 4].

Let $\mathrm{Inv}(W)$ denote the invariant part of an H-module. Since the maps described above are all intertwiners, we have the following characterization of observables.

Corollary 1. *The sum over all colorings of the images of $\bigotimes_{v \in V} \mathrm{Inv}(W'_v)$ is equal to $\mathfrak{O}$.*

2. Multitangles

The goal of this subsection is to develop a functor between two categories: the **category of multitangles**, $\mathcal{M}$, and the **category of connections**, $\mathcal{A}$. The objects of $\mathcal{M}$ are lattices, and a morphism is a set of tangles in one-to-one correspondence with the vertices of its domain. An object in $\mathcal{A}$ is the set of connections on a lattice, viewed as a left module over the gauge algebra of the lattice. The morphisms are pairs of maps, one from the connections in its domain to the connections in its range and the other between the gauge algebras. The second morphism allows us to pull back the connections on the range lattice to a module over the domain gauge algebra. The first map must intertwine this action with the standard one.

A multitangle is described by a collection of diagrams in one-to-one correspondence with the vertices of the domain lattice. Each diagram lies in a copy of $[0,1] \times [0,1]$ with the second factor determining a height function on the entire collection. Each such collection is built by stacking the elementary diagrams described below. A multitangle is an equivalence class of these collections under a relation that will be made explicit shortly.

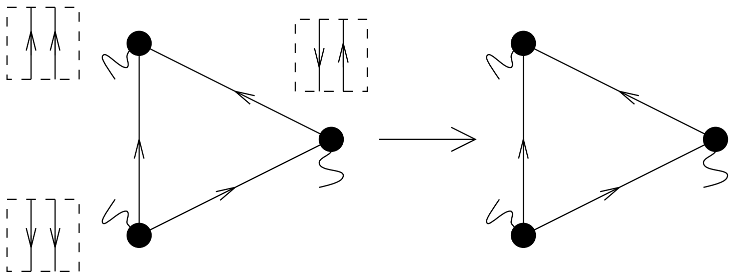


FIGURE 2. Identity multitangle.

2.1. Abstractly, a morphism in the category $\mathcal{M}$ has a specific domain and range lattice. In practice it will be given by a *picture* of a multitangle, from which one can easily determine E, O, and V^c for both range and domain, but not the involution. However, the specific details of the domain lattice are always clear from context, and the range lattice is then completely determined by the picture. We will now list several elementary morphisms in $\mathcal{M}$ by explicitly describing how a domain lattice is altered to produce the range lattice. We also describe an associated picture called an **elementary diagram**.

These are the elementary diagrams.

The identity: The domain and range are the same lattice. For each vertex v there is a diagram consisting of arcs with no crossings which are monotonic with respect to the height function. The arcs correspond, from left to right, to the cilial ordering of v. Arcs corresponding to edges in the orientation are directed downwards, others are directed upwards. Figure 2 shows an envelope and its identity morphism.

The convention for ordering and directing arcs used here is standard for all elementary diagrams. It can also be derived from the envelope of a lattice. Allow the band cores to protrude into a fat vertex and unroll it with a Möbius transformation to the upper half plane, cilium at infinity.

Crossings: The domain and range lattices differ only in the ciliation at a single vertex, v, where a pair of adjacent edges have been transposed. At each vertex other than v there is a trivial diagram as in the identity. The diagram at v has monotonic arcs and a single crossing between those arcs corresponding to the transposed edges. Either strand may pass over the other. A geometric example is given in Figure 3.

Triads: Consider an involutory pair of edges, e and $-e$, in the domain lattice with $e \in O$. In the range lattice, e is removed from its position in a ciliated vertex and replaced with e', e'' in that order. Similarly, $-e$ is replaced with $-e'', -e'$. The edges e' and e'' lie in the orientation of the new lattice, which is otherwise identical to the domain.

The diagrams have monotonic arcs without crossings corresponding to all the unchanged edges. The two arcs corresponding to e and $-e$ split at the same height into four arcs corresponding to e', e'', $-e''$ and $-e'$. In an envelope, this is the operation of doubling an edge (Figure 4).

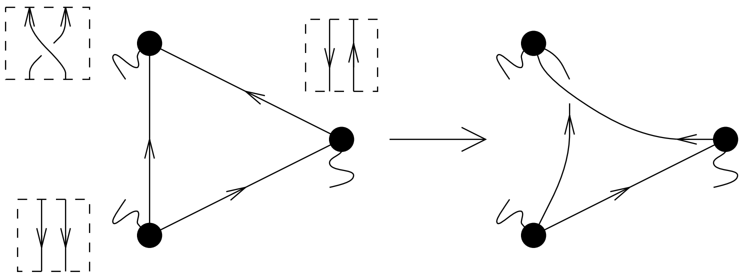


FIGURE 3. Negative crossing.

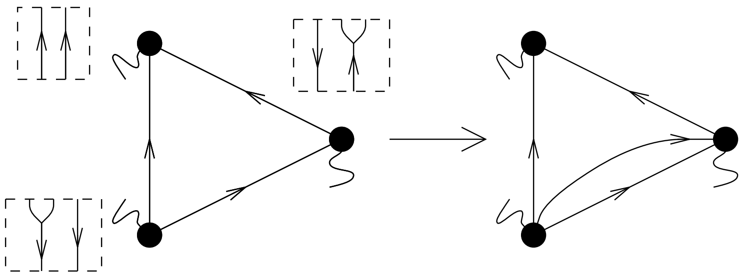


FIGURE 4. Triad.

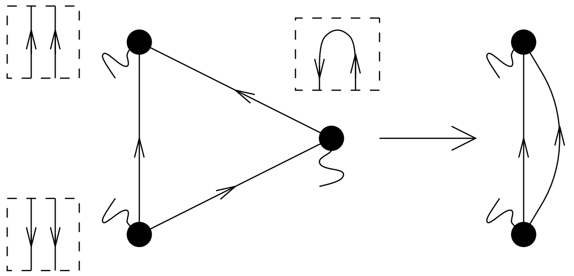


FIGURE 5. Cap joining a non-involutory pair of edges.

Caps: The edge set of the range will differ from the domain by deleting two adjacent edges from a vertex, exactly one of which lies in the orientation. If the two edges are not an involutory pair, then the two orphaned edges in the range become an involutory pair. The strands corresponding to the deleted edges meet at a local maximum. Otherwise the diagrams are trivial. The effect on envelopes is suggested in Figure 5.

Cups: The range lattice differs from the domain by introducing two new edges, e and $-e$ next to each other at a single vertex. The strands corresponding to the new edges originate in a local minimum and obey the usual directedness rule. Otherwise the diagrams are trivial. This creates a monogon at a vertex as shown in Figure 6.

Stumps: The range lattice is formed from the domain lattice by deleting an involutory pair of edges. The diagrams are trivial except for the two strands corresponding to the deleted edges, which simply terminate. Both stumps must occur at the same height. Figure 7 is an example.

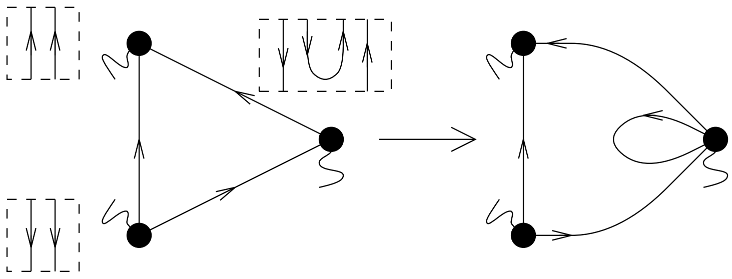


FIGURE 6. Cup creating an involutory pair at a single vertex.

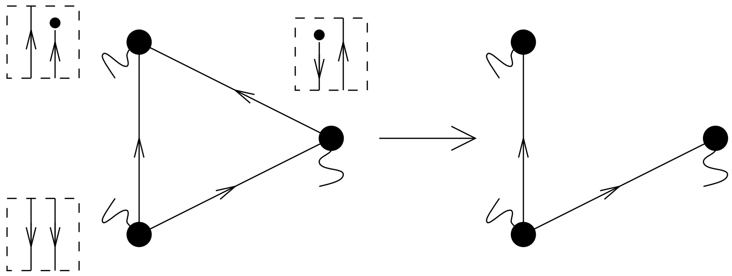


FIGURE 7. Stump eliminating an involutory pair.

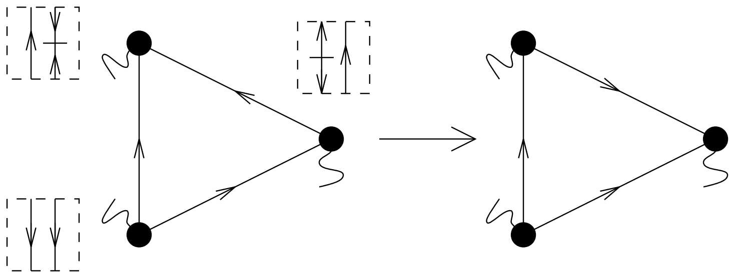


FIGURE 8. Switch exchanging an involutory pair.

Switches: The range differs from the domain by replacing some e with $-e$ in the orientation. This is indicated in the diagrams by a hash mark on each of the strands involved, with both marks lying at exactly the same height (Figure 8).

Cuts: The domain lattice is altered by dividing the ordered edges at some vertex into two non-empty consecutive sets, which form new ciliated vertices. The diagram for that vertex is trivial, except for a vertical mark at the top which indicates the cilium of the new vertex (Figure 9).

When the range lattice of one collection of diagrams matches the domain of another we may form a new set of diagrams by stacking the first two. It may be necessary to isotop the bases to get arcs to match, and if two diagrams are stacked atop a single diagram with a cut, the cut extends to the top of the new diagram. The height function is then uniformly rescaled. We define a **multidiagram** to be any such collection formed by stacking elementary diagrams.

A multidiagram is a picture of a framed embedding of a 1-dimensional CW-complex into a collection of cubes. One should *not* think of the hash marks of a switch as

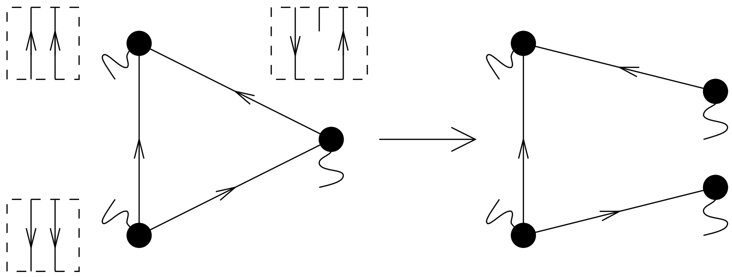


FIGURE 9. Cut splitting a vertex in two.

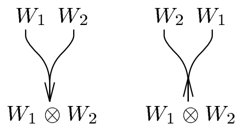


FIGURE 10. Triad coloring rule.

vertices; rather they are extra annotation on what is otherwise a smooth 1-cell. The 1-cells of this complex are called the **segments** of the multidiagram. Alternatively, segments are the components left behind if triads are removed, switches are ignored, and the arcs passing under a crossing are thought of as connected. The critical points, switches, stumps and triads of a multidiagram are collectively referred to as **events**.

A **coloring** of a multidiagram is an assignment of a finite dimensional H-module to each segment. Let $\mathfrak{b}$ denote the 1-dimensional module with H acting by its counit, and identify $W \otimes \mathfrak{b} = \mathfrak{b} \otimes W = W$. Except for critical points, one may think of a multidiagram as a 1-parameter family of lattices, Γ_t. A multidiagram is **consistently colored** if its coloring satisfies:

1. Segments creating an involutory pair of edges in any Γ_t must have the same color. (I.e., the coloring induces a coloring of Γ_t in the sense of Section 1.4).
2. Triads must obey the condition in Figure 10.
3. Any segment ending in a stump is colored by $\mathfrak{b}$.

2.2. We say that two colored multidiagrams are equivalent if one can be obtained from the other by a sequence of the following local moves.

Isotopies: We allow ambient isotopy of the diagrams subject to the following restriction: No two events sharing a segment may occur at the same height. A pair of marks indicating a stump, switch or triad must always remain at the same height. Cuts must remain vertical. Triads must maintain one segment below the horizontal and two above it. Finally, a cap may not occur at the same height as a stump, switch or triad if it shares an involutory pair of edges with the other event.

Generalized Reidemeister Moves: These are shown in Figure 11. The moves are valid regardless of the orientations of the arcs. We also allow the corresponding moves with reversed crossings.

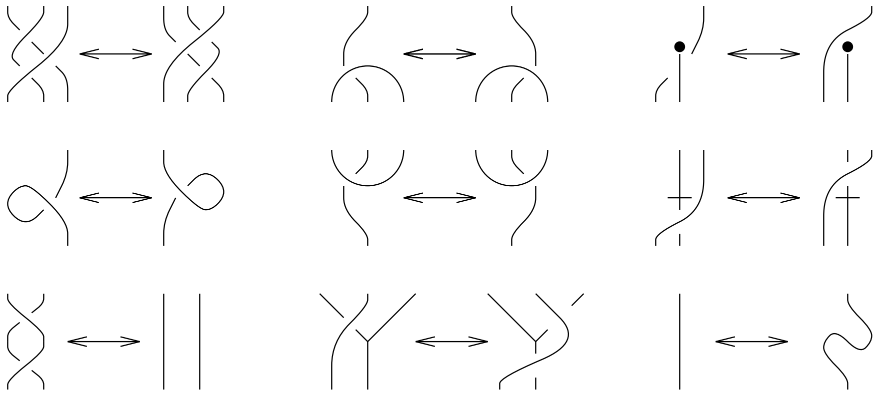


FIGURE 11. Generalized Reidemeister moves.

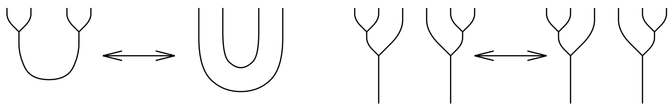


FIGURE 12. Triad moves.

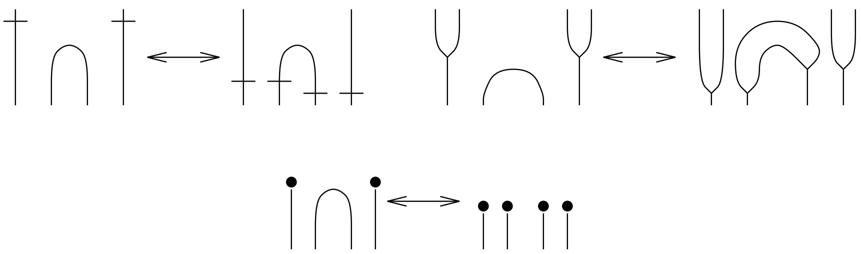


FIGURE 13. Cap moves.

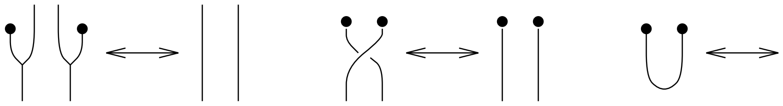


FIGURE 14. Stump moves.

Interacting Events: These moves describe the interaction of events that share either a common segment or two segments representing an involutory pair of edges. They are divided into triad moves (Figure 12), cap moves (Figure 13), stump moves (Figure 14) and switch moves (Figure 15). In each picture adjacent pairs of arcs, reading left to right along the base, represent involutory pairs of edges. Subject to that, any orientation of arcs is allowed, as are diagrams with all crossings reversed. The involutory pairs are shown as adjacent merely to conserve space; the moves are valid even for distant arcs.

Some of these moves alter the segments. In each case a consistent coloring of the multidiagram before the move induces an obvious consistent coloring of new multidiagram.

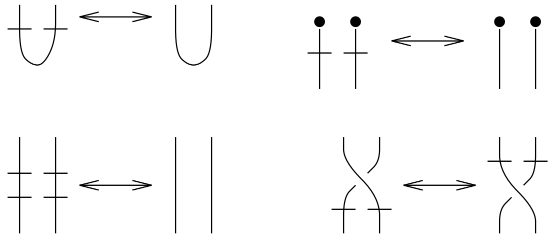


FIGURE 15. Switch moves.

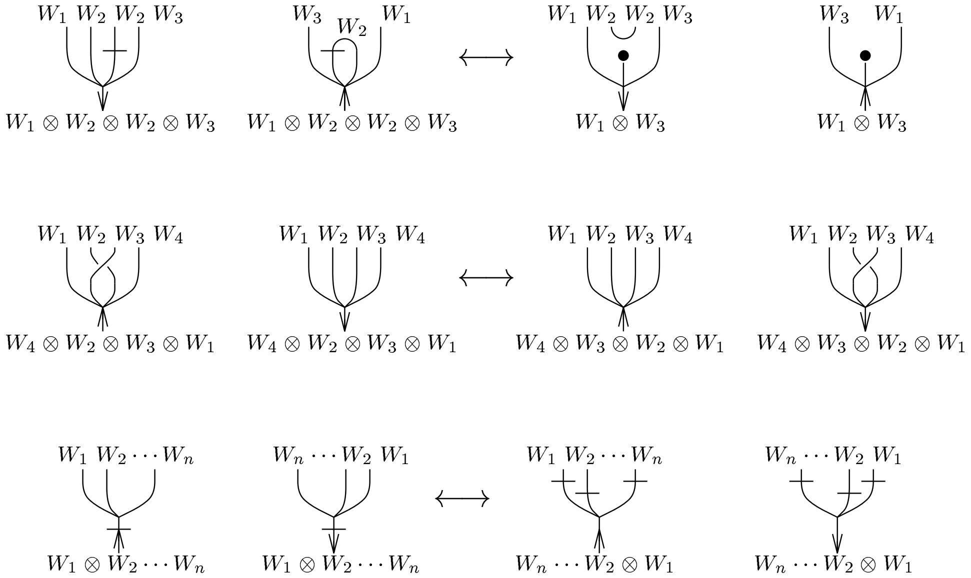


FIGURE 16. Specials

Specials: The three moves in Figure 16 all concern an involutory pair of edges entering a series of triads. In the first two moves three triads have been collapsed into a 5-valent vertex using a previously given triad move. In the third move this convention is extended to any number of triads. Note that the colors on the top of each diagram are as generic as possible and unchanged by the move. Consistency conditions then force the indicated colors on the bottom of each diagram.

The first move is allowed with the switch on the other side of the loop, as well as with opposite orientations. The second move is allowed with a negative crossing. In all cases, coloring is the same across the top and determined by consistency at the bottom.

Definition 1. *A* **multitangle** *is an equivalence class of consistently colored multidiagrams.*

There is a useful (although somewhat imprecise) topological way of understanding the equivalence of multidiagrams. Think of a multidiagram as a diagram of framed tangles in cubes. For the most part any isotopy relative to the boundary of the cubes is an equivalence, exceptions being the rigidities listed above. Since these involve

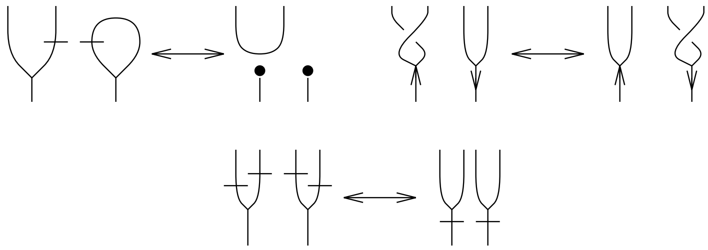


FIGURE 17. Special moves with trivial coloring.

involutory pairs, and since caps can alter the involution, it is best to be careful when isotoping an event past a cap.

A switch is a pair of marks that can slide up or down together unless obstructed by a cup, cap or triad. Any pair of switches that meets will cancel (and thus can be created), and a single switch can be canceled (or created) at a cup.

Triads, as long as they are moved about in pairs, can be manipulated as though performing edge slides on a graph provided the only other events encountered are crossings and triads. Interactions with cups are as expected, and in practice one can usually keep caps at the top of the tangle and switches at the bottom. When this is not possible, it is best to rely on the explicit moves as listed.

Stumps can be isotoped almost anywhere (only caps are a problem) so long as they never become horizontal. In particular, one can almost always shrink a segment ending in a stump to be very short, and if the other end is a triad or another stump, the segment can be eliminated.

In other words, as long as one avoids caps and keeps triads and stumps upright, any continuous deformation of a multitangle is an equivalence and, with the possible exception of the special moves, the behavior of cups, stumps, triads, and switches is fairly intuitive.

Finally, in many applications the coloring of a multidiagram is irrelevant. One may then treat every segment as carrying the color $\mathfrak{b}$ so that consistency is automatic, even for the special moves. In fact, for trivially colored multidiagrams, the special moves reduce to Figure 17.

2.3. We begin building a functor from $\mathcal{M}$ to $\mathcal{A}$ by sending a lattice to its connections and the elementary diagrams to the morphisms described below.

The identity: This diagram induces the identity on connections and on the gauge algebra.

Crossings: The map on connections is an action of the R-matrix or its inverse. There are 12 cases depending on the sign of the crossing, the directions of the arcs, and the possibility that they are an involutory pair. These are given in Figure 18, which describes the action in the factors corresponding to the crossing arcs. The map extends linearly on connections, using the identity in all other factors. The map on the gauge algebra is the identity.

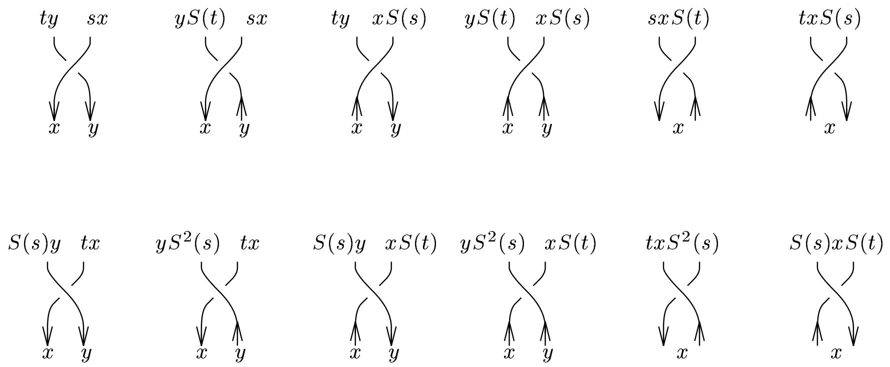


FIGURE 18. Action of R and R^{-1} at crossings.

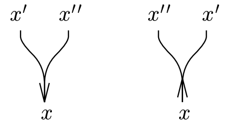


FIGURE 19. Action of a triad.

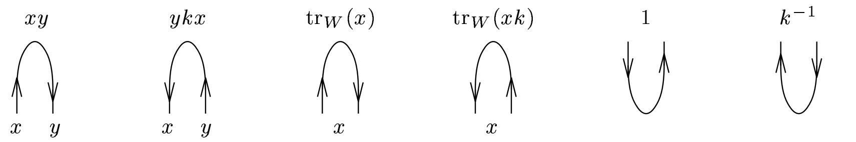


FIGURE 20. Operators corresponding to cups and caps.

Triads: Suppose that e and $-e$ are replaced by e', e'', $-e''$ and $-e'$, with $e \in O$ originally. The map on connections is comultiplication in the factor corresponding to e and the identity elsewhere, with the requirement that the image of the comultiplication take values in the tensor product of the factors corresponding to e' and e'' in that order. The distribution of the output, using Sweedler notation with summation suppressed, is illustrated in Figure 19. The diagram acts trivially on $\mathcal{G}$.

Caps: There are four cases at a local maximum, depending on orientations and on whether or not the incoming strands represent an involutory pair in the domain lattice. These are listed in Figure 20, where tr_W denotes the ordinary trace taken in the H-module W coloring that segment. If the lattice loses a vertex then the map on $\mathcal{G}$ is ϵ in that factor; otherwise it is the identity.

Cups: The action on $\mathbb{A}$ is either by the unit of H or by the unit followed by multiplication by k^{-1}, depending on the orientation of the cup. The two cases are shown in Figure 20. The map on gauge algebras is the identity.

Stumps: A stump acts as the counit in the corresponding factor of the connections. If the range lattice loses one or more vertices because of this, the map on gauge algebras is counit in those factors. Otherwise it is the identity.

Switches: For switches the map on connections is $x \mapsto S(xk)$ in the factor corresponding to the edge. The map on $\mathcal{G}$ is trivial.

Cuts: A cut has no effect on connections. It acts trivially on $\mathcal{G}$ except in the factor corresponding to the split vertex, where the map is $\Delta : H_v \to H_{v'} \otimes H_{v''}$. Here v' denotes the initial subset of v after the cut, and v'' the final subset.

A general multidiagram is a composition of elementary ones, so it is sent to the corresponding composition of maps.

Theorem 2. *Equivalent multidiagrams from Γ to Γ' induce identical maps on connections and gauge algebras. The map on connections intertwines the action of $\mathcal{G}_\Gamma$ on $\mathbb{A}_\Gamma$ with the one on $\mathbb{A}_{\Gamma'}$ pulled back via the map on gauge algebras.*

Proof. To check that equivalent multidiagrams induce the same morphisms one must evaluate both sides of each move under every possible arrangement of orientations and crossings. Number the moves in each of Figures 11–16, reading left to right and down the page. We will outline the identities and manipulations in H that make each move invariant on connections. That both sides induce the same map on gauge algebras is elementary.

Invariance of generalized Reidemeister moves:

1. This follows from the fact that R and R^{-1} solve the quantum Yang-Baxter equation.
2. Replacing any appearance of $S^2(x)$ with kxk^{-1} proves invariance in all cases.
3. This is essentially the identity $(\epsilon \otimes 1)(R) = (1 \otimes \epsilon)(R) = 1$. For some orientations the fact that $\epsilon(S(x)) = \epsilon(x)$ is also needed.
4. With the strand directed upwards the left hand side produces the following morphism, where subscripts indicate successive applications of R and implied summation.

$$\begin{aligned} x \mapsto & xS^2(s_1)kS(t_1) \\ =& xS(t_1k^{-1}S(s_1)) \\ =& xS(t_1\theta t_2S^2(s_2)S(s_1)) \\ =& xS(\theta)S(t_1t_2S(s_1S(s_2))) \\ =& \theta x \end{aligned}$$

   Similar computations show that, regardless of orientation, both sides act as multiplication by θ. If the crossings are reversed the action is by θ^{-1}.
5. This is similar to 2.
6. Those cases that are not immediate follow from an application of $S^2(x) = kxk^{-1}$.
7. $\sum_i S(s_i) \otimes t_i$ is R^{-1}.
8. Depending on crossings, use one of the identities $\Delta \otimes 1(R) = s_1 \otimes s_2 \otimes t_1t_2$ or $1 \otimes \Delta(R) = s_1s_2 \otimes t_2 \otimes t_1$.
9. Obvious.

Triad moves:

1. $\Delta(k^{-1}) = k^{-1} \otimes k^{-1}$.
2. Δ is coassociative.

Cap moves:

1. That S is an anti-algebra morphism suffices.
2. Δ is an algebra morphism.
3. ϵ is an algebra morphism and $\epsilon(k) = 1$.

Stump moves:

1. ϵ is the counit for Δ.
2. $\epsilon(k^{-1}) = 1$.
3. From earlier identities, $\epsilon \otimes \epsilon(R^{\pm 1}) = 1$

Switch moves:

1. $S(k) = k^{-1}$.
2. $\epsilon \circ S = \epsilon$ and $\epsilon(k) = 1$.
3. This is the claim that $x \mapsto S(xk)$ is an involution. It follows from $S^2(x) = kxk^{-1}$.
4. $(S \otimes S)(R) = R$.

Specials:

1. The definition of S. In Sweedler notation the identity is
$$x^{(1)} \otimes x^{(2)} = x^{(1)} \otimes S(x^{(2)})x^{(3)} \otimes x^{(4)} = x^{(1)} \otimes x^{(2)}S(x^{(3)}) \otimes x^{(4)}.$$
2. Constrained non-cocommutativity. In Sweedler notation
$$x^{(1)} \otimes sx^{(2)} \otimes tx^{(3)} \otimes x^{(4)} = x^{(1)} \otimes x^{(3)}s \otimes x^{(2)}t \otimes x^{(4)}.$$
3. S is an anti-coalgebra map and k is grouplike.

Checking that every multitangle is a $\mathcal{G}$-module intertwiner is again a matter of checking each elementary diagram under all orientations and crossings. As above, we will indicate the essential identity or manipulation on which the computation rests.

Identity: This is obvious.
Crossings: Apply the constrained non-cocommutativity identity to prove that a gauge transformation by y at a single vertex is an intertwiner. Any other gauge transformation can be expressed as sums of products of these.
Triads: Coassociativity of Δ again. The proof is trivial in Sweedler notation.
Caps: Apply the definition of S to check that gauge transformation by y at a single vertex is an intertwiner.
Cups: These work for pretty much the same reasons that caps do.
Stumps: Follows from the definition of ϵ.
Switches: S is an anti-algebra map.
Cuts: Coassociativity of Δ.

$\square$

Remark: We can think of and as single events called **positive** and **negative** **twists** respectively. It is worth remembering that a positive twist acts on a connection as multiplication by θ^{-1} in that factor. A negative twist acts by θ.

3. Comultiplication of Connections

Fix a lattice $\Gamma = (E, -, V^c, O)$. We define a multitangle whose domain is Γ by repeating the following construction at each vertex: Apply a triad to each arc. Then

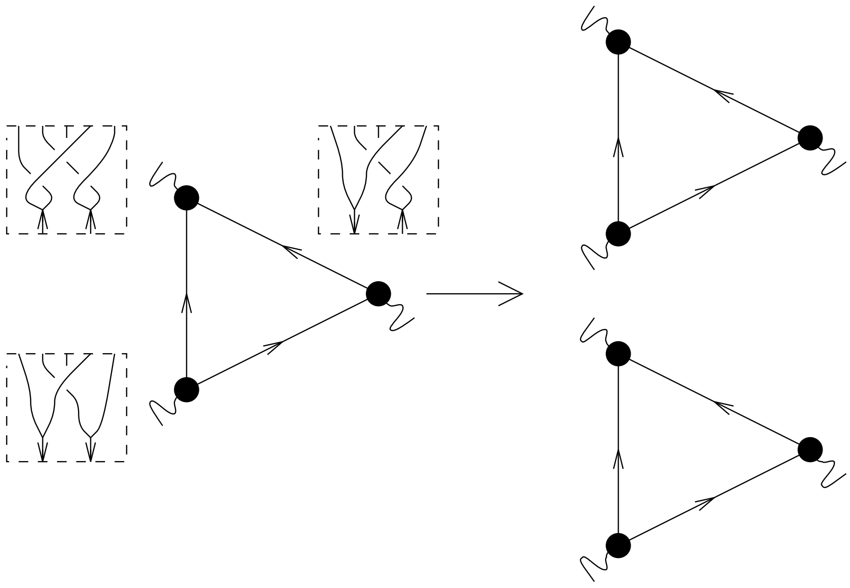


FIGURE 21. Multitangle for ∇.

move the strands corresponding to the x''s to the left of the strands corresponding to the x'''s so that the latter segments cross over. Finally, cut the diagrams to separate the x''s from the x'''s. An example is given in Figure 21.

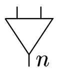
The multitangle determines a range lattice denoted $\Gamma^{\otimes 2}$. Its envelope is two disjoint copies of the envelope of Γ, but it is important to distinguish them as the **prime** and **double prime** copies. The induced morphism on connections is denoted

$$\nabla : \mathbb{A}_\Gamma \to \mathbb{A}_{\Gamma^{\otimes 2}}.$$

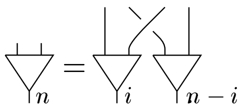
The map on gauge algebras is the standard comultiplication on a tensor power of H. We make the identification $\mathbb{A}_\Gamma \otimes \mathbb{A}_\Gamma = \mathbb{A}_{\Gamma^{\otimes 2}}$, and define $\epsilon_\Gamma = \otimes_{e \in O} \epsilon_e : \mathbb{A}_\Gamma \to \mathfrak{b}$.

Theorem 3. *The triple* $(\mathbb{A}, \nabla, \epsilon_\Gamma)$ *is a coalgebra.*

Proof. Figure 22 depicts diagrams for $(\nabla \otimes 1) \circ \nabla$ and $(1 \otimes \nabla) \circ \nabla$ at one possible trivalent vertex. To see that the left side is equivalent to the right, slide the higher triads down to the lower ones and then back up the other segments. This is possible because Δ is coassociative and because the diagrams separate into three disentangled layers. This phenomenon holds in general, and it is possible to organize this information into an inductive proof that ∇ is coassociative. We leave the details to the reader, with the suggestion that one use a coupon, say



to denote the diagram for ∇ at a generic n-valent vertex. It is also convenient that



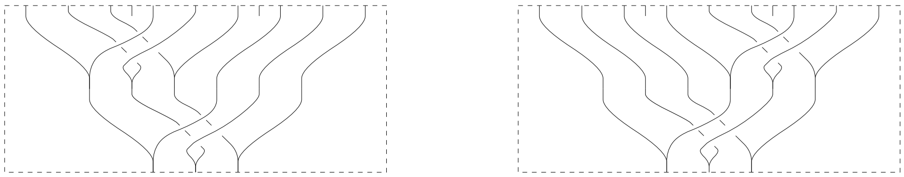


FIGURE 22. A suggestion of the coassociativity of ∇

The fact that stumps can be retracted and absorbed into triads gives a simple multitangle proof that $(\epsilon_\Gamma \otimes 1) \circ \nabla = (1 \otimes \epsilon_\Gamma) \circ \nabla = 1$. Thus ϵ_Γ is a counit for ∇. $\square$

The adjoint of ∇, restricted to observables, is denoted by $\star$: if $f, g \in \mathfrak{O}$ and $x \in \mathbb{A}$, then $(f \star g)(x) = (f \otimes g)(\nabla(x))$. Note that ϵ_Γ is itself an observable.

Corollary 2. $(\mathfrak{O}, \star, \epsilon_\Gamma)$ *is an algebra.*

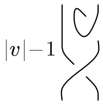
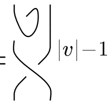
Proof. The intertwining property of morphisms induced by multitangles insures that $\star$ takes values in $\mathfrak{O}$. Linearity and associativity follow from linearity and coassociativity of ∇, and ϵ_Γ is the unit for $\star$ because it is the counit for ∇. $\square$

4. COMPUTING IN $\mathfrak{O}$

The algebras of observables for lattices that differ only by orientation or by the cyclic ordering of the ciliated vertices should be isomorphic. Furthermore, mimicking a classical phenomenon, the value of an observable on a connection should be computable from a suitable connection on the complement of a maximal tree in the graph. In the next two sections we will fix a lattice, $(E, -, V^c, O)$, and prove that these goals can be met, up to canonical isomorphisms. We will also address the interaction of multitangles with the algebra and coalgebra structures from the previous section. Throughout these two sections, all multitangles are assumed to have trivial coloring.

4.1. Given $e \in O$, let σ_e denote the map on connections induced by the multitangle which is trivial except for a switch on the strands $\pm e$. In an envelope of Γ, σ_e switches the orientation of the core of the corresponding band.

Given $v \in V$, let

$\tau_v =$ , and $\tau_v^{-1} =$ .

Here an integer next to an arc indicates that many parallel copies. The orientations are determined by the orientations of the edges at v, and the rest of the multitangle is trivial. The effect on an envelope of τ_v is to toggle the cilium at v one step counterclockwise, while τ_v^{-1} toggles it the other way.

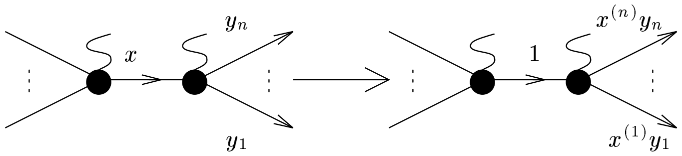
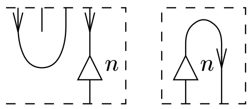


FIGURE 23. Effect of a push on a connection

Let $e \in O$ so that e and $-e$ lie in distinct vertices, e occupies the last position in the ciliation, and $-e$ occupies the first. Define

$$\pi_e =$$



where the coupon is Δ^{n-1} and the two strands entering it represent e and $-e$. The rest of the multitangle is trivial. The domain and range lattices are identical. The effect of the map on a connection is described in Figure 23, which also introduces the convention of writing a simple tensor in $\bigotimes_{e\in O} H_e$ by labeling the corresponding cores in an envelope. We call this map a **push**.

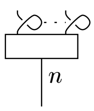
In order to avoid belaboring useless notation, we will assume that the domains of successive applications of switches, toggles and pushes are clear, provided the original domain was specified. We will also suppress the subscripts whenever possible. Any sequence of toggles, switches and pushes defines a $\mathcal{G}$-module map between connection algebras and thus an operator between observables as well.

We say that two connections are **gauge equivalent** if their difference lies in the span of $\{y \cdot x - \epsilon(y)x \mid x \in \mathbb{A}, y \in \mathcal{G}\}$. Observables cannot distinguish gauge equivalent connections. Two morphisms in $\mathcal{A}$ (with the same domain and range) are **gauge equivalent** if the images of every connection are gauge equivalent. The adjoints of a pair of gauge equivalent operators are identical maps on observables.

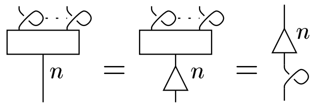
Proposition 1. *Let f be any sequence of toggles and switches that begins and ends at the same lattice. The induced operator on connections is gauge equivalent to the identity.*

Proof. It suffices to check the compositions σ^2, $\tau^{\pm}\tau^{\mp}$, $\sigma\tau^{\pm}\sigma\tau^{\mp}$, and $\tau_v^{\pm n}$, where n is the valence of v. Clearly σ is an involution. It follows easily from tangle equivalence that the next two are also the identity map.

The multitangle for τ^n is trivial away from v. At that vertex it is represented by



where the coupon denotes the generator (with positive crossings) of the center of the n-strand braid group. Note that it consists only of crossings and twists. Because such a tangle acts as multiplication in the factors corresponding to each segment, its behavior can be understood by its effect on the connection 1. We can evaluate this as follows:

$\tau^n(1) =$  $= \Delta^{n-1}(\theta^{-1}).$

Hence, the effect on an arbitrary connection is gauge action by θ^{-1} at v. The same proof with all crossings reversed shows that $\tau_v^{-|v|}$ is gauge action by θ. □

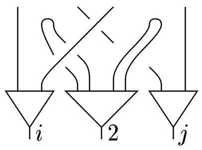
4.2. The standard tools for manipulating connections and observables are toggles, switches, pushes, triads (in succession), cups, caps (involving non-involutory pairs), stumps and cuts. We will need an understanding of how well they interact with the coalgebra and algebra structures.

We already have notation for the first three maps. Let Δ_e^n be the map induced by a succession of triads on the edges e and $-e$. We extend the notation to include Δ^0 for the identity and Δ^{-1} for a stump. A cup oriented from e_1 to e_2 (necessarily involutive) induces the map $\eta_{e_1}^{e_2}$. A cap oriented from e_1 to e_2, a non-involutive pair, induces $\mu_{e_1}^{e_2}$. A cut between e_1 and e_2 (in that order) induces $\kappa_{e_1}^{e_2}$. The notation is deliberately parallel with the defining maps of H, and when context permits we will suppress sub- and superscripts.

Theorem 4. *The maps σ, Δ^n, η and μ are coalgebra morphisms.*

Proof. A switch slides up through the multitangle for $\nabla \circ \sigma$ and, after a special move, becomes a pair of switches on the appropriate edges. Now apply the special move corresponding to $R\Delta = \Delta^{op}R$ to obtain a tangle for $(\sigma \otimes \sigma) \circ \nabla$.

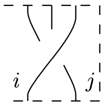
Consider the multitangle for $(\mu \otimes \mu) \circ \nabla$. At the vertex where the cap occurs we can expand this as



Since the caps do not involve involutory pairs, there is a cap move that makes this into a tangle for $\nabla \circ \mu$. The proof that $(\eta \otimes \eta) \circ \nabla = \nabla \circ \eta$ is similar.

If the lattice has only two edges, the proof that $(\Delta \otimes \Delta) \circ \nabla = \nabla \circ \Delta$ is just Figure 24. If there are more edges a similar proof works because of the layering phenomenon seen in the proof of Theorem 3. Higher powers of Δ follow immediately; Δ^0 is trivial; and Δ^{-1} comes from retracting stumps into triads. □

In many applications the output of a multitangle is needed only up to gauge equivalence. There is a particular occurrence which can greatly simplify computations. Let ϕ be the operator induced by a multitangle that is trivial except for

 .

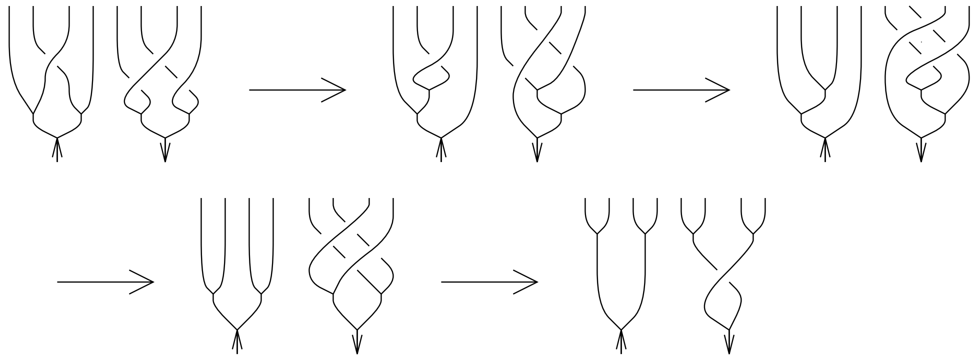


FIGURE 24. Triad commuting with ∇ at a valence two vertex.

The difference between the range and domain lattices is just that a subset of consecutive edges in some vertex has been split off to form a new one. The identity map on connections is an operator with the same domain and range as ϕ. However, it cannot be expressed via the trivial multitangle because the set of edges split off to make the new vertex is not at the end of the original ciliation.

Proposition 2. *With notation as above, ϕ is gauge equivalent to the identity.*

Proof. Given a connection x, we can express $\phi(x)$ as an action of $\phi(1)$ on x, as in the proof of Proposition 1. In the interest of computing $\phi(1)$, replace the i strands of the tangle with a single strand followed by Δ^{i-1}. Tangle equivalence allows the triads to slide over the crossing, after which we find that the action is by $\Delta^{i-1}(s_j \cdots s_2 s_1)$ in the i strands and by $t_1 \otimes t_2 \otimes \cdots \otimes t_j$ in the j strands.

Now consider the effect of ϕ on $x_1 \otimes \cdots \otimes x_j \otimes y_1 \otimes \cdots \otimes y_i$ in the factors corresponding to the non-trivial part of the tangle. The result is

$$t_1 \cdot x_1 \otimes \cdots \otimes t_j \cdot x_j \otimes \Delta^{i-1}(s_j \cdots s_1) \cdot (y_1 \otimes \cdots \otimes y_i),$$

where $t_k \cdot x_k$ means $t_k y_k$ or $y_k S(t_k)$, depending on orientation of the segment, and similarly for $\Delta^{i-1}(s_j \cdots s_1) \cdot (y_1 \otimes \cdots \otimes y_i)$. By definition, this is gauge action of $1 \otimes (s_j \cdots s_1)$ on the connection

$$t_1 \cdot x_1 \otimes \cdots \otimes t_j \cdot x_j \otimes y_1 \otimes \cdots \otimes y_i,$$

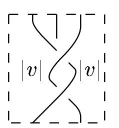
so the result is gauge equivalent to

$$\epsilon(s_j \cdots s_1) t_1 \cdot x_1 \otimes \cdots \otimes t_j \cdot x_j \otimes y_1 \otimes \cdots \otimes y_i.$$

Since $(\epsilon \otimes 1)(R) = 1$, this is indistinguishable from the behavior of the identity map. $\square$

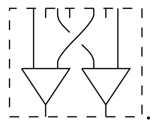
Theorem 5. *Up to gauge equivalence, $\tau^{\pm}$, κ and π are coalgebra maps.*

Proof. Draw the tangle for $(\tau_v \otimes \tau_v) \circ \nabla$. By Proposition 2 we can replace the cut with



The result is equivalent to the tangle for $\nabla \circ \tau_v$. Commutation of τ^{-1} derives from its gauge equivalence with some power of τ.

For $(\kappa \otimes \kappa) \circ \nabla$ the tangle at the cut vertex is



By Proposition 2, this is gauge equivalent to $\nabla \circ \kappa$. Since π is composed of triads, caps, a cut and a cup, it too commutes up to gauge equivalence. □

Corollary 3. *The adjoints of $\tau^{\pm}$, σ, π, Δ^n, η, μ and κ restricted to observables are algebra maps.*

Since τ and σ are invertible algebra morphisms, we can now see that $\mathfrak{O}$ is independent of orientation, up to isomorphism, and that it depends only on the cyclic ordering of edges at vertices.

5. Pushes

The goal of this section is to prove that a push is invisible to any observable. Let's begin with a basic fact about invariant tensors. Suppose that U and W are left H-modules. As in Subsection 1.4, $(U^* \otimes W)'$ is a right module. This can be identified with $\mathrm{Hom}(W, U)$ as follows: if $\phi \in U^*$, $w \in W$, then $h \in \mathrm{Hom}(W, U)$ becomes the functional sending $\phi \otimes w$ to $\phi(h(w))$. This isomorphism is an intertwiner if we use the following action: for $z \in H$ and $h \in \mathrm{Hom}(W, U)$, $(h \cdot z)(w) = S(z') \cdot h(z'' \cdot w)$.

Lemma 1. *If h is invariant then, for all $z \in H$ and all $w \in W$, $h(z \cdot w) = z \cdot h(w)$.*

Proof. Let $y = S^{-1}(z)$. Using Sweedler notation and invariance of h under the action of y', we have

$$\begin{aligned} h(z \cdot w) &= h(S(y) \cdot w) = h(\epsilon(y')S(y'') \cdot w) \\ &= \epsilon(y')h(S(y'') \cdot w) = S(y') \cdot h(y'' \cdot S(y''') \cdot w) \\ &= S(y') \cdot h(\epsilon(y'')w) = S(y) \cdot h(w) \\ &= z \cdot h(w). \end{aligned}$$

□

Now choose a coloring of the lattice with notation ρ_e, W_e, and W_v as in Subsection 1.4. Suppose that our lattice contains the configuration in Figure 23 and that the vertex on the right is $v_0 = \{-e_0, e_1, e_2, \dots, e_n\}$ (with each $e_i \in O$). Choose a gauge field of the form $f \otimes g$, where $f \in W'_{v_0}$ and $g \in \bigotimes_{v \neq v_0} W'_v$, and write the connection depicted in Figure 23 as $z \otimes x \otimes y_1 \otimes \cdots \otimes y_n$.

The usual method of evaluation is to convert the gauge field and the connection into tensors in $\bigotimes_{e \in E} W'_e$ and $\bigotimes_{e \in E} W_e$, then tensor them together and contract. (Meaning evaluate the functionals in the W'_e's on the vectors in corresponding W_e's.) Since these contractions can take place in any order, we can focus on just those taking place between W_e and W'_e for $e \in v_0$. To see the invariance of a push, however, we need to think of these contractions as a composition of morphisms.

Let W denote $W_{e_1} \otimes \cdots \otimes W_{e_n}$ and let $y = y_1 \otimes \cdots \otimes y_n$. Apply ρ_e's so that $x \in \mathrm{Hom}(W_{e_0}, W_{e_0})$ and $y \in \mathrm{Hom}(W, W)$. We can now view f, and hence $x \circ f \circ y$, as elements of $\mathrm{Hom}(W, W_{e_0})$. Using standard identifications, this becomes a tensor in $W_{e_0} \otimes W^*_{e_1} \otimes \cdots \otimes W^*_{e_n}$. The evaluation of the full gauge field is now completed by contracting g with $\alpha \otimes (x \circ f \circ y)$.

Lemma 2. *With notation as above, if f is invariant then $x \circ f \circ y = 1 \circ f \circ (x^{(1)} y_1 \otimes \cdots \otimes x^{(n)} y_n)$.*

Proof. Choose $w \in W$. Reinterpret $(x \circ f \circ y)(w)$ as $x \in H$ acting on $f(y(w))$. Then apply Lemma 1. □

In light of Corollary 1, we have established the following result:

Theorem 6. *The adjoint of a push is the identity on observables.*

Corollary 4. *Every sequence of toggles, switches and pushes from one lattice to another induces the same isomorphism on observables.*

Proof. Since pushes induce the identity, is suffices to prove this for just toggles and switches. By Proposition 1, any two sequences of toggles and switches are gauge equivalent. □

We have succeed in proving that $\mathfrak{O}$ is independent of orientation and that it depends only on cyclic ordering at vertices. Also, corollary 4 indicates that we can evaluate observables on whatever configuration is most favorable for the application at hand. There is one more task. We want to construct (as nearly as possible) a universal description of observables.

6. Quantum holonomy

Fix $\Gamma = (E, -, V^c, O)$, with envelope F. We will be quantizing the notion of holonomy along a loop in the lattice. In the classical world, a loop in Γ would be just that, the image of an oriented S^1 with base point. However, in order to quantize, we need the image to be generic and to introduce over- and under-crossings. That is, we need a knot diagram, not just a loop. Also, the base point introduces some technicalities.

We will refer to the disks in F representing the elements of V as vertices and the bands as edges. Let α be a proper, generic immersion from a disjoint, finite collection of oriented intervals into F, so that endpoints map to cilia, double points lie in vertices, and the image in each edge consists of arcs parallel to the core. Introduce over- and under-crossings at each interior double point. The resulting object is called a **q-net**. The special case when the domain is a single interval is called a **q-path** if the endpoints are distinct, and a **q-loop** if not.

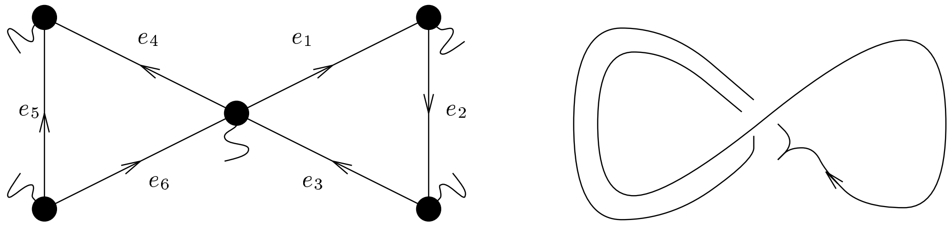


FIGURE 25. Envelope of Γ and a q-loop, α.

6.1. A q-net, α, determines an operator, hol_α, from $\mathbb{A}$ to a tensor power of H indexed by the components of α. It is defined as the composition of a pair of multitangles connecting a trio of lattices. The first lattice is Γ. The last is determined by the multitangles. The intermediate one comes from α, and it is best defined by constructing its envelope as a subsurface of F. Its fattened vertices are the vertices of F that α meets. Its fattened edges are obtained by thickening the arcs of α in the edges of F. Their cores are oriented according to the orientation of α, and the ciliation is inherited from F.

The multitangle connecting the first two lattices is formed as follows: Apply Δ^{m-1} to each pair $\pm e$, where m is the number of times γ meets the corresponding edge of F. The range of this morphism is a lattice identical to the intermediate one, described above, except possibly for orientation. Continue the multitangle by inserting switches whenever the orientations disagree. Color with trivial representations.

The second multitangle is formed from the oriented tangles created by α in each vertex of F. Given a vertex, apply a Möbius transformation mapping it to the upper half plane with the cilium at infinity. Choose a rectangular region that contains all of the image of α except for disjoint arcs running from the top edge to infinity. Rescale this to a unit square and color trivially. The composition of the two induces the operator

$$hol_\alpha : \mathbb{A} \to \bigotimes_{c \text{ a component of } \alpha} H_c.$$

One may think of envelope of the final lattice as the domain of α. Since each vertex is univalent, ciliation is irrelevant. Thus hol_α is an intertwiner from connections on Γ to connections on the domain of α.

6.2. The notation hol_α is meant to be read, "holonomy along alpha." To see it as a quantum analog of holonomy, and to illustrate some computational devices, we will work an example where Γ and α are as in Figure 25.

Consider the connection $x = \bigotimes_{e_i \in O} x_i$. Since there is no prescribed ordering to the factors, we express x by labeling each edge of Γ with the corresponding x_i as in Figure 26.

The evaluation of $hol_\alpha(x)$ begins with comultiplication applied to each of x_4, x_5 and x_6, followed by switches on the output of $\Delta(x_5)$. The resulting connection lives over the intermediate lattice described in the definition of hol_α. The envelope and connection together appear in Figure 27.

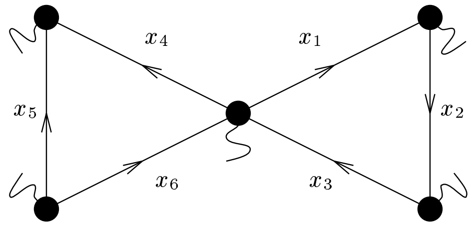


FIGURE 26. Connection $x = \otimes_{i=1}^{6} x_i$.

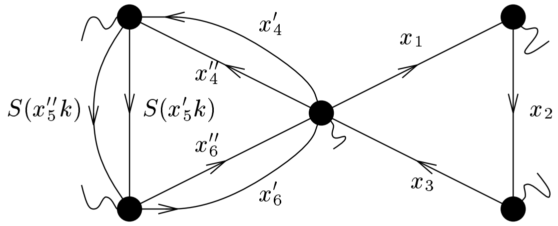


FIGURE 27. Intermediate envelope and connection for $hol_\alpha(x)$.

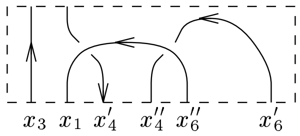


FIGURE 28. Central vertex tangle.

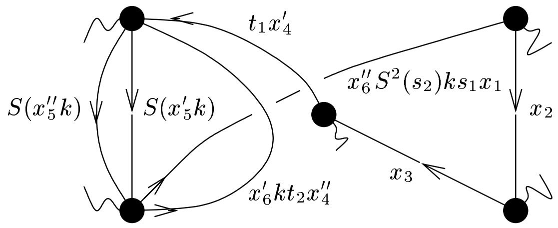


FIGURE 29. Evaluation of the central vertex tangle.

Next we evaluate the multitangle obtained by expanding the vertex tangles. Let us assume that the multitangle has been isotoped so that evaluation proceeds one vertex at a time. The tangle for the center vertex and the connection in its domain are shown in Figure 28.

Evaluation leads to the connection and envelope shown in Figure 29. Note that the two caps give rise to a a lattice with (abstractly) four fewer edges and a new involution. This is reflected in the envelope by the concatenation of pairs of edges into single edges.

The remaining vertex tangles are caps, none of which involve an involutory pair of strands. Each evaluation glues two edges of the envelope together at a vertex, creating a single edge that bypasses the vertex. The effect on the connection is multiplication in those two factors, with the cap's orientation determining whether or not k is inserted

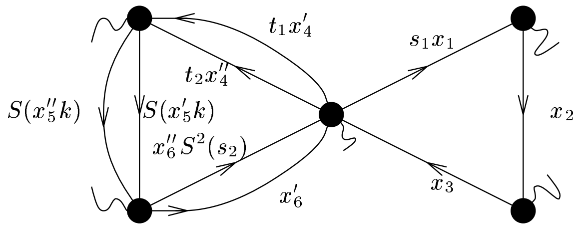


FIGURE 30. Schematic computation of $hol_\alpha(x)$.

between them. The final result is

$$hol_\alpha(x) = t_1x_4'kS(x_5''k)kx_6'kt_2x_4''kS(x_5'k)kx_6''S^2(s_2)ks_1x_1x_2x_3.$$

There is an alternative, more intuitive, method of computing holonomy. Begin with the connection drawn on the envelope as in Figure 26. Then draw the intermediate envelope. Label new edges with Sweedler notation, reading right to left with respect to the *original* orientation. If any edge is not traversed by the loop, remember to counit that factor of the connection. Apply $S(\,\cdot\,k)$ whenever the loop's orientation disagrees with that of the envelope, and the result is again Figure 27.

Next sketch each vertex tangle in which crossings occur. Evaluate only the crossings, writing the result on the intermediate lattice. In our example this gives Figure 30. Finally, traverse the q-loop from beginning to end, concatenating the symbols on each edge traversed. Additionally, each time the loop passes through a vertex insert k if the cilium lies to the right.

In the classical limit $k = 1$ and $R = 1 \otimes 1$, so hol_α is actually accumulating holonomy as the loop is traversed. Quantization occurs when self intersections become over- or under-crossings. For another indication that $hol_\alpha(x)$ behaves like holonomy, suppose that $z_{(1)} \otimes z_{(2)}$ denotes the tensor $sx_1x_2x_3 \otimes tx_4kS(x_5k)kx_6k$ with summation on both sides implied. The two factors resemble quantum holonomy around the two lobes of Γ, and $hol_\alpha(x) = z_{(2)}'z_{(2)}''z_{(1)}$.

In a classical setting holonomy is inverted if the direction of the path is reversed. In a cocommutative Hopf algebra inversion is replaced by the antipode. In general, however, S is not an involution. So, we have the following quantum analog of the reversing result for holonomy.

Proposition 3. *Let α be a q-net and $\overline{\alpha}$ the q-net with a component c reversed. The operator $hol_{\overline{\alpha}}$ is hol_α followed by a switch in the factor indexed by c.*

Proof. The map hol_α is defined by composing two multitangles: T_1, consisting of stumps and triads followed by some switches; and T_2 formed from the vertex tangles. If a switch is inserted between T_1 and T_2 on every edge coming from c, the resulting multitangle defines $hol_{\overline{\alpha}}$. The new switches pass upwards through T_2, canceling at caps, until only one remains. This will lie on a pair of strands corresponding to the component c in the range of the multitangle $T_2 \circ T_1$. □

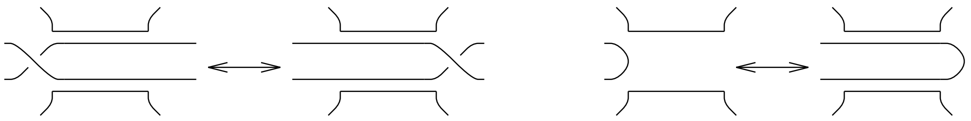


FIGURE 31. Slides of a q-loop.

7. WILSON OPERATORS

There is an equivalence relation on q-nets which, like the one on multidiagrams, mimics framed tangle isotopy. A **base point free** q-net is the result of smoothing the cusps at the base points of some (or all) the loops of a q-net. Two base point free q-nets are **equivalent** if they differ by a sequence of isotopies of F, Reidemeister moves of type II and III, and the framing equivalence = . If one thinks of a base point free q-net as a diagram of a framed tangle in $F \times I$, then this equivalence is ambient isotopy fixing the endpoints of the based components and their framing normals.

Let L be a base point free q-net with every closed component colored by a finite dimensional H-module and all other components colored by $\mathfrak{b}$. As with quantum holonomy, L determines an operator that is obtained from the composition of the multitangles, $T_2 \circ T_1$. This time, however, the segments of T_2 inherit the colors of the components of L. This coloring extends to T_1 via consistency conditions. The resulting color on an edge in the domain of T_1 can also be obtained as the right-to-left tensor product of the colors on arcs of L along that edge. The induced operator is denoted W_L. It takes values in a tensor power of H indexed by the arc components. Alternatively, one can think of the domain of L as a set of arcs and circles made into a lattice by ignoring the circles. The target of W_L is connections on the domain of L.

Proposition 4. *If L and L' are equivalent base point free q-nets with the same coloring then $W_L = W_{L'}$.*

Proof. Cerf theory implies L and L' differ by the moves of q-net equivalence, performed within vertices, together with the moves in Figure 31. The latter alter multitangles by special moves, and the former by generalized Reidemeister moves. $\square$

As is usual in knot theory, we use the L to denote both a base point free q-net (colored) and its equivalence class. The associated operator is called a **Wilson operator**. In the special cases when L consists entirely of base point free components, a single closed component, or a single based component, the operators are called, respectively, a Wilson **link**, **loop**, or **line**. In computing the output of a Wilson operator, we can use the same shortcut for caps as in 6.2. The starting point on a closed component does not matter because trace is invariant under cyclic permutation.

Wilson operators obey a reversing result: if the reversed component is an arc the effect is as in Proposition 3, but if it is closed its statement requires more care. Suppose that L and $\overline{L}$ differ by reversing a closeded component c colored by V. Let $\mathbb{A}_L$ denote the connections where W_L and $W_{\overline{L}}$ take values. Define a map $f : \mathbb{A} \to H_c \otimes \mathbb{A}_L$ as follows: Connect c to a base point to obtain a q-tangle $L_\bullet$. If the cilium at which c

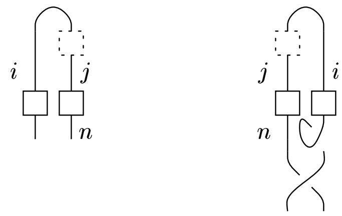


FIGURE 32. Tangle diagrams for W_L and $W_L \circ \tau_v$.

is based lies to its right, then f is $W_{L_\bullet}$ followed by right multiplication by k in the factor indexed by c. If it lies to the left, then $f = W_{L_\bullet}$. Finally, define functions tr_c and S_c on $H_c \otimes \mathbb{A}_L$ as $\mathrm{tr}_V \otimes 1$ and $S \otimes 1$ respectively.

Proposition 5. *With notation as above,* $W_L = \mathrm{tr}_c \circ f$ *and* $W_{\overline{L}} = \mathrm{tr}_c \circ S_c \circ f$

Remark: This is a bit easier to swallow if $L = c$. In that case the Wilson loop is computed by taking the trace of something, and the reversed loop is computed by taking the trace S of that something.

Proof. Repeat the proof of Proposition 3 up to the point where all but one of the new switches have been canceled. This time the switch will be on two involutory strands entering a cap colored by V. The multitangles for W_L and $W_{\overline{L}}$ differ by replacing a cap with a switch followed by a reversed cap. Since the rest of the multitangle induces $W_{L_\bullet}$, it is easy to verify the proposition for the two possible orientations of the cap. □

Since multitangles intertwine gauge transformations, Wilson links colored by adapted representations are necessarily elements of $\mathfrak{O}$. Furthermore, a Wilson link is determined by a geometric object associated to an envelope. While toggles and switches have no effect on the underlying q-tangle, the alterations to the lattice will effect the evaluation of the Wilson link. Fortunately the variance is natural.

Theorem 7. *Suppose that* Γ *and* Γ' *differ by toggles and switches and that* f *is the induced map on connections. If* W_L *and* W_L' *are Wilson links built on the same* L *in the two envelopes, then* $W_{L'} \circ f = W_L$.

Proof. Suppose Γ and Γ' differ by a switch σ. In the multitangle for $W_{L'} \circ \sigma$, pass the first switch up through the triads (if any) on that edge and cancel any resulting consecutive switches. The result is a multitangle for W_L. Assume now that the lattices differ by a counterclockwise toggle, τ. That part of the multitangle for W_L is represented by the left side of Figure 32, where the solid coupons are the switches and powers of Δ and the dashed coupon contains all the crossings of the transformed vertex. The portion of the multitangle for $W_{L'} \circ \tau$ at that vertex is now the right side of the figure. Since these are equivalent diagrams, $W_{L'} \circ \tau = W_L$. □

Corollary 5. *Let* L *be a colored, base point free q-net with no arc components. The unique isomorphism between* $\mathfrak{O}_\Gamma$ *and* $\mathfrak{O}_{\Gamma'}$ *identifies the Wilson links determined by* L *in each algebra.*

Observables produced by Wilson links can be graphically multiplied. Suppose that L and L' are equivalence classes of colored, entirely base point free q-nets. Laying L' over L and perturbing the result creates a new q-net denoted $L \star L'$. There are several ways to do this, but all are equivalent and the result is independent of the representatives used.

Theorem 8. $W_L \star W_{L'} = W_{L\star L'}$.

Proof. Let F denote the envelope of the lattice. If w_t is a tangent vector along a band core, the right hand side of the edge is determined by a normal vector w_n so that $\{w_n, w_t\}$ is in the orientation of F. By equivalence of q-nets, we may assume that L lies to the right of L' in every edge of F. At each vertex, the multitangle for $W_{L\star L'}$ begins with an arc for each edge of F, oriented accordingly. Apply a triad to each of these arcs so that it splits into a prime branch and a double prime branch. If L meets a given edge m times, apply Δ^{m-1} to the prime branch for that edge. If L' meets it n times, apply Δ^{n-1} to the double prime branch. Now insert the necessary switches to make this the first half of the multitangle.

Next, focus on the second part of the multitangle at a single vertex v of F. The transformed vertex fits onto the first part of the multitangle so that the arcs of $L \cap v$ meet the prime branches and the arcs of $L' \cap v$ meet the double primes. Since L lies under L' one can drag $L \cap v$ to the left and $L' \cap v$ to the right until they are disjoint. Finally, move everything between the initial triad on each arc and $L \cap v$ to just beneath $L \cap v$, and similarly for $L' \cap v$. This can be done for every vertex while preserving the simultaneous levels of triads and switches. Inserting a cut in between $L \cap v$ and $L' \cap v$ in each diagram, we have a multitangle that represents $(W_L \otimes W_{L'}) \circ \nabla$. $\square$

Let $\mathfrak{b}\mathcal{L}_F$ denote the linear space over the set of all framed, oriented, colored links in $F \times I$ (completed if $\mathfrak{b} = \mathbb{C}[[h]]$). This is an algebra under $\star$ which—for every Γ whose envelope is homeomorphic to F—is identified with a sub-algebra $\mathcal{W}_\Gamma \subset \mathfrak{O}_\Gamma$. Corollary 4 makes these algebras into a category. The content of Corollary 5 is that $\mathfrak{b}\mathcal{L}_F$ behaves somewhat like a universal object. One of the aims of the next section is to address how much of $\mathfrak{O}$ is generated by Wilson links.

Part 2. The Structure of the Algebra of Observables

In this part of the paper we investigate the structure of the algebra of observables for various choices of H and B. Suppose first that G is a connected affine algebraic group. We will show that the ring of G-characters of a free group is the algebra of observables for a theory in which H is the universal enveloping algebra of G and B is its coordinate ring. Next we consider the case when H is a Drinfeld-Jimbo deformation of the universal enveloping algebra. Here the observables become a deformation quantization of the classical character ring. Following Fock and Rosly, [10], we show that the Poisson structure with respect to which we are quantizing is the standard Poisson structure, as in the work of [2, 11]. We consider the extent to which one can generate the ring of observables using Wilson loops. Among the groups (and their quantum analogs) for which this is possible is $SL_2(\mathbb{C})$. We conclude with a demonstration that the observables based on $(U_h(sl_2),\ {}_qSL_2)$ are exactly the Kauffman bracket skein algebra of a cylinder over the envelope of the lattice.

8. Classical LGFT

8.1. Let's recap a few facts from [12]. An affine algebraic group G is a group equipped with a finitely generated algebra of "polynomial" functions that separate points—the coordinate ring, denoted B—so that multiplication and inversion are polynomial maps. Let H be the universal enveloping algebra of G. If the group is connected then its coordinate ring is naturally identified with a stable subalgebra of H^o. G acts on itself by conjugation, which adjointly induces an action of G on B. The most important fact for us is that the fixed part of B under this action is exactly the ring B^H as defined in Subsection 1.1.

Now suppose that π is any finitely presented group. A representation of π into G is a point in $G \times \cdots \times G$ whose coordinates are the images of the generators. These must satisfy polynomial identities corresponding to the relations of the presentation, so the space of all representations is an affine algebraic set. It has a coordinate ring on which G acts by the adjoint of conjugation in each factor. The fixed subring—which is independent of presentation (up to isomorphism)—is called the **affine G-characters of π**. We often shorten this to "characters" or "G-characters" when π or G (or both) are understood from context. The notation is $\mathcal{X}_G(\pi)$. If π is the fundamental group of a compact manifold M, we write $\mathcal{X}_G(M)$ for $\mathcal{X}_G(\pi_1(M))$.

8.2. Fix a connected affine algebraic group G and a lattice $\Gamma = (E, -, V^c, O)$. Let H be the universal enveloping algebra of G and B its coordinate ring, thought of as lying in H^o. In order to see what the observables of this theory look like, we rebuild it using groups instead of algebras.

Let the connections be the set of functions $A : O \to G$. The gauge group is the set of functions $g : V \to G$, and the gauge fields are the set $\otimes_{e \in O} B$. We can view the connections as the Cartesian product of copies of G indexed by O. The gauge fields are just the coordinate ring of the Cartesian product. The action of the gauge group on the connections is given by

$$g \bullet A(e) = g(\mathfrak{i}(e))A(e)g^{-1}(\mathfrak{t}(e)),$$

where $e \in O$, g is an element of the gauge group, and A is a connection. This induces a right action of the gauge group on the gauge fields by taking adjoints. Let $\mathfrak{O}_G$ denote the gauge fields fixed by this action.

Theorem 9. *Let G, H, B, $\Gamma = (E, -, O, V^c)$ and $\mathfrak{O}_G$ be as above, and let $\mathfrak{O}$ denote the usual observables.*

1. $\mathfrak{O} = \mathfrak{O}_G$.
2. *$\mathfrak{O}_G$ is the G-characters of π_1 of the geometric realization of $(E, -, V, O)$.*

Proof.

1. Note that the gauge group and the gauge algebra act on the same set of gauge fields. A gauge field is invariant under the action of the gauge algebra if and only if it is invariant under elements of the form $1\otimes\cdots\otimes y\otimes\cdots\otimes 1$, where y lies in the Lie algebra of G. The proof that these fields are exactly those fixed by the gauge group action is now a simple generalization of [12, Corollary IV.3.2].
2. If the lattice has one vertex containing all the edges then $\mathfrak{O}_G$ is, by definition, the G-characters of the fundamental group of the geometric realization. For arbitrary Γ consider another lattice Γ' that has a single vertex, and so that the two geometric realizations are homotopic. Choose a maximal tree in the geometric realization of Γ. There is an obvious map from the connections on Γ' into those on Γ that sends them to the edges of O not appearing in the tree. That this is an isomorphism at the level of observables is an elementary exercise.

□

9. Quantized Characters

For this subsection we assume that (H, B) is defined over $\mathbb{C}[[h]]$, and H is a Drinfeld-Jimbo quantization of a simple Lie algebra $\mathfrak{g}$ [13]. In this case, H/hH is the universal enveloping algebra of $\mathfrak{g}$. We will denote such H by $U_h(\mathfrak{g})$.

Suppose that B/hB is the coordinate ring of a connected affine algebraic group G. The equivalence of the representation theory of $U_h(\mathfrak{g})$ and $U(\mathfrak{g})$ makes it easy to see that $\mathfrak{O}/h\mathfrak{O}$ is the observables associated to the theory based on $(U(\mathfrak{g}), B/hB)$. We use topological tensor products as in [13], so the gauge fields are topologically free. Since observables form a closed subalgebra, it is also topologically free. Therefore the observables based on $(U_h(\mathfrak{g}), B)$ are a deformation quantization of the G-characters of the fundamental group of the geometric realization of Γ.

The only question that remains unanswered is which Poisson structure on the ring of characters is the tangent vector to this deformation. We can assume that $U_h(\mathfrak{g})$ has an R-matrix of the form $1\otimes 1 + h\mathbf{r} + h^2\mathbf{a}$, where $\mathbf{a}$ is some formal power series with coefficients in $U(\mathfrak{g})\otimes U(\mathfrak{g})$, and $\mathbf{r}$ is a solution of the modified classical Yang-Baxter equation. In specific, $\mathbf{r}\in\mathfrak{g}\otimes\mathfrak{g}$. Such solutions were classified by Belavin and Drinfeld [14].

The standard presentation of $U(\mathfrak{g})$ is in terms of generators X_i, Y_i, H_i, where i runs over some index set, and each triple generates a copy of $U(sl_2)$. The Killing form is an element of $\mathfrak{g}^*\otimes\mathfrak{g}^*$, but being nondegenerate, you can contract it with respect to itself to give an element of $\mathfrak{g}\otimes\mathfrak{g}$. This element can be expressed in the form $a_iX_i\otimes Y_i + b_iH_i\otimes H_i + c_iY_i\otimes X_i$. The element $\mathbf{r}$ can be assumed to be of the form

$$\mathbf{r} = 2a_iX_i\otimes Y_i + b_iH_i\otimes H_i.$$

There are two actions of $\mathbf{r}$ on $B\otimes B$. You can act on the left, by letting $\mathbf{r}\cdot(f\otimes g)(Z_1\otimes Z_2) = f\otimes g(\mathbf{r}\, Z_1\otimes Z_2)$, or you can act on the right by letting $(f\otimes g)\cdot\mathbf{r}(Z_1\otimes Z_2) = f(Z_1\otimes Z_2\,\mathbf{r})$. The Poisson bracket on G is given by

$$\{f, g\} = (f\otimes g)\cdot\mathbf{r} - \mathbf{r}\cdot(f\otimes g).$$

In order to write out a formula for the Poisson structure, we need to create a version of $\mathbf{r}$ that operates on the gauge fields on a lattice. We also need to distinguish left actions and right actions: if Z acts by right multiplication in the factor corresponding to the edge e, we denote it by $Z'(e)$; if it acts by multiplication on the left, we denote it by $Z(e)$. Since the formula we derive involves antisymmetrization, we let $Z_1 \wedge Z_2 = Z_1 \otimes Z_2 - Z_2 \otimes Z_1$.

Now suppose that f and g are gauge fields in the quantum theory. One may compute $f \star g - g \star f$ by writing it as $f \otimes g \circ \nabla - g \otimes f \circ \nabla$, and expanding the R matrix as a power series in h. The linear term is

$$\sum_{e \in O} 2a_i X_i'(e) \otimes Y_i'(e) + b_i H_i'(e) \otimes H_i'(e) - 2a_i X_i(e) \otimes Y_i(e) - b_i H_i(e) \otimes H_i(e) +$$

$$\sum_{v \in V} \Bigg(- \sum_{\alpha < \beta} 2a_i X_i'(\alpha) \wedge Y_i'(\beta) + b_i H_i'(\alpha) \wedge H_i'(\beta)$$

$$- \sum_{-\alpha < -\beta} 2a_i X_i(-\alpha) \wedge Y_i(-\beta) + b_i H_i(-\alpha) \wedge H_i(-\beta)$$

$$+ \sum_{\alpha < -\beta} 2a_i X_i'(\alpha) \wedge Y_i(-\beta) + b_i H_i'(\alpha) \wedge H_i(-\beta)$$

$$+ \sum_{-\alpha < \beta} 2a_i X_i(-\alpha) \wedge Y_i'(\beta) + b_i H_i(-\alpha) \wedge H_i'(\beta) \Bigg)$$

This is the same as the formula derived by Fock and Rosly. Hence the Poisson structure on our algebra of classical observables is the complex linear extension of the standard Poisson structure on the characters of the surface with respect to the compact group.

10. The Kauffman Bracket Skein Module

At the end of Section 7 we introduced the algebra of links, $\mathfrak{b}\mathcal{L}_F$, which maps naturally into the observables for any lattice with envelope F. For lattice gauge field theory based on one of the groups $GL_n(\mathbb{C})$, $O_n(\mathbb{C})$, $Sp_n(\mathbb{C})$, the map is onto. This is due to a theorem of Procesi [15], stating that the invariant theory for Cartesian products of these groups is generated by traces. It follows that Wilson links generate observables for theories built on Drinfeld-Jimbo deformations of these groups as well. Furthermore, since each of these groups has a fundamental representation in whose tensor powers one may find all irreducible representations, a single color suffices.

The invariant theory of $SL_n(\mathbb{C})$ is a quotient of the invariant theory for $GL_n(\mathbb{C})$, so here again Wilson links with a single color suffice. If $n = 2$, we can even specify the kernel of the map from links to observables in terms of skein relations. A space of links divided by skein relations is a **skein module**. We will show that the observables for a theory built on $(U_h(sl_2),_q SL_2)$ are the Kauffman bracket skein module. (Actually, there is a sign change involved in the morphism. This could be eliminated by redefining the skein module, but we prefer to keep its original form.)

To obtain this result we need an explicit formula for the R-matrix in the fundamental representation. The generators of $U_h(sl_2)$ are X, Y and H. Let $\underline{\frac{1}{2}}$ be the vector space spanned by $e_{-\frac{1}{2}}, e_{\frac{1}{2}}$. The standard representation

$$\rho : U_h(sl_2) \to End(\underline{\frac{1}{2}}[[h]])$$

is given by

$$\rho(X) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \rho(Y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \rho(H) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

One may now expand the R-matrix in [13] using these matrices and writing $e^{h/2}$ for each appearance of q. The trace on $U_h(sl_2)$ is just the trace under this representation, and takes on values in $\mathbb{C}[[h]]$. The action of S in the fundamental representation is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -e^{h/2}b \\ -e^{h/2}c & a \end{pmatrix},$$

So $\mathrm{tr} \circ S = \mathrm{tr}$.

In $SL_2(\mathbb{C})$, the Cayley-Hamilton identity is:

$$A^2 - \mathrm{tr}(A)A + I = 0.$$

Putting the term in trace on the other side of the equation, and multiplying by A^{-1} we get a linear homogeneous equation.:

$$A + A^{-1} = \mathrm{tr}(A)I.$$

Now multiply by B to get back a bilinear equation:

$$AB + A^{-1}B = \mathrm{tr}(A)B.$$

Finally take the trace:

$$\mathrm{tr}(AB) + \mathrm{tr}(A^{-1}B) = \mathrm{tr}(A)\mathrm{tr}(B).$$

This formula is the Cayley-Hamilton identity as a trace identity for $SL_2(\mathbb{C})$. It is fundamental for $SL_2(\mathbb{C})$ in the sense that every other identity between traces follows from evaluation of this one.

The same identity persists in $U(sl_2)$, except that you need to use the antipode instead of the inverse:

$$\mathrm{tr}(ZW) + \mathrm{tr}(S(Z)W) = \mathrm{tr}(Z)\mathrm{tr}(W).$$

Finally the fundamental trace identity for $U_h(sl_2)$ is:

$$t\,\mathrm{tr}(ZW) + t^{-1}\,\mathrm{tr}(S(Z)W) = \sum_i \mathrm{tr}(s_i Z)\mathrm{tr}(t_i W),$$

where $\sum_i s_i \otimes t_i$ is the R-matrix for $U_h(sl_2)$ and $t = e^{h/4}$.

Let $\mathcal{L}_M$ be the set of framed links (including $\emptyset$) in a 3-manifold M. Let $\mathbb{C}\mathcal{L}_M[[h]]$ denote formal power series in h with coefficients in the vector space over $\mathcal{L}_M$. We define $S(M)$ to be the topological submodule of $\mathbb{C}\mathcal{L}_M[[h]]$ generated by all expressions of the form

1.  $+\,t$  $+\,t^{-1}$)(, and
2. $\bigcirc + t^2 + t^{-2}$.

These formulas indicate relations that hold among links which can be isotoped in M so that they are identical except in the neighborhood shown. The Kauffman bracket skein module is the quotient

$$K(M) = \mathbb{C}\mathcal{L}_M[[h]]/S(M).$$

The Kauffman bracket skein module of $F \times I$ is an algebra with multiplication as in the space of links in Section 7. We denote it $K(F)$. From [7] we know that $K(F)$ is a deformation quantization of the $SL_2(\mathbb{C})$-characters of the fundamental group of F.

Theorem 10. *Let F be a compact, connected surface with boundary. If the surface underlying the envelope of Γ is F then the observables of a lattice gauge field theory based on $(U_h(sl_2),_q SL_2)$ are canonically isomorphic to $K(F)$.*

Proof. We define a map $\zeta : K(F) \to \mathfrak{O}$ as follows. Let L be a framed link in $F \times I$. Represent it as a diagram in the envelope of Γ with the blackboard framing. Orient the components arbitrarily and color them with the fundamental representation. Now perturb it so that it is a base point free q-net, also denoted L. Finally, map it to the observable $(-1)^{|L|}W_L$. This is well defined at the level of $\mathcal{L}_{F\times I}$ by Propositions 4 and 5 and the fact that $\mathrm{tr} \circ S = \mathrm{tr}$. That ζ sends elements of $S(F \times I)$ to zero follows from the quantum Cayley-Hamilton identity.

By Theorem 8, ζ is an algebra map. To see that it is an isomorphism, consider the map induced between $K(F)/hK(F)$ and $\mathfrak{O}/h\mathfrak{O}$. This is known to be an isomorphism ([6]), so ζ must be one as well. ∎

References

[1] A. Y. Alekseev, H. Grosse, V. Schomerus, *Combinatorial quantization of the Hamiltonian Chern-Simons theory I-II*, hep-th/9403066 and hep-th/9408097.

[2] I. Biswas and K. Guruprasad, *Principle bundles on open surfaces and invariant functions on Lie groups*, Int. J. Math. **4** (1993) 535–544.

[3] E. Buffenoir, *Chern-Simons theory on a lattice and a new description of 3-manifold invariants*, q-alg/9509020.

[4] E. Buffenoir and Ph. Roche, *Two dimensional lattice gauge theory based on a quantum group*, Comm. Math. Phys. **170** (1995).

[5] E. Buffenoir and Ph. Roche, *Link invariants and combinatorial quantization of Hamiltonian Chern-Simons theory*, q-alg/9507001.
[6] D. Bullock, *Rings of $SL_2(\mathbb{C})$-characters and the Kauffman bracket skein module*, q-alg/9604014.
[7] D. Bullock, C. Frohman and J. Kania-Bartoszyńska, *Understanding the Kauffman bracket skein module*, q-alg/9604013.
[8] D. Bullock, C. Frohman and J. Kania-Bartoszyńska, *Skein quantization and lattice gauge field theory*, q-alg/9609024.
[9] V. G. Drinfeld, *Quantum groups*, Proc. Int. Congress Math., Berkeley 1986, American Mathematical Society (1987) 798–820.
[10] V. V. Fock and A. A. Rosly, *Flat connections and polyubles*, Teoreticheskaya i Matematicheskaya Fizika **95** no. 2, 228-238.
[11] W. Goldman, *Invariant functions on Lie groups and Hamiltonian flows of surface group representations*, Invent. Math. **85** (1986) 263–302.
[12] G. Hochschild, *Basic Theory of Algebraic Groups and Lie Algebras*, Springer Verlag.
[13] C. Kassel, *Quantum Groups*, Springer-Verlag (1995).
[14] L. Korgodsky and Y. Soibelman, *Algebras of functions on quantum groups*, preprint.
[15] C. Procesi, *The Invariant Theory of $n \times n$ Matrices*, Advances in Mathematics, **19**(1976), 306-381.
[16] J. H. Przytycki and A. Sikora, *On skein algebras and $SL_2(\mathbb{C})$-character varieties*, q-alg/9705011.
[17] A. Sikora and C. Frohman, *A new geometric method for SL_n-representations,* preprint.
[18] M. Sweedler, *Hopf Algebras*, W.A. Benjamin, New York (1969).

Department of Mathematics, Boise State University, Boise, ID 83725, USA

E-mail address: bullock@math.idbsu.edu

Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA

E-mail address: frohman@math.uiowa.edu

Department of Mathematics, Boise State University, Boise, ID 83725, USA

E-mail address: kania@math.idbsu.edu