THE YANG-MILLS MEASURE IN THE $SU(3)$ SKEIN MODULE

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ABSTRACT. Let $A \neq 0$ be a complex number so that $|A| \neq 1$. Let F be a compact smooth oriented surface, the $SU(3)$-skein space of F, $S_A(F)$, is the vector space over $\mathbb{C}$ generated by framed oriented links (including framed oriented trivalent graphs in $F \times I$) quotiented by the $SU(3)$-skein relations due to Kuperberg. For closed, F, with genus greater than 1. We construct a local diffeomorphism invariant trace on $S_A(F)$.

1. INTRODUCTION

Throughout this paper, three manifolds and surfaces will be compact and oriented. A **framed oriented trivalent graph** is a space that is homeomorphic to a closed regular neighborhood of an oriented trivalent graph embedded in an orientable surface, along with an embedding of that oriented graph in the space. As these are oriented, each edge of the graph carries a direction. In diagrams, we will just draw the graph and the reader can imagine its regular neighborhood running parallel to the graph in the plane of the paper. We always have the same "side" of the neighborhood facing up. By a **framed oriented link** in a three-manifold M we mean an embedding of such a space in M. Two framed oriented links are equivalent if there is an isotopy of M taking one to the other that preserves the orientations of the edges. We will also work with **relative** framed oriented links. These are graphs that also have some monovalent vertices but they are exactly the points of intersection of the graph with the boundary of the manifold. Of course, the framed graph intersects the boundary of M in arcs so that each arc has a monovalent vertex in its interior.

Let $A \neq 0$ be a complex number so that $|A| \neq 1$.

Define,

$$[n] = \frac{A^{3n} - A^{-3n}}{A^3 - A^{-3}},$$

and when $A = \pm 1$, we define $[n] = n$. Let $[n]! = [n][n-1]\cdots[1]$. Finally,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}.$$

If M is a compact oriented three-manifold let $\mathcal{L}$ be the set of equivalence classes of framed oriented links so that all the vertices are *sources* or *sinks*. That is, at each

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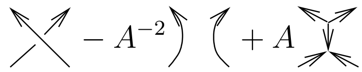
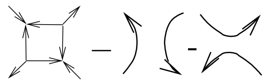
vertex either all three edges point in or they all point out. It is worth noting that the empty link is included in this collection. Let $\mathbb{C}\mathcal{L}$ denote the vector space having $\mathcal{L}$ as a basis. Let R_A be the subspace of $\mathbb{C}\mathcal{L}$ spanned by the following five skein relations from Kuperberg [5]:

- *positive crossing*

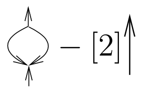
- *negative crossing*



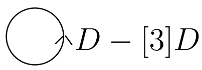
- *square*



- *bubble*

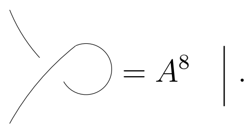


- *trivial component*



Definition 1. *The $SU(3)$ skein space of M at A, denoted by $S_A(M)$, is the quotient space $\mathbb{C}\mathcal{L}/R_A$.*

There is another skein relation that can be easily derived from these, which indicates change of framing.



It is convenient to note that the skein relations for the two crossings, the change of framing and the trivial component generate all the skein relations. In the case that $A = \pm 1$, this skein module has been studied by Adam Sikora [8]. At these values of A, crossings are irrelevant as the two crossing relations reduce to show that the skeins are equal. Consequently there is a well defined product structure. If γ and β are two framed oriented graphs whose vertices are sources and sinks in M, we perturb them so that they are disjoint and take their union as the product. The product structure on $S_{\pm 1}(M)$ is the one induced by this. Sikora constructs a natural homomorphism from $S_{\pm 1}(M)$ onto the $SL_3(\mathbb{C})$ characters of $\pi_1(M)$. The kernel of this homomorphism is the nilradical of $S_{\pm 1}(M)$.

We will also use relative skein spaces. For such, choose a collection of arcs in the boundary of M along with a sign $[+]$ or $[-]$ for each arc. The relative skein space is the vector space spanned by equivalence classes of relative framed oriented links whose vertices are sources and sinks that intersect the boundary of M in those arcs so that if the sign of the arc is $[+]$ then edge of the graph points into M and if the sign of the is $[-]$ then the edge of the graph points out of M.

An especially important class of relative skein spaces are cylinders over a disk, where we have indicated a family of arcs on the boundary of the disk. You can think of assigning plusses and minuses to the arcs, and either by enhancing the arguments of Sikora or imitating the work of Kuperberg. The associated relative module is isomorphic to $\mathrm{Inv}(V^p \otimes V^{*n})$, where V is the fundamental representation of $SL_3\mathbb{C}$ and V^* is its dual, and p is the number of positive arcs and n the number of negative arcs.

If $M = F \times [0,1]$, we represent framed oriented links by drawing oriented trivalent graphs with overcrossings and undercrossings in F and using the blackboard framing. In this case $S_A(F \times [0,1])$ is an algebra. The multiplication is defined by laying one skein over the other. To emphasize that the algebra structure comes from the surface, we denote such a skein space by $S_A(F)$.

In section 2, we recall some related results in $SU(3)$ skein and study the $SU(3)$-skein modules of the solid torus $S^1 \times D^2$, $S^1 \times S^2$ and for the connected sum of two 3-manifolds. We list some results as follows. If $A \neq 0$ and $|A| \neq 1$,

(1) $S_A(S^1 \times D^2)$ has a countable basis indexed by the set of all ordered pairs of nonnegative integers.
(2) $S_A(S^1 \times S^2) = \mathbb{C}\emptyset$, i.e., $S_A(S^1 \times S^2)$ is generated by the empty framed link.
(3) $S_A(M_1 \# M_2) \cong S_A(M_1) \otimes S_A(M_2)$. This says that the $SU(3)$-skein module of the connected sum of two 3-manifolds is isomorphic to the tensor product of the $SU(3)$-skein modules of the manifolds.
(4) From (2) and (3), we conclude that the $SU(3)$-skein module of the connected sum of g copies of $S^1 \times S^2$ is also generated by the empty skein.

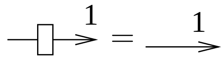
$$S_A(\#^g S^1 \times S^2) = \mathbb{C}\emptyset.$$

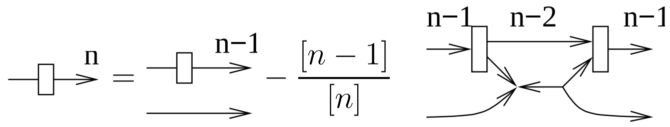
In sections 3, 4, we define and study the Yang-mills measure in a handlebody and on a closed surface.

2. Basics in $SU(3)$-skein theory

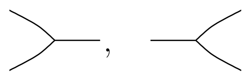
2.1. **Related results from Ohtsuki and Yamada [7]—Magic Elements.**

Definition 2. *A magic element of type* $(n, 0)$ *is inductively defined by the following formula:*





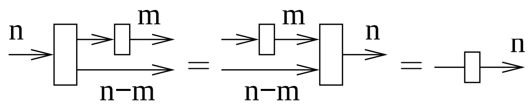
The following diagrams are called a left-Y and a right-Y:



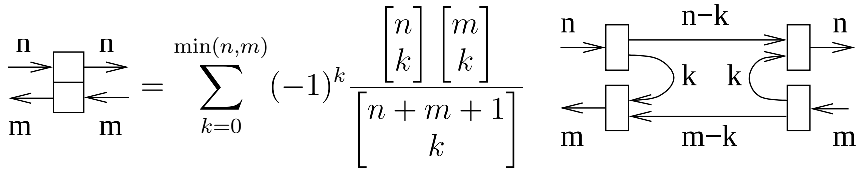
Properties of the magic element of type $(n, 0)$*:*

(1) When attached a left-Y to the right side or a right-Y to the left side, the magic element of type $(n, 0)$ vanishes.

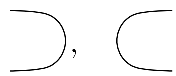
(2) The magic element of type $(n, 0)$ absorbs any magic elements of type $(m, 0)$ with $m \leq n$.



Definition 3. *A magic element of type* (n, m) *is defined by the following formula:*

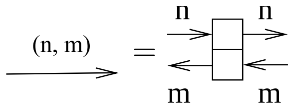


We illustrate the left-U and right-U as follows:

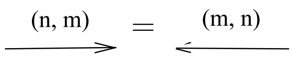


Properties of the magic element of type (n, m): When attached a left-Y or a left-U to the right side, or attached a right-Y or a right-U to the left side, the magic element of type (n, m) vanishes.

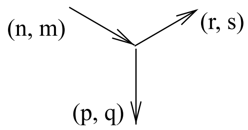
2.2. **Coloring a trivalent graph with magic elements.** The coloring of an oriented edge $\longrightarrow$ by a pair of nonnegative integers (n, m) is by replacing the edge in the graph by the magic element of type (n, m):



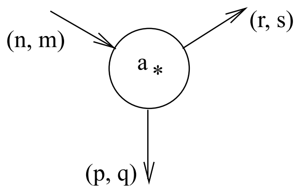
Then



A vertex with acceptable labels becomes a triad with three edges colored by (acceptable) nonnegative integer pairs (n, m), (r, s) and (p,q). Here we illustrate a triad with indicated choice of orientations of the edges:



A triad represents a skein element in the relative skein space of the disk with $(n+s+q)$ input points and $(m+r+p)$ output points. Since there are possibly many different ways that strands can intertwine in the middle, a triad with edges colored by (n, m), (r, s) and (p,q) is not uniquely defined. Therefore we introduce a label by an a_* inside a circle to represent a specific skein in the middle of a triad and indicate it as



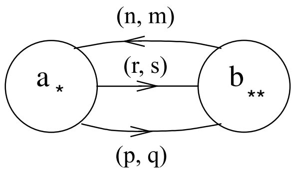
In the case that A is either ± 1 or not a root of unity, the skein module can be interpreted in terms of invariant tensors in the representation theory of $U_q(sl_3))$ (or $U(sl_3)$ when $A = \pm 1$). This can be seen in the works of Kuperberg [5], Kuperberg-Khovanov [4], or Sikora [8]. We summarize some conclusions of their work that we need for this development. Let V be the fundamental representation of $U_q(sl_3)$ and let V^* be its dual. Let $V_{p,q}$ be the highest weight irreducible representation in $V^{\otimes p} \otimes V^{*\otimes q}$. It has quantum dimension

$$\Delta_{p,q} = \frac{[p+1][q+1][p+q+2]}{[2]}.$$

There are invariant tensors in $V \otimes V \otimes V$ and $V^* \otimes V^* \otimes V^*$ that correspond to the two trivalent vertices. There are invariant pairings $V \otimes V^* \to \mathbb{C}$, and $V^* \otimes V \to \mathbb{C}$, that can be used to "stitch" two trivalent vertices together along an edge so that each "web" in a disk, that is an embedded graph with trivalent vertices in the interior of the disk and monovalent vertices on the boundary, so that the trivalent vertices are sources and sinks, corresponds to an invariant tensor in the tensor product of copies of V and V^* corresponding to choosing a basepoint on the boundary of the disk and keeping track of the arrows going in and out as you go around the disk. Modding out by the skein relations corresponding to removing trivial simple closed curves, bubbles and four-sided regions yields a vector space that is isomorphic to the space of invariants. There is a further refinement where you group bunches of edges together to form a clasped web space. Some of these "clasped web" spaces correspond to $S_A(D^2, (n, m), (r, s), (p, q))$, the relative skein space generated by the triads attached with the three magic elements of types (n, m), (r, s) and (p, q). We call $(n, m), (r, s), (p, q)$ an admissible (acceptable) coloring of a vertex if $S_A(D^2, (n, m), (r, s), (p, q)) \neq 0$. From now on, we will only consider admissible triads. There are three conclusions that we need to draw from this work.

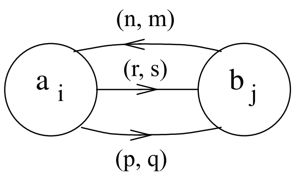
(1) The relative skein $S_A(D^2, (n, m), (r, s), (p, q))$ is up to cyclic permutation canonically isomorphic to $\mathrm{Inv}(V_{n,m} \otimes V_{r,s} \otimes V_{p,q})$. Since $\dim(V_{m,n}) \leq 9mn$, we see that there is a polynomial $p(m, n, r, s, p, q)$ so that
$$\dim(S_A(D^2, (n, m), (r, s), (p, q))) \leq p(m, n, r, s, p, q).$$

(2) The pairing $S_A(D^2, (n, m), (r, s), (p, q)) \otimes S_A(D^2, (q, p), (s, r), (m.n)) \to \mathbb{C}$ corresponding to gluing the two disks together to form a sphere is nondegenerate as it corresponds to pairing $\mathrm{Inv}(V_{m,n} \otimes V_{r,s} \otimes V_{p,q})$ with its dual. We can express this pairing in a diagram theoretic fashion as:



Note that $S_A(D^2) = \mathbb{C}\emptyset$, so it induces a pairing $<, >$ on
$$S_A(D^2, (n, m), (r, s), (p, q)) \otimes S_A(D^2, (q, p), (s, r), (m, n)) \to \mathbb{C}$$
by
$$< \alpha, \beta >= f(\alpha, \beta)$$
where $f(\alpha, \beta)$ is the complex number such that $\alpha \otimes \beta = f(\alpha, \beta)\emptyset$ in $S_A(D^2)$. In this sense, we have $S_A(D^2, (n, m), (r, s), (p, q))^* = S_A(D^2, (q, p), (s, r), (m, n))$.

Following [2], we choose bases $\{a_i\}$ for the space $S_A(D^2, (n, m), (r, s), (p, q)$ and dual bases $\{b_i\}$ for $S_A(D^2, (n, m), (r, s), (p, q))^*$, so that



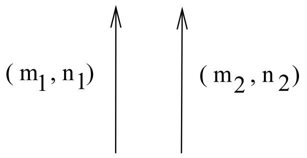
$= \delta_i^j$

When one of the labels is $(0, 0)$ then depending on the direction of the other two arrows, the other two labels are either the same (n, m) or its dual. In this case the skein of the disk with two clasps is one dimensional so everything can be written as a multiple of the skein s obtained by filling in the disk with straight lines from one clasp to the other. Surprisingly, our choice of normalization leads to the peculiar realization that if our dual bases are chosen as $a = \alpha s$ and $b = \beta s$ then $\alpha * \beta = \frac{1}{\Delta_{m,n}}$.

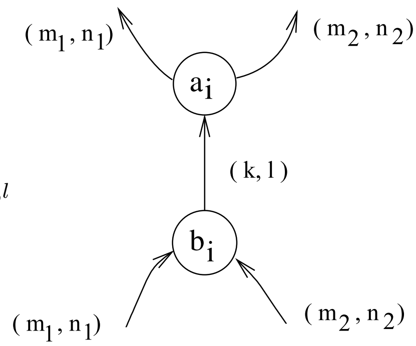
(3) The skein space $S_A(D^2,(m,n),(r,s),(p,q),(u,v))$ is isomorphic to the result of "stitching" the sum below along the (k,l) and (l,k) factors.

$$\oplus S_A(D^2,(m,n),(r,s),(k,l)) \otimes S_A(D^2,(l,k),(p,q),(u,v)),$$

where the sum is over all (k,l) so that $\mathrm{Inv}(V_{m,n}\otimes V_{r,s}\otimes V_{k,l})$ is nonzero and $\mathrm{Inv}(V_{l,k}\otimes V_{r,s}\otimes V_{u,v})$ is nonzero. Using the dual bases chosen above we get the fusion formula [2]:



$$= \sum \Delta_{k,l}$$



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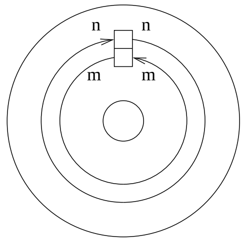
where the sum is over all admissible triples $(m_1,n_1),(m_2,n_2),(k,l)$ and dual bases a_i and b_i.

Let $P_{n,m}$ be the closure of the magic element of type (n,m) in the solid torus $S^1\times D^2$:

$$P_{n,m} =$$

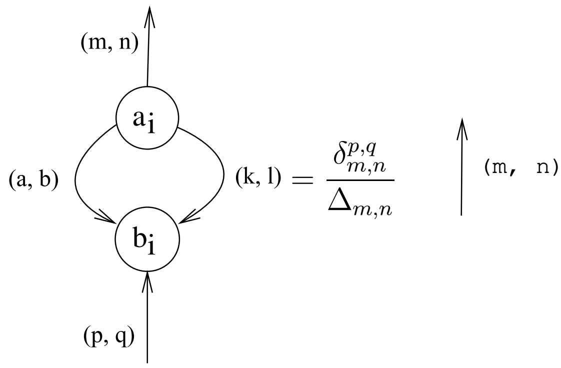


As pictured, the solid torus is the cylinder over $S^1\times[0,1]$. The inclusion of $S^1\times[0,1]$ into the plane $\mathbb{R}^2$ induces a corresponding inclusion of cylinders. Let $\Delta_{n,m}$ be the complex scalar multiple of $P_{n,m}$ by writing $P_{n,m}=\Delta_{n,m}\emptyset$ in $S_A(\mathbb{R}^2)=\mathbb{C}\emptyset$ by including $S_A(S^1\times[0,1]$ into $S_A(\mathbb{R}^2)$.

The following identities hold:

(1) From [7], when m,n are nonnegative integers, $\Delta_{n,m}=[n+1][m+1][n+m+2]/[2]$. When at least one of m,n is a negative integer, we define $\Delta_{n,m}=0$. Note that if A is a real number and is not a root of unity, then $\Delta_{n,m}\geq 1$ for all nonnegative integers m,n.

(2)

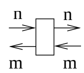


where a_i and b_i are dual bases for the admissible triple (m, n), (a, b), (k, l).

Proof. (I) When $(m, n) \neq (p, q)$, we can prove that the skein element on the left hand side is zero. This follows from the non-convexity of the basis of the clasped web space of Kuperberg [5]. To explain, we consider the clasped web space $W(C)$ where C is given by the sequence $[(+\cdots+)_n(-\cdots-)_m(+\cdots+)_p(-\cdots-)_q]$ (the subscript indicates the number of plusses and minuses in the sequence. When $(m, n) \neq (p, q)$, the clasped web space $W(C)$ is zero, as there will be a minimal cut path with lower weight separating the clasps $[(+\cdots+)_n(-\cdots-)_m]$ and $(+\cdots+)_p(-\cdots-)_q]$ which causes convex clasps.

(II) When $(m, n) = (p, q)$, we prove the skein element on the left hand side is a scalar multiple of the magic element of type (m, n).

According to the Lemma 3.3 of [7], if D is a diagram in the disk with $2n + 2m$ boundary points with neither biangles nor squares as shown:

$D =$ 

Then D satisfies at least one of the following three conditions:

(i) There is a left-Y or a left-U attached to the left side,

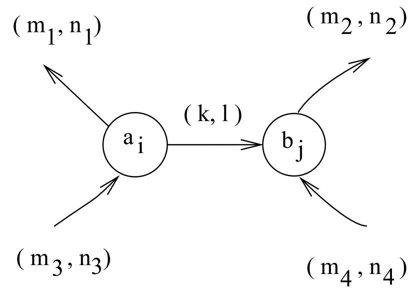
(ii) There is a right-Y or a right-U attached to the right side,

(iii) The diagram D is $n + m$ parallel lines from the left side to the right side.

We can rewrite the middle part of the skein diagram of the given identity as a linear sum of diagrams fitting in the above lemma, then all three possible cases will give a multiple of the magic element of type (m, n). Note that cases (i) and (ii) will contribute 0 when attached to the magic element of type (m, n).

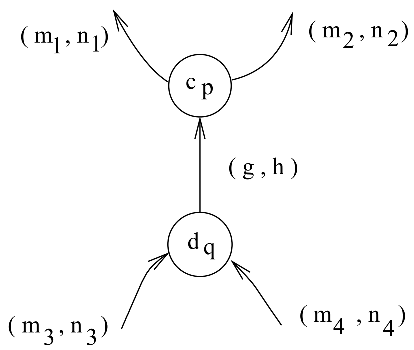
(III) To find the scalar multiple, we close both sides, only when a_i and b_i are dual bases, the closure of the left hand side is nonzero and equals 1, the closure on the right hand side of the magic element of type (m,n) contributes $\Delta_{m,n}$. Therefore the scalar multiple is $\frac{1}{\Delta_{m,n}}$, the identity holds. $\square$

(3) Let (k,l) be a pair of nonnegative integers such that the triples $(m_3,n_3),\ (m_1,n_1),\ (k,l)$ and $(m_2,n_2),\ (m_4,n_4),\ (k,l)$ are admissible, then the collection of elements



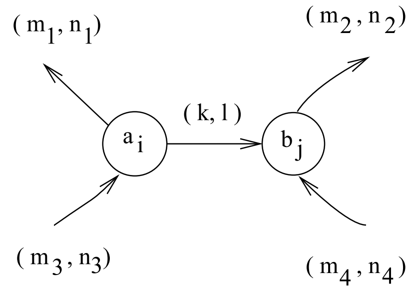
over all such (k,l) and all basis elements a_i, b_j of the corresponding triads, forms a basis for the skein space $S_A(D^2,(m_3,n_3),\ (m_1,n_1),\ (m_2,n_2),\ (m_4,n_4))$.

On the other hand, over all pairs of nonnegative integers (g,h) such that the triples $(m_1,n_1),\ (m_2,n_2),\ (g,h)$ and $(m_3,n_3),\ (m_4,n_4),\ (g,h)$ are admissible and basis elements c_p, d_q of the corresponding triads, the collection of elements

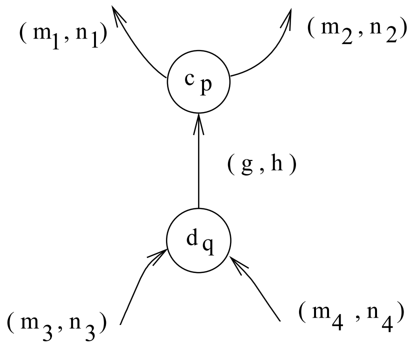


also forms a basis for the skein space $S_A(D^2,(m_3,n_3),\ (m_1,n_1),\ (m_2,n_2),\ (m_4,n_4))$.

(4) There is a change of bases on $S_A(D^2, (m_3, n_3),\ (m_1, n_1),\ (m_2, n_2),\ (m_4, n_4))$:

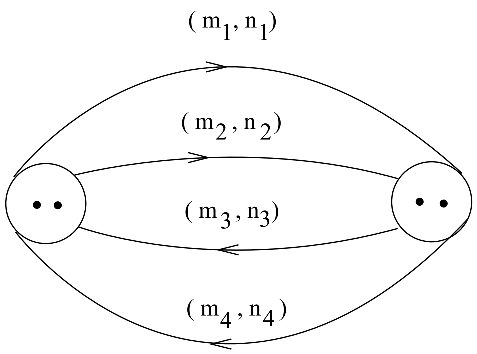


$$= \sum \left\{ \begin{matrix} (m_3, n_3),\ (m_1, n_1)\ (g, h) \\ (m_2, n_2)\ (m_4, n_4)\ (k, l) \end{matrix}\ a_i, b_j, c_p, d_q \right\}$$



where the summation is over all bases c_p, d_q and admissible triples $(m_1, n_1), (m_2, n_2), (g, h)$ and $(m_3, n_3), (m_4, n_4), (g, h)$.

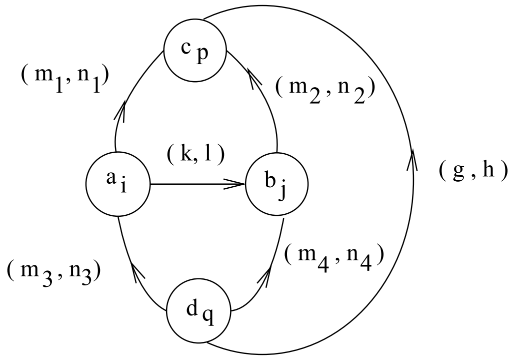
Similarly, we have a pairing on $S_A(D^2, (m_1, n_1),\ (m_2, n_2),\ (m_3, n_3),\ (m_4, n_4)) \otimes S_A(D^2, (n_1, m_1),\ (n_2, m_2),\ (n_3, m_3),\ (n_4, m_4)) \to \mathbb{C}$ induced by the bilinear form of attaching skein elements along the boundary of D^2 through the inclusion into $S_A(D^2)$.



Definition 4. *We define*

$$\mathrm{Tet} \left\{ \begin{matrix} ((m_3, n_3),\ (m_1, n_1)\ (g, h) \\ (m_2, n_2)\ (m_4, n_4)\ (k, l) \end{matrix}\ a_i, b_j, c_p, d_q \right\}$$

to be the coefficient of the empty skein corresponding to the following skein element as a multiple of the empty skein $\emptyset$ *in* $S_A(D^2) = \mathbb{C}\emptyset$.



Theorem 1.

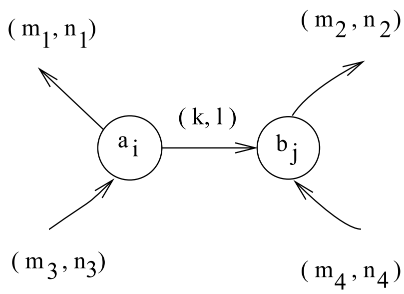
$$\left\{\begin{matrix}(m_3,n_3), & (m_1,n_1) & (g,h) \\ (m_2,n_2) & (m_4,n_4) & (k,l)\end{matrix}\ a_i,\ b_j,\ c_p,\ d_q\right\} = \mathrm{Tet}\left\{\begin{matrix}(m_3,n_3), & (m_1,n_1) & (g,h) \\ (m_2,n_2) & (m_4,n_4) & (k,l)\end{matrix}\ a_i,\ b_j,\ c_p,\ d_q\right\}\Delta_{g,h}$$

The proof of this theorem is similar to the computation in [6].

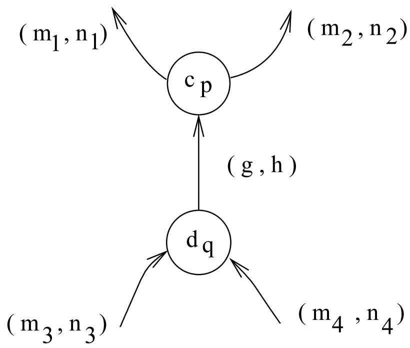
Theorem 2. *If* $A > 0$ *and* A *is not a root of unity,*

$$\left|\ \mathrm{Tet}\left\{\begin{matrix}(m_3,n_3), & (m_1,n_1) & (g,h) \\ (m_2,n_2) & (m_4,n_4) & (k,l)\end{matrix}\ a_i,\ b_j,\ c_p,\ d_q\right\}\ \right| \leq \frac{1}{\sqrt{\Delta_{k,l}\Delta_{g,h}}}$$

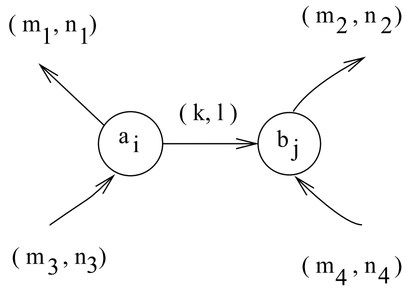
Proof. The key is using the change of bases identity twice.



$$= \sum \left\{\begin{matrix} (m_3,n_3), & (m_1,n_1) & (g,h) \\ (m_2,n_2) & (m_4,n_4) & (k,l) \end{matrix} \; a_i, b_j, c_p, d_q \right\}$$



$$= \sum \alpha_*$$



where

$$\alpha_* = \left\{\begin{matrix} (m_3,n_3), & (m_1,n_1) & (g,h) \\ (m_2,n_2) & (m_4,n_4) & (k,l) \end{matrix} \; a_i, b_j, c_p, d_q \right\} \left\{\begin{matrix} (m_4,n_4) & (m_3,n_3) & (k,l) \\ (m_1,n_1) & (m_2,n_2) & (g,h) \end{matrix} \; a_i, b_j, c_p, d_q \right\}$$

Therefore,

$$\sum \left\{\begin{matrix} (m_3,n_3), & (m_1,n_1) & (g,h) \\ (m_2,n_2) & (m_4,n_4) & (k,l) \end{matrix} \; a_i, b_j, c_p, d_q \right\} \left\{\begin{matrix} (m_3,n_3), & (m_1,n_1) & (g,h) \\ (m_2,n_2) & (m_4,n_4) & (k,l) \end{matrix} \; a_i, b_j, c_p, d_q \right\} = 1$$

i.e.,

$$\sum \mathrm{Tet}^2 \left\{\begin{matrix} (m_3,n_3), & (m_1,n_1) & (g,h) \\ (m_2,n_2) & (m_4,n_4) & (k,l) \end{matrix} \; a_i \; b_j \; c_p \; d_q \right\} \Delta_{k,l}\Delta_{g,h} = 1$$

We observe that each term in the summation is positive, hence

$$0 \leq \mathrm{Tet}^2 \left\{\begin{matrix} (m_3,n_3), & (m_1,n_1) & (g,h) \\ (m_2,n_2) & (m_4,n_4) & (k,l) \end{matrix} \; a_i \; b_j \; c_p \; d_q \right\} \Delta_{k,l}\Delta_{g,h} \leq 1$$

The result follows. □

Corollary 1. *When $A > 0$ and A is not a root of unity,*

$$\left| \; \mathrm{Tet} \left\{\begin{matrix} (m_3,n_3), & (m_1,n_1) & (g,h) \\ (m_2,n_2) & (m_4,n_4) & (k,l) \end{matrix} \; a_i, \; b_j, \; c_p, \; d_q \right\} \; \right| \leq \frac{1}{\sqrt{\Delta_{k,l}}}$$

and

$$\left| \mathrm{Tet}\begin{Bmatrix} (m_3,n_3), & (m_1,n_1) & (g,h) & \\ (m_2,n_2) & (m_4,n_4) & (k,l) & a_i,\ b_j,\ c_p,\ d_q \end{Bmatrix} \right| \le \frac{1}{\sqrt{\Delta_{g,h}}}$$

Proof. This follows from the fact that $\Delta_{g,h} \ge 1$ and $\Delta_{k,l} \ge 1$ for all nonnegative integer pairs (g,h) and (k,l). □

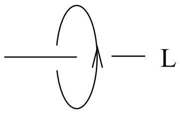
2.3. **Some fundamental examples.** Assume A is not a root of unity.

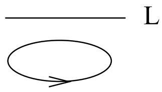
Proposition 1. [7] *$S_A(S^1 \times D^2)$ has a basis given by the collection $\{P_{n,m} \mid n, m$ are nonnegative integers$\}$.*

Theorem 3.

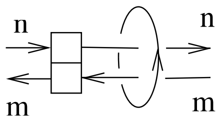
$$S_A(S^1 \times S^2) = \mathbb{C}\emptyset.$$

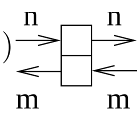
Proof. As $S^1 \times S^2$ can be obtained from the solid torus $S^1 \times D^2$ by adding a 2-handle. There is an epimorphism $S_A(S^1 \times D^2) \to S_A(S^1 \times S^2)$ induced by embedding $S^1 \times D^2$ into $S^1 \times S^2$. Adding a 2 handle results in adding relations to the generators. Here we prove it suffices to consider only the following sliding relation:



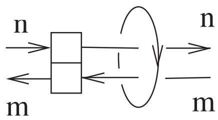
=


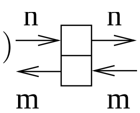
the equality holds in $S_A(S^1 \times S^2)$, where L is any skein element in $S_A(S^1 \times D^2)$. We only need to consider the sliding relation on the generators $P_{n,m}$ of $S_A(S^1 \times D^2)$. From [7], we have the following skein relation:



$= (A^{4n+2m+6} + A^{-2n+2m} + A^{-2n-4m-6})$


While the product of $P_{n,m}$ with a trivial component is $[3]P_{n,m} = (A^6 + A^{-6} + 1)P_{n,m}$, we conclude that $(A^{4n+2m+6} + A^{-2n+2m} + A^{-2n-4m-6} - A^6 - A^{-6} - 1)P_{n,m} = 0$ in $S_A(S^1 \times S^2)$. On the other hand, it's easy to obtain the following skein relation similar to the above:



$= (A^{4m+2n+6} + A^{-2m+2n} + A^{-2m-4n-6})$


Then $(A^{4m+2n+6}+A^{-2m+2n}+A^{-2m-4n-6}-A^6-A^{-6}-1)P_{n,m} = 0$ in $S_A(S^1 \times S^2)$. When A is not a root of unity and m, n are not both zero, one can prove that $(A^{4n+2m+6} + A^{-2n+2m} + A^{-2n-4m-6} - A^6 - A^{-6} - 1)$ and $(A^{4m+2n+6} + A^{-2m+2n} + A^{-2m-4n-6} - A^6 - A^{-6} - 1)$ are not both zero. Therefore $P_{n,m} = 0$ in $S_A(S^1 \times S^2)$ when n, m are not both zero. While $P_{0,0} = \emptyset$ doesn't involve the sliding relation, it survives. Hence $S_A(S^1 \times S^2) =< \emptyset >$. □

Theorem 4.

$$S_A(M_1 \# M_2) = S_A(M_1) \otimes S_A(M_2)$$

The proof follows the same outline as [3].

Corollary 2.

$$S_A(\#^g S^1 \times S^2) = \mathbb{C}\emptyset.$$

3. The Yang-Mills measure in $S_A(F \times [0,1])$ with $\partial F \neq \emptyset$

Let F be a compact oriented surface and $I = [0,1]$. Assume A is not a root of unity. We denote the skein algebra of $S_A(F \times I)$ by $S_A(F)$ to emphasize that the algebra structure depends on F. Notice that $F \times I$ is a handlebody. If you choose a family K of proper arcs on F that cut it down to a disk, then $K \times I$ is a family of disks that cut $F \times I$ into a ball. The double of $F \times I$, denoted by $D(F \times I)$, is the result of gluing two copies of $F \times I$ together using the identity map on their boundary. The disks $K \times I$ in each copy are glued together to form a system of spheres in $D(F \times I)$ that cut it down to a punctured ball. Therefore $D(F \times I)$ is homeomorphic to a connected sum of copies of $S^1 \times S^2$. From the Preliminaries in 2.4, $S_A(D(F \times I)) = \mathbb{C}\emptyset$. This induces a linear functional $\mathcal{YM} : S_A(F \times I) \to \mathbb{C}$ by the inclusion of $F \times I$ into $D(F \times I)$, i.e., if $\alpha \in S_A(F \times I) \subset S_A(D(F \times I))$, we can write $\alpha = f(\alpha)\emptyset$ for some complex number $f(\alpha)$ in $S_A(D(F \times I))$, then $\mathcal{YM}(\alpha) = f(\alpha)$.

Proposition 2.

$$\mathcal{YM}(\alpha * \beta) = \mathcal{YM}(\beta * \alpha)$$

Proof. If you remove $\partial F \times I$ from $F \times I$, then the double of the resulting object is homeomorphic to $\mathring{F} \times S^1$ where the product structure coincides with the product $\mathring{F} \times I$ on each half. Also

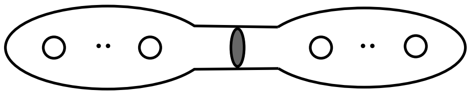
$$F \times I \subset \mathring{F} \times S^1 \subset D(F \times I),$$

we can perturb representations of α and β so that they miss ∂F. Now it is clear that $\alpha * \beta$ and $\beta * \alpha$ are isotopic in $\mathring{F} \times S^1$. □

Let Γ be an oriented trivalent spine of F. An admissible coloring of Γ is by attaching the magic elements of types (m_i, n_i) along the edges and acceptable labels at the vertices, such a coloring of Γ is an element of $S_A(F)$.

Theorem 5. *The admissible colorings of* Γ *form a spanning set for* $S_A(F)$.

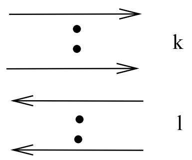
Proof. Let $\mathcal{H}_g$ be the handlebody $F \times I$, let D be a separating meridian disk of $\mathcal{H}_g$:



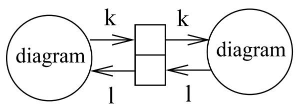
D

Let $V_D = [-1,1] \times D$ be a regular neighborhood of D in $\mathcal{H}_g$, V_D can be projected into a disk $D_p = [-1,1] \times [0,1]$.

Let α be a framed link in $\mathcal{H}_g$ in general position to V_D, let $\alpha' = \alpha \cap V_D$,

$\alpha' =$ 

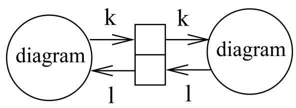
In the next lemma, we show that we can write α' as a linear sum of skein elements in V_D which have the magic elements in the middle such as the following:



where k, l are some nonnegative integers.

By induction on the number of separating meridian discs of $\mathcal{H}_g$, we can write α as a linear sum of skein elements in $S_A(\mathcal{H}_g)$ which have the magic elements at the regular neighborhood of each separating disk. Note that such an element corresponds to an admissible coloring of the trivalent spine of $\mathcal{H}_g$. Therefore, the admissible colorings of Γ spans $S_A(\mathcal{H}_g)$. $\square$

Lemma 1. *An element in the relative skein module of the cylinder with n parallel strands going to the right and m parallel strands going to the left can be written as a linear sum of elements in the form*

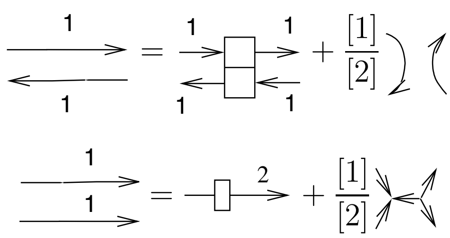


with $(k + l) \leq (m + n)$.

Proof. We proceed by induction on $n + m$.

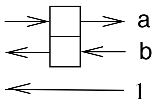
(1) When $n + m = 1$, i.e, $n = 1, m = 0$ or $m = 1, n = 0$, it is trivial.

When $n + m = 2$, there are three cases: (i) $n = m = 1$; (ii) $n = 2, l = 0$; (iii) $m = 2, n = 0$. We illustrate cases (i) and (ii) by the following. Case (iii) is similar to case (ii).



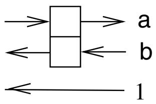
(2) Assume the result is true for $(n+m) \leq k$ for some natural number k, we prove the result is true for the case $(n+m) = (k+1)$. By induction, the result is true for all nonnegative integers n, m.

When $n + m = k + 1$, without loss of generality, we can assume that $m \geq 1$. By the induction assumption, the result is true for $n + (m-1) = k$, i.e., we can consider the part with n parallel strands going to the right and $(m-1)$ parallel strand going to the left and write it as a linear sum of elements of the assumed form with the size of the magic elements in the middle of $\leq k$; now it suffices to prove that the following element is a linear sum of the assumed form,

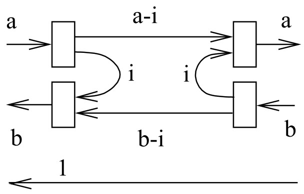


where $(a+b) = k$.

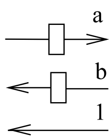
By the definition of the magic element of type (a, b),



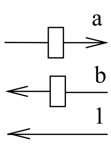
$$= \sum_{i=0}^{\min(a,b)} (-1)^i \frac{\begin{bmatrix} a \\ i \end{bmatrix} \begin{bmatrix} b \\ i \end{bmatrix}}{\begin{bmatrix} a+b+1 \\ i \end{bmatrix}}$$



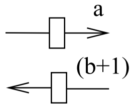
Note that all terms in the summation corresponding to $i \geq 1$ intersect the separating disk at no more than k times, so by the induction assumption, these can be written as the sum of the assumed forms. Therefore we only need to worry about the first term which corresponds to $i = 0$, i.e.,



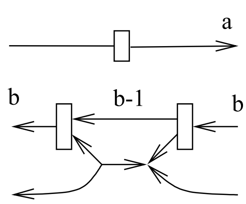
Now use the definition of the magic element of type $(0, b)$,



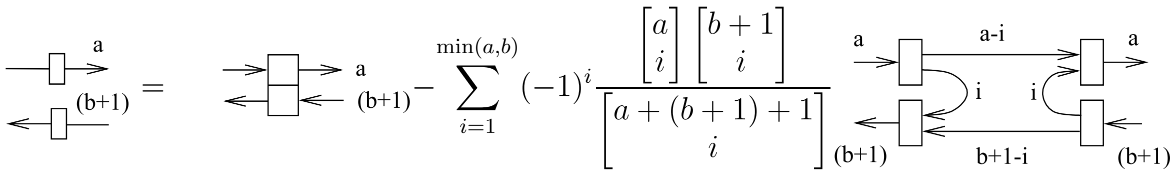
$$=$$



$$+ \frac{[b]}{[b+1]}$$



Similarly, the second term intersects the separating disk at $(a+b) = k$ times, so we only need to consider the first term on the right hand side, again we use the definition of the magic element of type $(a, b+1)$:



We observe again that each term in the summation has $i \geq 1$, so the skein elements intersect the separating disk at $\leq k$ times; by the induction assumption, these can be written as a linear sum of elements which have the magic elements at the middle of size $\leq k$. Therefore we conclude that the result is true for $n + m = k + 1$. □

Lemma 2. *If k is a properly embedded arc in F, then $S_A(F)$ is spanned by trivalent colored graphs so that each graph intersects k in one transverse point of intersection in the interior of one of its edges.*

The preceeding lemmas give a method of computing the Yang-Mills measure. Let k be a proper arc in F. By Lemma 1 any skein can be written as a colored trivalent graph intersecting k in a single point of transverse intersection in the interior of an edge. By the argument in Theorem 3 the Yang-Mills measure of such a graph is zero unless that edge carries the label $(0,0)$. So, choose a system of proper arcs that cut F into a family of disks. Using Lemma 2 repeatedly write the skein as a colored trivalent graph that intersects each of the arcs at most once in a point of transverse intersection in the interior of an edge. Throw out all terms where a graph carries a nonzero label on an edge intersecting one of the arcs. Next erase the edges and renormalize to take into account the peculiarity of the normalization. Finally, evaluate the invariant of the remaining skeins in the disks.

We formalize this:

Proposition 3. Locality *Let F be a compact oriented surface and let k be a proper arc. Let F' be the result of cutting F along k. If s is a skein that is represented by a sum of colored trivalent graphs that each intersects k in at most a single point of transverse intersection in the interior of an edge. Let s_0 be sum of all terms where the graph is either disjoint from k or intersects in an edge labeled $(0,0)$. The skein s_0 corresponds to a skein s_0' in the image of the inclusion of $S_A(F') \to S_A(F)$. Then,*

$$\mathcal{YM}(s) = \mathcal{YM}(s_0').$$

□

4. The Yang-Mills measure on a closed surface

In this section F will be a closed oriented surface of genus greater than 1. Further we suppose that A is a positive real number not equal to 1. We prove that there is a linear functional

$$\mathcal{YM} : S_A(F) \to \mathbb{C}.$$

that is a trace in the sense that $\mathcal{YM}(\alpha * \beta) = \mathcal{YM}(\beta * \alpha)$ for all $\alpha, \beta \in S_A(F)$.

If we remove an open disk with nice boundary from F we get a compact surface with one boundary component F' which is a subsurface of F. The inclusion map $i : F' \to F$ induces a surjective map,

$$S_A(F') \to S_A(F).$$

Let $\partial_{(m,n)}$ be the skein that is the result of coloring a framed knot that is parallel to ∂F with the (m,n) type magic element. We define $\mathcal{YM} : S_A(F) \to \mathbb{C}$. If $\alpha \in S_A(F)$ then choose α' to be a skein in $S_A(F')$ that gets mapped onto α by the inclusion. We denote the Yang-Mills measure on F' by $\mathcal{YM}'_F$ and define,

$$(*) \qquad \mathcal{YM}(\alpha) = \lim_{N\to\infty} \sum_{(m,n)}^{m+n\leq N} \Delta_{m,n} \mathcal{YM}_{F'}(\partial_{(m,n)} * \alpha').$$

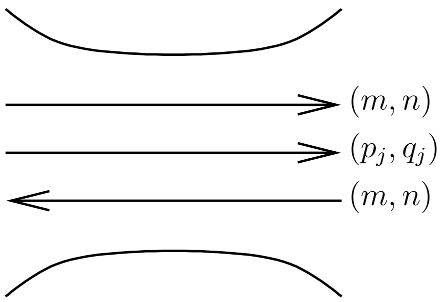
We need to prove two things. The first is that the series we gave above is convergent and the second is to prove that is independent of the choice of α'. To prove convergence we just need to prove it converges on a collection of skeins that spans $S_A(F)$. Luckily we have such a family, admissibly colored spines.

Let s_c denote a trivalent spine of a compact oriented surface F with one boundary component that has been colored admissibly. If the spine has v_i vertices and e_j edges, then

$$v_i = -2\chi(F); \quad e_j = -3\chi(F).$$

where $\chi(F)$ is the Euler characteristic of F.

We need a global estimate on $\mathcal{YM}(\partial_{m,n} * s_c)$. Let (p_j, q_j) be the label on the jth edge of s_c and let a_i be the skein in the ith vertex. To compute this we fuse along the handles, that look like this



We compute by fusing one (m,n) strand with the (p_i, q_i) strand, then fusing with the other (m,n) strand, and throwing out everything that the central edge is not labeled $(0,0)$. The number of terms is equal to the multiplicity of $V_{(n,m)}$ in $V_{(m,n)} \otimes V_{(p_i,q_i)}$. We call the set of pairs of skeins that appear in the two vertices along this edge, $\mathcal{A}_j$. We then need to erase the $(0,0)$ edges, which entails dividing by $\Delta_{m,n}$ for each edge.

We are then just computing the value of sum of the products of some tetrahedral coefficients. The result is

$$\sum_{\mathcal{A}_j}\prod_{v_i}\mathrm{Tet}\left\{\begin{matrix}(m,n) & (m,n) & (m,n)\\ (p_{i_1},q_{i_1}) & (p_{i_2},q_{i_2}) & (p_{i_3},q_{i_3})\end{matrix}\; a_j,b_j,c_j,d_j\right\}$$

The a_j,b_j,c_j,d_j are skeins in vertices coming from the fusions along the edges. The number of terms is less than or equal to a polynomial evaluated on the labels, and the size of each term in the sum is less than $\prod_i \frac{1}{\sqrt{\Delta_{m,n}}}$. Therefore we have the following Proposition.

Proposition 4. *There is a polynomial in variables (m,n) that only depends on the colors assigned to the edges, $p(m,n)$ so that*

$$\mathcal{YM}(\partial_{(m,n)} * s_c) \leq \frac{p(m,n)}{\Delta_{m,n}^{-\chi(F)}},$$

where $\chi(F)$ is the Euler characteristic of F.

Proposition 5. *The formula given by the equation (*) for the Yang-Mills measure converges.*

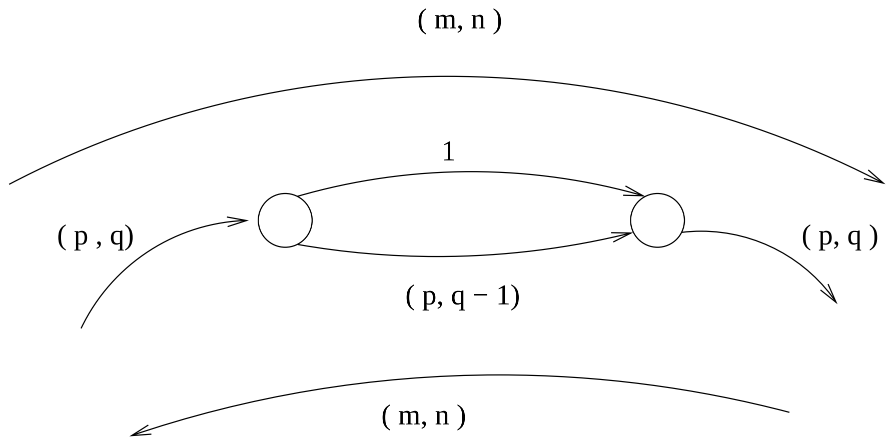
Proof. The proof is by comparison with the series $\sum_{m,n}\frac{p(m,n)}{\Delta_{m,n}^{1/2}}$. We know from its formula that $\Delta_{m,n}$ grows exponentially in m and n. Hence the series we just mentioned converges. By the estimate given by the previous proposition and since the euler characteristic of F' is ≤ -3 we see that the terms in the series for $\mathcal{YM}(s_c)$ are bounded in absolute value by the series we just gave. □

The final step of the argument is to show that $\mathcal{YM}(\alpha)$ is independent of the choice of α'. Since we can pass from any skein α' to any other skein α'' that is sent to α under

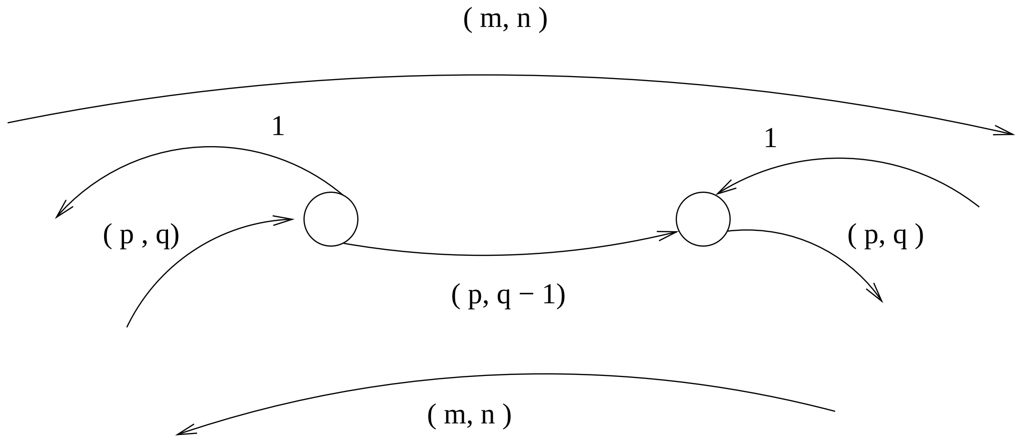
$$S_A(F') \rightarrow S_A(F)$$

by handleslides. By fusing, we can reduce this to check this is true for the result of sliding one string of a trivalent colored spine of F' across the boundary disk. Without loss of generality, let $\alpha' = s_c$ be the trivalent spine of a compact oriented surface F with one boundary component that has been colored admissibly, let $\partial_{m,n}$ be the framed knot corresponding to ∂F oriented with the boundary orientation from F colored with the (m,n) magic element.

Locally $\partial_{(m,n)} * \alpha'$ looks like



Let α'' be the skein obtained from $\alpha' = s_c$ by sliding one strand over the added disk, locally the diagram $\partial_{(m,n)} * \alpha''$ looks like



In the following, we will show that the Yang-Mills measure defined by the equation (*) is well-defined by proving

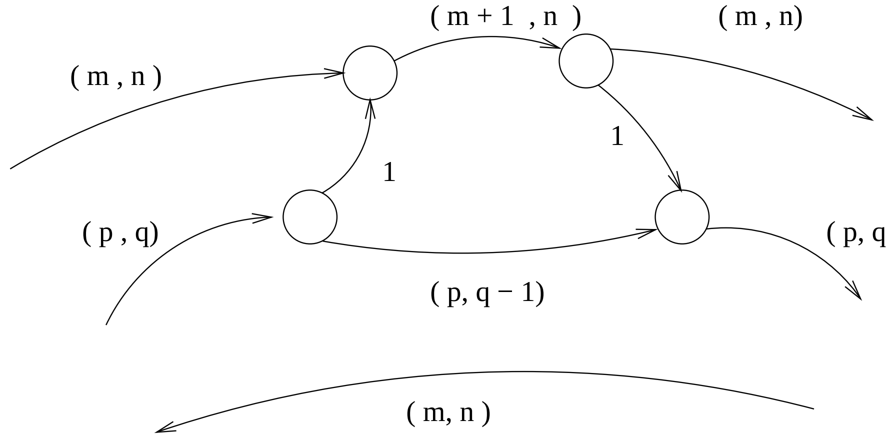
$$\lim_{N\to\infty} \sum_{(m,n)}^{m+n\leq N} \Delta_{m,n}(\mathcal{YM}_{F'}(\partial_{(m,n)} * \alpha') - \mathcal{YM}_{F'}(\partial_{(m,n)} * \alpha'')) = 0.$$

Lemma 3.

$$\sum_{(m,n)}^{m+n\leq N} \Delta_{m,n}((\partial_{(m,n)} * \alpha') - (\partial_{(m,n)} * \alpha'')) = \sum_{(m,n)}^{m+n=N} \Delta_{m,n}(\Delta_{m+1,n}s_1 - \Delta_{n,m+1}s_2)$$

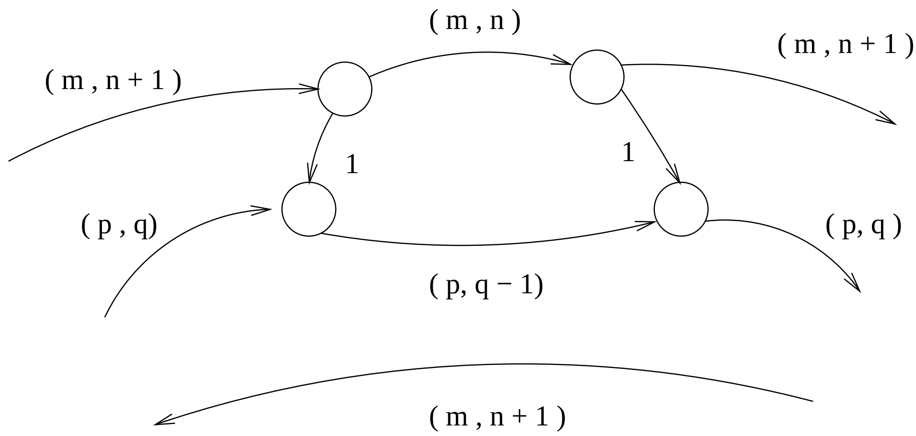
where s_1 *illustrated below is the skein element which is almost the same as* $\partial_{(m,n)} * \alpha'$ *except where it's shown,*

$s_1 =$



s_2 *is the skein element which is almost the same as* $\partial_{(m,n+1)} * \alpha''$ *except where it's shown,*

$s_2 =$

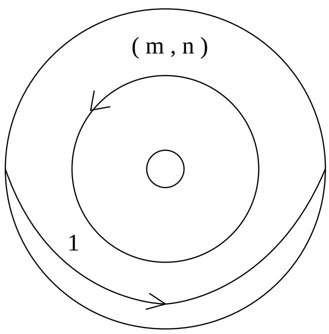


.

The proof follows from the following lemma.

Lemma 4.

$$\sum_{(m,n)}^{m+n\leq N} \Delta_{m,n}($$



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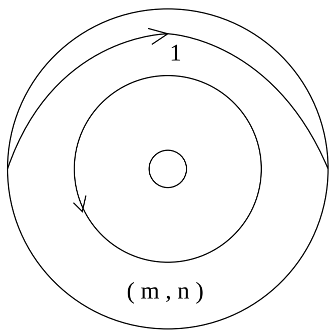
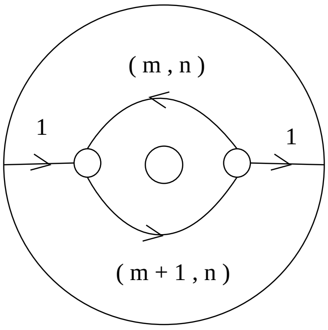
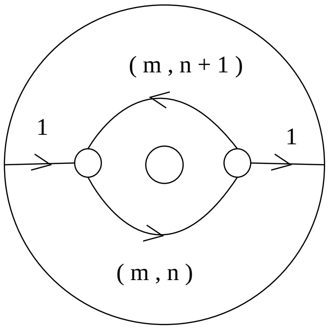


$$) =$$

$$\sum_{(m,n)}^{m+n=N} \Delta_{m,n}(\Delta_{m+1,n}$$



$$- \Delta_{m,n+1}$$



$$)$$

Proof. We consider the tensor product of the magic element of type (m, n) and the magic element of type $(1, 0)$. From the representation theory, we have

$$V_{m,n} \otimes V_{1,0} = V_{m,n-1} \oplus V_{m-1,n+1} \oplus V_{m+1,n}.$$

Similarly,

$$V_{m,n} \otimes V_{0,1} = V_{m-1,n} \oplus V_{m+1,n-1} \oplus V_{m,n+1}.$$

These give the corresponding fusion identities in the $SU(3)$-skein. When we apply these fusion identities to the left hand side of the identity in the lemma, almost all terms are canceled except the terms left on the right hand side. $\square$

Theorem 6.

$$\lim_{N\to\infty} \sum_{(m,n)}^{m+n=N} \Delta_{m,n}\Delta_{m+1,n}\mathcal{YM}_{F'}(s_1) = 0$$

Proof. To compute the Yang-Mills measure of the skein s_1, we fuse to isolate the vertices. Notice that fusing s_1 will require two more cross cuts than that of $\partial_{(m,n)} * \alpha'$. After throwing out everything that the central edge is not labeled $(0, 0)$ and erasing the $(0, 0)$ edges, the Yang-Mills measure of s_1 is the product of

(I) $$\sum_{\mathcal{A}'_j} \frac{1}{\Delta_{m,n}\Delta_{m+1,n}} \mathrm{Tet}\begin{Bmatrix} (m,n) & (m,n) & (m+1,n) & \\ (1,0) & (p,q-1) & (p,q) & a_j, b_j, c_j, d_j \end{Bmatrix} \times$$

$$Tet\begin{Bmatrix} (m,n) & (m+1,n) & (m,n) & \\ (1,0) & (p,q) & (p,q-1) & a'_j, b'_j, c'_j, d'_j \end{Bmatrix}$$

(the a_i, b_j, c_j, d_j and a'_i, b'_j, c'_j, d'_j are skeins in vertices coming from the fusions along the edges.)
with the standard product of fusion on $\partial_{(m,n)} * \alpha'$,

(II) $$\sum_{\mathcal{A}_j} \prod_{v_i \text{ of } \alpha'} \mathrm{Tet}\begin{Bmatrix} (m,n) & (m,n) & (m,n) & \\ (p_{i_1}, q_{i_1}) & (p_{i_2}, q_{i_2}) & (p_{i_3}, q_{i_3}) & a_i, b_j, c_j, d_j \end{Bmatrix}$$

First the product in (I) is less than or equal to

$$\sum_{\mathcal{A}_j} \frac{1}{\Delta_{m,n}\Delta_{m+1,n}} \frac{1}{\sqrt{\Delta_{m,n}}\sqrt{\Delta_{m+1,n}}} \leq p'(m,n)\Delta_{m,n}^{-\frac{3}{2}}\Delta_{m+1,n}^{-\frac{3}{2}}$$

where $p'(m,n)$ is another polynomial depending on the colors (m,n).

Secondly, by a previous proposition, there exist polynomials $p(m,n)$ in variables (m,n) that depend only on the colors assigned to the edges so that the above standard product is less than or equal to

$$\frac{p(m,n)}{\Delta_{m,n}^{-\chi(F)}}.$$

Therefore

$$\mathcal{YM}_{F'}(s_1) \leq p'(m,n)p(m,n)\Delta_{m,n}^{-\frac{3}{2}}\Delta_{m+1,n}^{-\frac{3}{2}}\Delta_{m,n}^{\chi(F)},$$

and

$$\lim_{N\to\infty}\sum_{(m,n)}^{m+n=N}\Delta_{m,n}\Delta_{m+1,n}\mathcal{YM}_{F'}(s_1) = \lim_{N\to\infty}\sum_{(m,n)}^{m+n=N} p'(m,n)p(m,n)\Delta_{m,n}^{-\frac{1}{2}+\chi(F)}\Delta_{m+1,n}^{-\frac{1}{2}} = 0$$

□

Corollary 3.

$$\lim_{N\to\infty}\sum_{(m,n)}^{m+n=N}\Delta_{m,n}\Delta_{n,m+1}\mathcal{YM}_{F'}(s_2) = 0$$

Corollary 4.

$$\lim_{N\to\infty}\sum_{(m,n)}^{m+n\leq N}\Delta_{m,n}(\mathcal{YM}_{F'}(\partial_{(m,n)} * \alpha') - \mathcal{YM}_{F'}(\partial_{(m,n)} * \alpha'')) =$$

$$\lim_{N\to\infty}\sum_{(m,n)}^{m+n=N}\Delta_{m,n}(\Delta_{m+1,n}\mathcal{YM}_{F'}(s_1) - \Delta_{n,m+1}\mathcal{YM}_{F'}(s_2) = 0$$

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