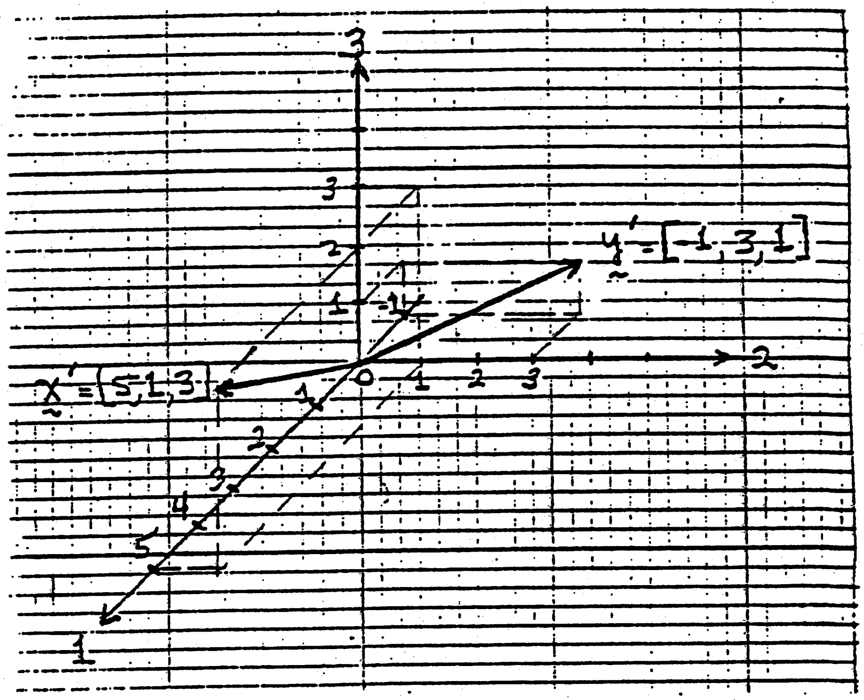
Chapter 2

2.1 a)



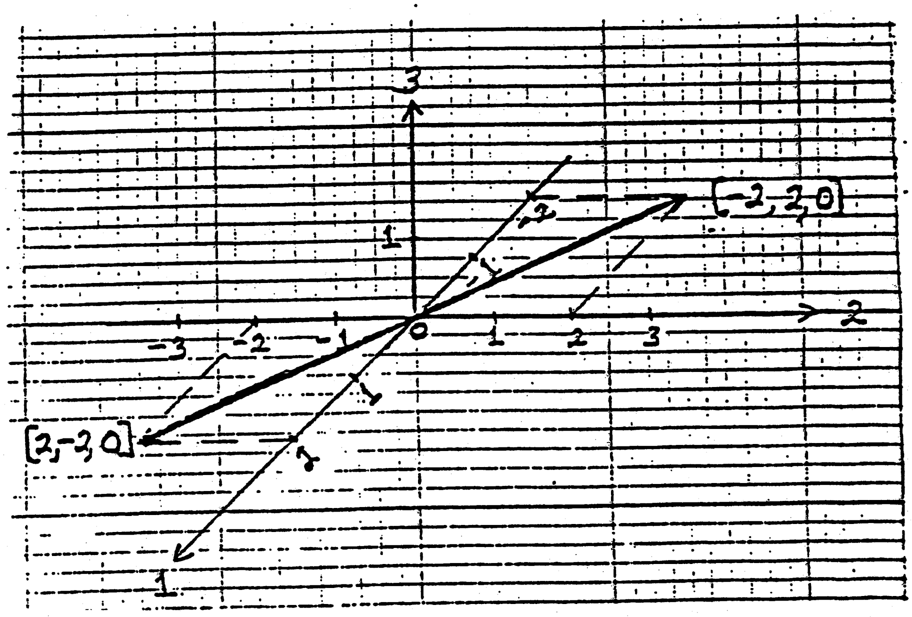
b) i) $L_{\underset{\sim}{x}} = \sqrt{\underset{\sim}{x}'\underset{\sim}{x}} = \sqrt{35} = 5.916$

ii) $\cos(\theta) = \dfrac{\underset{\sim}{x}'\underset{\sim}{y}}{L_{\underset{\sim}{x}} L_{\underset{\sim}{y}}} = \dfrac{1}{19.621} = .051$

$\theta = \arc\cos\ (.051) \doteq 87°$

iii) projection of $\underset{\sim}{y}$ on $\underset{\sim}{x}$ is $\left|\dfrac{\underset{\sim}{y}'\underset{\sim}{x}}{\underset{\sim}{x}'\underset{\sim}{x}}\right| \underset{\sim}{x} = \dfrac{1}{35}\underset{\sim}{x} = \left[\dfrac{1}{7}, \dfrac{1}{35}, \dfrac{3}{35}\right]'$

c)



2.2 a) $5A = \begin{bmatrix} -5 & 15 \\ 20 & 10 \end{bmatrix}$ b) $BA = \begin{bmatrix} -16 & 6 \\ -9 & -1 \\ 2 & -6 \end{bmatrix}$

c) $A'B' = \begin{bmatrix} -16 & -9 & 2 \\ 6 & -1 & -6 \end{bmatrix}$ d) $C'B = [12, -7]$

e) No.

2.3 a) $A' = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = A$ so $(A')' = A' = A$

b) $C' = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$; $(C')^{-1} = \begin{bmatrix} -\frac{2}{10} & \frac{3}{10} \\ \frac{4}{10} & -\frac{1}{10} \end{bmatrix}$

$C^{-1} = \begin{bmatrix} -\frac{2}{10} & \frac{4}{10} \\ \frac{3}{10} & -\frac{1}{10} \end{bmatrix}$; $(C^{-1})' = \begin{bmatrix} -\frac{2}{10} & \frac{3}{10} \\ \frac{4}{10} & -\frac{1}{10} \end{bmatrix} = (C')^{-1}$

c)

$(AB)' = \begin{bmatrix} 7 & 8 & 7 \\ 16 & 4 & 11 \end{bmatrix}' = \begin{bmatrix} 7 & 16 \\ 8 & 4 \\ 7 & 11 \end{bmatrix}$

$B'A' = \begin{bmatrix} 1 & 5 \\ 4 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 16 \\ 8 & 4 \\ 7 & 11 \end{bmatrix} = (AB)'$

d) AB has (i,j)th entry

$$a_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj} = \sum_{\ell=1}^{k} a_{i\ell}b_{\ell j}$$

Consequently, $(AB)'$ has $(i,j)^{th}$ entry

$$c_{ji} = \sum_{\ell=1}^{k} a_{j\ell}b_{\ell i} .$$

Next B' has i^{th} row $[b_{1i}, b_{2i}, \cdots, b_{ki}]$ and A' has j^{th}

column $[a_{j1}, a_{j2}, \cdots, a_{jk}]'$ so $B'A'$ has $(i,j)^{th}$ entry

$$b_{1i}a_{j1} + b_{2i}b_{j2} + \cdots + b_{ki}a_{jk} = \sum_{\ell=1}^{k} a_{j\ell}b_{\ell i} = c_{ji}$$

Since i and j were arbitrary choices, $(AB)' = B'A'$.

2.4 a) $I = I'$ and $AA^{-1} = I = A^{-1}A$. Thus $I' = I = (AA^{-1})' = (A^{-1})'A'$ and $I = (A^{-1}A)' = A'(A^{-1})'$. Consequently, $(A^{-1})'$ is the inverse of A' or $(A')^{-1} = (A^{-1})'$.

b) $(B^{-1}A^{-1})AB = B^{-1}\underbrace{(A^{-1}A)}_{I}B = B^{-1}B = I$ so AB has inverse $(AB)^{-1} = B^{-1}A^{-1}$. It was sufficient to check for a left inverse but we may also verify $AB(B^{-1}A^{-1}) = A\underbrace{(BB^{-1})}_{I}A^{-1} = AA^{-1} = I$.

2.5

$$QQ' = \begin{bmatrix} \frac{5}{13} & \frac{12}{13} \\ \frac{-12}{13} & \frac{5}{13} \end{bmatrix} \begin{bmatrix} \frac{5}{13} & \frac{-12}{13} \\ \frac{12}{13} & \frac{5}{13} \end{bmatrix} = \begin{bmatrix} \frac{169}{169} & 0 \\ 0 & \frac{169}{169} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = Q'Q .$$

2.6 a) Since $A = A'$, A is symmetric.

b) Since the quadratic form

$$\underset{\sim}{x}'A\underset{\sim}{x} = [x_1, x_2]\begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 9x_1^2 - 4x_1x_2 + 6x_2^2$$

$$= (2x_1 - x_2)^2 + 5(x_1^2 + x_2^2) > 0 \quad \text{for} \quad [x_1, x_2] \neq [0,0]$$

we conclude that A is positive definite.

2.7 a) Eigenvalues: $\lambda_1 = 10$, $\lambda_2 = 5$.

Normalized eigenvectors: $\underset{\sim}{e}_1' = [2/\sqrt{5}, -1/\sqrt{5}] = [.894, -.447]$

$\underset{\sim}{e}_2' = [1/\sqrt{5}, 2/\sqrt{5}] = [.447, .894]$

b) $A = \begin{bmatrix} 9 & -2 \\ -2 & 9 \end{bmatrix} = 10 \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} [2/\sqrt{5}, \ -1/\sqrt{5}] + 5 \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} [1/\sqrt{5}, \ 2/\sqrt{5}]$

c) $A^{-1} = \dfrac{1}{9(6)-(-2)(-2)} \begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix} = \begin{bmatrix} .12 & .04 \\ .04 & .18 \end{bmatrix}$

d) Eigenvalues: $\lambda_1 = .2, \ \lambda_2 = .1$

Normalized eigenvectors: $\underset{\sim}{e}_1' = [1/\sqrt{5}, \ 2/\sqrt{5}]$

$\underset{\sim}{e}_2' = [2/\sqrt{5}, \ -1/\sqrt{5}]$

2.8 Eigenvalues: $\lambda_1 = 2, \ \lambda_2 = -3$

Normalized eigenvectors: $\underset{\sim}{e}_1' = [2/\sqrt{5}, \ 1/\sqrt{5}]$

$\underset{\sim}{e}_2' = [1/\sqrt{5}, -2/\sqrt{5}]$

$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} = 2 \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} [2/\sqrt{5}, \ 1/\sqrt{5}] - 3 \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} [1/\sqrt{5}, \ -2/\sqrt{5}]$

2.9 a) $A^{-1} = \dfrac{1}{1(-2)-2(2)} \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{6} \end{bmatrix}$

b) Eigenvalues: $\lambda_1 = 1/2, \ \lambda_2 = -1/3$

Normalized eigenvectors: $\underset{\sim}{e}_1' = [2/\sqrt{5}, \ 1/\sqrt{5}]$

$\underset{\sim}{e}_2' = [1/\sqrt{5}, \ -2/\sqrt{5}]$

c) $A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{6} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} [2/\sqrt{5}, \ 1/\sqrt{5}] - \frac{1}{3} \begin{bmatrix} 1/\sqrt{5} \\ -2/\ 5 \end{bmatrix} [1/\sqrt{5}, \ -2/\sqrt{5}]$

2.10

$$B^{-1} = \frac{1}{4(4.002001)-(4.001)^2}\begin{bmatrix} 4.002001 & -4.001 \\ -4.001 & 4 \end{bmatrix}$$

$$= 333,333 \begin{bmatrix} 4.002001 & -4.001 \\ -4.001 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4(4.002)-(4.001)^2}\begin{bmatrix} 4.002 & -4.001 \\ -4.001 & 4 \end{bmatrix}$$

$$= -1,000,000 \begin{bmatrix} 4.002 & -4.001 \\ -4.001 & 4 \end{bmatrix}$$

Thus $A^{-1} \doteq (-3)B^{-1}$

2.11 With $p = 1$, $|a_{11}| = a_{11}$ and with $p = 2$

$$\begin{vmatrix} a_{11} & 0 \\ 0 & a_{22} \end{vmatrix} = a_{11}a_{22} - 0(0) = a_{11}a_{22}$$

Proceeding by induction, we assume the result holds for any $(p-1)\times(p-1)$ diagonal matrix A_{11}. Then writing

$$\underset{(p\times p)}{A} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A_{11} & \\ 0 & & & \end{bmatrix}.$$

we expand $|A|$ according to Definition 2A.24 to find $|A| = a_{11}\,|A_{11}| + 0 + \cdots + 0$. Since $|A_{11}| = a_{22}a_{33}\cdots a_{pp}$ by the induction hypothesis, $|A| = a_{11}(a_{22}a_{33}\cdots a_{pp}) = a_{11}a_{22}a_{33}\cdots a_{pp}$.

2.12 By (2-20), $A = P\Lambda P'$ with $PP' = P'P = I$. From Result 2A.11(e) $|A| = |P|\ |\Lambda|\ |P'| = |\Lambda|$. Since Λ is a diagonal matrix with diagonal elements $\lambda_1, \lambda_2, \ldots, \lambda_p$, we can apply Exercise 2.11 to get $|A| = |\Lambda| = \prod_{i=1}^{p} \lambda_i$.

2.14 Let λ be an eigenvalue of A. Thus $0 = |A-\lambda I|$. If Q is orthogonal, $QQ' = I$ and $|Q||Q'| = 1$ by Exercise 2.13. Using Result 2A.11(e) we can then write

$$0 = |Q|\ |A-\lambda I|\ |Q'| = |QAQ'-\lambda I|$$

and it follows that λ is also an eigenvalue of QAQ' if Q is orthogonal.

2.16 $(A'A)' = A'(A')' = A'A$ showing $A'A$ is symmetric.

$$\underset{\sim}{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} = A\underset{\sim}{x}. \text{ Then } 0 \le y_1^2+y_2^2+ \cdots + y_p^2 = \underset{\sim}{y}'\underset{\sim}{y} = \underset{\sim}{x}'A'A\underset{\sim}{x}$$

and $A'A$ is non-negative definite by definition.

2.18 Write $c^2 = \underset{\sim}{x}'A\underset{\sim}{x}$ with $A = \begin{bmatrix} 4 & -\sqrt{2} \\ -\sqrt{2} & 3 \end{bmatrix}$. The eigenvalue-normalized eigenvector pairs for A are:

$$\lambda_1 = 2, \quad \underset{\sim}{e}_1' = [.577,\ \ .816]$$

$$\lambda_2 = 5, \quad \underset{\sim}{e}_2' = [.816,\ -.577]$$

For $c^2 = 1$, the half lengths of the major and minor axes of the ellipse of constant distance are

$$\frac{c}{\sqrt{\lambda_1}} = \frac{1}{\sqrt{2}} = .707 \quad \text{and} \quad \frac{c}{\sqrt{\lambda_2}} = \frac{1}{\sqrt{5}} = .447$$

respectively. These axes lie in the directions of the vectors $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$ respectively.

For $c^2 = 4$, the half lengths of the major and minor axes are

$$\frac{c}{\sqrt{\lambda_1}} = \frac{2}{\sqrt{2}} = 1.414 \quad \text{and} \quad \frac{c}{\sqrt{\lambda_2}} = \frac{2}{\sqrt{5}} = .894 .$$

As c^2 increases the lengths of the major and minor axes increase.

2.20 Using matrix A in Exercise 2.3, we determine

$$\lambda_1 = 1.382, \quad \underset{\sim}{e}_1 = [.8507, \quad -.5257]'$$

$$\lambda_2 = 3.618, \quad \underset{\sim}{e}_2 = [.5257, \quad .8507]'$$

We know

$$A^{1/2} = \sqrt{\lambda_1}\, \underset{\sim}{e}_1 \underset{\sim}{e}_1' + \sqrt{\lambda_2}\, \underset{\sim}{e}_2 \underset{\sim}{e}_2' = \begin{bmatrix} 1.376 & .325 \\ .325 & 1.701 \end{bmatrix}$$

$$A^{-1/2} = \frac{1}{\sqrt{\lambda_1}} \underset{\sim}{e}_1 \underset{\sim}{e}_1' + \frac{1}{\sqrt{\lambda_2}} \underset{\sim}{e}_2 \underset{\sim}{e}_2' = \begin{bmatrix} .7608 & -.1453 \\ -.1453 & .6155 \end{bmatrix}$$

We check

$$A^{1/2} A^{-1/2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A^{-1/2} A^{1/2}$$

2.21 (a)

$$\mathbf{A'A} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 1 \\ 1 & 9 \end{bmatrix}$$

$0 = |\mathbf{A'A} - \lambda\,\mathbf{I}\,| = (9-\lambda)^2 - 1 = (10-\lambda)(8-\lambda)$, so $\lambda_1 = 10$ and $\lambda_2 = 8$. Next,

$$\begin{bmatrix} 1 & 1 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 10 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad \text{gives} \quad e_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 8 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad \text{gives} \quad e_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

(b)

$$\mathbf{AA'} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 1 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 8 \end{bmatrix}$$

$$0 = |\mathbf{AA'} - \lambda\mathbf{I}\,| = \begin{vmatrix} 2-\lambda & 0 & 4 \\ 0 & 8-\lambda & 0 \\ 4 & 0 & 8-\lambda \end{vmatrix}$$

$= (2-\lambda)(8-\lambda)^2 - 4^2(8-\lambda) = (8-\lambda)(\lambda-10)\lambda$ so $\lambda_1 = 10, \lambda_2 = 8$, and $\lambda_3 = 0$.

$$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 8 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = 10 \begin{bmatrix} e_1 \\ e_2 \\ e_2 \end{bmatrix}$$

$$\text{gives} \quad \begin{aligned} 4e_3 &= 8e_1 \\ 8e_2 &= 10e_2 \end{aligned} \quad \text{so} \quad e_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 8 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = 8 \begin{bmatrix} e_1 \\ e_2 \\ e_2 \end{bmatrix}$$

$$\text{gives} \quad \begin{aligned} 4e_3 &= 6e_1 \\ 4e_1 &= 0 \end{aligned} \quad \text{so} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Also, $e_3 = [-2/\sqrt{5}, 0, 1/\sqrt{5}\,]'$.

(c)

$$\begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 2 & 2 \end{bmatrix} = \sqrt{10}\begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{2}{\sqrt{5}} \end{bmatrix} \left[\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}} \right] + \sqrt{8}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \left[\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}} \right]$$

2.22 (a)

$$\mathbf{AA}' = \begin{bmatrix} 4 & 8 & 8 \\ 3 & 6 & -9 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 8 & 6 \\ 8 & -9 \end{bmatrix} = \begin{bmatrix} 144 & -12 \\ -12 & 126 \end{bmatrix}$$

$0 = |\mathbf{AA}' - \lambda\, \mathbf{I}\, | = (144-\lambda)(126-\lambda) - (12)^2 = (150-\lambda)(120-\lambda)$, so $\lambda_1 = 150$ and $\lambda_2 = 120$. Next,

$$\begin{bmatrix} 144 & -12 \\ -12 & 126 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 150 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad \text{gives} \quad e_1 = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$$

and $\lambda_2 = 120$ gives $e_2 = [1/\sqrt{5}, 2/\sqrt{5}\,]'$.

(b)

$$\mathbf{A}'\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 8 & 6 \\ 8 & -9 \end{bmatrix} \begin{bmatrix} 4 & 8 & 8 \\ 3 & 6 & -9 \end{bmatrix} = \begin{bmatrix} 25 & 50 & 5 \\ 50 & 100 & 10 \\ 5 & 10 & 145 \end{bmatrix}$$

$$0 = |\mathbf{A}'\mathbf{A} - \lambda\mathbf{I}\,| = \begin{vmatrix} 25-\lambda & 50 & 5 \\ 50 & 100-\lambda & 10 \\ 5 & 10 & 145-\lambda \end{vmatrix} = (150-\lambda)(\lambda-120)\lambda$$

so $\lambda_1 = 150$, $\lambda_2 = 120$, and $\lambda_3 = 0$. Next,

$$\begin{bmatrix} 25 & 50 & 5 \\ 50 & 100 & 10 \\ 5 & 10 & 145 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = 150 \begin{bmatrix} e_1 \\ e_2 \\ e_2 \end{bmatrix}$$

$$\text{gives} \quad \begin{array}{rcrl} -120e_1 & + & 60e_2 & = 0 \\ -25e_1 & + & 5e_3 & = 0 \end{array} \quad \text{or } e_1 = \frac{1}{\sqrt{30}} \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 50 & 5 \\ 50 & 100 & 10 \\ 5 & 10 & 145 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = 120 \begin{bmatrix} e_1 \\ e_2 \\ e_2 \end{bmatrix}$$

$$\text{gives} \quad \begin{array}{rcrl} 60e_1 & + & 60e_3 & = 0 \\ -120e_2 & + & -240e_3 & = 0 \end{array} \quad \text{or} \quad e_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Also, $e_3 = [2/\sqrt{5}, -1/\sqrt{5}, 0]'$.

(c)

$$\begin{bmatrix} 4 & 8 & 8 \\ 3 & 6 & -9 \end{bmatrix}$$

$$= \sqrt{150} \begin{bmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & \frac{5}{\sqrt{30}} \end{bmatrix} + \sqrt{120} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

2.24

a) $\Sigma^{-1} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b) $\lambda_1 = 4, \quad \mathbf{e}_1 = [1,0,0]'$
$\lambda_2 = 9, \quad \mathbf{e}_2 = [0,1,0]'$
$\lambda_3 = 1, \quad \mathbf{e}_3 = [0,0,1]'$

c) For Σ^{-1}: $\lambda_1 = 1/4, \quad \mathbf{e}_1' = [1,0,0]'$
$\lambda_2 = 1/9, \quad \mathbf{e}_2' = [0,1,0]'$
$\lambda_3 = 1, \quad \mathbf{e}_3' = [0,0,1]'$

2.25

a) $V^{1/2} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$; $\boldsymbol{\rho} = \begin{bmatrix} 1 & -1/5 & 4/15 \\ -1/5 & 1 & 1/6 \\ 4/15 & 1/6 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -.2 & .267 \\ -.2 & 1 & .167 \\ .267 & .167 & 1 \end{bmatrix}$

b) $V^{1/2} \boldsymbol{\rho} V^{1/2} =$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1/5 & 4/15 \\ -1/5 & 1 & 1/6 \\ 4/15 & 1/6 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 4/3 \\ -2/5 & 2 & 1/3 \\ 4/5 & 1/2 & 3 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix} = \boldsymbol{\Sigma}$$

2.26

a) $\rho_{13} = \sigma_{13}/\sigma_{11}^{1/2}\sigma_{22}^{1/2} = 4/\sqrt{25}\,\sqrt{9} = 4/15 = .267$

b) Write $X_1 = 1 \cdot X_1 + 0 \cdot X_2 + 0 \cdot X_3 = \mathbf{c}_1'\mathbf{X}$ with $\mathbf{c}_1' = [1,0,0]$

$$\frac{1}{2}X_2 + \frac{1}{2}X_3 = \mathbf{c}_2'\mathbf{X} \quad \text{with} \quad \mathbf{c}_2' = [0, \tfrac{1}{2}, \tfrac{1}{2}]$$

Then $\mathrm{Var}(X_1) = \sigma_{11} = 25$. By (2-43),

$$\mathrm{Var}(\tfrac{1}{2}X_2 + \tfrac{1}{2}X_3) = \mathbf{c}_2' \boldsymbol{\Sigma} \mathbf{c}_2 = \frac{1}{4}\sigma_{22} + \frac{2}{4}\sigma_{23} + \frac{1}{4}\sigma_{33} = 1 + \frac{1}{2} + \frac{9}{4}$$

$$= \frac{15}{4} = 3.75$$

By (2-45), (see also hint to Exercise 2.28),

$$\mathrm{Cov}(X_1, \tfrac{1}{2}X_1 + \tfrac{1}{2}X_2) = \mathbf{c}_1' \boldsymbol{\Sigma} \mathbf{c}_2 = \frac{1}{2}\sigma_{12} + \frac{1}{2}\sigma_{13} = -1 + 2 = 1$$

so

$$\mathrm{Corr}(X_1, \tfrac{1}{2}X_1 + \tfrac{1}{2}X_2) = \frac{\mathrm{Cov}(X_1, \frac{1}{2}X_1 + \frac{1}{2}X_2)}{\sqrt{\mathrm{Var}(X_1)}\ \sqrt{\mathrm{Var}(\frac{1}{2}X_1 + \frac{1}{2}X_2)}} = \frac{1}{5\sqrt{3.75}} = .103$$

2.27

a) $\mu_1 - 2\mu_2$, $\sigma_{11} + 4\sigma_{22} - 4\sigma_{12}$

b) $-\mu_1 + 3\mu_2$, $\sigma_{11} + 9\sigma_{22} - 6\sigma_{12}$

c) $\mu_1 + \mu_2 + \mu_3$, $\sigma_{11} + \sigma_{22} + \sigma_{33} + 2\sigma_{12} + 2\sigma_{13} + 2\sigma_{23}$

d) $\mu_1 + 2\mu_2 - \mu_3$, $\sigma_{11} + 4\sigma_{22} + \sigma_{33} + 4\sigma_{12} - 2\sigma_{13} - 4\sigma_{23}$

e) $3\mu_1 - 4\mu_2$, $9\sigma_{11} + 16\sigma_{22}$ since $\sigma_{12} = 0$.

2.29

$$\boldsymbol{\Sigma} = \left[\begin{array}{cc|ccc} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{15} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} & \sigma_{25} \\ \hline \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} & \sigma_{35} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} & \sigma_{45} \\ \sigma_{51} & \sigma_{52} & \sigma_{53} & \sigma_{54} & \sigma_{55} \end{array}\right] = \left[\begin{array}{c|c} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \hline \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{array}\right]$$

2.31 (a)

$$E[\boldsymbol{X}^{(1)}] = \boldsymbol{\mu}^{(1)} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \quad \text{(b)} \quad \mathrm{A}\boldsymbol{\mu}^{(1)} = [\,1 \quad -1\,] \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 1$$

(c)

$$\text{Cov}(\boldsymbol{X}^{(1)}\,) = \Sigma_{11} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

(d)

$$\text{Cov}(\mathrm{A}\boldsymbol{X}^{(1)}\,) = \mathrm{A}\Sigma_{11}\mathrm{A}' = [\,1 \quad -1\,] \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 4$$

(e)

$$E[\boldsymbol{X}^{(2)}] = \boldsymbol{\mu}^{(2)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{(f)} \quad \mathrm{B}\boldsymbol{\mu}^{(2)} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

(g)

$$\text{Cov}(\boldsymbol{X}^{(2)}\,) = \Sigma_{22} = \begin{bmatrix} 9 & -2 \\ -2 & 4 \end{bmatrix}$$

(h)

$$\text{Cov}(\mathrm{B}\boldsymbol{X}^{(2)}\,) = \mathrm{B}\Sigma_{22}\mathrm{B}' = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 9 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 48 & -8 \\ -8 & 4 \end{bmatrix}$$

(i)

$$\text{Cov}(\boldsymbol{X}^{(1)}, \boldsymbol{X}^{(2)}) = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}$$

(j)

$$\text{Cov}(\mathrm{A}\boldsymbol{X}^{(1)}, \mathrm{B}\boldsymbol{X}^{(2)}) = \mathrm{A}\Sigma_{12}\mathrm{B}' = [\,1 \quad -1\,] \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} = [\,0 \quad 2\,]$$

2.32 (a)

$$E[\boldsymbol{X}^{(1)}] = \boldsymbol{\mu}^{(1)} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \text{(b)} \quad \mathbf{A}\boldsymbol{\mu}^{(1)} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

(c)

$$\text{Cov}(\boldsymbol{X}^{(1)}) = \boldsymbol{\Sigma}_{11} = \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}$$

(d)

$$\text{Cov}(\mathbf{A}\boldsymbol{X}^{(1)}) = \mathbf{A}\boldsymbol{\Sigma}_{11}\mathbf{A}' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 1 \\ 1 & 5 \end{bmatrix}$$

(e)

$$E[\boldsymbol{X}^{(2)}] = \boldsymbol{\mu}^{(2)} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} \quad \text{(f)} \quad \mathbf{B}\boldsymbol{\mu}^{(2)} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \end{bmatrix}$$

(g)

$$\text{Cov}(\boldsymbol{X}^{(2)}) = \boldsymbol{\Sigma}_{22} = \begin{bmatrix} 6 & 1 & -1 \\ 1 & 4 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

(h)

$$\text{Cov}(\mathbf{B}\boldsymbol{X}^{(2)}) = \mathbf{B}\boldsymbol{\Sigma}_{22}\mathbf{B}'$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 6 & 1 & -1 \\ 1 & 4 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 12 & 9 \\ 9 & 24 \end{bmatrix}$$

(i)

$$\text{Cov}(\boldsymbol{X}^{(1)}, \boldsymbol{X}^{(2)}) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

(j)

$$\text{Cov}(\mathbf{A}\boldsymbol{X}^{(1)}, \mathbf{B}\boldsymbol{X}^{(2)}) = \mathbf{A}\boldsymbol{\Sigma}_{12}\mathbf{B}'$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2.33 (a)

$$E[\boldsymbol{X}^{(1)}] = \boldsymbol{\mu}^{(1)} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} \quad \text{(b)} \ \mathbf{A}\boldsymbol{\mu}^{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

(c)

$$\text{Cov}(\boldsymbol{X}^{(1)}) = \boldsymbol{\Sigma}_{11} = \begin{bmatrix} 4 & -1 & \frac{1}{2} \\ -1 & 3 & 1 \\ \frac{1}{2} & 1 & 6 \end{bmatrix}$$

(d)

$$\text{Cov}(\mathbf{A}\boldsymbol{X}^{(1)}) = \mathbf{A}\boldsymbol{\Sigma}_{11}\mathbf{A}'$$

$$= \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 4 & -1 & \frac{1}{2} \\ -1 & 3 & 1 \\ \frac{1}{2} & 1 & 6 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 23 & 4 \\ 4 & 63 \end{bmatrix}$$

(e)

$$E[\boldsymbol{X}^{(2)}] = \boldsymbol{\mu}^{(2)} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \text{(f)} \ \mathbf{B}\boldsymbol{\mu}^{(2)} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

(g)

$$\text{Cov}(\boldsymbol{X}^{(2)}) = \boldsymbol{\Sigma}_{22} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

(h)

$$\text{Cov}(\mathbf{B}\boldsymbol{X}^{(2)}) = \mathbf{B}\boldsymbol{\Sigma}_{22}\mathbf{B}' = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 12 & 0 \\ 0 & 6 \end{bmatrix}$$

(i)

$$\mathrm{Cov}(\boldsymbol{X}^{(1)}, \boldsymbol{X}^{(2)}) = \begin{bmatrix} -\frac{1}{2} & 0 \\ -1 & 0 \\ 1 & -1 \end{bmatrix}$$

(j)

$$\mathrm{Cov}(\mathbf{A}\boldsymbol{X}^{(1)}, \mathbf{B}\boldsymbol{X}^{(2)}) = \mathbf{A}\boldsymbol{\Sigma}_{12}\mathbf{B}'$$

$$= \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 0 \\ -1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -4.5 & 4.5 \end{bmatrix}$$

2.34 $\underset{\sim}{b}'\underset{\sim}{b} = 4 + 1 + 16 + 0 = 21$, $\underset{\sim}{d}'\underset{\sim}{d} = 15$ and $\underset{\sim}{b}'\underset{\sim}{d} = -2-3-8+0 = -13$

$$(\underset{\sim}{b}'\underset{\sim}{d})^2 = 169 \le 21(15) = 315$$

2.35 $\underset{\sim}{b}'\underset{\sim}{d} = -4+3 = -1$

$$\underset{\sim}{b}'B\underset{\sim}{b} = [-4, 3]\begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} -4 \\ 3 \end{bmatrix} = [-14 \quad 23]\begin{bmatrix} -4 \\ 3 \end{bmatrix} = 125$$

$$\underset{\sim}{d}'B^{-1}\underset{\sim}{d} = [1,1]\begin{bmatrix} 5/6 & 2/6 \\ 2/6 & 2/6 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 11/6$$

so $1 = (\underset{\sim}{b}'\underset{\sim}{d})^2 \le 125\ (11/6) = 229.17$

2.36

$$4x_1^2 + 4x_2^2 + 6x_1x_2 = \mathbf{x}'A\mathbf{x} \quad \text{where } A = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}.$$

$(4-\lambda)^2 - 3^2 = 0$ gives $\lambda_1 = 7, \lambda_2 = 1$. Hence the maximum is 7 and the minimum is 1.

2.37 From (2-51),

$$\max_{\underset{\sim}{x}'\underset{\sim}{x}=1} \underset{\sim}{x}'A\underset{\sim}{x} = \max_{\underset{\sim}{x}\neq\underset{\sim}{0}} \frac{\underset{\sim}{x}'A\underset{\sim}{x}}{\underset{\sim}{x}'\underset{\sim}{x}} = \lambda_1$$

where λ_1 is the largest eigenvalue of A. For A given in Exercise 2.6, we have from Exercise 2.7, $\lambda_1 = 10$ and $\underset{\sim}{e}_1' = [.894, -.447]$. Therefore $\max_{\underset{\sim}{x}'\underset{\sim}{x}=1} \underset{\sim}{x}'A\underset{\sim}{x} = 10$ and this maximum is attained for $\underset{\sim}{x} = \underset{\sim}{e}_1$.

2.38

Using computer, $\lambda_1 = 18$, $\lambda_2 = 9$, $\lambda_3 = 9$. Hence the maximum is 18 and the minimum is 9.

2.41 (a) $E(\mathbf{AX}) = \mathbf{A}E(\mathbf{X}) = \mathbf{A}\mu_X = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

(b) $Cov(\mathbf{AX}) = \mathbf{A}Cov(\mathbf{X})\mathbf{A}' = \mathbf{A}\Sigma_X\mathbf{A}' = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 36 \end{bmatrix}$

(c) All pairs of linear combinations have zero covariances.

2.42 (a) $E(\mathbf{AX}) = \mathbf{A}E(\mathbf{X}) = \mathbf{A}\mu_X = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

(b) $Cov(\mathbf{AX}) = \mathbf{A}Cov(\mathbf{X})\mathbf{A}' = \mathbf{A}\Sigma_X\mathbf{A}' = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 24 \end{bmatrix}$

(c) All pairs of linear combinations have zero covariances.