Chapter 3

3.1

a) $\bar{\mathbf{x}} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

b) $\mathbf{e}_1 = \mathbf{y}_1 - \bar{x}_1 \mathbf{1} = [4, 0, -4]'$

$\mathbf{e}_2 = \mathbf{y}_2 - \bar{x}_2 \mathbf{1} = [-1, 1, 0]'$

c) $L_{\mathbf{e}_1} = \sqrt{32}$; $L_{\mathbf{e}_2} = \sqrt{2}$

Let θ be the angle between $\mathbf{e}_1$ and $\mathbf{e}_2$, then $\cos(\theta) = -4/\sqrt{32 \times 2} = -.5$

Therefore $n\, s_{11} = L^2_{\mathbf{e}_1}$ or $s_{11} = 32/3$; $n\, s_{22} = L^2_{\mathbf{e}_2}$ or $s_{22} = 2/3$; $n\, s_{12} = \mathbf{e}_1' \mathbf{e}_2$ or $s_{12} = -4/3$. Also, $r_{12} = \cos(\theta) = -.5$. Consequently $S_n = \begin{bmatrix} 32/3 & -4/3 \\ -4/3 & 2/3 \end{bmatrix}$ and $R = \begin{bmatrix} 1 & -.5 \\ -.5 & 1 \end{bmatrix}$.

3.2

a) $\bar{\mathbf{x}} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

b) $\mathbf{e}_1 = \mathbf{y}_1 - \bar{x}_1 \mathbf{1} = [-1, 2, -1]'$

$\mathbf{e}_2 = \mathbf{y}_2 - \bar{x}_2 \mathbf{1} = [3, -3, 0]'$

c) $L_{\mathbf{e}_1} = \sqrt{6}$; $L_{\mathbf{e}_2} = \sqrt{18}$

Let θ be the angle between $\mathbf{e}_1$ and $\mathbf{e}_2$, then $\cos(\theta) = -9/\sqrt{6 \times 18} = -.866$.

Therefore $n\, s_{11} = L^2_{\mathbf{e}_1}$ or $s_{11} = 6/3 = 2$; $n\, s_{22} = L^2_{\mathbf{e}_2}$ or $s_{22} = 18/3 = 6$; $n\, s_{12} = \mathbf{e}_1' \mathbf{e}_2$ or $s_{12} = -9/3 = -3$. Also, $r_{12} = \cos(\theta) = -.866$. Consequently $S_n = \begin{bmatrix} 2 & -3 \\ -3 & 6 \end{bmatrix}$ and $R = \begin{bmatrix} 1 & -.866 \\ -.866 & 1 \end{bmatrix}$

3.3 $\underset{\sim}{y}_1 = [1, 4, 4]'$; $\bar{x}_1 \underset{\sim}{1} = [3, 3, 3]$; $\underset{\sim}{y}_1 - \bar{x}_1 \underset{\sim}{1} = [-2, 1, 1]'$

Thus

$$\underset{\sim}{y}_1 = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \bar{x}_1 \underset{\sim}{1} + (\underset{\sim}{y}_1 - \bar{x}_1 \underset{\sim}{1})$$

3.5 a) $\mathbf{X}' = \begin{bmatrix} 9 & 5 & 1 \\ 1 & 3 & 2 \end{bmatrix}$; $\bar{\underset{\sim}{x}} \underset{\sim}{1}' = \begin{bmatrix} 5 & 5 & 5 \\ 2 & 2 & 2 \end{bmatrix}$

$$2\,S = (\mathbf{X} - \bar{\underset{\sim}{x}} \underset{\sim}{1}')(\mathbf{X} - \bar{\underset{\sim}{x}} \underset{\sim}{1}')' = \begin{bmatrix} 4 & 0 & -4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 0 & 1 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 32 & -4 \\ -4 & 2 \end{bmatrix}$$

so $S = \begin{bmatrix} 16 & -2 \\ -2 & 1 \end{bmatrix}$ and $|S| = 12$

b) $\mathbf{X}' = \begin{bmatrix} 3 & 6 & 3 \\ 4 & -2 & 1 \end{bmatrix}$; $\bar{\underset{\sim}{x}} \underset{\sim}{1}' = \begin{bmatrix} 4 & 4 & 4 \\ 1 & 1 & 1 \end{bmatrix}$

$$2\,S = (\mathbf{X} - \underset{\sim}{1}\,\bar{\underset{\sim}{x}}')'(\mathbf{X} - \underset{\sim}{1}\,\bar{\underset{\sim}{x}}') = \begin{bmatrix} -1 & 2 & -1 \\ 3 & -3 & 0 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & -3 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & -9 \\ -9 & 18 \end{bmatrix}$$

so $S = \begin{bmatrix} 3 & -9/2 \\ -9/2 & 9 \end{bmatrix}$ and $|S| = 27/4$

3.6 a) $\mathbf{X}' - \underset{\sim}{1}\,\bar{\underset{\sim}{x}}' = \begin{bmatrix} -3 & 0 & -3 \\ 0 & 1 & 1 \\ 3 & -1 & 2 \end{bmatrix}$. Thus $\underset{\sim}{d}'_1 = [-3, 0, -3]$,

$\underset{\sim}{d}'_2 = [0, 1, -1]$ and $\underset{\sim}{d}'_3 = [-3, 1, 2]$.

Since $\underset{\sim}{d}_1 = \underset{\sim}{d}_2 = \underset{\sim}{d}_3$, the matrix of deviations is not of full rank.

b)

$$2\,S = (\underset{\sim}{X}-\underset{\sim}{1}\,\bar{\underset{\sim}{x}}')'(\underset{\sim}{X}-\underset{\sim}{1}\,\bar{\underset{\sim}{x}}') = \begin{bmatrix} 18 & -3 & 15 \\ -3 & 2 & -1 \\ 15 & -1 & 14 \end{bmatrix}$$

So

$$S = \begin{bmatrix} 9 & -3/2 & 15/2 \\ -3/2 & 1 & -1/2 \\ 15/2 & -1/2 & 7 \end{bmatrix}$$

$|S| = 0$ (Verify). The 3 deviation vectors lie in a 2-dimensional subspace. The 3-dimensional volume enclosed by the deviation vectors is zero.

c) Total sample variance = 9 + 1 + 7 = 17 .

3.7 All ellipses are centered at $\bar{\underset{\sim}{x}}$.

i) For $S = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$, $S^{-1} = \begin{bmatrix} 5/9 & -4/9 \\ -4/9 & 5/9 \end{bmatrix}$

Eigenvalue-normalized eigenvector pairs for S^{-1} are:

$\lambda_1 = 1,\ \underset{\sim}{e}_1' = [.707,\ \ -.707]$

$\lambda_2 = 1/9,\ \underset{\sim}{e}_2' = [.707,\ \ .707]$

Half lengths of axes of ellipse $(\underset{\sim}{x}-\bar{\underset{\sim}{x}})'S^{-1}(\underset{\sim}{x}-\bar{\underset{\sim}{x}}) \le 1$ are $1/\sqrt{\lambda_1} = 1$ and $1/\sqrt{\lambda_2} = 3$ respectively. The major axis of ellipse lies in the direction of $\underset{\sim}{e}_2$; the minor axis lies in the direction of $\underset{\sim}{e}_1$.

ii) For $S = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$, $S^{-1} = \begin{bmatrix} 5/9 & 4/9 \\ 4/9 & 5/9 \end{bmatrix}$

Eigenvalue-normalized eigenvectors for S^{-1} are:

$\lambda_1 = 1,\ \underset{\sim}{e}_1' = [.707,\ \ .707]$

$\lambda_2 = 1/9,\ \underset{\sim}{e}_2' = [.707,\ \ -.707]$

Half lengths of axes of ellipse $(\underset{\sim}{x}-\bar{\underset{\sim}{x}})'S^{-1}(\underset{\sim}{x}-\bar{\underset{\sim}{x}}) \le 1$ are, again, $1/\sqrt{\lambda_1} = 1$ and $1/\sqrt{\lambda_2} = 3$. The major axes of the ellipse lies in the direction of $\underset{\sim}{e}_2$; the minor axis lies in the direction of $\underset{\sim}{e}_1$. Note that $\underset{\sim}{e}_2$ here is $\underset{\sim}{e}_1$ in part (i) above and $\underset{\sim}{e}_1$ here is $\underset{\sim}{e}_2$ in part (i) above.

iii) For $S = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, $S^{-1} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix}$

Eigenvalue-normalized eigenvector pairs for S^{-1} are:

$$\lambda_1 = 1/3; \quad \underset{\sim}{e}_1' = [1, 0]$$

$$\lambda_2 = 1/3, \quad \underset{\sim}{e}_2' = [0, 1]$$

Half lengths of axes of ellipse $(\underset{\sim}{x}-\bar{\underset{\sim}{x}})'S^{-1}(\underset{\sim}{x}-\bar{\underset{\sim}{x}}) \le 1$ are equal and given by $1/\sqrt{\lambda_1} = 1/\sqrt{\lambda_2} = \sqrt{3}$. Major and minor axes of ellipse can be taken to lie in the directions of the coordinate axes. Here, the solid ellipse is, in fact, a solid sphere.

Notice for all three cases $|S| = 9$.

3.8 a) Total sample variance in both cases is 3 .

b) For $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $|S| = 1$

For $S = \begin{bmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{bmatrix}$, $|S| = 0$

3.9 **(a)** We calculate $\bar{x} = [16, 18, 34\,]'$ and

$$X_c = \begin{bmatrix} -4 & -1 & -5 \\ 2 & 2 & 4 \\ -2 & -2 & -4 \\ 4 & 0 & 4 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and we notice } \text{col}_1(\, X_c\,) + \text{col}_2(\, X_c\,) = \text{col}_1(\, X_c\,)$$

so $a = [1, 1, -1\,]'$ gives $X_c a = 0$.

(b)

$$S = \begin{bmatrix} 10 & 3 & 13 \\ 3 & 2.5 & 5.5 \\ 13 & 5.5 & 18.5 \end{bmatrix} \quad \text{so } |\,S\,| = \begin{matrix} 10(2.5)(18.5) & + & 39(15.5) & + & 39(15.5) \\ -(13)^2(2.5) & - & 9(18.5) & - & 55(5.5) \end{matrix} = 0$$

As above in a)

$$Sa = \begin{bmatrix} 10 + 3 - 13 \\ 3 + 2.5 - 5.5 \\ 13 + 5.5 - 18.5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(c) Check.

3.10 **(a)** We calculate $\bar{x} = [5, 2, 3\,]'$ and

$$X_c = \begin{bmatrix} -2 & -1 & -3 \\ 1 & 2 & 3 \\ -1 & 0 & -1 \\ 2 & -2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and we notice } \text{col}_1(\, X_c\,) + \text{col}_2(\, X_c\,) = \text{col}_1(\, X_c\,)$$

so $a = [1, 1, -1\,]'$ gives $X_c a = 0$.

(b)

$$S = \begin{bmatrix} 2.5 & 0 & 2.5 \\ 0 & 2.5 & 2.5 \\ 2.5 & 2.5 & 5 \end{bmatrix} \quad \text{so } |\,S\,| = \begin{matrix} 5(2.5)^2 & + & 0 & + & 0 \\ -(2.5)^3 & - & 0 & - & (2.5)^3 \end{matrix} = 0$$

Using the save coefficient vector a as in Part a) $Sa = 0$.

(c) Setting $\boldsymbol{X}\boldsymbol{a} = 0$,

$$\begin{array}{lllll} 3a_1 & + & a_2 & & = 0 \\ 7a_1 & & & + \; 3a_3 & = 0 \\ 5a_1 & + & 3a_2 & + \; 4a_3 & = 0 \end{array} \quad \text{so} \quad \begin{array}{lll} a_1 & = & -\tfrac{3}{7}a_3 \\ 5a_1 & - & 3(3a_1) + 4a_3 = 0 \end{array}$$

so we must have $a_1 = a_3 = 0$ but then, by the first equation in the first set, $a_2 = 0$. The columns of the data matrix are linearly independent.

3.11

$$S = \begin{bmatrix} 14808 & 14213 \\ 14213 & 15538 \end{bmatrix}.$$ Consequently

$$R = \begin{bmatrix} 1 & .9370 \\ .9370 & 1 \end{bmatrix}; \quad D^{1/2} = \begin{bmatrix} 121.6881 & 0 \\ 0 & 124.6515 \end{bmatrix}$$

and $$D^{-1/2} = \begin{bmatrix} .0082 & 0 \\ 0 & .0080 \end{bmatrix}$$

The relationships $R = D^{-1/2}\, S\, D^{-1/2}$ and $S = D^{1/2}\, R\, D^{1/2}$ can now be verified by direct matrix multiplication.

3.14 a) From first principles we have

$$\underset{\sim}{b}'\,\underset{\sim}{x}_1 = [2 \quad 3]\begin{bmatrix} 9 \\ 1 \end{bmatrix} = 21$$

Similarly $\underset{\sim}{b}'\,\underset{\sim}{x}_2 = 19$ and $\underset{\sim}{b}'\,\underset{\sim}{x}_3 = 8$ so

$$\text{sample mean} = \frac{21+19+8}{3} = 16$$

$$\text{sample variance} = \frac{(21-16)^2+(19-16)^2+(8-16)^2}{2} = 49$$

Also $\underset{\sim}{c}'\,\underset{\sim}{x}_1 = [-1 \quad 2]\begin{bmatrix} 9 \\ 1 \end{bmatrix} = -7$; $\underset{\sim}{c}'\,\underset{\sim}{x}_2 = 1$ and $\underset{\sim}{c}'\,\underset{\sim}{x}_3 = 3$

so

sample mean = -1

sample variance = 28

Finally sample covariance = $\frac{(21-16)(-7+1)+(19-16)(1+1)+(8-16)(3+1)}{2}$ =

-28.

b) $\underset{\sim}{\bar{x}}' = [5 \quad 2]$ and $S = \begin{bmatrix} 16 & -2 \\ -2 & 1 \end{bmatrix}$

Using (3-36)

sample mean of $\underset{\sim}{b}'\underset{\sim}{X} = \underset{\sim}{b}'\bar{\underset{\sim}{x}} = [2 \quad 3]\begin{bmatrix}5\\2\end{bmatrix} = 16$

sample mean of $\underset{\sim}{c}'\underset{\sim}{X} = [-1 \quad 2]\begin{bmatrix}5\\2\end{bmatrix} = -1$

sample variance of $\underset{\sim}{b}'\underset{\sim}{X} = \underset{\sim}{b}'S\underset{\sim}{b} = [2 \quad 3]\begin{bmatrix}16 & -2\\-2 & 1\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix} = 49$

sample variance of $\underset{\sim}{c}'\underset{\sim}{X} = \underset{\sim}{c}'S\underset{\sim}{c} = [-1 \quad 2]\begin{bmatrix}16 & -2\\-2 & 1\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix} = 28$

sample covariance of $\underset{\sim}{b}'\underset{\sim}{X}$ and $\underset{\sim}{c}'\underset{\sim}{X}$

$= \underset{\sim}{b}'S\underset{\sim}{c} = [2 \quad 3]\begin{bmatrix}16 & -2\\-2 & 1\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix} = -28$

Results same as those in part (a).

3.15

$$\bar{\underset{\sim}{x}} = \begin{bmatrix}5\\3\\4\end{bmatrix}, \quad S = \begin{bmatrix}13 & -2.5 & 1.5\\-2.5 & 1 & -1.5\\1.5 & -1.5 & 3\end{bmatrix}$$

sample mean of $\underset{\sim}{b}'\underset{\sim}{X} = 12$

sample mean of $\underset{\sim}{c}'\underset{\sim}{X} = -1$

sample variance of $\underset{\sim}{b}'\underset{\sim}{X} = 12$

sample variance of $\underset{\sim}{c}'\underset{\sim}{X} = 43$

sample covariance of $\underset{\sim}{b}'\underset{\sim}{X}$ and $\underset{\sim}{c}'\underset{\sim}{X} = -3$

3.16 Since
$$\Sigma_V = E(\mathbf{V}-\boldsymbol{\mu}_V)(\mathbf{V}-\boldsymbol{\mu}_V)'$$
$$= E(\mathbf{V}\mathbf{V}' - \mathbf{V}\boldsymbol{\mu}_V' - \boldsymbol{\mu}_V\mathbf{V}' + \boldsymbol{\mu}_V\boldsymbol{\mu}_V')$$
$$= E(\mathbf{V}\mathbf{V}') - E(\mathbf{V})\boldsymbol{\mu}_V' - \boldsymbol{\mu}_V E(\mathbf{V}') + \boldsymbol{\mu}_V\boldsymbol{\mu}_V'$$
$$= E(\mathbf{V}\mathbf{V}') - \boldsymbol{\mu}_V\boldsymbol{\mu}_V' - \boldsymbol{\mu}_V\boldsymbol{\mu}_V' + \boldsymbol{\mu}_V\boldsymbol{\mu}_V'$$
$$= E(\mathbf{V}\mathbf{V}') - \boldsymbol{\mu}_V\boldsymbol{\mu}_V' ,$$

we have $E(\mathbf{V}\mathbf{V}') = \Sigma + \boldsymbol{\mu}_V\boldsymbol{\mu}_V'$.

3.18 **(a)** Let $y = x_1+x_2+x_3+x_4$ be the total energy consumption. Then

$$\bar{y} = [1 \quad 1 \quad 1 \quad 1]\bar{\mathbf{x}} = 1.873$$
$$s_y^2 = [1 \quad 1 \quad 1 \quad 1]\mathbf{S}[1 \quad 1 \quad 1 \quad 1]' = 3.913$$

(b) Let $y = x_1-x_2$ be the excess of petroleum consumption over natural gas consumption. Then

$$\bar{y} = [1 \quad -1 \quad 0 \quad 0]\bar{\mathbf{x}} = .258$$
$$s_y^2 = [1 \quad -1 \quad 0 \quad 0]\mathbf{S}[1 \quad -1 \quad 0 \quad 0]' = .154$$