On the norm of the hyperinterpolation operator on the unit disk and its use for the solution of the nonlinear Poisson equation

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Abstract

In this article we study the properties of the hyperinterpolation operator on the unit disk D in $\mathbb{R}^2$. We show how the hyperinterpolation can be used in connection with the Kumar-Sloan method to approximate the solution of a nonlinear Poisson equation on the unit disk (discrete Galerkin method). A bound for the norm of the hyperinterpolation operator in the space $C(D)$ is derived. Our results prove the convergence of the discrete Galerkin method in the maximum norm if the solution of the Poisson equation is in the class $C^{1,\delta}(D)$, $\delta > 0$. Finally we present numerical examples which show that the discrete Galerkin method converges faster than $O(n^{-k})$, for every $k \in \mathbb{N}$, if the solution of the nonlinear Poisson equation in is $C^\infty(D)$.

Keywords: Hyperinterpolation operator, discrete Galerkin method, projector norm, nonlinear Poisson equation

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1 Introduction

The most common approach to solving numerically integral and partial differential equations over a planar region is to use piecewise polynomial approximations of the solution. An approach that has been used much less is to use polynomial approximations of the solution. We used this recently in [6] to solve the nonlinear Poisson equation $\Delta u = f(\cdot, u)$ over the closed unit disk $D \subseteq \mathbb{R}^2$ [By a simple change of variables this extends a wide variety of other planar regions; e.g. see [5] for a conformal transformation of the ellipse onto the disk.] Methods based on polynomial approximations are often more rapidly convergent, and usually they lead to solving linear or nonlinear systems that are of much smaller order than numerical methods based on piecewise polynomial approximations.

Numerical methods for approximating differential or integral equations are usually of collocation or Galerkin type. We have been considering methods of Galerkin type because collocation methods require results on multivariate polynomial interpolation theory, and this theory is still being developed (e.g. see [20]). With Galerkin methods, there are many integrals to be evaluated numerically. In attempting to minimize the order of the quadrature rule being used, one is led to the idea of *hyperinterpolation*, a concept and term introduced by Ian Sloan in [15]. In this paper we investigate hyperinterpolation in connection with Galerkin's method over the unit disk.

The Galerkin method with polynomial approximations makes use of the orthogonal projection from $L^2(D)$ onto the subspace of polynomials of degree $\leq n$. The hyperinterpolation operator is based on approximating the integrations that appear in the formula for the orthogonal projection operator. In §2 we introduce the hyperinterpolation operator and discuss bounding its norm as an operator on $C(D)$. The main result of this paper is given at the conclusion of the section, and its proof is given as a series of lemmata in §§3,4.

The bound which we derive for the hyperinterpolation operator implies that the resulting discrete Galerkin method is convergent in the maximum norm if the solution of the equation has a smoothness of $C^{1,\delta}(D)$, $\delta > 0$. In previous articles, see [6], we assumed that certain integrals are known or can be approximated with a sufficiently accurate integration rule. Our analysis here shows that the quadrature rules used in the construction of the hyperinterpolation operator are sufficient to guarantee the convergence and that the resulting convergence rate is at most $O(\log(n))$ slower than the optimal rate which we can expect. In the final Section we apply the method to nonlinear Poisson equations where the solution is in $C^\infty(D)$. Here we expect that the convergence is faster than $O(n^{-k})$ for every $k \in \mathbb{N}$. Our numerical results confirm this.

2 The hyperinterpolation operator on the disk

We consider the disk $D := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$ and the space $L^2(D)$ with the scalar product

$$(f, g) := \frac{1}{\pi} \int_D f(x) g(x)\, dx \tag{1}$$

$$= \frac{1}{\pi} \int_0^1 \int_0^{2\pi} f(r, \phi) g(r, \phi) r \; d\phi dr \tag{2}$$

We always identify the two representations of a function f on D: $f(x) = f(r, \phi)$. By Π_n we denote the space of all polynomials in two variables of degree n or less

$$\Pi_n := \left\{ \sum_{j=0}^{n} \sum_{k=0}^{j} a_{j,k} x_1^k x_2^{j-k} \,\middle|\, a_{j,k} \in \mathbb{R} \right\}$$

and the dimension of Π_n is $\binom{n+2}{2}$.

Let $\{\Psi_{j,k}\}_{j=0\ldots n, k=0\ldots j}$ be an orthonormal basis for Π_n. The orthogonal projection of $L^2(D)$ onto Π_n is given by

$$(\mathcal{P}_n f)(x) := \sum_{j=0}^{n} \sum_{k=0}^{j} (f, \Psi_{j,k}) \Psi_{j,k}(x)$$

The property

$$\|\mathcal{P}_n\|_{L^2(D) \mapsto L^2(D)} = 1$$

is well known; and Xu proved in [19] the important result

$$\|\mathcal{P}_n\|_{C(D) \mapsto C(D)} \sim n, \tag{3}$$

here $A(n) \sim n$, means that there are constants $c_1 > 0$ and $c_2 > 0$ such that $c_1 n \leq A(n) \leq c_2 n$

To approximate the projection $\mathcal{P}_n$ we can replace the scalar product by a finite sum

$$\begin{aligned}
(f, g)_d :=& \frac{1}{\pi} \sum_{l=0}^{n} \sum_{m=0}^{2n} f\left(r_l, \frac{2\pi\, m}{2n+1}\right) g\left(r_l, \frac{2\pi\, m}{2n+1}\right) \omega_l \frac{2\pi}{2n+1} r_l \\
=& \sum_{l=0}^{n} \sum_{m=0}^{2n} f \underbrace{\left(r_l, \frac{2\pi\, m}{2n+1}\right)}_{=:\xi_{l,m}} g\left(r_l, \frac{2\pi\, m}{2n+1}\right) \underbrace{\omega_l \frac{2}{2n+1} r_l}_{=:w_{m,l}}
\end{aligned} \tag{4}$$

we use the trapezoidal rule for the azimuthal direction and a Gaussian quadrature rule for the radial direction. This quadrature is exact for all polynomials $f, g \in \Pi_n$. Here the numbers ω_l are the weights of Gauss-Legendre quadrature on $[0, 1]$:

$$\int_0^1 p(x) dx = \sum_{l=0}^{n} p(r_l) \omega_l,$$

for all single-variable polynomials $p(x)$ with $\deg(p) \leq 2n+1$. [Another possible choice would be the Gauss quadrature on $[0, 1]$ with measure rdr; the term r_l in formula (4) would then disappear.] The discrete semi-definite scalar product $(\cdot, \cdot)_d$ depends on n but we do not indicate explicitly this dependency.

With the help of the discrete scalar product we can now define an approximation to the orthogonal projection $\mathcal{P}_n f$ when f is restricted to being continuous over D:

$$(\mathcal{L}_n f)(x) := \sum_{j=0}^{n} \sum_{k=0}^{j} (f, \Psi_{j,k})_d \Psi_{j,k}(x)$$

The operator $\mathcal{L}_n$ is the *hyperinterpolation operator* of Sloan and Womersley [16]; and it can also be considered as a *discrete orthogonal projection operator*, as in [4]. Methods using this approximating operator are sometimes called *discrete Galerkin methods.*

2.1 Bounds on the projection error

When using the Galerkin method or the discrete Galerkin method for solving integral equations, the rate of convergence is generally related to the error in the orthogonal projection. For a discussion of a general framework for analyzing Galerkin methods and discrete Galerkin methods for solving integral equations, see [3, Chap. 3]. The error analysis often leads to a formula

$$\|f - f_n\| \le c\|f - \mathcal{Q}_n f\| \tag{5}$$

that is shown to hold for all sufficiently large n. We consider both the cases $\mathcal{Q}_n = \mathcal{P}_n$ or $\mathcal{Q}_n = \mathcal{L}_n$. If the analysis is being done within $L^2(D)$, with the norm being $\|\cdot\|_{L^2}$, then we know $\mathcal{P}_n f \to f$ as $n \to \infty$ for any $f \in L^2(D)$. However when using $\mathcal{L}_n$ we must work in $C(D)$, and we also are often interested in doing so for $\mathcal{P}_n$; the function space norm is then $\|\cdot\|_\infty$. In both of the cases $C(D)$ and $L^2(D)$, we are interested in examining the rate of convergence $\mathcal{Q}_n f$ to f as it is affected by the smoothness of the function f.

To examine this question, we can use results on best polynomial approximations, and for this, we use results from Ragozin [14, p. 164] as summarized below. Assume $f \in C^k(D)$ with $k \ge 0$ an integer. For the norm on $C^k(D)$, we use the standard definition

$$\|f\|_{C^k} = \sum_{i+j \le k} \left\| \frac{\partial^{i+j} f}{\partial x^i \partial y^j} \right\|_\infty$$

In addition, define various moduli of continuity by

$$\omega(f; h) = \sup\{|f(x_1, y_1) - f(x_2, y_2)| : \|(x_1, y_1) - (x_2, y_2)\| \le h\}$$

$$\omega_k(f; h) = \sum_{i+j=k} \omega\left(\frac{\partial^{i+j} f}{\partial x^i \partial y^j}; h\right), \qquad k \ge 1$$

Then there exists a sequence of polynomials p_n of degree $\le n$ such that

$$\|f - p_n\|_\infty \le \frac{B_k}{n^k}\left[\frac{\|f\|_{C^k}}{n} + \omega_k\left(f; \frac{1}{n}\right)\right], \qquad d \ge 1 \tag{6}$$

where each constant B_k depends only on $k \ge 0$.

To bound $\|f - \mathcal{Q}_n f\|$, note that with both choices for $\mathcal{Q}_n$, we have

$$\mathcal{Q}_n p = p \qquad \forall p \in \Pi_n$$

Thus,

$$f - \mathcal{Q}_n f = (f - p_n) - \mathcal{Q}_n(f - p_n)$$

$$\|f - \mathcal{Q}_n f\| \le (1 + \|\mathcal{Q}_n\|) \|f - p_n\| \tag{7}$$

When the function space is $L^2(D)$ and $\mathcal{Q}_n = \mathcal{P}_n$, we know $\|\mathcal{P}_n\|_{L^2 \to L^2} = 1$ and thus

$$\begin{aligned} \|f - \mathcal{P}_n f\|_{L^2} &\le 2\|f - p_n\|_{L^2} \\ &\le 2\pi \|f - p_n\|_\infty \end{aligned} \tag{8}$$

With the space $C(D)$ and either choice for $\mathcal{Q}_n$, we have

$$\|f - \mathcal{Q}_n f\|_\infty \le (1 + \|\mathcal{Q}_n\|_{C \to C}) \|f - p_n\|_\infty \tag{9}$$

With $\mathcal{Q}_n = \mathcal{P}_n$, we know that $\|\mathcal{P}_n\|_{C \to C} = O_{n\to\infty}(n)$ (cf. (3)). When this is combined with (6), we lose one power of n in the rate of uniform convergence of $\mathcal{P}_n f$ to f. We need $k > 1$ to insure uniform convergence; although using other results from [14] this can be weakened to requiring f to have first derivatives that are Hölder continuous with some exponent $\alpha > 0$.

If $\mathcal{Q}_n = \mathcal{L}_n$, we need to know $\|\mathcal{L}_n\|_{C \to C}$ in order to bound $\|f - \mathcal{L}_n f\|_\infty$ Estimating this norm is the focus of the present paper.

2.2 The reproducing kernel for Π_n

To obtain another useful formula for $\mathcal{L}_n f$, we need a result from Xu in [19] where he derived formulas for the reproducing kernel G_n for Π_n. We specialize his formulas to our case, obtaining the following:

$$\begin{aligned} G_n(x, y) &= \sum_{j=0}^{n} \sum_{k=0}^{j} \Psi_{j,k}(x) \Psi_{j,k}(y) \\ &= \int_0^\pi \Big[C_n^{(2)} \left(x \cdot y + \sqrt{1 - \|x\|^2} \sqrt{1 - \|y\|^2} \cos(\psi) \right) \\ &\quad + C_{n-1}^{(2)} \left(x \cdot y + \sqrt{1 - \|x\|^2} \sqrt{1 - \|y\|^2} \cos(\psi) \right) \Big] \, d\psi \\ &= \frac{2\Gamma\left(\frac{5}{2}\right) \Gamma(n+3)}{\Gamma(4)\Gamma\left(n + \frac{3}{2}\right)} && (10) \\ &\quad \times \int_0^\pi P_n^{\left(\frac{3}{2}, \frac{1}{2}\right)} \left(x \cdot y + \sqrt{1 - \|x\|^2} \sqrt{1 - \|y\|^2} \cos(\psi) \right) d\psi && (11) \end{aligned}$$

Here $P_n^{(\lambda,\mu)}$ denotes a standard Jacobi polynomial of degree n (cf. [1, p. 774], [2, §6.3], [17, Chap. 4]); and we remark that the multiplying constant in (10) satisfies

$$\frac{2\Gamma\left(\frac{5}{2}\right) \Gamma(n+3)}{\Gamma(4)\Gamma\left(n + \frac{3}{2}\right)} \sim n^{\frac{3}{2}}. \tag{12}$$

Using $G_n(x, y)$, we have the following reproducing kernel property over Π_n:

$$\int_D G_n(x, y) f(x) \, dx = f(y), \quad y \in D, \quad \text{for all } f \in \Pi_n \tag{13}$$

Now we can proceed as in Sloan and Womersley [16, (4.13)] and derive a representation for $\mathcal{L}_n$ with the help of G_n:

$$\begin{aligned}
(\mathcal{L}_n f)(x) &= \sum_{j=0}^{n}\sum_{k=0}^{j}\left(\sum_{l=0}^{n}\sum_{m=0}^{2n} w_{l,m} f(\xi_{l,m})\Psi_{j,k}(\xi_{l,m})\right)\Psi_{j,k}(x) \\
&= \sum_{l=0}^{n}\sum_{m=0}^{2n} w_{l,m} f(\xi_{l,m})\left(\sum_{j=0}^{n}\sum_{k=0}^{j}\Psi_{j,k}(\xi_{l,m})\Psi_{j,k}(x)\right) \\
&= \sum_{l=0}^{n}\sum_{m=0}^{2n} w_{l,m} f(\xi_{l,m}) G_n(\xi_{l,m}, x) \qquad (14)
\end{aligned}$$

It is straightforward to show that

$$\|\mathcal{L}_n\|_{C(D)\mapsto C(D)} = \max_{x\in D}\sum_{l=0}^{n}\sum_{m=0}^{2n} w_{l,m}\left|G_n(\xi_{l,m}, x)\right|$$

Because the kernel function G_n is continuous, we know that there is a $\xi_0 = \hat{r}(\cos(\alpha_0), \sin(\alpha_0)) \in D$ (we omit the dependency on n) such that

$$\begin{aligned}
&\|\mathcal{L}_n\|_{C(D)\mapsto C(D)} \\
&= \sum_{l=0}^{n}\sum_{m=0}^{2n} w_{l,m}\left|G_n(\xi_{l,m}, \xi_0)\right| \\
&\leq \frac{2\Gamma\left(\frac{5}{2}\right)\Gamma(n+3)}{\Gamma(4)\Gamma\left(n+\frac{5}{2}\right)}\sum_{l=0}^{n}\sum_{m=0}^{2n} w_{l,m}\times \qquad (15) \\
&\int_0^{\pi}\left|P_n^{\left(\frac{3}{2},\frac{1}{2}\right)}\left(r_l\hat{r}\cos(\alpha_0 - \frac{2m\pi}{2n+1}) + \sqrt{1-\hat{r}^2}\sqrt{1-r_l^2}\cos(\psi)\right)\right| d\psi
\end{aligned}$$

2.3 Bounding the hyperinterpolation operator

We can obtain a simple bound for $\|\mathcal{L}_n\|_{C(D)\mapsto C(D)}$ by modifying an argument given in [16, Thm 5.5.2]. We begin by using the Cauchy-Schwartz inequality

$$\begin{aligned}
\|\mathcal{L}_n\|_{C(D)\mapsto C(D)} &= \sum_{l=0}^{n}\sum_{m=0}^{2n} w_{l,m}\left|G_n(\xi_{l,m},\xi_0)\right| \\
&= \sum_{l=0}^{n}\sum_{m=0}^{2n} \sqrt{w_{l,m}}\sqrt{w_{l,m}}\left|G_n(\xi_{l,m},\xi_0)\right| \\
&\le \left(\sum_{l=0}^{n}\sum_{m=0}^{2n}\left(\sqrt{w_{l,m}}\right)^2\right)^{\frac{1}{2}}\left(\sum_{l=0}^{n}\sum_{m=0}^{2n}\left(\sqrt{w_{l,m}}\left|G_n(\xi_{l,m},\xi_0)\right|\right)^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{l=0}^{n}\sum_{m=0}^{2n} w_{l,m}\left[G_n(\xi_{l,m},\xi_0)\right]^2\right)^{\frac{1}{2}} \\
&= \left(\int_D \left[G_n(x,\xi_0)\right]^2 dx\right)^{\frac{1}{2}}
\end{aligned}$$

The last equality follows from the exactness of the quadrature formula for polynomials of degree $\le 2n$ and from $\left[G_n(x,\xi_0)\right]^2$ being a polynomial of degree $2n$ in the integration variable x. Using the reproducing kernel property (13) of G_n with $f(x) = G_n(x,\xi_0)$, the integral term simplifies to

$$\int_D \left[G_n(x,\xi_0)\right]^2 dx = G_n(\xi_0,\xi_0),$$

and thus,

$$\|\mathcal{L}_n\|_{C(D)\mapsto C(D)} \le \sqrt{\pi G_n(\xi_0,\xi_0)} \tag{16}$$

Next we use the following bound on Jacobi polynomials, taken from [1, 22.14.1]:

$$\left|P_n^{(\lambda,\mu)}(t)\right| \le \binom{n+q}{n} \approx n^q, \qquad q = \max\left(\lambda,\mu\right), \quad -1 \le t \le 1 \tag{17}$$

provided $q \ge -\frac{1}{2}$ and $\lambda,\mu > -1$. In the integral of (15),

$$\left|\int_0^{\pi} P_n^{\left(\frac{3}{2},\frac{1}{2}\right)}\left(x\cdot y + \sqrt{1-\|x\|^2}\sqrt{1-\|y\|^2}\cos(\psi)\right) d\psi\right| \le \pi\binom{n+\frac{3}{2}}{n}, \quad x,y \in B$$

When combined with (12) and (17), we have

$$|G_n(x,y)| = O_{n\to\infty}\left(n^3\right)$$

and then from (16),

$$\|\mathcal{L}_n\|_{C(D)\mapsto C(D)} \le O_{n\to\infty}\left(n^{\frac{3}{2}}\right) \tag{18}$$

Can this result be improved to the growth rate given in (3) for $\|\mathcal{P}_n\|_{C(D)\mapsto C(D)}$? The main result of this paper shows something quite close to this.

Theorem 1 *The hyperinterpolation operator* $\mathcal{L}_n$ *satisfies*

$$\|\mathcal{L}_n\|_{C(D)\mapsto C(D)} = O_{n\to\infty}(n\ln(n)).$$

The proof of this result is given as a series of lemmata that are given in the following two sections of the paper. Before taking up the proof, we note the following.

1. The proof shows that not only our special choice of trapezoidal and Gaussian rule for the hyperinterpolation operator leads to the above norm estimate, but that every choice of a quadrature rule that is a convergent Riemann sum with maximal stepwidth proportional to $\frac{1}{n}$ will lead to the above estimate. Even so, we still need a quadrature rule in (4) which calculates the inner product in (1) exactly for functions in Π_n in order to have $\mathcal{L}_n$ be a projection operator from $C(D)$ onto Π_n.

2. The proof of Theorem 1 includes also a proof of $\|\mathcal{P}_n\|_{C(D)\mapsto C(D)} = O_{n\to\infty}(n)$. The proof given here is independent of the proof by Xu in [18].

3 Bounds for the angular quadrature

The proof of Theorem 1 is shown through a series of lemmata. We begin by looking at the angular quadrature portion of the formula (15),

$$\frac{2\pi}{2n+1}\sum_{m=0}^{2n}\int_0^{\pi}\left|P_n^{(\frac{3}{2},\frac{1}{2})}\left(r_l\hat{r}\cos(\alpha_0-\frac{2m\pi}{2n+1})+\sqrt{1-\hat{r}^2}\sqrt{1-r_l^2}\cos(\psi)\right)\right|d\psi \tag{19}$$

To bound this sum, we begin with a result from Xu [19, Lemma 3.2].

Lemma 2 *For* $\alpha,\beta>-1$, $t\in[0,1]$,

$$\left|P_n^{(\alpha,\beta)}(t)\right| \le \frac{c_0}{\sqrt{n}}(1+\frac{1}{n^2}-t)^{-\frac{\alpha+\frac{1}{2}}{2}},$$

where c_0 *does not depend on* t *or* n.

Because of $\left|P_n^{(\alpha,\beta)}(-t)\right| = \left|P_n^{(\beta,\alpha)}(t)\right|$ we get a similar estimate for $t\in[-1,0]$, namely

$$\left|P_n^{(\alpha,\beta)}(t)\right| \le \frac{c_0}{\sqrt{n}}(1+\frac{1}{n^2}+t)^{-\frac{\beta+\frac{1}{2}}{2}}$$

Let $\alpha=\frac{3}{2}$ and $\beta=\frac{1}{2}$, add the two bounds to obtain the overall bound

$$\left|P_n^{(\frac{3}{2},\frac{1}{2})}(t)\right| \le \frac{c_0}{\sqrt{n}}\left(\frac{1}{1+\frac{1}{n^2}-t}+\frac{1}{\sqrt{1+\frac{1}{n^2}+t}}\right), \qquad -1\le t\le 1 \tag{20}$$

For a general $a > 0$, we have

$$\frac{1}{\sqrt{a}} \le \frac{c}{a}$$

provided $c \ge \sqrt{a}$. Using this with the final fraction in (20), we find that we need $c \ge \sqrt{3}$ when allowing $-1 \le t \le 1$ and $n \ge 1$. Thus

$$\left| P_n^{\left(\frac{3}{2},\frac{1}{2}\right)}(t) \right| \le \frac{\sqrt{3}c_0}{\sqrt{n}} \left(\frac{1}{1+\frac{1}{n^2}-t} + \frac{1}{1+\frac{1}{n^2}+t} \right) \tag{21}$$

Using the estimate (21) in (15) and the definition of $w_{m,l}$ we get

$$\begin{aligned} \|L_n\|_{C(D)\mapsto C(D)} &\le c_1(n) \sum_{l=0}^{n} w_l r_l \\ &\times \sum_{m=0}^{2n} \frac{2\pi}{2n+1} \int_0^\pi \left[\frac{1}{1+\frac{1}{n^2} - t(n,\hat{r},\alpha_0,r_l,m,\psi)} \right. \\ &\qquad \left. + \frac{1}{1+\frac{1}{n^2} + t(n,\hat{r},\alpha_0,r_l,m,\psi)} \right] d\psi \end{aligned} \tag{22}$$

where we have introduced

$$t(n,\hat{r},\alpha_0,r,m,\psi) := \hat{r}r\cos\left(\alpha_0 - \frac{2m\pi}{2n+1}\right) + \sqrt{1-\hat{r}^2}\sqrt{1-r^2}\cos(\psi) \tag{23}$$

and

$$c_1(n) := \frac{8c_0\Gamma\left(\frac{5}{2}\right)\Gamma(n+3)}{\Gamma(4)\Gamma\left(n+\frac{5}{2}\right)\sqrt{n}} = O_{n\to\infty}(n) \tag{24}$$

To estimate the trapezoidal rule in the square brackets of (22), we start by calculating the integral. For this, use [10, 2.553],

$$\int_0^\pi \frac{1}{a+b\cos(\psi)} d\psi = \frac{\pi}{\sqrt{a^2-b^2}}, \qquad |a| > |b|. \tag{25}$$

Apply this to (22) with

$$\begin{aligned} a &= 1 + \frac{1}{n^2} \pm \hat{r}r_l\cos\left(\alpha_0 - \frac{2m\pi}{2n+1}\right), \\ b &= \pm\sqrt{1-\hat{r}^2}\sqrt{1-r_l^2} \end{aligned}$$

Note that

$$\begin{aligned} a^2 - b^2 &\ge (1 + \frac{1}{n^2} - \hat{r}r_l)^2 - (1-\hat{r}^2)(1-r_l^2) \\ &= (\hat{r}-r_l)^2 + \frac{1}{n^4} + \frac{2}{n^2}(1-\hat{r}r_l) \\ &\ge \frac{1}{n^4}, \qquad \hat{r}, r_l \in [0,1]. \end{aligned}$$

Introduce the notation

$$H_1^{\pm}(n,\hat{r},r,\alpha)$$
$$:= \frac{\pi}{\sqrt{(\hat{r}^2+r_l^2-\hat{r}^2r_l^2+\frac{1}{n^4}+\frac{2}{n^2})\pm 2\hat{r}r_l(1+\frac{1}{n^2})\cos(\alpha)+\hat{r}^2r_l^2\cos(\alpha)^2}} \quad (26)$$

For (22), introduce the notation

$$T_n(\hat{r},\alpha_0,r) := \sum_{m=0}^{2n}\frac{2\pi}{2n+1}\times \quad (27)$$
$$\int_0^{\pi}\left(\frac{1}{1+\frac{1}{n^2}-t(n,\hat{r},\alpha_0,r_l,m,\psi)}+\frac{1}{1+\frac{1}{n^2}+t(n,\hat{r},\alpha_0,r_l,m,\psi)}\right)d\psi$$

Applying (25),

$$T_n(\hat{r},\alpha_0,r)$$
$$=\sum_{m=0}^{2n}\frac{2\pi}{2n+1}\left[H_1^-(n,\hat{r},r,\alpha_0-\frac{2m\pi}{2n+1})+H_1^+(n,\hat{r},r,\alpha_0-\frac{2m\pi}{2n+1})\right]$$

Introduce

$$I_1(n,\hat{r},r) := \int_0^{2\pi}\left[H_1^+(n,\hat{r},r,\alpha)+H_1^-(n,\hat{r},r,\alpha)\right]\,d\alpha, \quad (28)$$

Then (27) is nothing more than a trapezoidal rule, shifted by α_0, for the approximation of $I_1(n,\hat{r},r_l)$. To further simplify our analysis, note that the integrand in the definition of I_1 has period π. Also, the simple identity $\cos(\pi-\alpha) = -\cos(\alpha)$ can be used to show that the integral over $[0,\pi]$ of H_1^+ equals that of H_1^-. Consequently,

$$I_1(n,\hat{r},r) = 4\int_0^{\pi} H_1^-(n,\hat{r},r,\alpha)\,d\alpha$$

To estimate $T_n(\hat{r},\alpha_0,r)$ we need to find estimates for the integral I_1 and for the quadrature error of the shifted trapezoidal rule. We use the following result. A proof for the case of the trapezoidal rule is given in the Brass [7], but it works the same for every Riemann sum.

Lemma 3 *Let $f : [a,b] \mapsto \mathbb{R}$ be a continuous function with bounded variation and Q_n a quadrature rule which is also a Riemann sum:*

$$Q_n(f) := \sum_{i=0}^{n} f(\xi_i^{[n]})(x_i^{[n]}-x_{i-1}^{[n]}),$$

$a = x_0^{[n]} < x_1^{[n]} < \ldots < x_n^{[n]} = b$, $\xi_i^{[n]} \in [x_{i-1}^{[n]}, x_i^{[n]}]$. *Then*

$$\left|\int_a^b f(x)dx - Q_n(f)\right| \le Var(f)\max_{i=1}^{n}(x_i^{[n]}-x_{i-1}^{[n]}),$$

where $Var(f)$ is the variation of f over $[a,b]$. This implies

$$|Q_n(f)| \leq \left| \int_a^b f(x)dx \right| + Var(f) \max_{i=1}^{n} (x_i^{[n]} - x_{i-1}^{[n]}).$$

Because the shifted trapezoidal rule is a Riemann sum we can use Lemma 3 to estimate $T_n(\hat{r}, \alpha_0, r)$.

Lemma 4 *For the sum T_n, defined in (27), we get*

$$\begin{aligned} T_n(\hat{r}, \alpha_0, r) \leq & \frac{8\pi}{\sqrt{A(n,\hat{r},r)}} K \left(\sqrt{\frac{4\hat{r}r\sqrt{1-\hat{r}^2}\sqrt{1-r^2}}{A(n,\hat{r},r)}} \right) \\ & + \frac{2\pi^2}{n} \frac{1}{\sqrt{(\hat{r}-r)^2 + \frac{1}{n^4} + \frac{2}{n^2}(1-\hat{r}r)}}, \\ A(n,\hat{r},r) := & \hat{r}^2 + r^2 - 2\hat{r}^2 r^2 + \frac{1}{n^4} + \frac{2}{n^2} + 2\hat{r}r\sqrt{1-\hat{r}^2}\sqrt{1-r^2} \end{aligned} \tag{29}$$

Remark 5 *Here K denotes the complete elliptic integral of the first kind, see [1, §17.3], [2, p. 132], [8].*

Proof. By Lemma 3 we have to calculate $\int_0^\pi H_1^-(n,\hat{r},r,\alpha)d\alpha$ and estimate $\mathrm{Var}(H_1^-(n,\hat{r},r,\alpha))$. We start with the integral and introduce the abbreviations

$$\begin{aligned} \kappa &= \frac{1}{n^4} + \frac{2}{n^2} \\ a &= \hat{r}^2 + r^2 - \hat{r}^2 r^2 + \kappa \\ b &= -2\hat{r}r(1 + \frac{1}{n^2}) \\ c &= \hat{r}^2 r^2 \end{aligned} \tag{30}$$

Now we can write

$$\begin{aligned} \int_0^\pi H_1^-(n,\hat{r},r,\alpha)d\alpha &= \int_0^\pi \frac{\pi}{\sqrt{a + b\cos(\alpha) + c\cos(\alpha)^2}} d\alpha \\ &= 2\pi \int_0^\infty \frac{1}{\sqrt{a + b\frac{1-\beta^2}{1+\beta^2} + c\left(\frac{1-\beta^2}{1+\beta^2}\right)^2}} \frac{d\beta}{1+\beta^2} \\ &= 2\pi \int_0^\infty \frac{1}{\sqrt{c_1 + b_1\beta^2 + a_1\beta^4}} d\beta \end{aligned}$$

Here we have used the substitution

$$\alpha = 2\arctan(\beta)$$

and have introduced the abbreviations

$$\begin{aligned}
a_1 &= a - b + c \\
&= (\hat{r} + r)^2 + \frac{1}{n^4} + \frac{2}{n^2}(1 + \hat{r}r) \\
&\geq \kappa \\
b_1 &= 2(a - c) \\
&= 2(\hat{r}^2 + r^2 - 2\hat{r}^2 r^2 + \kappa) \\
&\geq \kappa \\
c_1 &= a + b + c \\
&= (\hat{r} - r)^2 + \frac{2}{n^2}(1 - \hat{r}r) + \frac{1}{n^4} \\
&\geq \frac{1}{n^4}
\end{aligned}$$

Finally we use the substitution

$$\gamma = \beta^2$$

and obtain the following formula for $I_1(n, \hat{r}, r)$,

$$I_1(n, \hat{r}, r) = 4\pi \int_0^\infty \frac{1}{\sqrt{\gamma}\sqrt{c_1 + b_1\gamma + a_1\gamma^2}} d\gamma$$

The above way to transform the integral is the common way to transform these kind of integrals in order to bring them into a standard form connected to elliptic integrals; see [10, 2.580]. Finally we calculate the zeros $0 > \gamma_1 > \gamma_2$ of

$$0 = a_1\gamma^2 + b_1\gamma + c_1$$

and get

$$\begin{aligned}
\gamma_1 &= \frac{-(\hat{r}^2 + r^2 - 2\hat{r}^2 r^2 + \kappa) + 2\hat{r}r\sqrt{1 - \hat{r}^2}\sqrt{1 - r^2}}{(\hat{r} + r)^2 + \frac{1}{n^4} + \frac{2}{n^2}(1 + \hat{r}r)} \\
\gamma_2 &= \frac{-(\hat{r}^2 + r^2 - 2\hat{r}^2 r^2 + \kappa) - 2\hat{r}r\sqrt{1 - \hat{r}^2}\sqrt{1 - r^2}}{(\hat{r} + r)^2 + \frac{1}{n^4} + \frac{2}{n^2}(1 + \hat{r}r)}
\end{aligned}$$

This leads us to

$$\begin{aligned}
I_1(n, \hat{r}, r) &= \frac{4\pi}{\sqrt{a_1}} \int_0^\infty \frac{1}{\sqrt{(\gamma - 0)(\gamma - \gamma_1)(\gamma - \gamma_2)}} d\gamma \\
&= \frac{4\pi}{\sqrt{a_1}} \frac{2}{\sqrt{-\gamma_2}} K\left(\sqrt{\frac{\gamma_1 - \gamma_2}{-\gamma_2}}\right)
\end{aligned}$$

with the complete elliptic integral of first kind K, see [10, 3.131.8]. Plugging in the expressions for a_1, γ_1, and γ_2 gives us the first term in our estimate for the trapezoidal rule in (29).

Finally we calculate the total variation of the function $H_1^-(n, \hat{r}, r, \alpha)$ over $[0, 2\pi]$. We define the function

$$f(\alpha) := -2\hat{r}r(1 + \frac{1}{n^2})\cos(\alpha) + \hat{r}^2 r^2 \cos(\alpha)^2$$

Then $f'(\alpha) = 0$ for $\alpha \in \{0, \pi, 2\pi\}$, and checking these numbers for H_1^- we find that $H_1^-(n, \hat{r}, r, 0) = H_1^-(n, \hat{r}, r, 2\pi)$ is the maximum and $H_1^-(n, \hat{r}, r, \pi)$ is the minimum. In between the function is monotone, so the variation is bounded by

$$\begin{aligned}\mathrm{Var}(H_1^-(n, \hat{r}, r, \cdot)) &= 2(H_1^-(n, \hat{r}, r, 0) - H_1^-(n, \hat{r}, r, \pi)) \\ &\le 2H_1^-(n, \hat{r}, r, 0)\end{aligned}$$

Together with the stepwidth of $\frac{2\pi}{2n+1}$ of the trapezoidal rule, Lemma (3) gives the second term in formula (29). ■

4 Bounds for the radial quadrature

Combining (22), Lemma 4, and the notation

$$J_1(n, \hat{r}, r) := \frac{8\pi}{\sqrt{A(n, \hat{r}, r)}} K\left(\sqrt{\frac{4\hat{r}r\sqrt{1-\hat{r}^2}\sqrt{1-r^2}}{A(n, \hat{r}, r)}}\right) \tag{31}$$

$$A(n, \hat{r}, r) := \hat{r}^2 + r^2 - 2\hat{r}^2 r^2 + \frac{1}{n^4} + \frac{2}{n^2} + 2\hat{r}r\sqrt{1-\hat{r}^2}\sqrt{1-r^2} \tag{32}$$

$$J_2(n, \hat{r}, r) := 2\pi^2 \frac{1}{\sqrt{(\hat{r}-r)^2 + \frac{1}{n^4} + \frac{2}{n^2}(1-\hat{r}r)}} \tag{33}$$

we can estimate

$$\begin{aligned}\|L_n\|_{C(D)\mapsto C(D)} &\le c_1(n) \sum_{l=0}^{n} w_l r_l \left(J_1(n, \hat{r}, r_l) + \frac{1}{n} J_2(n, \hat{r}, r_l)\right) \\ &= c_1(n) Q_n^G[r(J_1(n, \hat{r}, r) + \frac{1}{n} J_2(n, \hat{r}, r))] \qquad (34)\end{aligned}$$

where Q_n^G denotes the $n+1$ point Gaussian quadrature rule on $[0, 1]$. For these quadrature rules we use the following result.

Theorem 6 *The Gaussian quadrature rules are Riemann sums, and if we denote by $a < x_0^{[n]} < x_1^{[n]} < \ldots < x_n^{[n]} < b$ the knots of the Gaussian quadrature rule on $[a, b]$, then*

$$\max_{i=1}^{n}(x_i^{[n]} - x_{i+1}^{[n]}) \le c_2 \frac{b-a}{n}$$

where $c_2 > 0$ is independent of n.

Proof. See for example [7, Thms. 53 and 85], or [17, 3.41.1 and 6.21.3] ■

Theorem 6 allows us to use Lemma 3 to estimate the right hand side of equation (33), in a similar way to the proof of Lemma 4. We prove the required estimates in the next four lemmata.

Lemma 7 *For the function J_1, defined in formula (31), we get*

$$\int_0^1 rJ_1(n, \hat{r}, r)\, dr \le c_3, \tag{35}$$

where c_3 does not depend on n or $\hat{r}$.

Proof. First we define the angle $\psi \in [0, \frac{\pi}{2}]$ by $\hat{r} = \cos(\psi)$ and then we substitute $r = \cos(\phi)$ in the integral in (35). We remark

$$A(n, cos(\psi), \cos(\phi)) = \kappa + \sin(\phi + \psi)^2.$$

See (32) for the definition of A and (30) for the definition of κ. Using these results and the substitution we get

$$\int_0^1 rJ_1(n, \cos(\psi), r)\, dr = 8\pi \int_0^{\frac{\pi}{2}} \frac{\cos(\phi)\sin(\phi)}{\sqrt{\kappa + \sin(\psi+\phi)^2}} K\left(\sqrt{\frac{\sin(2\psi)\sin(2\phi)}{\kappa + \sin(\psi+\phi)^2}}\right) d\phi$$

For $\phi, \psi \in [0, \frac{\pi}{2}]$ we can estimate

$$\begin{aligned}
\frac{\cos(\phi)\sin(\phi)}{\sqrt{\kappa + \sin(\psi+\phi)^2}} &\le \frac{\cos(\phi)\sin(\phi)}{\sin(\phi+\psi)} \\
&= \frac{\cos(\phi)\sin(\phi)}{\sin(\psi)\cos(\phi) + \cos(\psi)\sin(\phi)} \\
&= \frac{1}{\frac{\sin(\psi)}{\sin(\phi)} + \frac{\cos(\psi)}{\cos(\phi)}} \\
&\le \frac{1}{\sin(\psi) + \cos(\psi)} \\
&\le 1
\end{aligned}$$

and therefore

$$\int_0^1 rJ_1(n, \cos(\psi), r)\, dr \le 8\pi \int_0^{\frac{\pi}{2}} K\left(\sqrt{\frac{\sin(2\psi)\sin(2\phi)}{\kappa + \sin(\psi+\phi)^2}}\right) d\phi$$

To estimate the integral we first rewrite the complete elliptic integral as a hypergeometric function (see [2, 3.2.3]),

$$K(z) = \frac{\pi}{2} F_{2,1}\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; z^2\right).$$

Furthermore the function $F_{2,1}(\frac{1}{2}, \frac{1}{2}; 1; \cdot)$ is monotone increasing on $[0, 1]$ and ([2, Th. 2.1.3])

$$\lim_{z \to 1-} \frac{F_{2,1}\left(\frac{1}{2}, \frac{1}{2}; 1; z\right)}{-\ln(1-z)} = \frac{1}{\pi}.$$

This implies that there is a constant c_4 such that

$$F_{2,1}\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; z\right) \leq c_4(1 - \ln(1-z)), \qquad 0 \leq z < 1. \tag{36}$$

Using this estimate and the fact that

$$F_{2,1}\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \frac{\sin(2\psi)\sin(2\phi)}{\kappa + \sin(\psi+\phi)^2}\right) \leq F_{2,1}\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \frac{\sin(2\psi)\sin(2\phi)}{\sin(\psi+\phi)^2}\right)$$

shows us that we need only bound

$$-\int_0^{\frac{\pi}{2}} \ln\left(1 - \frac{\sin(2\psi)\sin(2\phi)}{\sin(\psi+\phi)^2}\right) d\phi$$

independently of ψ to finish the proof of the lemma. We define

$$f(\psi, \phi) := \frac{\sin(2\psi)\sin(2\phi)}{\sin(\psi+\phi)^2}, \qquad 0 \leq \psi, \phi \leq \frac{\pi}{2}$$

Then

$$\begin{aligned} f(\psi, \psi) &= 1, \\ f(\psi, 0) &= f\left(\psi, \frac{\pi}{2}\right) = 0. \end{aligned}$$

We derive some properties of f:

$$\begin{aligned} \frac{\partial f(\psi, \phi)}{\partial \phi} &= \frac{2\sin(2\psi)}{\sin^3(\psi+\phi)} \sin(\psi - \phi) \\ &= 0 \quad \Leftrightarrow \quad \psi = \phi \end{aligned}$$

so all functions $f(\psi, \cdot)$ are increasing between 0 and ψ and then decreasing between ψ and $\frac{\pi}{2}$. Furthermore

$$f\left(\frac{\pi}{2} - \psi, \frac{\pi}{2} - \phi\right) = f(\psi, \phi),$$

so we only have to consider $\psi \in [0, \frac{\pi}{4}]$. For $\varepsilon \in [0, \psi]$ we get

$$f(\psi, \psi - \varepsilon) = \frac{\sin(2\psi)\sin(2\psi - 2\varepsilon)}{\sin(2\psi - \varepsilon)^2}$$

and

$$\begin{aligned} \frac{\partial f(\psi, \psi - \varepsilon)}{\partial \psi} &= \frac{2\sin(\varepsilon)}{\sin(2\psi - \varepsilon)^3} \left[\sin(2\psi) - \sin(2\psi - 2\varepsilon)\right] \\ &\geq 0 \end{aligned}$$

This proves

$$\begin{aligned} f(\psi, \psi - \varepsilon) &\le f\left(\frac{\pi}{4}, \frac{\pi}{4} - \varepsilon\right) \\ &= 1 - \tan(\varepsilon)^2, \qquad 0 \le \varepsilon \le \psi \end{aligned}$$

and therefore

$$f(\psi, \phi) \le 1 - \tan(\psi - \phi)^2, \qquad 0 \le \phi \le \psi \le \frac{\pi}{4} \tag{37}$$

Now we study $\varepsilon \in [0, \frac{\pi}{4}]$. Noting

$$f(\psi, \psi + \varepsilon) = \frac{\sin(2\psi)\sin(2\psi + 2\varepsilon)}{\sin(2\psi + \varepsilon)^2}$$

we get

$$\begin{aligned} \frac{\partial f(\psi, \psi + \varepsilon)}{\partial \psi} &= \frac{2\sin(\varepsilon)}{\sin(2\psi + \varepsilon)^3}(\sin(2\psi + 2\varepsilon) - \sin(2\psi)) \\ &= 0 \quad \Leftrightarrow \quad \psi = \frac{\pi}{4} - \frac{\varepsilon}{2} \end{aligned}$$

Therefore

$$\begin{aligned} f(\psi, \psi + \varepsilon) &\le f\left(\frac{\pi}{4} - \frac{\varepsilon}{2}, \frac{\pi}{4} + \frac{\varepsilon}{2}\right) \\ &= \cos(\varepsilon)^2 \end{aligned}$$

and this proves the estimate

$$f(\psi, \phi) \le \cos(\phi - \psi)^2, \qquad \psi \le \phi \le \psi + \frac{\pi}{4} \tag{38}$$

But $f(\psi, \phi)$ is monotone decreasing for $\phi > \psi$, so we get also

$$\begin{aligned} f(\psi, \phi) &\le f\left(\psi, \psi + \frac{\pi}{4}\right) \\ &= \cos\left(\frac{\pi}{4}\right)^2 \\ &= \frac{1}{2}, \qquad \psi + \frac{\pi}{4} \le \phi \le \frac{\pi}{2} \end{aligned} \tag{39}$$

Using the estimates (37)–(39) we get

$$
\begin{aligned}
-\int_0^{\frac{\pi}{2}} \ln\left(1-f(\psi,\phi)\right) d\phi &\le -\int_0^{\psi} \ln(\tan(\psi-\phi)^2)\, d\phi \\
&\quad -\int_{\psi}^{\psi+\frac{\pi}{4}} \ln(1-\cos(\phi-\psi)^2)\, d\phi - \int_{\psi+\frac{\pi}{4}}^{\frac{\pi}{2}} \ln(\frac{1}{2})\, d\phi \\
&\le -2\int_0^{\psi} \ln(\tan(\tau))\, d\tau \\
&\quad -2\int_0^{\frac{\pi}{4}} \ln(\sin(\tau))\, d\tau + \int_{\psi+\frac{\pi}{4}}^{\frac{\pi}{2}} \ln(2)\, d\phi \\
&\le -2\int_0^{\frac{\pi}{4}} \ln(\tan(\tau))\, d\tau \\
&\quad -2\int_0^{\frac{\pi}{4}} \ln(\sin(\tau))\, d\tau + \frac{\pi\ln(2)}{4} \\
&< \infty
\end{aligned}
$$

which is a bound for the integral, independent of ψ. This proves the lemma. ■

Lemma 8 *The function* J_2 *defined by (33) satisfies*

$$
\int_0^1 rJ_2(n,\hat{r},r)\, dr \le c_5 \ln(n), \tag{40}
$$

where c_5 *does not depend on* n *or* $\hat{r}$.

Proof. First rewrite the integral in (40),

$$
\begin{aligned}
\int_0^1 rJ_2(n,\hat{r},r)\, dr &= 2\pi^2 \int_0^1 \frac{r}{\sqrt{(1-\hat{r}^2)\kappa + \left(r-\hat{r}\left(1+\frac{1}{n^2}\right)\right)^2}} dr \\
&= 2\pi^2 \int_{-\hat{r}(1+1/n^2)}^{1-\hat{r}(1+1/n^2)} \frac{\hat{r}(1+1/n^2)+u}{\sqrt{(1-\hat{r}^2)\kappa+u^2}} du \\
&= 2\pi^2 \hat{r}(1+1/n^2) \int_{-\hat{r}(1+1/n^2)}^{1-\hat{r}(1+1/n^2)} \frac{1}{\sqrt{(1-\hat{r}^2)\kappa+u^2}} du \\
&+ 2\pi^2 \int_{-\hat{r}(1+1/n^2)}^{1-\hat{r}(1+1/n^2)} \frac{u}{\sqrt{(1-\hat{r}^2)\kappa+u^2}} du \\
&=: K_1(n,\hat{r}) + K_2(n,\hat{r})
\end{aligned}
$$

where we have used the substitution $u = r - \hat{r}(1+1/n^2)$ and the definition of κ in (30). Note that $K_2(n,\hat{r})$ is bounded

$$
\begin{aligned}
K_2(n,\hat{r}) &= 2\pi^2 / \sqrt{(1-\hat{r}^2)\kappa + u^2}\Big|_{-\hat{r}(1+1/n^2)}^{1-\hat{r}(1+1/n^2)} \\
&= 2\pi^2 \left[\sqrt{(1-\hat{r}^2)\kappa + (1-\hat{r}(1+1/n^2))^2} - \sqrt{(1-\hat{r}^2)\kappa + (-\hat{r}(1+1/n^2))^2}\right] \\
&\leq 2\sqrt{2}\,\pi^2
\end{aligned}
$$

if we assume $n \geq 2$ which implies $\kappa < 1$.

Before we start to estimate $K_1(n,\hat{r})$ we calculate $K_1(n,1)$,

$$
\begin{aligned}
K_1(n,1) &= 2\pi^2 \left(1 + \frac{1}{n^2}\right) \int_{-(1+1/n^2)}^{-1/n^2} \frac{1}{|u|}\, du \\
&= 2\pi^2 \left(1 + \frac{1}{n^2}\right) \ln(1+n^2) \\
&= O_{n\to\infty}(\ln(n))
\end{aligned}
$$

Thus we cannot expect to find a finite bound for the function K_1. To estimate K_1 we consider three cases:

1. $0 \leq \hat{r} \leq \frac{1}{2}\frac{1}{1+1/n^2}$. In this case the upper limit of the integral for K_1 is larger than zero; and in particular, it is larger than $\hat{r}(1+1/n^2)$.
2. $\frac{1}{2}\frac{1}{1+1/n^2} \leq \hat{r} \leq \frac{1}{1+1/n^2}$. This implies that the upper limit in the integral for K_1 is still positive. The upper limit is also smaller than $\hat{r}(1+1/n^2)$.
3. $\frac{1}{1+1/n^2} < \hat{r} < 1$. Now zero is no longer in the interval of integration for K_1.

In each case we are able to find a logarithmic bound for K_1.

Case 1. Here we use the fact that $\hat{r} \leq 1/2$, $\kappa > 2/n^2$, and $1 + 1/n^2 \leq 2$ to estimate

$$
\begin{aligned}
K_1(n,\hat{r}) &\leq 2\pi^2 \cdot 2 \int_{-\frac{1}{2}(1+1/n^2)}^{1} \frac{1}{\sqrt{\frac{1}{2}\cdot\frac{2}{n^2} + u^2}}\, dy \\
&\leq 4\pi^2 \int_{-1}^{1} \frac{1}{\sqrt{\frac{1}{n^2} + u^2}}\, du \\
&= 4\pi^2 \int_{-n}^{n} \frac{1}{\sqrt{1+v^2}}\, dv \\
&\leq 8\pi^2 \left(\int_0^1 1\, dv + \int_1^n \frac{1}{v}\, dv\right) \\
&= O(\ln(n))
\end{aligned}
$$

Case 2. In addition to the above estimates for κ and $(1+1/n^2)$, we use $\hat{r} < 1$ and the fact that now $\hat{r}(1+1/n^2) \geq 1 - \hat{r}(1+1/n^2)$. It then follows that

$$\begin{aligned} K_1(n,\hat{r}) &\leq 4\pi^2 \int_{-\hat{r}(1+1/n^2)}^{\hat{r}(1+1/n^2)} \frac{1}{\sqrt{\frac{1-\hat{r}^2}{n^2} + u^2}}\, du \\ &\leq 4\pi^2 \int_{-2\hat{r}}^{2\hat{r}} \frac{1}{\sqrt{\frac{1-\hat{r}^2}{n^2} + u^2}}\, du \\ &\leq 8\pi^2 \int_0^{\frac{2n\hat{r}}{\sqrt{1-\hat{r}^2}}} \frac{1}{\sqrt{1+v^2}}\, dv \end{aligned}$$

where we used the substitution $v = nu/\sqrt{1-\hat{r}^2}$. It is easy to see that the function $\hat{r} \mapsto \frac{\hat{r}}{\sqrt{1-\hat{r}^2}}$ is monotone increasing on $[0,1)$, so the upper limit of the above integral is (in case b) smaller than

$$\begin{aligned} \frac{2n\hat{r}}{\sqrt{1-\hat{r}^2}}\bigg|_{\hat{r}=\frac{1}{1+1/n^2}} &= \frac{2n\frac{1}{1+1/n^2}}{\sqrt{1-(\frac{1}{1+1/n^2})^2}} \\ &= \frac{2n}{\sqrt{(1+1/n^2)^2-1}} \\ &\leq \frac{2n}{\sqrt{1/n^2}} \\ &= 2n^2 \end{aligned}$$

Similar to case 1 we estimate

$$\begin{aligned} K_1(n,\hat{r}) &\leq 8\pi^2 \left(\int_0^1 1 dv + \int_1^{2n^2} \frac{1}{v}\, dv \right) \\ &= O(\ln(n)) \end{aligned}$$

Case 3. Here the integral is only over negative numbers. We again use the above mentioned estimates for κ and $(1+1/n^2)$, and we use $\hat{r} < 1$ to derive

$$\begin{aligned} K_1(n,\hat{r}) &\leq 4\pi^2 \int_{-\hat{r}(1+1/n^2)}^{1-\hat{r}(1+1/n^2)} \frac{1}{\sqrt{\frac{1-\hat{r}^2}{n^2} + u^2}}\, du \\ &= 4\pi^2 \int_{\frac{n}{\sqrt{1-\hat{r}^2}}(\hat{r}(1+1/n^2)-1)}^{\frac{n\hat{r}}{\sqrt{1-\hat{r}^2}}(1+1/n^2)} \frac{1}{\sqrt{1+v^2}},\, dv \end{aligned}$$

We have again used the substitution $v = nu/\sqrt{1-\hat{r}^2}$. This time we calculate the integral in order to get our estimate. First we remember

$$\int \frac{1}{\sqrt{1+v^2}}\, dv = \ln(v + \sqrt{1+v^2})$$

Then we get

$$K_1(n,\hat{r}) \leq 4\pi^2 \ln\left(\frac{f_n(\hat{r})}{g_n(\hat{r})}\right)$$

where

$$g_n(\hat{r}) := n(\hat{r}(1+\frac{1}{n^2})-1) + \sqrt{(1-\hat{r})^2 + n^2\left(\hat{r}\left(1+\frac{1}{n^2}\right)-1\right)^2}$$

$$f_n(\hat{r}) := n\hat{r}(1+\frac{1}{n^2}) + \sqrt{(1-\hat{r})^2 + n^2\hat{r}^2\left(1+\frac{1}{n^2}\right)^2}$$

The function

$$\psi(n,\hat{r}) := (1-\hat{r})^2 + n^2\left(\hat{r}\left(1+\frac{1}{n^2}\right)-1\right)^2$$

has its minimum at

$$r^* := \frac{n^2+1}{n^2+1+\frac{1}{n^2}} \in \left[\frac{1}{1+\frac{1}{n^2}}, 1\right]$$

and one can derive

$$\psi(n,r^*) \geq \frac{1}{2n^2}$$

This implies

$$g_n(\hat{r}) \geq \frac{1}{\sqrt{2}\,n}$$

For $f_n(r)$ it is easy to see

$$\begin{aligned} 1 &\leq f_n(\hat{r}) \\ &\leq 4n \end{aligned}$$

So we finally get

$$K_1(n,\hat{r}) \leq 4\pi^2 \ln(8n^2)$$

Using the results from the cases (1)-(3) together with the estimate for K_2 proves the existence of a constant $c_5 > 0$ in the statement of the lemma. ∎

In the next two lemmata we study the total variation of $rJ_1(n,\hat{r},r)$ and $rJ_2(n,\hat{r},r)$ on $[0,1]$.

Lemma 9 *There is a constant c_6 independent of n and $\hat{r}$ such that*

$$Var(rJ_1(n,\hat{r},r)) \leq c_6 n \ln(n).$$

Proof. As in the proof of Lemma 7 we introduce the notation $\hat{r} = \cos(\psi)$, $\psi \in [0,\pi/2]$ and substitute $r = \cos(\phi)$, this will not change the maximum values of our function and the monotonicity is just reversed, but this does not change the total variation. So we study the function

$$\begin{aligned} f(n,\psi,\phi) &:= f_1(n,\psi,\phi) f_2(n,\psi,\phi) \\ f_1(n,\psi,\phi) &:= \frac{\cos(\phi)}{\sqrt{\kappa+\sin(\psi+\phi)^2}} \\ f_2(n,\psi,\phi) &:= K\left(\sqrt{\frac{\sin(2\psi)\sin(2\phi)}{\kappa+\sin(\phi+\psi)^2}}\right) \\ &= \frac{\pi}{2} F_{2,1}\left(\tfrac{1}{2},\tfrac{1}{2};1;\frac{\sin(2\psi)\sin(2\phi)}{\kappa+\sin(\phi+\psi)^2}\right) \end{aligned}$$

See Lemma 7 and formula (30) for the definition of κ. Note that we also neglected the constant in the function $rJ_1(n,\hat{r},r)$. The following observation allows us to treat f_1 and f_2 separately

$$\mathrm{Var}(f_1 f_2) \le \|f_1\|_\infty \mathrm{Var}(f_2) + \|f_2\|_\infty \mathrm{Var}(f_1)$$

First we study f_1.

$$\begin{aligned} f_1(n,\psi,0) &= \frac{1}{\sqrt{\kappa+\sin(\psi)^2}} \\ f_1(n,\psi,\frac{\pi}{2}) &= 0 \\ \frac{\partial f_1(n,\psi,\phi)}{\partial\phi)} &= -\frac{\kappa\sin(\phi)+\sin(\psi+\phi)\cos(\psi)}{(\kappa+\sin(\psi+\phi))^{3/2}} \\ &\le 0, \qquad \psi,\phi\in\left[0,\frac{\pi}{2}\right]. \end{aligned}$$

This implies

$$\begin{aligned} \|f_1\|_\infty &= \mathrm{Var}(f_1) \\ &= \frac{1}{\sqrt{\kappa+\sin(\psi)^2}} \\ &\le \frac{1}{\sqrt{\kappa}} \\ &\le n. \end{aligned}$$

Now we turn to f_2, and remember that $F_{2,1}(\frac{1}{2},\frac{1}{2};1;\cdot)$ is a monotone increasing function on $[0,1]$, $F_{2,1}(\frac{1}{2},\frac{1}{2};1;0)=1$ and (see (36))

$$F_{2,1}\left(\tfrac{1}{2},\tfrac{1}{2};1;z\right) \le c_4(1-\ln(1-z)), \qquad 0\le z<1.$$

So we first have to understand the behavior of the function inside the logarithmic term:

$$
\begin{aligned}
f_3(n,\psi,\phi) &:= 1 - \frac{\sin(2\psi)\sin(2\phi)}{\kappa + \sin(\phi+\psi)^2} \\
f_3(n,\psi,0) &= 1 \\
f_3\left(n,\psi,\frac{\pi}{2}\right) &= 1 \\
\frac{\partial f_3(n,\psi,\phi)}{\partial \phi} &= -\frac{2\sin(2\psi)}{(\kappa+\sin(\psi+\phi)^2)^2}\left[\kappa\cos(2\psi) + \sin(\psi+\phi)\sin(\psi-\phi)\right]
\end{aligned}
$$

It is easy to see that both terms in the square brackets are decreasing. The values range from $\kappa + \sin(\psi)^2 > 0$ to $-\kappa - \cos(\psi)^2 < 0$, so f_3 is first decreasing and then increasing. We estimate the minimum value of f_3 (see also the proof of Lemma 7)

$$
\begin{aligned}
f_3(n,\psi,\phi) &= \frac{\kappa + \sin(\psi+\phi)^2 - \sin(2\psi)\sin(2\phi)}{\kappa + \sin(\psi+\phi)^2} \\
&\geq \frac{\kappa}{\kappa + \sin(\psi+\phi)^2} \\
&\geq \frac{\kappa}{\kappa+1} \\
&\geq \frac{\kappa}{2} \geq \frac{1}{n^2}
\end{aligned}
$$

if $n \geq 2$. Together with the monotonicity of $F_{2,1}$ this proves

$$
\begin{aligned}
\|f_2\|_\infty &\leq c_7 \ln(1/\kappa) \leq c_7 \ln(n^2) \\
\mathrm{Var}(f_2) &\leq c_7 \ln(1/\kappa) \leq c_7 \ln(n^2)
\end{aligned}
$$

with a suitable constant c_7 independent of n and $\hat{r} = \cos(\psi)$. This finishes the proof of Lemma (9). ■

Lemma 10 *The function $rJ_2(n,\hat{r},r)$, defined in (33) satisfies*

$$
Var(rJ_2(n,\hat{r},r)) \leq c_8 n^2
$$

where $c_8 > 0$ is independent of n and $\hat{r}$.

Proof. We define

$$
f(n,\hat{r},r) := \frac{r}{\sqrt{(\hat{r}-r)^2 + \frac{1}{n^4} + \frac{2}{n^2}(1-\hat{r}r))}}
$$

to estimate the total variation of $rJ_2(n, \hat{r}, r)$, where we neglect the constant in (33). We have

$$\begin{aligned} f(n, \hat{r}, 0) &= 0, \qquad \text{and} \\ \max_{r \in [0,1]} f(n, \hat{r}, r) &\leq n^2 \\ \frac{\partial f(n, \hat{r}, r)}{\partial r} &= \frac{-\hat{r}r(1+\frac{1}{n^2}) + \hat{r}^2 + \frac{1}{n^4} + \frac{2}{n^2}}{\left((\hat{r}-r)^2 + \frac{1}{n^4} + \frac{2}{n^2}(1-\hat{r}r)\right)^{3/2}} \end{aligned}$$

So we have

$$\begin{aligned} \frac{\partial f(n, \hat{r}, r^*)}{\partial r} = 0 \quad &\Leftrightarrow \\ r^* &= \frac{\hat{r}^2 + \frac{1}{n^4} + \frac{2}{n^2}}{\hat{r}\left(1 + \frac{1}{n^2}\right)} \end{aligned}$$

This implies that $f(n, \hat{r}, \cdot)$ is either increasing and then decreasing, or only increasing on $[0, 1]$. This allows to conclude

$$\mathrm{Var}(f(n, \hat{r}, \cdot)) \leq 2n^2$$

which proves the lemma. ■

Proof of Theorem 1:
Theorem 6 shows that we can use Lemma 3 to estimate the sum (34). This gives us

$$\begin{aligned} \|\mathcal{L}_n\|_{C(D)\to C(D)} &\leq c_1(n) \Bigg[\int_0^1 rJ_1(n, \hat{r}, r)\, dr + \frac{1}{n} \int_0^1 rJ_2(n, \hat{r}, r)\, dr \\ &\quad + \frac{c_2}{n} \mathrm{Var}(rJ_1(n, \hat{r}, r)) + \frac{c_2}{n^2} \mathrm{Var}(rJ_2(n, \hat{r}, r)) \Bigg] \\ &\leq c_1(n) \Big[c_3 + c_5 \frac{\ln(n)}{n} + c_2 c_6 \ln(n) + c_2 c_8 \Big] \\ &= O_{n\to\infty}(n \ln(n)) \end{aligned}$$

where we have used Lemma 7–10 and the fact that $c_1(n) = O_{n\to\infty}(n)$.

Remark 11 *The proof shows that* $\|\mathcal{P}_n\|_{C(D)\to C(D)} \leq c_3 c_1(n)$*, because this is the estimate for the iterated integral.*

5 Numerical examples

We solve a semi-linear Poisson problem of the form

$$\begin{aligned} -\Delta u(x) &= f(x, u(x)), \quad x \in \overset{\circ}{D} \\ u(x) &= 0, \qquad\qquad\; x \in \partial D \end{aligned} \tag{41}$$

Let $G(x; y)$ be the Green's function where $x, y \in D$. The solution u to (41) satisfies

$$u(x) = \int_D G(x,y) f(y, u(y) dy, \qquad x \in D \tag{42}$$

As in Kumar and Sloan [12], introduce $v(x) = f(x, u(x))$. The function v is a solution of

$$v(x) = f\left(x, \int_D G(x,y) v(y) dy\right), \qquad x \in D. \tag{43}$$

This is the equation we solve with Galerkin's method. After finding v, we calculate

$$u(x) = \int_D G(x,y) v(y) dy, \qquad x \in D. \tag{44}$$

Let Π_n denote the space of all polynomials in two variables of degree n or less, as described in Section 2 and let $\{\Lambda_n : 1 \le n \le N := \frac{(n+1)(n+2)}{2}\}$ be a basis for Π_n. We choose Λ_n to be the "ridge polynomials" introduced by Logan and Shepp [13]. We approximate v by v_n:

$$v(x) \approx v_n(x) = \sum_{m=1}^{N} \alpha_m \Lambda_m(x)$$

The Galerkin method for solving (43) consists of determining the coefficients $\{\alpha_m\}$ by solving the nonlinear system

$$\sum_{m=1}^{N} \alpha_m (\Lambda_m, \Lambda_n) - \left(f\left(x, \int_D G(x,y) v_n(y) dy\right), \Lambda_n\right) = 0 \tag{45}$$

for $n = 1, \ldots, N$. Since Λ_n's are the ridge polynomials,

$$(\Lambda_m, \Lambda_n) = \delta_{mn},$$

and

$$\int_D G(x,y) \Lambda_m(y) dy = \Psi_m(x)$$

where Ψ_m's are polynomials defined in Atkinson and Hansen [6]. The term

$$\left(f\left(x, \int_D G(x,y) v_n(y) dy\right), \Lambda_n\right) = \left(f\left(x, \sum_{m=1}^{N} \alpha_m \Psi_m\right), \Lambda_n\right)$$

is also approximated by

$$\left(f\left(x, \sum_{m=1}^{N} \alpha_m \Psi_m\right), \Lambda_n\right)_d$$

as defined by (4). Thus, the nonlinear system (45) is simplified as

$$\alpha_n - \left(f\left(x, \sum_{m=1}^{N} \alpha_m \Psi_m\right), \Lambda_n\right)_d = 0 \quad \text{for } n = 1, \ldots, N. \tag{46}$$

Newton's method was used to solve the nonlinear system (46). For the solution of equation (41),

$$u_n(x) = \sum_{m=1}^{N} \alpha_m \Psi_m(x).$$

The first numerical example we solve is the problem as seen in Atkinson and Hansen [6]. Note that D is the unit disk in $\mathbb{R}^2$.

$$\begin{aligned} -\Delta u(x) &= e^{u(x,y)} + \beta(x,y), \quad & x \in \overset{\circ}{D} \\ u(x) &= 0, & x \in \partial D \end{aligned}$$

with $\beta(x,y)$ chosen such that the true solution is

$$u(x,y) = (1 - x^2 - y^2)e^{x \cos y}, \qquad (x,y) \in D$$

In Table 1, we give numerical results for $n = 1, \ldots, 20$. The error was evaluated using a polar coordinate mesh of approximately 2500 points. The linearity of the semi-log graph in Figure 1 shows that the convergenc is exponential in n. From (6) and (9) we expect that the convergent rate is faster than $O(n^{-k})$, for every $k \in \mathbb{N}$, if the solution is $C^\infty(D)$.

deg	N	$\|u - u_n\|_\infty$	$\|\widetilde{u} - \widetilde{u}_n\|_\infty$	deg	N	$\|u - u_n\|_\infty$	$\|\widetilde{u} - \widetilde{u}_n\|_\infty$
1	3	7.33E-1	7.84E-2	11	78	5.93E-7	1.13E-6
2	6	7.61E-2	2.74E-2	12	91	1.42E-7	4.02E-7
3	10	2.10E-2	7.19E-3	13	105	3.67E-8	1.42E-7
4	15	4.92E-3	2.15E-3	14	120	9.53E-9	5.08E-8
5	21	1.44E-3	7.18E-4	15	136	2.26E-9	1.81E-8
6	28	4.04E-4	2.32E-4	16	153	5.67E-10	6.52E-9
7	36	9.28E-5	7.84E-5	17	171	1.36E-10	2.33E-9
8	45	3.21E-5	2.66E-5	18	190	3.20E-11	8.46E-10
9	55	8.03E-6	9.30E-6	19	210	7.75E-12	3.06E-10
10	66	2.05E-6	3.24E-6	20	231	1.80E-12	1.10E-10

Table 1: Maximum error in u_n

The second numerical example we solve is the Debye-Hückel equation, see [9]

$$\begin{cases} -\Delta \widetilde{u}(x,y) = -\sinh(\widetilde{u}(x,y)), & (x,y) \in \overset{\circ}{D} \\ \widetilde{u}(x,y) = \widetilde{g}(x,y), & (x,y) \in \partial D \end{cases} \tag{47}$$

We assume that $\widetilde{g}$ is given as a function on D. Define

$$u(x,y) = \widetilde{u}(x,y) - \widetilde{g}(x,y).$$

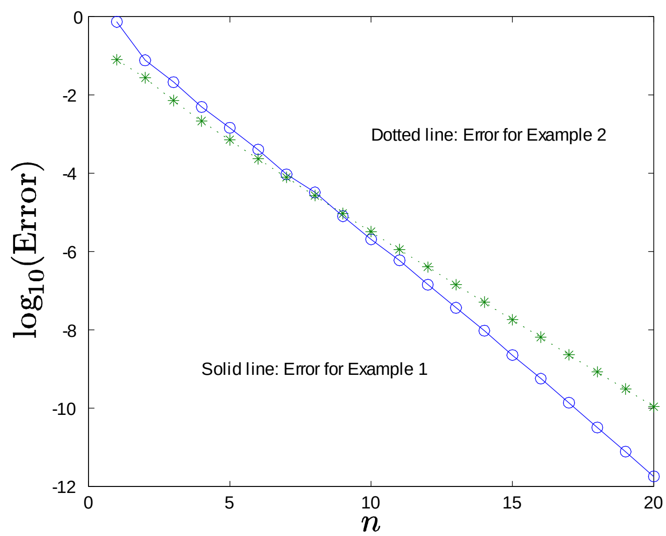

Figure 1: $\log_{10}(\text{Error})$ vs. n

Then, $u(x,y) = 0$ for $(x,y) \in \partial D$ and

$$\begin{aligned} -\Delta u &= -\Delta(\widetilde{u} - \widetilde{g}) = -\Delta\widetilde{u} + \Delta\widetilde{g} = -\sinh(\widetilde{u}) + \Delta\widetilde{g} \\ &= -\sinh\left(\widetilde{u}(x,y) - \widetilde{g}(x,y) + \widetilde{g}(x,y)\right) + \Delta\widetilde{g}(x,y) \\ &= f(x,y,u(x,y)). \end{aligned}$$

Thus, instead of solving the Debye-Hückel equation, we solve the equation (41). Then, the approximated solution $\widetilde{u}_n$ of the equation (47) is

$$\widetilde{u}_n(x,y) = u_n(x,y) + \widetilde{g}(x,y), \qquad (x,y) \in D.$$

As a test case, we choose

$$\widetilde{g}(x,y) = \exp\left(x + \frac{y}{\pi}\right).$$

The true solution of $\widetilde{u}$ is unknown, so we use $\widetilde{u}_{25}$ as our true solution, and it is illustrated in Figure 2.

As in Example 1, we give numerical results for $n = 1, \ldots, 20$ in Table 1. The error was evaluated using a polar coordinate mesh of approximately 2500 points. The linearity of the semi-log graph in Figure 1 shows that the convergence is exponential in n, as in Example 1.

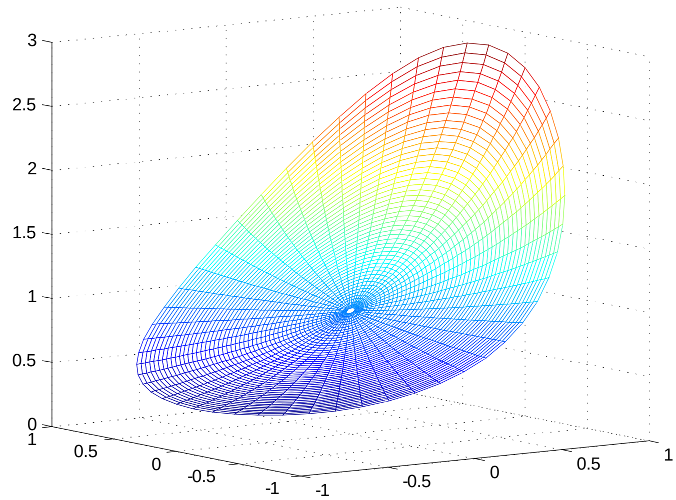

Figure 2: The true solution $\widetilde{u}_{25}$ for Example 2.

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