Completion of Discrete Cluster Categories of type $\mathbb{A}$.

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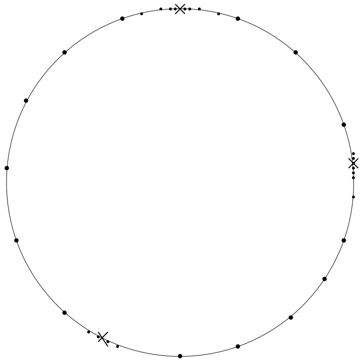
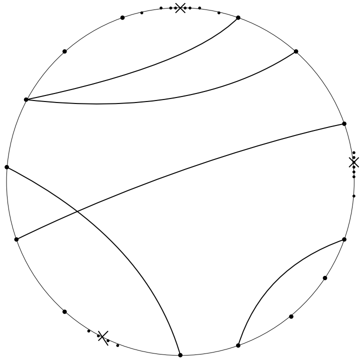
Setting



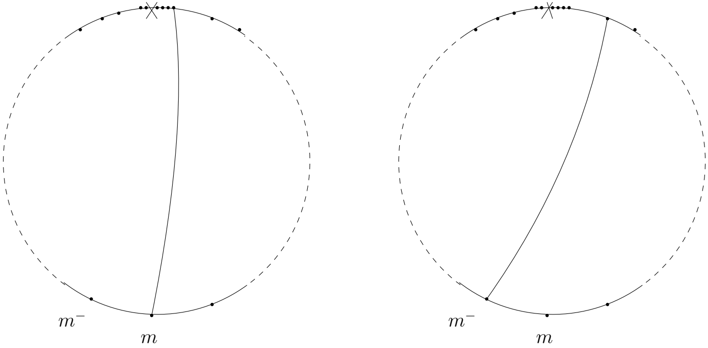
- M = a discrete set of infinitely many marked points with finitely many accumulation points
- $\mathrm{acc(M)}$ = a set of accumulation points which are two-sided.

Igusa-Todorov discrete cluster category of type A

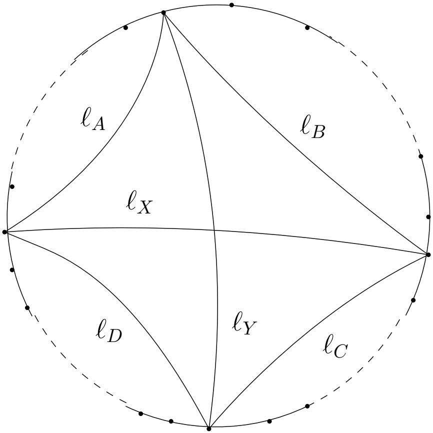


$\mathcal{C}_{(S,M)}$

- Indecomposable objects $\leftrightarrow$ arcs between marked points in M
- $\mathrm{Ext}^1(X, Y) \neq 0 \Leftrightarrow \ell_X$ and ℓ_Y cross.

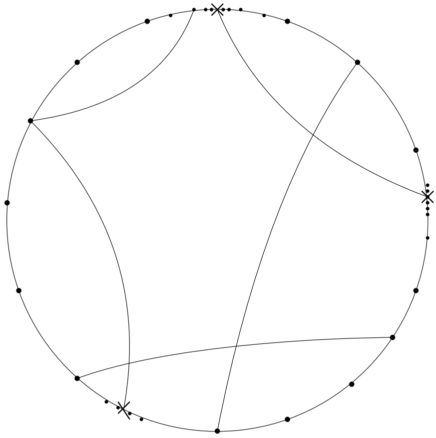


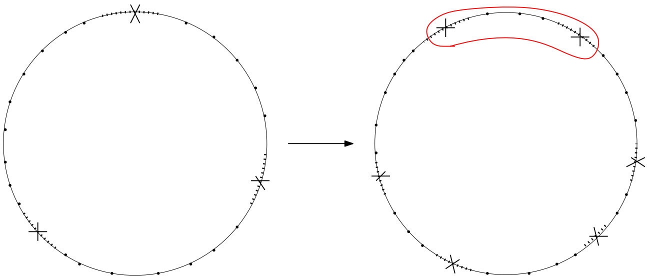
- $\mathcal{C}_{(S,M)}$ is a Hom-finite 2-Calabi-Yau triangulated category.
- $m \mapsto m^-$ is a bijection in M.
- Let $\ell_X : m - m'$, then $\ell_X[1] : m^- - m'^-$.



- $X \to A \oplus C \to Y \to X[1]$
- $Y \to B \oplus D \to X \to Y[1]$

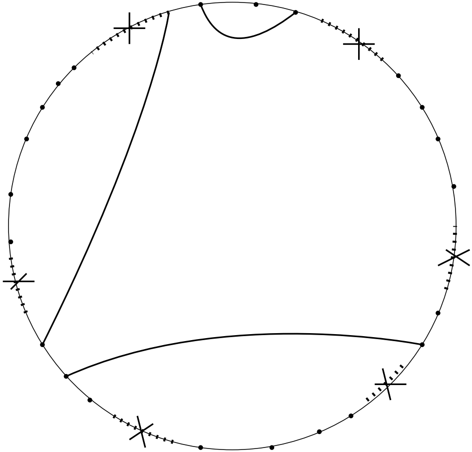
A completion of $\mathcal{C}_{(S,M)}$





- Replace each accumulation point z_i by a closed interval $[z_i^-, z_i^+]$ with marked points $\{z_{ij} \mid j \in \mathbb{Z}\}$ where $\lim_{j \to -\infty} z_{ij} = z_i^-$, $\lim_{j \to +\infty} z_{ij} = z_i^+$
- We obtain a new discrete cluster category $\mathcal{C}(S', M')$.

A subcategory $\mathcal{D}$



- We let $\mathcal{D}$ be the full additive subcategory generated by the objects where both endpoints belong to an added interval.
- Then $\mathcal{D}$ is a triangulated subcategory.

Verdier quotient of $\mathcal{C}(S', M')$

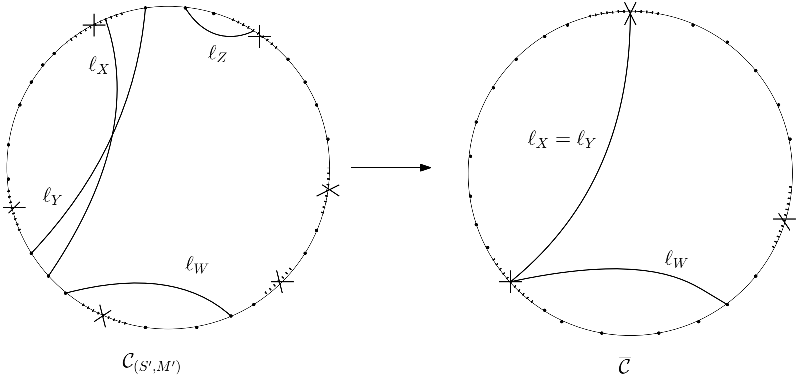
- $\Sigma = \{f : M \to N \mid \mathrm{cone}(f) \in \mathcal{D}\}$.
- Σ is a multiplicative system compatible with triangulated structure.
- We have a quotient category

$$\overline{\mathcal{C}} \quad := \quad \mathcal{C}(S', M')/\mathcal{D} \quad = \quad \mathcal{C}(S', M')(\Sigma^{-1})$$

- $\overline{\mathcal{C}}$ is a triangulated category.

Geometric description of $\overline{\mathcal{C}}(S, M)$

- Objects in $\overline{\mathcal{C}}$ are the same as objects in $\mathcal{C}(S', M')$.
- $\ell_X \sim \ell_Y$ in (S', M') if ℓ_X, ℓ_Y become the same when we collapse all the added intervals.
- Morphisms are some equivalence classes of left fractions $X \to Z \leftarrow Y$.



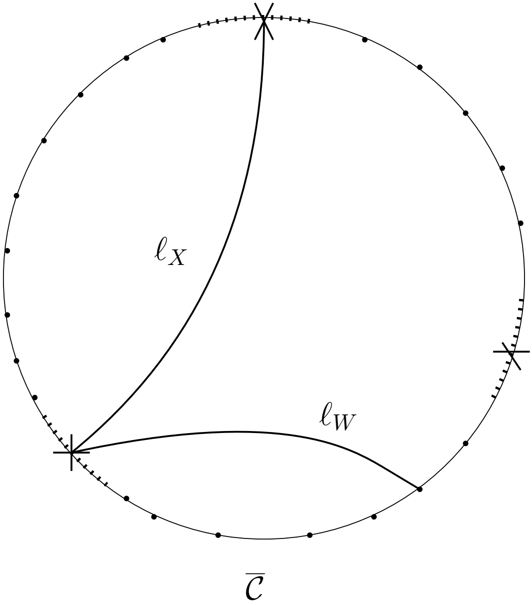
ℓ_X
ℓ_Z
ℓ_Y
ℓ_W
$\mathcal{C}_{(S',M')}$
$\ell_X = \ell_Y$
ℓ_W
$\overline{\mathcal{C}}$

- Therefore, indecomposable objects correspond to arcs of $(S, \overline{M})$.
- If a is an accumulation point, then $a^+ = a$.
- What is $\mathrm{Hom}(X, Y)$ in $\overline{\mathcal{C}}$?

Let X, Y be indecomposable objects in $\overline{\mathcal{C}}$. Then

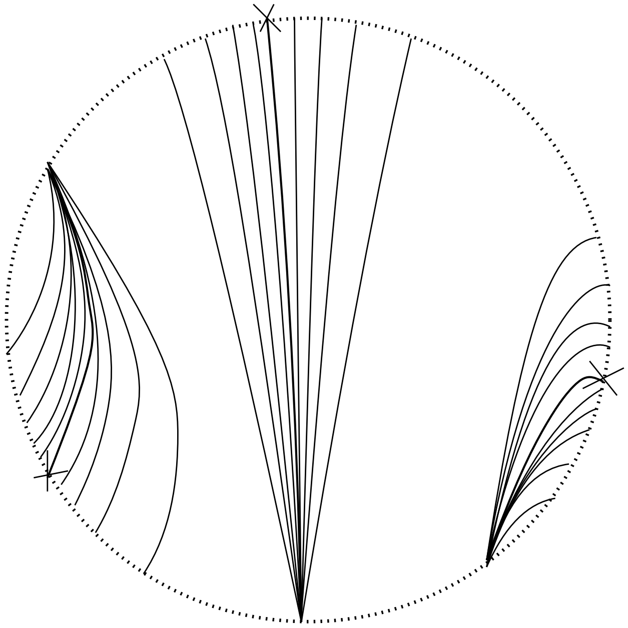
$$\mathrm{Hom}(X, Y[1]) = \begin{cases} \mathbb{k}, & \text{if } \ell_X, \ell_Y \text{ cross;} \\ \mathbb{k}, & \text{if } \ell_X, \ell_Y \text{ share an acc. pt. and } \ell_X \to \ell_Y; \\ 0, & \text{otherwise.} \end{cases}$$

Example



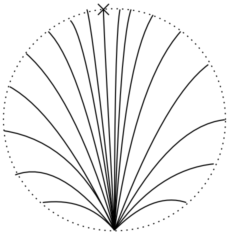
Cluster-tilting subcategories

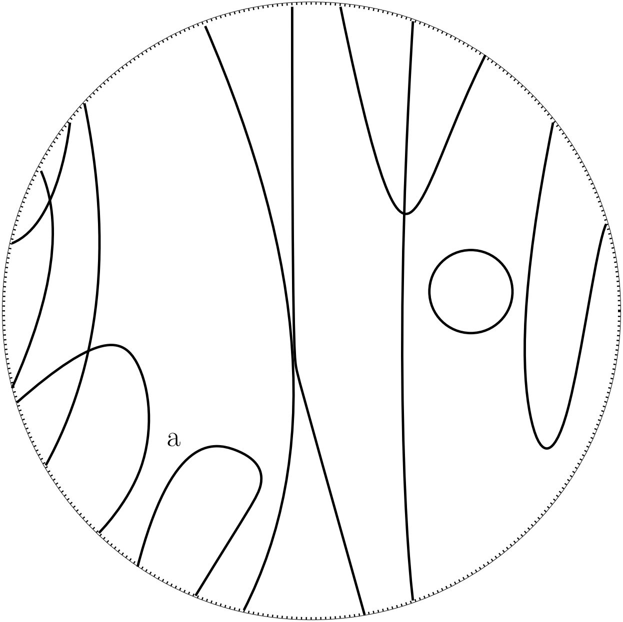
- A full additive subcategory $\mathcal{T}$ of $\overline{\mathcal{C}}$ is cluster tilting if
  (i) For $X \in \overline{\mathcal{C}}$, we have
  $$X \in \mathcal{T} \Leftrightarrow \mathrm{Hom}(X, \mathcal{T}[1]) = 0 \Leftrightarrow \mathrm{Hom}(\mathcal{T}, X[1]) = 0$$
  (ii) The subcategory $\mathcal{T}$ is functorially finite in $\overline{\mathcal{C}}$.
- We have a description of cluster-tilting subcategories for $\overline{\mathcal{C}}$.



Link with representation theory

- If $\mathcal{T}$ is cluster-tilting, then we have an equivalence $\overline{\mathcal{C}}/\mathcal{T}[1] \cong \mathrm{mod}^{\mathrm{fp}}\mathcal{T}$.

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