THE A-POLYNOMIAL FROM THE NONCOMMUTATIVE VIEWPOINT

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1. Introduction

This paper places the A-polynomial of a knot into the framework of noncommutative geometry. The A-polynomial was introduced in [CCGLS]. It describes how the $Sl_2\mathbb{C}$-characters of a knot lie inside the $Sl_2\mathbb{C}$-characters of its boundary torus, and can be related to the Alexander polynomial of the knot and to the structure of essential surfaces in the complement of the knot [CL].

The noncommutative generalization is constructed by replacing the geometric definition of the A-polynomial by a definition based on spaces of functions, and then deforming these spaces with respect to a complex parameter t. As such, $Sl_2\mathbb{C}$-characters of surfaces and 3-manifolds are replaced by Kauffman bracket skein module of cylinders over surfaces and 3-manifolds. The Kauffman bracket skein module depends on a parameter t, so that when t is set equal to -1 the $Sl_2\mathbb{C}$-characters are recovered. The A-polynomial can be derived from the noncommutative invariant at $t = -1$.

Our construction uses the isomorphism between the Kauffman bracket skein module of the cylinder over a torus and the subalgebra of the noncommutative torus generated by noncommutative cosines, which was already explicated in [FG]. The noncommutative torus is a fundamental example in noncommutative geometry [Co].

The invariant is a left ideal of polynomials in the noncommutative plane. Since the noncommutative plane is a Nötherian algebra which admits a Gröbner basis algorithm for the lexical order, this ideal is finitely generated and admits a minimal reduced Gröbner basis. The use of Gröbner bases is an important theme in modern commutative algebra. For instance, algorithms associated with Gröbner bases are the heart of most symbolic manipulation programs. The use of Gröbner bases to study the noncommutative plane is foreshadowed by the work of Cohn on skew polynomial rings [C].

There is a complex valued pairing induced by the tensor product of the skein module of the knot complement with the skein module of the solid torus over the skein algebra of the boundary torus. Topologically the pairing is determined by the gluing of the solid torus to the knot complement. This allows us to compare the noncommutative invariant with data coming from colored Jones polynomials. Specifically, the matrix associated to the noncommutative invariant annihilates the vector whose entries are the colored Jones polynomials. This should lead to a deeper understanding of the relationship between the Jones polynomial and the representation theory of the fundamental group of the complement of a knot.

In section 2 the basic definitions associated with the A-polynomial are recalled. In section 3 we introduce the noncommutative analogues of the spaces and maps used in section 2 and then define the noncommutative A-ideal and the A-basis. In section 4 the examples of the unknot and the left- and right-handed trefoils are described. In section 5 we derive the orthogonality relationship between the noncommutative A-ideal and the Jones polynomial. In the final section we pose some questions and suggest avenues of future investigation.

2. Characters and Knot Invariants

2.1. **$Sl_2\mathbb{C}$-Characters.** Let $G =< a_i | r_j >$ be a finitely generated group. A representation $\rho : G \to Sl_2\mathbb{C}$ is determined by a choice of matrices A_i in $Sl_2\mathbb{C}$ so that when the relations r_j are rewritten with the a_i replaced with the A_i they are equal to the identity in $Sl_2\mathbb{C}$. In other words, the representations can be identified with the subset of $\prod_{i=1}^n Sl_2\mathbb{C}$, where n is the number of generators of G, that satisfies the equations obtained by requiring the r_j to evaluate to the identity. Denote this subset $\mathrm{Rep}(G)$, and call it the representations of G. Since we only deal with $Sl_2\mathbb{C}$-representations, we suppress references to $Sl_2\mathbb{C}$.

Let $a_{rs}(i)$, $r, s \in \{1, 2\}$, and $i \in \{1, \ldots , n\}$ be the function on $\prod_{i=1}^n Sl_2\mathbb{C}$ which yields the entry in the r-th row and s-th column of the i-th matrix of each element of $\prod_{i=1}^n Sl_2\mathbb{C}$. The coordinate ring $C[\prod_{i=1}^n Sl_2\mathbb{C}]$ consists of the polynomials in the $a_{rs}(i)$ modulo the ideal generated by

$$a_{11}(i)a_{22}(i) - a_{12}(i)a_{21}(i) - 1,$$

for $i \in \{1, \ldots , n\}$. Each relation r_j gives rise to four polynomials corresponding to the four entries of a matrix, and coming from the condition that the relations evaluate to the identity. Let $I(G)$ be the ideal in $C[\prod_{i=1}^n Sl_2\mathbb{C}]$ generated by the polynomials coming from the relations. Finally, $\mathcal{R}(G) = C[\prod_{i=1}^n Sl_2\mathbb{C}]/I(G)$ is the *affine representation ring* of G. In more generality, Lubotsky and Magid [LM] proved that the isomorphism class of $\mathcal{R}(G)$ is an invariant of the group. The word "affine" refers to the fact that $\mathcal{R}(G)$ is the unreduced coordinate ring of the representations. To get the classical representation ring, take the quotient of $\mathcal{R}(G)$ by its nilradical $\sqrt{0} = \{p | p^n = 0 \text{ for some } n\}$.

There is a left action of $Sl_2\mathbb{C}$ on $\prod_{i=1}^n Sl_2\mathbb{C}$ by conjugation,

$$A \bullet (A_1, \dots , A_n) = (AA_1A^{-1}, \dots , AA_nA^{-1}).$$

This action induces a right action on $C[\prod_{i=1}^n Sl_2\mathbb{C}]$. It is easy to check that the action leaves $I(G)$ invariant, hence the action descends to a right action on $\mathcal{R}(G)$. The invariant subring of this action, denoted $\chi(G) = \mathcal{R}(G)^{Sl_2\mathbb{C}}$, is called the *affine characters* of G. This ring is an invariant of the group G. Once again, to obtain what is classically referred to as the characters, take the quotient of $\chi(G)$ by its nilradical.

For the most frequently encountered groups we are being too careful, the affine representation ring and the affine character ring have trivial nilradical. However, there are examples of groups where the distinction is real [KM].

The affine characters have recently been the subject of scrutiny in works of Bullock, Brumfiel-Hilden, Przytycki-Sikora, and Sikora, [B, BH, PS, S1]. Here is an intrinsic definition due to Sikora. Let $\hat{G}$ be the set of conjugacy classes of G. If W is an element of G we denote the conjugacy class of W by $< W >$. Let $S(\hat{G})$ be the symmetric algebra on $\hat{G}$, that is, polynomials where the variables are conjugacy classes in G and the coefficients are complex numbers. Let J be the ideal generated by all polynomials , $< Id > +2$, and $< AB > + < A^{-1}B >$ $+ < A >< B >$ where A and B range over the elements of G. An isomorphism $S(G)/J \to \chi(G)$ is induced by sending each $< A >$ into the polynomial corresponding to $-\mathrm{tr}(A)$.

There is a space corresponding to the characters. The action of $Sl_2\mathbb{C}$ on $C[\prod_{i=1}^n Sl_2\mathbb{C}]$ is not good, in the sense that the quotient is not Hausdorff. We sidestep this by defining an equivalence relation on $\mathrm{Rep}(G)$, that yields a Hausdorff space. Let $\rho, \eta \in \mathrm{Rep}(G)$ be equivalent if for every $t \in \chi(G)$, $t(\rho) = t(\eta)$. The resulting quotient space is the character variety of G. An *algebraic subset* of an $\mathbb{C}^n$ is the solution set of a system of polynomial equations. A set of polynomials S *cuts out* an algebraic subset V, if $x \in V$ if and only if every function in S is zero at x. The ideal $I(V)$ of V is the set of all polynomials that vanish on V. It is a consequence of the Nullstellensatz that the ideal of V is the radical of the smallest ideal containing any set of polynomials that cut V out. The coordinate ring of V is the quotient of the ring of polynomials in n variables by $I(V)$. Sometimes it is nice to have a set $X(G)$ corresponding to the characters $\chi(G)$. Of course the set only matters to the extent that its points correspond to maximal ideals in $\chi(G)$. To this end, define a *realization* of $X(G)$ to be an algebraic subset of some $\mathbb{C}^n$, where the coordinates on $\mathbb{C}^n$ correspond to traces of elements of G, whose coordinate ring is isomorphic to $\chi(G)/\sqrt{0}$. The set $X(G)$ is in one to one correspondence with the character variety of G [CS].

2.2. **Characters of the Torus.** Concentrate on the case of the fundamental group of a torus. Let us think of it as the free Abelian group on λ and μ. The representations correspond exactly to pairs of matrices in $Sl_2\mathbb{C}$ that commute. Let x be the trace of the image of λ, y be the trace of the image of μ and let z be the trace of the image of their product. The affine character ring, $\chi(\pi_1(T^2))$, is generated by x, y, z with relation

$$x^2+y^2+z^2-xyz-4=0.$$

The ring has trivial nilradical. The space $X(T^2)=X(\pi_1(T^2))$ is realized as the set of points in $\mathbb{C}^3$ satisfying the equation above. There is a two-fold branched cover of $X(T^2)$ that is used in the definition of knot invariants. Consider $\mathbb{C}^*\times\mathbb{C}^*$, where $\mathbb{C}^*$ denotes the nonzero complex numbers, give it coordinates l and m. In order to see that its coordinate ring is $\mathbb{C}[l,l^{-1},m,m^{-1}]$, it is helpful to think of $\mathbb{C}^*\times\mathbb{C}^*$ as ordered pairs of diagonal matrices in $Sl_2\mathbb{C}$.

$$\left(\begin{pmatrix} l & 0\\ 0 & l^{-1}\end{pmatrix},\begin{pmatrix} m & 0\\ 0 & m^{-1}\end{pmatrix}\right)$$

There is a map $p:\mathbb{C}^*\times\mathbb{C}^*\to X(T^2)$ given by sending each pair of points to its equivalence class. The map p is a two-fold branched cover whose singular points are the four ordered pairs chosen from $\begin{pmatrix}\pm1 & 0\\ 0 & \pm1\end{pmatrix}$. In terms of our realization of $X(T^2)$, the map sends

$$\left(\begin{pmatrix} l & 0\\ 0 & l^{-1}\end{pmatrix},\begin{pmatrix} m & 0\\ 0 & m^{-1}\end{pmatrix}\right)$$

to the triple $(l+l^{-1},m+m^{-1},lm+l^{-1}m^{-1})$. Being a two-fold branched cover, p has a deck transformation,

$$\theta:\mathbb{C}^*\times\mathbb{C}^*\to\mathbb{C}^*\times\mathbb{C}^*,$$

$\theta(l,m)=(l^{-1},m^{-1})$.

Dual to p, and θ are maps, that we denote $\hat{p}$ and $\hat{\theta}$,

$$\hat{p}:C[X(T^2)]\to\mathbb{C}[l,l^{-1},m,m^{-1}],$$

and

$$\hat{\theta}:\mathbb{C}[l,l^{-1},m,m^{-1}]\to\mathbb{C}[l,l^{-1},m,m^{-1}].$$

One can show that the image of $\hat{p}$ is the fixed subalgebra of $\hat{\theta}$.

Proposition 1. *Suppose that $V \subset X(T^2)$ is algebraic and S is a set of functions that cuts out V. The ideal of $p^{-1}(V)$ is the radical of the smallest ideal of $\mathbb{C}[l, l^{-1}, m, m^{-1}]$ containing $\hat{p}(S)$.*

Proof. The set $\hat{p}(S)$ cuts out $p^{-1}(V)$, the rest is a standard characterization of the ideal of an algebraic set. □

The coordinate ring $\mathbb{C}[l, l^{-1}, m, m^{-1}]$ can be seen as the ring of fractions of $\mathbb{C}[l, m]$ with respect to the set of monomials in l and m.

Proposition 2. *If $I \subset \mathbb{C}[l, l^{-1}, m, m^{-1}]$ is an ideal then its contraction J to $\mathbb{C}[l, m]$ is $I \cap \mathbb{C}[l, m]$. The extension of J is just the ideal of $\mathbb{C}[l, l^{-1}, m, m^{-1}]$ generated by J. Furthermore, the extension of J is I.*

2.3. **The A-polynomial.** The A-polynomial is defined in [CCGLS]. Here is a brief overview of that definition. Let K be a knot in S^3, or more generally in an oriented homology 3-sphere. Let M be the complement of a regular neighborhood of K. The space M is a compact manifold with boundary homeomorphic to T^2. To choose the meridian and the longitude of the knot, orient the knot and choose the meridian to bound a disk in the regular neighborhood and to have the linking number with the knot equal to $+1$. The longitude bounds a Seifert surface and has intersection number with the meridian $+1$, thus we choose the positive orientation on the boundary torus. This determines the longitude and the meridian up to sign. The inclusion map $T^2 = \partial M \subset M$ induces a map $r : X(M) \to X(T^2)$. Consider the closure of the 1-dimensional part of the peripheral characters $r(X(M))$. Its pull-back through the map $p : \mathbb{C}^* \times \mathbb{C}^* \to X(T^2))$ is still 1-dimensional since p is sujective and is $2-1$ except at four points. Viewed as a curve in $\mathbb{C} \times \mathbb{C}$, this set is the zero set of a two-variable polynomial $B(l, m)$, which is unique if we require that all of its factors are simple and that it is monic. This polynomial always factors as $B(l, m) = (l-1)A(l, m)$, where the factor $l - 1$ stands for the curve of characters of reducible representations. The factor $A(l, m)$ is by definition the A-polynomial of K.

3. Skein modules and the noncommutative torus

3.1. **The Noncommutative Torus.** The noncommutative torus was first introduced by Rieffel [Ri1] as an example in the study of nocommutative C^*-algebras, its physical significance and geometric properties were only later discovered [Co], [Ri2], [Ri3], [We]. The noncommutative torus is a "virtual" geometric space whose algebra of continuous functions is the (irrational) rotation algebra A_θ. It is customary to call the algebra of functions itself the noncommutative torus.

The algebra A_θ is usually defined for a real angle of rotation θ, however we consider θ to be any complex number, and let $t = e^{\pi i\theta}$. This algebra can be introduced abstractly by exponentiating the Heisenberg noncommutation relation. That is, A_θ is the closure in a certain C^*-norm of the algebra spanned by l, m, l^{-1}, m^{-1}, subject to the relation $lm = t^2 ml$.

One can also define A_θ by its multiplication rule. Consider the elements $e_{p,q} = t^{-pq} l^p m^q$, $p, q \in \mathbb{Z}$. They span over $\mathbb{C}$ a dense subalgebra. Defines the multiplication via the formula

$$e_{p,q} * e_{r,s} = t^{\left|\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right|} e_{p+r,q+s}$$

which, from our approach, is just a consequence of the defining relation. The irrational rotation algebra is the closure of the algebra spanned by the $e_{p,q}$'s in the norm determined by the left regular representation (with this new product) on $L^2(T^2)$.

For the purpose of this paper we are interested only in the subalgebra of the noncommutative torus consisting of Laurent polynomials in l and m, which we denote by $\mathbb{C}_t[l, l^{-1}, m, m^{-1}]$. There is an automorphism

$$\Theta : \mathbb{C}_t[l, l^{-1}, m, m^{-1}] \to \mathbb{C}_t[l, l^{-1}, m, m^{-1}],\ \Theta(e_{p,q}) = e_{-p,-q}$$

Let $\mathcal{C}_t$ be its invariant part. The notation is motivated by the fact that this subalgebra is spanned by the noncommutative cosines $\frac{1}{2}(e_{p,q} + e_{-p,-q})$, $p, q \in \mathbb{Z}$.

In addition, let $\mathbb{C}_t[l, m]$ be the subalgebra of $\mathbb{C}_t[l, l^{-1}, m, m^{-1}]$ spanned by $e_{p,q}$, with $p, q \geq 0$. This is nothing but the ring of noncommuting polynomials in two variables l and m satisfying the noncommutation relation $lm = t^2 ml$. This ring is frequently referred to as the quantum plane.

Proposition 3. *([K], Proposition IV.1.1) The ring $\mathbb{C}_t[l, m]$ is both left and right Nötherian and has no zero divisors.*

We now need to broach the subject of Gröbner bases in $\mathbb{C}_t[l, m]$. As $\mathbb{C}_t[l, m]$ is so close to being commutative, the concepts translate over very easily from the case of two variable polynomials.

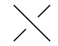
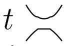
We lexicographically order the $l^p m^q$. Hence $l^p m^q < l^r m^s$ if either $p < r$ or $p = r$ and $q < s$. Given $f \in \mathbb{C}_t[l, m]$ we can write $f = \sum \alpha_{p,q} e_{p,q}$ where the sum is finite. The leading term $lt(f)$ of f is the $\alpha_{p,q} l^p m^q$ where the $l^p m^q$ is largest in the lexicographical ordering among those terms with $\alpha_{p,q} \neq 0$. The leading power product is $l^p m^q$ and the leading coefficient is $\alpha_{p,q}$.

Suppose that $u, v, w \in \mathbb{C}_t[l, m]$ and $u = vw$, then we say w divides u on the right and we let $\frac{u}{w} = v$. A *Gröbner basis* for a left (respectively right) ideal I is a collection f_i of elements of the ideal I so that the ideal generated by the leading terms of the f_i is equal to the ideal generated by the leading terms of elements of I. We say the Gröbner basis is *minimal* if no two f_i have the same leading power product. We say the Gröbner basis is *reduced* if no power product in each f_i is divisible by the leading power product of any other f_i.

Proposition 4. *For a left (or right) ideal of polynomials in $\mathbb{C}_t[l, m]$ there exists a unique minimal, reduced Gröbner basis, consisting of monic polynomials.*

Proof. By changing any statements in [AL] about ideals to statements about left (or right) ideals, the proof there goes through verbatim. □

3.2. **The Kauffman Bracket Skein Module.** Let M be an orientable 3-manifold. A framed link in M is an embedding of a disjoint union of annuli into M. In diagrams we will draw only the core of an annulus lying parallel to the plane of the paper (i.e. with blackboard framing).

Two framed links in M are equivalent if there is an isotopy of M taking one to the other. Let $\mathcal{L}$ denote the set of equivalence classes of framed links in M, including the empty link. Fix a complex number t. Consider the vector space, $\mathbb{C}\mathcal{L}$ with basis $\mathcal{L}$. Define $S(M)$ to be the smallest subspace of $\mathbb{C}\mathcal{L}$ containing all expressions of the form  $- t$  $- t^{-1}$  and  $+ t^2 + t^{-2}$, where the framed links in each expression are identical outside balls, in which they look like pictured in the diagrams. The Kauffman bracket skein module $K_t(M)$ is the quotient

$$\mathbb{C}\mathcal{L}/S(M).$$

In the case of the cylinder over the torus, $T^2 \times I$, $K_t(T^2 \times I)$ has the structure of an algebra with multiplication given by laying one link over the other. More precisely, to multiply skeins corresponding to links α and β, isotope them so that α lies in $T^2 \times [\frac{1}{2}, 1]$ and β in $T^2 \times [0, \frac{1}{2}]$. Then $\alpha \cdot \beta$ is the element of the skein module represented by the class of the union of these two links in $T^2 \times [0, 1]$. Extend this to a distributive product.

Oriented simple closed curves on the torus up to isotopy are indexed by pairs of relatively prime integers (p, q). Corresponding to (p, q) is a framed link in $T^2 \times I$. Take an annulus in $T^2 \times I$ whose core projects to a (p, q) curve, so that the annulus runs parallel to the boundary of $T^2 \times I$. As the framed links are unoriented, (p, q) and $(-p, -q)$

give rise to the same link, which we also denote by (p,q). A standard argument based on the proof that the Kauffman bracket in S^3 is well defined shows that as a vector space, $K_t(T^2\times I)$ has as basis all links consisting of parallel copies of the (p,q).

Let x be $(0,1)$, $y=(1,0)$ and $z=(1,1)$. It is a theorem of Bullock and Przytycki [BP] that $K_t(T^2\times I)$ is isomorphic to polynomials in three noncommutative variables, x, y and z modulo the ideal generated by

$$t^2x^2+t^{-2}y^2+t^2z^2-txyz-2(t^2+t^{-2}),$$

$$txy-t^{-1}yx-(t^2-t^{-2})z,$$

$$tzx-t^{-1}xz-(t^2-t^{-2})y,$$

and

$$tyz-t^{-1}zy-(t^2-t^{-2})x.$$

When $t=-1$ the Kauffman bracket skein module of an arbitrary three-manifold can be made into an algebra. The point is that at $t=-1$ the skein relation allows us to change crossings. To multiply two links, perturb them so that they miss one another and take their union. As crossings don't count, the answer is independent of the perturbation. This extends to make $K_{-1}(M)$ into an algebra for any M. It is a theorem of Bullock [B] that $K_{-1}(M)/\sqrt{0}$ is naturally isomorphic to $\chi(\pi_1(M))/\sqrt{0}$. You can see this isomorphism from the description of $\chi(\pi_1(M))$ due to Sikora given above. The correspondence at $t=-1$ is slightly tricky as the x, y and z given here correspond to $-x$, $-y$ and $-z$ in the relation we gave for the $SL_2\mathbb{C}$-characters of the torus.

In [FG] the following theorem is proved.

Theorem 1. *There exists an isomorphism of algebras*

$$\hat{p}: K_t(T^2\times I)\to \mathcal{C}_t$$

determined by

$$\hat{p}((p,q))=e_{(p,q)}+e_{(-p,-q)},\ p,q\in\mathbb{Z}.$$

4. The noncommutative A-ideal

4.1. **Definition.** In this section we introduce a noncommutative generalization of the A-polynomial, depending on the complex parameter t, from which the classical case can be recovered by letting $t=-1$. To define this invariant, we replace the geometric definition of the A-polynomial, based on character varieties, with a definition based on spaces of functions on character varieties, then deform these spaces with respect to the parameter. Through this procedure, the algebra of functions on the character variety of the torus deforms to a noncommutative algebra, while the algebra of functions on the knot complement deforms to a right module over the algebra of the torus.

Denote by $\mathcal{I}(K)$ the kernel of

$$\hat{r} : \chi(\pi_1(T^2))) \to \chi(\pi_1(S^3 - K)).$$

Notice that $\mathcal{I}(K)$ cuts out an algebraic subset V of $X(T^2)$.

Proposition 5. *Let J be the ideal generated by $\hat{p}(\mathcal{I}(K))$ in $\mathbb{C}[l, l^{-1}, m, m^{-1}]$, and J^c its contraction to $\mathbb{C}[l, m]$. Then the product of the generators of the principal minimal prime ideals associated to J^c is equal to $B(l, m)$ (modulo multiplication by a unit). That is, the ideal $\mathcal{I}(K)$ determines the A-polynomial.*

Proof. The ideal J cuts out the preimage in $\mathbb{C}^* \times \mathbb{C}^*$ through the map p, of the variety of peripheral characters, and consequently J^c cuts out this variety as viewed in $\mathbb{C} \times \mathbb{C}$. By restricting ourselves to the 1-dimensional part, the conclusion follows. $\square$

You can think of the extension of the ideal $\mathcal{I}(K)$ as the holomorphic sections of a line bundle over $\mathbb{C}^* \times \mathbb{C}^*$. Specifically, the sections of the line bundle associated to the divisor of $\frac{1}{B(l,m)}$.

There is a map $\hat{r} : K_t(T^2 \times I) \to K_t(S^3 - K)$ obtained by gluing the cylinder over a torus into the complement of the knot at the $T^2 \times \{0\}$ end so that the meridian goes to the meridian and the longitude goes to the longitude. Let $\mathcal{I}_t(K)$ be the kernel of $\hat{r}$. This ideal is called the peripheral ideal of the knot. Recall for any 3-manifold $K_{-1}(M)/\sqrt{0}$ is isomorphic to $\chi(\pi_1(M))/\sqrt{0}$. For a cylinder over a torus the radical of both rings is trivial, hence

$$K_{-1}(T^2 \times I) = \chi(\pi_1(T^2)) = C(X(T^2)).$$

Under this identification $\mathcal{I}_{-1}(K) = \mathcal{I}(K)$. From the above proposition it follows that if we extend the ideal $\mathcal{I}_{-1}(K)$ to $K_{-1}(\mathbb{C}^* \times \mathbb{C}^*) = \mathbb{C}_{-1}[l, l^{-1}, m, m^{-1}]$ and then contract it to $K_{-1}(\mathbb{C} \times \mathbb{C}) = \mathbb{C}_{-1}[l, m]$, we get an ideal for which the product of the generators of associated minimal primes which are principal is equal to $B(-l, -m)$ (eventually multiplied by a unit). The negatives are determined by the fact that the skein relation corresponds to an identity satisfied by the negative of the trace.

This motivates the following

Definition 1. *The noncommutative A-ideal $\mathcal{A}_t(K)$ of a knot K is the left ideal equal to the contraction to $\mathbb{C}_t[l,m]$ of the extension to $\mathbb{C}_t[l,l^{-1},m,m^{-1}]$ of $\mathcal{I}_t(K)$.*

As a consequence of Proposition 5 we get

Proposition 6. *The noncommutative A-ideal at $t=-1$ determines the A-polynomial.*

4.2. **Examples.** 1. *K is the unknot.* From [FG] we get that the peripheral ideal $\mathcal{I}_t(K)$ is generated by $(1,0)+t^2+t^{-2}$ and $(1,1)+t^3(0,1)$. The noncommutative A-ideal is generated by $(l+t^2)(l+t^{-2})$ and $lm^2(l+t^2)+t^2(l+t^{-2})$. An easy application of Buchberger's Algorithm yields the minimal reduced Gröbner basis $\{l^2+(t^2+t^{-2})l+1, lm^2-t^4l+t^6m^2-t^2\}$.

For $t=-1$, the ideal is generated by $(l+1)^2$ and $(m^2+1)(l+1)$. To get the classical A-polynomial take the radical and get the ideal generated by $B(-l,-m)=l+1$. Thus $B(l,m)=l-1$ and $A(l,m)=1$ which is what we expected to get. Note that in addition to the A-polynomial, the A-ideal detects the embedded primes coresponding to the points $(1,i)$ and $(1,-i)$.

2. K is the right-handed trefoil. In [Ge], as a consequence of the formula for the the action of $K_t(T^2\times I)$ on the the basis of the skein module of the complement of the right-handed trefoil, one gets that the noncommutative A-ideal at all but finitely many values of the parameter t is generated by the polynomials $m^4(l+t^{10})(lm^6-t^6)$ and $(l+t^2)(lm^6-t^6)$. Away from 8th roots of unity the minimal reduced Gröbner basis is $\{l^2m^6+t^2lm^6-t^6l-t^8, lm^{10}-t^{14}m^4\}$.

The value $t=-1$ is, however, among the finitely many exceptional values. In this case, the A-ideal is generated by $(l+1)(lm^6-1)(m^2+1)$, and $(l+1)^2(lm^6-1)$. The radical of this ideal is generated by $(l+1)(lm^6-1)$. After the change of variable we get $B(l,m)=(l-1)(lm^6+1)$, that is $A(l,m)=(lm^6+1)$.

3. K is the left-handed trefoil.

Except for finitely many values of t, the noncommutative A-ideal is generated by $m^4(l+t^{10})(l-t^6m^6)$ and $(l+t^2)(l-t^6m^6)$. Away from 8th roots of unity the minimal reduced Gröbner basis is $\{l^2+t^2l-t^{24}m^{12}-t^8m^6, lm^4-t^{14}m^{10}\}$.

At $t=-1$ the ideal is generated by $(l+1)(l-m^6)m^2$ and $(l+1)^2(l-m^6)$. In this case one gets $A(l,m)=l-m^6$.

4.3. **Relation with the Jones polynomial.** Assume in this section that t is not a root of unity. The definition and normalization of the

Jones-Wenzl idempotents will be used as in [Li]. Let S_c be the skein in the solid torus obtained by plugging the c-th Jones-Wenzl idempotent into the core of the solid torus. The Kauffman bracket skein module of the solid torus has the set $\{S_c\}$ as a basis. Let $\hat{K}_t(S^1 \times D^2)$ be the vector space of formal sums $\sum_c z_c S_c$ where the z_c are complex numbers and the c range over the natural numbers starting at 0.

The double of the solid torus is $S^1 \times S^2$. Any skein α in $S^1 \times S^2$ can be represented by a linear combination of framed links that miss $1 \times S^2$. Hence, any skein in $S^1 \times S^2$ can be represented as a skein in a punctured ball, and it has a Kauffman bracket. It follows that $K_t(S^1 \times S^2)$ is canonically isomorphic to $\mathbb{C}$. There is a pairing between skeins in $K_t(S^1 \times D^2)$. If $\alpha, \beta \in K_t(S^1 \times D^2)$ are represented by a single framed link each, take the union of two copies of the solid torus, identified along their boundaries, with the link representing α in one and the link representing β in the other. As this yields a skein in $S^1 \times S^2$ we get a complex number by taking the Kauffman bracket as above. This can be extended bilinearly to give a pairing,

$$K_t(S^1 \times D^2) \otimes \hat{K}_t(S^1 \times D^2) \to \mathbb{C},$$

for although the sum is infinite only finitely many terms are nonzero. In this way we identify $\hat{K}_t(S^1 \times D^2)$ with $K_t(S^1 \times D^2)^*$.

There is a representation of the Kauffman bracket skein algebra of the cylinder over the torus into endomorphisms of $\hat{K}_t(S^1 \times D^2)$. Glue the cylinder onto $S^1 \times D^2$ along the 0-end of the cylinder so that longitudes and meridians go to longitudes and meridians. The matrix of a skein in the cylinder over the torus as a matrix with respect to the basis S_c, is of bounded width. That is, there is an integer n so that if $|i-j| > n$ the ij-entry of the matrix is zero. This can be seen, as the matrix induced is the adjoint of the matrix corresponding to the endomorphism of $K_t(S^1 \times D^2)$ induced by gluing the 1-end of the cylinder to $S^1 \times D^2$. Notice that if $\hat{Z} \in \hat{K}_t(S^1 \times D^2)$ then the annihilator of $\hat{Z}$ in the Kauffman bracket skein module of the cylinder over the torus is a left ideal.

Let $K \subset S^3$ be a framed knot. Let X be the complement of an open regular neighborhood of the knot. There is a pairing,

$$K_t(S^1 \times D^2) \otimes K_t(X) \to \mathbb{C},$$

obtained by gluing the solid torus into the knot so that the meridian of the solid torus goes to the meridian of the knot and the blackboard longitude goes to the framing of the knot. To pair two skeins, take their union and then take the Kauffman bracket in S^3 of the result. By using the empty skein in X we get a linear functional,

$$Z(K) : K_t(S^1 \times D^2) \to \mathbb{C}.$$

Let $\kappa(K, c)$ be the value of $Z(K)$ on S_c. We can then represent $Z(K)$ by

$$\hat{Z}(K) = \sum_c \kappa(K, c)\sigma_c$$

where $\{\sigma_c\}$ is the basis dual to $\{S_c\}$. It is worth noting that the $\kappa(K, c)$ are the colored Kauffman brackets of the knot (which are a version of the colored Jones polynomials of the knot). Indeed, the c-th coefficient of the series expansion is computed by plugging S_c along the core of the regular neighborhood of the knot and evaluating in S^3, that is by coloring the knot with the c-th Jones-Wenzl idempotent and evaluating the result in the skein space of the plane.

Let $\mathcal{F}_t(K)$ be the annihilator of $\hat{Z}(K)$ in $K_t(T^2 \times I)$. This is a left ideal that we call the *formal ideal* of K.

Theorem 2. *The ideal $\mathcal{I}_t(K)$ lies in $\mathcal{F}_t(K)$.*

Proof. Recall that a skein α in $T^2 \times I$ is in $\mathcal{I}_t(K)$ if when you glue the 0 end of $T^2 \times I$ to X, the skein is equivalent to 0 in $K_t(X)$. Let $\alpha \in \mathcal{I}_t(K)$ and $x \in K_t(S^1 \times K^2)$. Then $\alpha Z(K)$ is a functional on $K_t(S^1 \times D^2)$ and its value on x is computed by embedding the skein a in the knot complement by gluing the cylinder over the torus to the knot complement, and then gluing the solid torus with the skein x in it to the knot complement. But the skein α can be transformed into the zero skein by skein moves taking place entirely in the complement of x, hence the value of the functional is zero. Thus the functional itself is zero. □

If α is a skein in $K_t(T^2 \times I)$, then α has a matrix representation coming from the action on the skein module of the solid torus, with basis $\{S_c\}$. The theorem shows that the rows of this matrix are orthogonal to the vector $\hat{Z}(K)$

5. The view beyond

The first question about this set up is whether $\mathcal{I}_t(K) = \mathcal{F}_t(K)$. Even in restricted cases, like when the colored Jones polynomials of the knot are all trivial this is a provocative question. This gives the first glimmer of how one might go about proving that the Jones polynomial distinguishes the unknot.

The fact that the peripheral ideal annihilates the vector whose coordinates are the colored Kauffman brackets of the knot gives rise to a family of relations for the generators which depend on the parameter t. Do these depend smoothly on t? If yes, they can be differentiated and evaluated at $t = -1$ to yield derivatives of the A-polynomial of a knot. How are those to be understood in terms of the $Sl_2\mathbb{C}$-representation of the knot? The pairing used in the orthogonality relation is clearly some sort of averaging process. Are there integral formulas for the colored Jones polynomials, where the integral is taken over the character variety of the knot?

What kind of information about incompressible surfaces in a knot complement can be extracted from the noncommutative invariant. Can you see the placement of incompressible surfaces in the knot complement from the colored Jones polynomials of the knot?

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