Amalgamation, unamalgamation and the phi-dimension conjecture

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Introduction

This is joint work with Eric Hanson based on ongoing joint work with Gordana Todorov on "amalgamation" and "unamalgamation" which in turn originated in joint work with Dani Álvarez-Gavela on Legendrian embeddings using plabic diagrams.

(a) Plabic diagrams and amalgamation.

(b) Counterexample to the ϕ-dimension conjecture.

phi-dimension conjecture

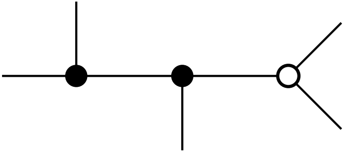
The ϕ-dimension conjecture states that, for any artin algebra Λ, there is a uniform bound on the ϕ-dimension of the f.g. Λ-modules. For modules of finite projective dimension, the projective dimension is equal to the ϕ-dimension. Therefore, findim-$\Lambda \leq \phi$-dim-Λ. So, ϕ-dim $\Lambda < \infty$ implies findim-$\Lambda < \infty$.

ϕ-dim-Λ is defined to be the supremum of $\phi(M)$ for all Λ-modules M. To get a lower bound on $\phi(M)$ we use the following.

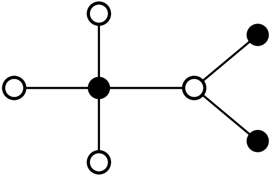
Lemma Let X, Y be Λ-modules so that $\Omega^k X \ncong \Omega^k Y$ but $\Omega^{k+1} X \cong \Omega^{k+1} Y$. Then $\phi(X \oplus Y) \geq k$.

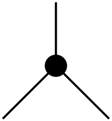
Plabic diagrams

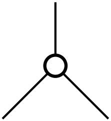
Here is a plabic diagram (planar bicolored graph).



Standard bipartite version
(1) Coalescing vertices of same color
(2) Add boundary vertices of opposite color. (We skip this step.)



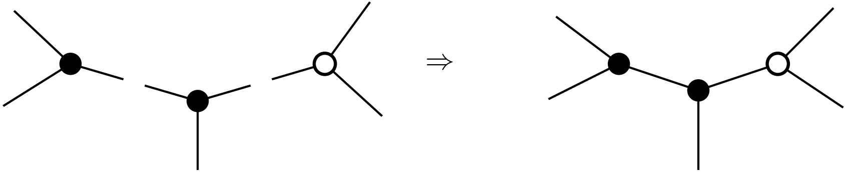




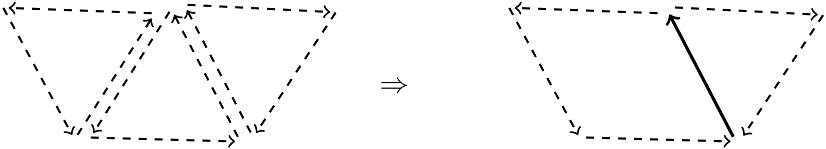
Plabic diagrams are assembled from the pieces on the left.

Jacobian algebra given by dual quiver

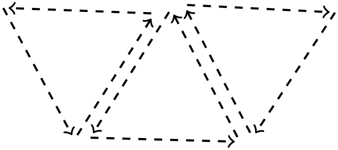
We assemble the plabic diagram out of pieces:



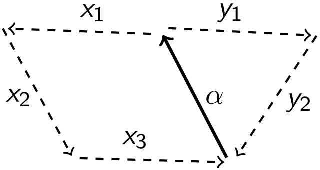
The quiver is also assembled from pieces:



Amalgamation (Fock-Goncharov)



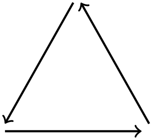
$\Rightarrow$



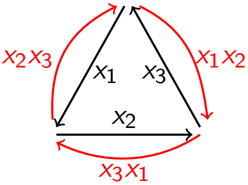
$\partial_\alpha W = 0$ (using only solid arrows) gives $x_1 x_2 x_3 = y_1 y_2$.

Take triangles with dotted arrows. These are half-arrows. When you add two half-arrows you get either a solid arrow or no arrow. For the Jacobian algebra, only derivatives with respect to solid arrows are set equal to zero.

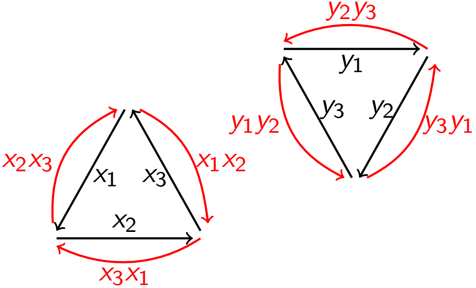
Amalgamation: an equivalent version

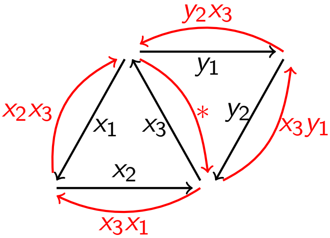


Take triangles with solid arrows.



Add "redundant" arrows (in red).

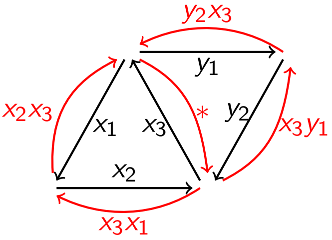




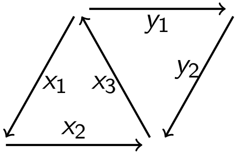
Amalgamate by identifying.

Red arrow $* = x_1x_2 = y_1y_2$

Result of amalgamation



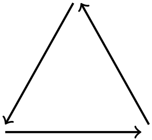
$* = x_1x_2 = y_1y_2$



with relation: $x_1x_2 = y_1y_2$

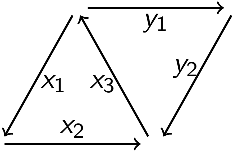
Summary: Adding redundant arrows, identifying arrows, then removing redundant arrows gives the Jacobian algebra of the F-G amalgamation.

We call this "amalgamation" (adding redundant arrow and identifying arrows).



Let C be the algebra given by the triangular quiver on the left modulo $rad^2 = 0$.

Let A be the algebra given by the quiver on the right modulo $rad^2 = 0$. (This is the amalgamation of two copies of C.)



Theorem $\Lambda = A \otimes C$ has infinite ϕ-dimension.[1]

[1] Two days after us, Barrios and Mata also posted an example (arXiv:1911.02325).

Outline of proof

To prove this we construct a sequence of pair of Λ-modules WX_k, WY_k so that

$$\Omega^{3k} WX_k \ncong \Omega^{3k} WY_k$$

but

$$\Omega^{3k+1} WX_k \cong \Omega^{3k+1} WY_k.$$

This implies that $\phi(WX_k \oplus WY_k) \geq 3k$. So, ϕ-dim $\Lambda \geq 3k$ for all k. So, ϕ-dim $\Lambda = \infty$.

The modules WX_k, WY_k are constructed out of chain complexes of A-modules X_k, Y_k.

Wrapping chain complexes

A $\Lambda = A \otimes C$ module M is a triple of modules M_1, M_2, M_3 and maps $d : M_i \to M_{i-1}$ so that $d^2 = 0$. Conversely, given any chain complex of A-modules

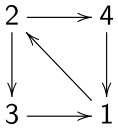
$$V_* : \qquad 0 \leftarrow V_0 \leftarrow V_1 \leftarrow V_2 \leftarrow V_3 \leftarrow \cdots$$

define WV_* to be the triple of A-modules $M_1 = V_1 \oplus V_4 \oplus V_7 \oplus \cdots$, $M_2 = V_2 \oplus V_5 \oplus \cdots$, $M_3 = V_0 \oplus V_3 \oplus V_6 \oplus \cdots$ with boundary maps $M_i \to M_{i-1}$ given by the boundary maps of V_*.

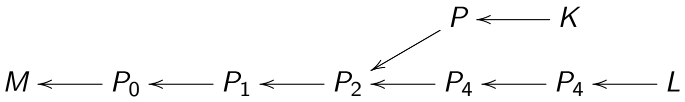
Lemma W is an exact functor which commutes with Ω and takes exact sequences to exact sequences.

The chain complexes X_k, Y_k

The A-chain complexes X_k, Y_k are truncated projective resolutions of different simple A-module S_3, S_4 of length $3k$.



The branch point moves to the left under syzygy:



Branch point moves to the left under syzygy

$$
\begin{array}{ccccccccccccc}
 & & & & & & & & P & \longleftarrow & K & & \\
 & & & & & & & \swarrow & & & & & \\
M & \longleftarrow & P_0 & \longleftarrow & P_1 & \longleftarrow & P_2 & \longleftarrow & P_3 & \longleftarrow & P_4 & \longleftarrow & L
\end{array}
$$

$$
\begin{array}{ccccccccccccc}
 & & & & & & P & \longleftarrow & P' & \longleftarrow & \Omega K & & \\
 & & & & & \swarrow & & & & & & & \\
\Omega M & \longleftarrow & P_1 & \longleftarrow & P_2 & \longleftarrow & P_3 & \longleftarrow & P_4 & \longleftarrow & P_5 & \longleftarrow & \Omega L
\end{array}
$$

$$
\begin{array}{ccccccccccccc}
 & & & & P & \longleftarrow & P' & \longleftarrow & P'' & \longleftarrow & \Omega^2 K & & \\
 & & & \swarrow & & & & & & & & & \\
\Omega^2 M & \longleftarrow & P_2 & \longleftarrow & P_3 & \longleftarrow & P_4 & \longleftarrow & P_5 & \longleftarrow & P_6 & \longleftarrow & \Omega^2 L
\end{array}
$$

$$
\begin{array}{ccccccccccccc}
 & & P & \longleftarrow & P' & \longleftarrow & P'' & \longleftarrow & P''' & \longleftarrow & \Omega^3 K & & \\
 & \swarrow & & & & & & & & & & & \\
\Omega^3 M & \longleftarrow & P_3 & \longleftarrow & P_4 & \longleftarrow & P_5 & \longleftarrow & P_6 & \longleftarrow & P_7 & \longleftarrow & \Omega^3 L
\end{array}
$$

Once more and we loose the information of how the resolution was truncated.

Branches fall off

The chain complexes $\Omega X_k, \Omega Y_k$ are both truncated projective resolution of the same module S_1. They are truncated in different ways in degree $3k$. So, after taking $3k$ syzygies, when the "branch" "falls off" we cannot tell the difference and they become isomorphic:

$$\Omega^{3k+1} X_k \cong \Omega^{3k+1} Y_k$$

Since the wrapping functor is exact and takes projectives to projectives we get

$$\Omega^{3k+1} WX_k \cong \Omega^{3k+1} WY_k.$$

Since $\Omega^{3k} X_k$ and $\Omega^{3k} Y_k$ are truncated differently we can show that

$$\Omega^{3k} WX_k \ncong \Omega^{3k} WY_k.$$

So, ϕ-dim Λ is unbounded.

THANK YOU!

Figure from ribbon Legendrians paper

(with D. Álvarez-Gavela)

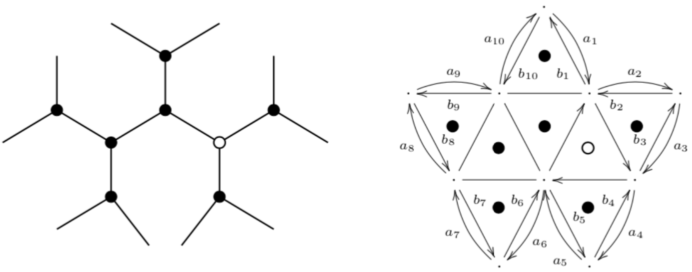


FIGURE 9. A spanning tree in a bicolored trivalent ribbon graph (left) produces, on the perimeter, a cyclic quiver with n clockwise arrow a_i and n counterclockwise arrows

(example used to illustrate the proof of the main theorem)