Chapter 6

6.1 Eigenvalues and eigenvectors of S_d are:

$$\lambda_1 = 449.778, \quad \mathbf{e}_1' = [.333, \ .943]$$
$$\lambda_2 = 168.082, \quad \mathbf{e}_2' = [.943, \ -.333]$$

Ellipse centered at $\bar{\mathbf{d}}' = [-9.36, \ 13.27]$. Half length of major axis is 20.57 units. Half length of minor axis is 12.58 units. Major and minor axes lie in $\mathbf{e}_1$ and $\mathbf{e}_2$ directions, respectively.

Yes, the test answers the question: Is $\boldsymbol{\delta} = \mathbf{0}$ inside the 95% confidence ellipse?

6.2 Using a critical value $t_{n-1}(\alpha/2p) = t_{10}(0.0125) = 2.6338$,

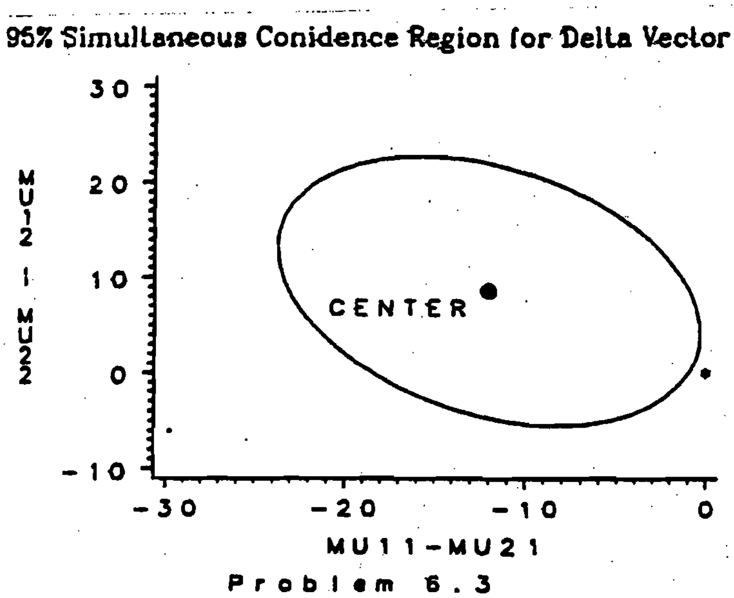
	LOWER	UPPER
Bonferroni C. I.:	-20.57	1.85
	-2.97	29.52
Simultaneous C. I.:	-22.45	3.73
	-5.70	32.25

Simultaneous confidence intervals are larger than Bonferroni's confidence intervals.

6.3 The 95% Bonferroni intervals are

	LOWER	UPPER
Bonferroni C. I.:	-21.92	-2.08
	-3.36	20.56
Simultaneous C. I.:	-23.70	-0.30
	-5.50	22.70

Since the hypothesized vector $\delta = 0$ (denoted as $*$ in the plot) is outside the joint confidence region, we reject $H_0 : \delta = 0$. Bonferroni C.I. are consistent with this result. After the elimination of the outlier, the difference between pairs became significant.



6.4

(a). Hotelling's $T^2 = 10.215$. Since the critical point with $\alpha = 0.05$ is 9.459, we reject $H_0 : \underset{\sim}{\delta} = \underset{\sim}{0}$.

(b).

	Lower	Upper
Bonferroni C. I.:	−1.09	−0.02
	−0.04	0.64
T^2 Simultaneous C. I.:	−1.18	0.07
	−0.10	0.69

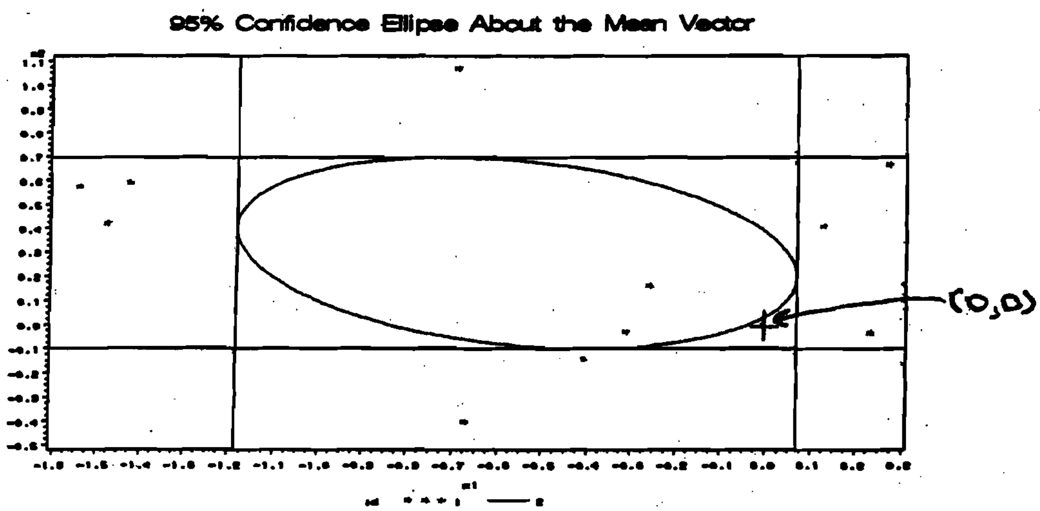
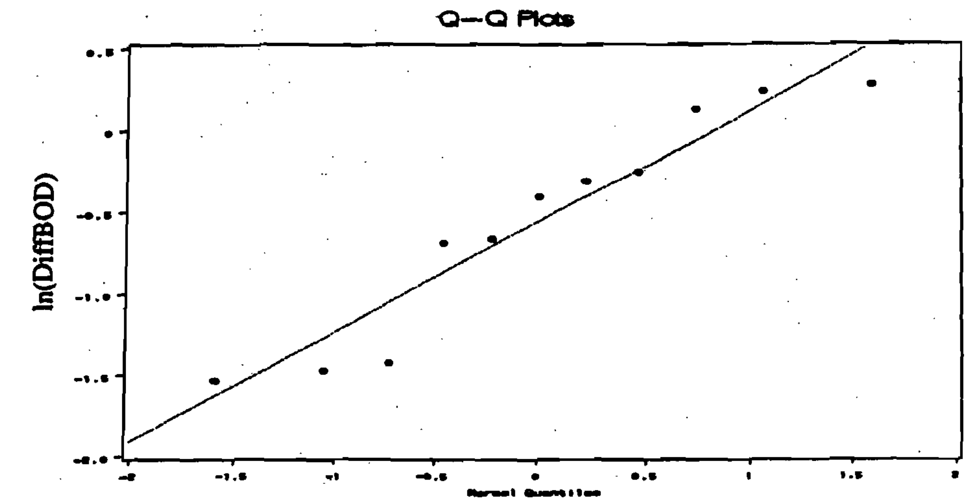
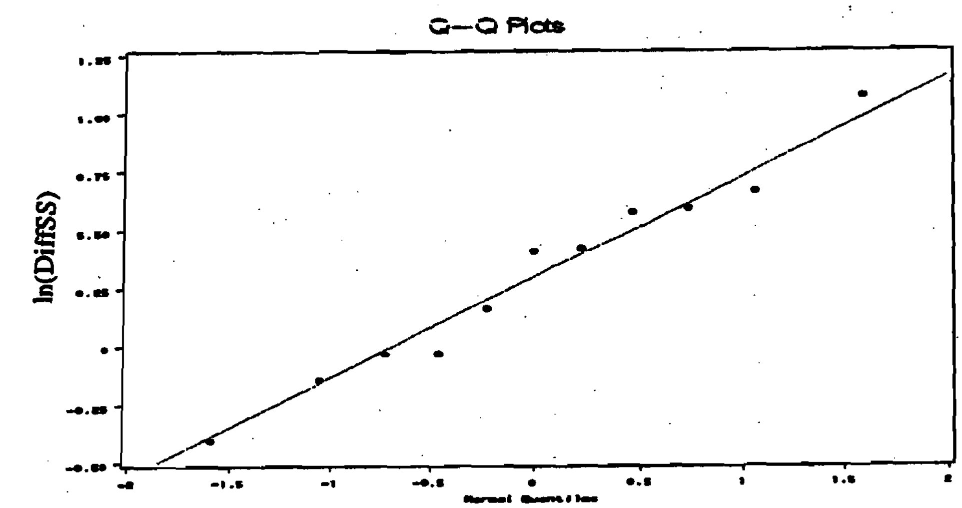
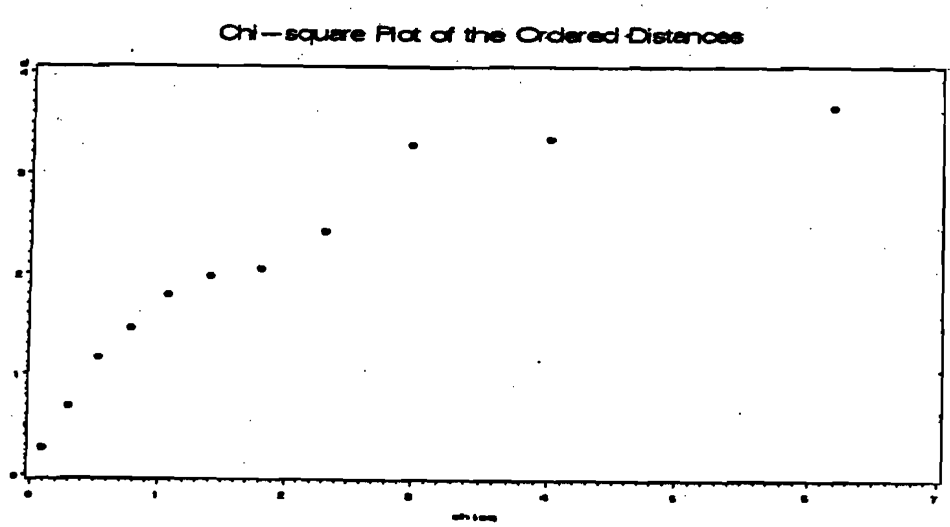


Figure 1: 95% Confidence Ellipse and Simultaneous T^2 Intervals for the Mean Difference

(c) The Q-Q plots for ln(DiffBOD) and ln(DiffSS) are shown below. Marginal normality cannot be rejected for either variable. The χ^2 plot is not straight (with at least one apparent bivariate outlier) and, although the sample size (n =11) is small, it is difficult to argue for bivariate normality.







6.5 a) H_0: $C\underset{\sim}{\mu} = \underset{\sim}{0}$ where $C = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$, $\underset{\sim}{\mu}' = [\mu_1, \mu_2, \mu_3]$.

$$C\bar{\underset{\sim}{x}} = \begin{bmatrix} -11.2 \\ 6.9 \end{bmatrix}, \quad CSC' = \begin{bmatrix} 55.5 & -32.6 \\ -32.6 & 66.4 \end{bmatrix}$$

$$T^2 = n(C\bar{\underset{\sim}{x}})'(CSC')^{-1}(C\bar{\underset{\sim}{x}}) = 90.4; \quad n = 40; \quad q = 3$$

$$\frac{(n-1)(q-1)}{(n-q+1)} F_{q-1,n-q+1}(.05) = \frac{(39)2}{38}(3.25) = 6.67$$

Since $T^2 = 90.4 > 6.67$ reject H_0: $C\underset{\sim}{\mu} = \underset{\sim}{0}$

b) 95% simultaneous confidence intervals:

$\mu_1 - \mu_2$: $(46.1 - 57.3) \pm \sqrt{6.67}\sqrt{\frac{55.5}{40}} = -11.2 \pm 3.0$

$\mu_2 - \mu_3$: 6.9 ± 3.3

$\mu_1 - \mu_3$: -4.3 ± 3.3

The means are all different from one another.

6.6 a) Treatment 2: Sample mean vector $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$; sample covariance matrix

$$\begin{bmatrix} 1 & -3/2 \\ -3/2 & 3 \end{bmatrix}$$

Treatment 3: Sample mean vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$; sample covariance matrix

$$\begin{bmatrix} 2 & -4/3 \\ -4/3 & 4/3 \end{bmatrix}$$

$$S_{pooled} = \begin{bmatrix} 1.6 & \\ -1.4 & \end{bmatrix}$$

b) $$T^2 = [2-3,\ 4-2] \left[\left(\frac{1}{3} + \frac{1}{4}\right) \begin{bmatrix} 1.6 & -1.4 \\ -1.4 & 2 \end{bmatrix} \right]^{-1} \begin{bmatrix} 2-3 \\ 4-2 \end{bmatrix} = 3.88$$

$$\frac{(n_1+n_2-2)p}{(n_1+n_2-p-1)} F_{p,n_1+n_2-p-1}(.01) = \frac{(5)2}{4}(18) = 45$$

Since $T^2 = 3.88 < 45$ do not reject $H_0: \underset{\sim}{\mu}_2 - \underset{\sim}{\mu}_3 = \underset{\sim}{0}$ at the $\alpha = .01$ level.

c). 99% simultaneous confidence intervals:

$$\mu_{21} - \mu_{31}: (2-3) \pm \sqrt{45}\ \sqrt{\left(\frac{1}{3}+\frac{1}{4}\right)1.6} = -1 \pm 6.5$$

$$\mu_{22} - \mu_{32}: 2 \pm 7.2$$

6.7 $$T^2 = [74.4\ \ 201.6] \left[\left(\frac{1}{45} + \frac{1}{55}\right) \begin{bmatrix} 10963.7 & 21505.5 \\ 21505.5 & 63661.3 \end{bmatrix} \right]^{-1} \begin{bmatrix} 74.4 \\ 201.6 \end{bmatrix} = 16.1$$

$$\frac{(n_1+n_2-2)p}{n_1+n_2-p-1} F_{p,n_1+n_2-p-1}(.05) = 6.26$$

Since $T^2 = 16.1 > 6.26$ reject $H_0: \underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2 = \underset{\sim}{0}$ at the $\alpha = .05$ level.

$$\hat{\underset{\sim}{a}} = S^{-1}_{pooled}(\bar{\underset{\sim}{x}}_1 - \bar{\underset{\sim}{x}}_2) = \begin{bmatrix} .0017 \\ .0026 \end{bmatrix}$$

6.8 a) For first variable:

observation = mean + treatment effect + residual

$$\begin{bmatrix} 6 & 5 & 8 & 4 & 7 \\ 3 & 1 & 2 & & \\ 2 & 5 & 3 & 2 & \end{bmatrix} = \begin{bmatrix} 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & & \\ 4 & 4 & 4 & 4 & \end{bmatrix} + \begin{bmatrix} 2 & 2 & 2 & 2 & 2 \\ -2 & -2 & -2 & & \\ -1 & -1 & -1 & -1 & \end{bmatrix} + \begin{bmatrix} 0 & -1 & 2 & -2 & 1 \\ 1 & -1 & 0 & & \\ -1 & 2 & 0 & -1 & \end{bmatrix}$$

$SS_{obs} = 246$ $\quad SS_{mean} = 192$ $\quad SS_{tr} = 36$ $\quad SS_{res} = 18$

For second variable:

$$\begin{bmatrix} 7 & 9 & 6 & 9 & 9 \\ 3 & 6 & 3 & & \\ 3 & 1 & 1 & 3 & \end{bmatrix} = \begin{bmatrix} 5 & 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & & \\ 5 & 5 & 5 & 5 & \end{bmatrix} + \begin{bmatrix} 3 & 3 & 3 & 3 & 3 \\ -1 & -1 & -1 & & \\ -3 & -3 & -3 & -3 & \end{bmatrix} + \begin{bmatrix} -1 & 1 & -2 & 1 & 1 \\ -1 & 2 & -1 & & \\ 1 & -1 & -1 & 1 & \end{bmatrix}$$

$SS_{obs} = 402$ $\quad SS_{mean} = 300$ $\quad SS_{tr} = 84$ $\quad SS_{res} = 18$

Cross product contributions:

275 $\quad$ 240 $\quad$ 48 $\quad$ -13

b) MANOVA table:

Source of Variation	SSP	d.f.
Treatment	$B = \begin{bmatrix} 36 & 48 \\ 48 & 84 \end{bmatrix}$	$3 - 1 = 2$
Residual	$W = \begin{bmatrix} 18 & -13 \\ -13 & 18 \end{bmatrix}$	$5 + 3 + 4 - 3 = 9$
Total (corrected)	$\begin{bmatrix} 54 & 35 \\ 35 & 102 \end{bmatrix}$	11

c) $\Lambda^* = \frac{|W|}{|B+W|} = \frac{155}{4283} = .0362$

Using Table 6.3 with $p = 2$ and $g = 3$

$$\left(\frac{1-\sqrt{\Lambda^*}}{\sqrt{\Lambda^*}}\right)\left(\frac{\Sigma n_\ell - g - 1}{g-1}\right) = 17.02 .$$

Since $F_{4,16}(.01) = 4.77$ we conclude that treatment differences exist at $\alpha = .01$ level.

Alternatively, using Bartlett's procedure,

$$-\left(n - 1 - \frac{(p+g)}{2}\right) \ln \Lambda^* = -(12 - 1 - \tfrac{5}{2})\ln(.0362) = 28.209$$

Since $\chi^2_4(.01) = 13.28$ we again conclude treatment differences exist at $\alpha = .01$ level.

6.9 For <u>any</u> matrix C

$$\bar{\underset{\sim}{d}} = \frac{1}{n}\Sigma\, \underset{\sim}{d}_j = C(\frac{1}{n}\Sigma\, \underset{\sim}{x}_j) = C\,\bar{\underset{\sim}{x}}$$

and $\underset{\sim}{d}_j - \bar{\underset{\sim}{d}} = C(\underset{\sim}{x}_j - \bar{\underset{\sim}{x}})$

so $S_d = \frac{1}{n-1}\Sigma(\underset{\sim}{d}_j-\bar{\underset{\sim}{d}})(\underset{\sim}{d}_j-\bar{\underset{\sim}{d}})' = C(\frac{1}{n-1}\Sigma(\underset{\sim}{x}_j-\bar{\underset{\sim}{x}})(\underset{\sim}{x}_j-\bar{\underset{\sim}{x}})')C' = CSC'$

6.10 $(\bar{x}\,\underset{\sim}{1})'[(\bar{x}_1-\bar{x})\underset{\sim}{u}_1 + \dots + (\bar{x}_g-\bar{x})\underset{\sim}{u}_g]$

$$= \bar{x}[(\bar{x}_1-\bar{x})n_1 + \dots + (\bar{x}_g-\bar{x})n_g]$$

$$= \bar{x}[n_1\bar{x}_1 + \dots + n_g\bar{x}_g - \bar{x}(n_1 + \dots + n_g)]$$

$$= \bar{x}[(n_1 + \dots + n_g)\bar{x} - \bar{x}(n_1 + \dots + n_g)] = 0$$