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CONVERGENCE RATES FOR APPROXIMATE EIGENVALUES OF COMPACT INTEGRAL OPERATORS*

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Abstract. Let $\mathscr{K}$ be an integral operator and $\{\mathscr{K}_n\}$ a sequence of numerical integral operators approximating $\mathscr{K}$. Let $\lambda_0 \neq 0$ be an eigenvalue of $\mathscr{K}$ of multiplicity m and index ν, and let σ_n be the eigenvalues of $\mathscr{K}_n$ within some small fixed neighborhood of λ_0. Then for some $c > 0$ and all sufficiently large n,

$$|\lambda - \lambda_0| \leqq c \max \{\|\mathscr{K}\varphi_i - \mathscr{K}_n\varphi_i\|^{1/\nu} | 1 \leqq i \leqq m\}$$

for all $\lambda \in \sigma_n$. The set $\{\varphi_1, \cdots, \varphi_m\}$ is a basis for null $(\lambda_0 - \mathscr{K})^\nu$.

1. Introduction. We shall consider the eigenvalue problem for the compact integral operator

$$\mathscr{K}x(s) = \int_D K(s,t)x(t)\,dt, \qquad s \in D, \quad x \in C(D), \tag{1.1}$$

with D a closed, bounded region in $\mathbb{R}^m$, $m \geqq 1$. The use of numerical integration to approximate $\mathscr{K}x$ leads to the sequence of operators

$$\mathscr{K}_n x(s) = \sum_{j=1}^{n} w_{j,n}(s)x(t_{j,n}), \qquad s \in D, \quad x \in C(D), \tag{1.2}$$

with all $t_{j,n} \in D$ and appropriate weights $w_{j,n}(s)$.

For $\lambda \neq 0$, the eigenvalue problem for $\mathscr{K}_n$,

$$\lambda x_n = \mathscr{K}_n x_n, \qquad n \geqq 1, \tag{1.3}$$

can be reduced to an equivalent finite-dimensional eigenvalue problem,

$$\lambda x_n(t_{i,n}) = \sum_{j=1}^{n} w_{j,n}(t_{i,n})x(t_{j,n}), \qquad i = 1, \cdots, n. \tag{1.4}$$

The equivalence is accomplished by using (1.3) as an interpolation formula for the solution of (1.4); this idea is due originally to Nyström [11].

Let $\lambda_0 \neq 0$ be an eigenvalue of $\mathscr{K}$, and let $\varepsilon > 0$ be less than the distance from λ_0 to the remaining part of the spectrum of $\mathscr{K}$. Let σ_n denote the set of eigenvalues of $\mathscr{K}_n$ which are within ε of λ_0. In [4] it was shown that for all sufficiently large n, the sum of the multiplicities of the eigenvalues in σ_n equals the multiplicity of λ_0, and the elements of σ_n all converge to λ_0,

$$\max_{\lambda \in \sigma_n} |\lambda - \lambda_0| \to 0 \quad \text{as } n \to \infty. \tag{1.5}$$

There were also results on the rates of convergence for the associated eigenfunctions. The major result of the present paper is a bound on the rate of convergence in (1.5) in terms of the quadrature error for the approximation (1.2).

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For an abstract framework for (1.1)–(1.3), we use the hypotheses of Anselone and Moore [1], [2].

A1. $\mathscr{K}$ and $\mathscr{K}_n$, $n \geqq 1$, are linear operators on the Banach space X into itself.

A2. $\mathscr{K}_n x \to \mathscr{K} x$ as $n \to \infty$, for all $x \in X$.

A3. The family $\{\mathscr{K}_n | n \geqq 1\}$ is collectively compact, i.e., $\{\mathscr{K}_n x | n \geqq 1$ and $\|x\| \leqq 1\}$ has compact closure in X.

For a review of the resulting theory, see [1], [2], [5].

THEOREM. *Assume* A1–A3. *Let* $\lambda_0 \neq 0$ *be an eigenvalue of* $\mathscr{K}$ *of index* ν, *i.e.*, ν *is the smallest integer for which*

$$\text{null}\,((\lambda_0 - \mathscr{K})^\nu) = \text{null}\,((\lambda_0 - \mathscr{K})^{\nu+1}).$$

Then for some constant $c > 0$ *and for all sufficiently large* n,

$$\max_{\lambda \in \sigma_n} |\lambda_0 - \lambda| \leqq c \max_{1 \leqq i \leqq m} \|\mathscr{K}\varphi_i - \mathscr{K}_n\varphi_i\|^{1/\nu}, \tag{1.6}$$

where $\{\varphi_1, \cdots, \varphi_m\}$ *is a basis for* null $((\lambda_0 - \mathscr{K})^\nu)$.

Some preliminary lemmas for eigenvalues of matrices are given in § 2. The theorem is proved in § 3, and some consequences of it are discussed.

Previous convergence results have restricted $\mathscr{K}$ to be self-adjoint or normal, e.g., [6], [7], [10], [12], [13]. Also, the kernel function was assumed to be smooth and there were some limitations on the quadrature formula. But our result (1.6) does not give a constructive bound, in contrast with some of the earlier work.

2. Preliminary lemmas on matrices.

LEMMA 1. *Let* A *and* B *be square matrices of order* m, *and assume*

$$|A_{ij}| \leqq B_{ij}, \qquad i,j = 1, \cdots, m. \tag{2.1}$$

Then

$$r_\sigma(A) \leqq r_\sigma(B), \tag{2.2}$$

where $r_\sigma(A)$ *is the spectral radius of* A, *i.e., the maximum of the moduli of the eigenvalues of* A.

Proof. Introduce the operator matrix norm

$$\|A\| = \max_i \sum_j |A_{ij}|,$$

which is induced by the vector norm $\|x\| = \max |x_i|$. Then from [8, p. 567],

$$r_\sigma(A) = \lim_{r \to \infty} \|A^r\|^{1/r}. \tag{2.3}$$

From (2.1) it follows easily that

$$|(A^r)_{ij}| \leqq (B^r)_{ij}, \qquad i,j = 1, \cdots, m, \quad r \geqq 1.$$

Thus

$$\|A^r\| \leqq \|B^r\|, \qquad r \geqq 1,$$

and (2.2) follows from (2.3).

LEMMA 2. *Let* A *be a square matrix of order* m, *and let it have the single eigen-*

value λ_0 *of multiplicity* m *and index* ν. *Let* $\{A_n | n \geqq 1\}$ *be a sequence of* $m \times m$ *matrices for which*

$$\|A - A_n\| \to 0 \qquad \text{as } n \to \infty, \tag{2.4}$$

for some matrix norm. Then

$$\max_{\lambda \in \sigma(A_n)} |\lambda_0 - \lambda| \leqq c\|A - A_n\|^{1/\nu}, \qquad n \geqq 1, \tag{2.5}$$

for some $c > 0$. *The notation* $\sigma(A_n)$ *is the set of all eigenvalues of* A_n.

Proof. Without loss of generality, we assume A is in Jordan canonical form. Otherwise, for some nonsingular P, $P^{-1}AP = J$ is in canonical form, and $P^{-1}A_nP \equiv J_n$ will still be close to J,

$$\|J - J_n\| \leqq \|P\| \, \|P^{-1}\| \, \|A - A_n\|.$$

Also, $\sigma(J_n) = \sigma(A_n)$ because A_n and J_n are similar.

Write $A = \lambda_0 I + U$, with U a matrix whose superdiagonal is all zeros and ones with all other elements equal to zero. Define $E_n = A_n - A$. We wish to solve

$$0 = \det(A_n - \lambda I) = \det(U + E_n - (\lambda - \lambda_0)I).$$

To bound $\lambda - \lambda_0$, we want to bound the eigenvalues of $U + E_n$. Define

$$\delta_n = \max_{i,j} |(E_n)_{ij}|.$$

Using Lemma 1, we have

$$r_\sigma(U + E_n) \leqq r_\sigma(U + \delta_n K),$$

with K the $m \times m$ matrix every element of which is one.

We shall bound the eigenvalues of $U + \delta_n K$. At this point, we could cite [14, p. 81] to conclude the proof. But the following derivation, together with the above, is a shorter proof of that result, and thus is of some interest in itself. Let

$$U = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_r \end{bmatrix}, \qquad J_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \ddots & \\ \vdots & & & & \ddots & 1 \\ 0 & \cdots & & & \cdots & 0 \end{bmatrix}$$

with U of order m and J_i of order ν_i. By hypothesis,

$$\max \nu_i = \nu \geqq 1.$$

Let

$$e = (1, 1, \cdots, 1)^T \in \mathbb{R}^m.$$

Then

$$(U + \delta_n K)x = \lambda x, \qquad x \in \mathbb{R}^m, \quad x \neq 0,$$

implies

$$Ux + \delta_n S e = \lambda x, \qquad S = \sum_1^m x_i. \tag{2.6}$$

For $v = 1$, $U = 0$; and it follows easily that

$$\sigma(U + \delta_n K) = \{0, m\delta_n\},$$

from which (2.5) follows.

For $v > 1$, use partitioned matrices to write

$$x = (x^{(1)}, \cdots, x^{(r)})^T, \qquad x^{(j)} \in \mathbb{R}^{v_j}.$$

From (2.6),

$$J_i x^{(i)} + \delta_n S\, e^{(i)} = \lambda x^{(i)}, \qquad i = 1, \cdots, r. \tag{2.7}$$

In system form,

$$\begin{aligned} x_{l+1}^{(i)} + \delta_n S &= \lambda x_l^{(i)}, \qquad & l = 1, 2, \cdots, v_i - 1, \\ \delta_n S &= \lambda x_{v_i}^{(i)}, \qquad & i = 1, \cdots, r. \end{aligned} \tag{2.8}$$

For $S = 0$, (2.8) implies $\lambda = 0$. If also $r \geqq 2$, then $S = 0$ can be satisfied with $x \neq 0 \in \mathbb{R}^m$ and $\lambda = 0$ will be an eigenvalue of $U + \delta_n K$.

For $S \neq 0$, (2.8) implies $\lambda \neq 0$. We first show that $\lambda = 1$ is not possible for all sufficiently large n. If $\lambda = 1$, then solving (2.8) yields

$$x_l^{(i)} = (v_i + 1 - l)\delta_n S, \qquad 1 \leqq l \leqq v_i, \quad i = 1, \cdots, r.$$

Summing over l, we obtain

$$S_i \equiv \sum_{l=1}^{v_i} x_l^{(i)} = \frac{v_i(v_i + 1)}{2} \delta_n S.$$

Summing over i and cancelling S, we have

$$1 = \frac{\delta_n}{2} \sum_1^r v_i(v_i + 1).$$

But as $\delta_n \to 0$, this cannot be satisfied. For the remainder of the proof we can assume $\lambda \neq 1$.

From (2.8) with S_i defined as above,

$$S_i - x_1^{(i)} + v_i \delta_n S = \lambda S_i, \qquad i = 1, \cdots, r,$$

and summing over i, we obtain

$$x_1^{(1)} + x_1^{(2)} + \cdots + x_1^{(r)} = [1 - \lambda + m\delta_n]S. \tag{2.9}$$

Solving in (2.8) for $x_1^{(i)}$, we obtain

$$\lambda^{v_i} x_1^{(i)} = \delta_n S \frac{1 - \lambda^{v_i}}{1 - \lambda}.$$

Dividing by λ^{ν_i}, summing over i, substituting into (2.9), and then multiplying by λ^ν, we obtain that λ must satisfy the polynomial equation

$$-\lambda^{\nu+1} + \lambda^\nu(1 + m\delta_n) - \delta_n \sum_1^r \lambda^{\nu-\nu_i}\left(\frac{1 - \lambda^{\nu_i}}{1 - \lambda}\right) = 0. \tag{2.10}$$

The last term has degree $\nu - 1$. Since one root of (2.10) is $\lambda = 1$, we can divide by $\lambda - 1$ to obtain

$$-\lambda^\nu + \delta_n q(\lambda) = 0, \tag{2.11}$$

with $q(\lambda)$ a polynomial of degree $\nu - 1$.

Since the roots $\lambda(\delta_n)$ of (2.11) will be continuous functions of δ_n [9, p. 136], we can assume that for some $B_1 > 0$,

$$|\lambda(\delta_n)| \leqq B_1, \qquad |\delta_n| \leqq 1,$$

for all the roots of (2.11). Using this in (2.11), we have $B_2 > 0$ with

$$|\lambda(\delta_n)|^\nu \leqq B_2\delta_n,$$

which completes the proof of (2.5), for all sufficiently large n. It can be made true for all n by merely making the bound B_2 larger.

3. Rates of convergence for approximate eigenvalues. We begin by proving the theorem stated in § 1. Let $\lambda_0 \neq 0$ be an eigenvalue of $\mathscr{K}$ of index $\nu \geqq 1$ and multiplicity $m \geqq \nu$. Let $\varepsilon > 0$ be less than the distance from λ_0 to the remainder of $\sigma(\mathscr{K})$, the spectrum of $\mathscr{K}$. Associated with the eigenspace

$$X(\lambda_0) \equiv \text{null}\,(\lambda_0 - \mathscr{K})^\nu$$

is the projection operator

$$E(\lambda_0, \mathscr{K}) = \frac{1}{2\pi i}\int_{|\lambda - \lambda_0| = \varepsilon} (\lambda - \mathscr{K})^{-1}\, d\lambda$$

which maps X onto $X(\lambda_0)$; the finite-dimensional space $X(\lambda_0)$ of dimension m is invariant under $\mathscr{K}$. See [8, pp. 566–580] for a complete treatment of the operator calculus for compact operators.

From [4], the set σ_n of eigenvalues of $\mathscr{K}_n$ which are within ε of λ_0 will equal m in the sum of their multiplicities, for all sufficiently large $n \geqq N$. Moreover, we can define the projection operator

$$E(\sigma_n, \mathscr{K}_n) = \frac{1}{2\pi i}\int_{|\lambda - \lambda_0| = \varepsilon} (\lambda - \mathscr{K}_n)^{-1}\, d\lambda, \qquad n \geqq N.$$

Its range is

$$X(\sigma_n) \equiv \text{null}\,(\lambda_1 - \mathscr{K}_n)^{\nu(\lambda_1)} \oplus \cdots \oplus \text{null}\,(\lambda_{r(n)} - \mathscr{K}_n)^{\nu(\lambda_r)},$$

with

$$\sigma_n = \{\lambda_1, \cdots, \lambda_{r(n)}\}$$

and where $\nu(\lambda_i)$ denotes the index of λ_i. Then there is a constant $c > 0$ with

$$\|x - E(\sigma_n, \mathscr{K}_n)x\| \leqq c\|x\|\rho_n, \qquad x \in X(\lambda_0), \quad n \geqq N, \tag{3.1}$$

$$\rho_n = \max\{\|(\mathscr{K} - \mathscr{K}_n)\mathscr{K}\|, \|(\mathscr{K} - \mathscr{K}_n)\mathscr{K}_n\|\}.$$

From A1–A3, $\rho_n \to 0$ as $n \to \infty$, and its size is related to the quadrature error in (1.2); see [1], [5]. In addition, the family $\{E(\sigma_n, \mathscr{K}_n) | n \geqq N\}$ is uniformly bounded.

Consider $E(\sigma_n, \mathscr{K}_n)$ as an operator restricted to $X(\lambda_0)$ into $X(\sigma_n)$. We shall show it is invertible. Define $S_n : X(\lambda_0) \to X(\lambda_0)$,

$$S_n x = x - E(\lambda_0, \mathscr{K})E(\sigma_n, \mathscr{K}_n)x, \qquad x \in X(\lambda_0). \tag{3.2}$$

Then

$$\|S_n x\| \leqq \|E(\lambda_0, \mathscr{K})\| \, \|x - E(\sigma_n, \mathscr{K}_n)x\|$$
$$\leqq c\|E(\lambda_0, \mathscr{K})\|\rho_n\|x\|.$$

Regarded as an operator from $X(\lambda_0)$ to $X(\lambda_0)$,

$$\|S_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.3}$$

As a consequence,

$$(I - S_n)^{-1} : X(\lambda_0) \to X(\lambda_0)$$

exists and is uniformly bounded for all sufficiently large n. Then the operator

$$E(\sigma_n, \mathscr{K}_n)^{-1} \equiv (I - S_n)^{-1}E(\lambda_0, \mathscr{K})$$

is easily shown to be the inverse of $E(\sigma_n, \mathscr{K}_n) | X(\lambda_0)$, and moreover, it is uniformly bounded for all large n.

Define $\tilde{\mathscr{K}}_n : X(\lambda_0) \to X(\lambda_0)$ by

$$\tilde{\mathscr{K}}_n x = E(\sigma_n, \mathscr{K}_n)^{-1}\mathscr{K}_n E(\sigma_n, \mathscr{K}_n)x, \qquad x \in X(\lambda_0).$$

The spectrum of $\tilde{\mathscr{K}}_n$ on $X(\lambda_0)$ is the same as that of $\mathscr{K}_n$ on $X(\sigma_n)$, namely σ_n. Now consider $\mathscr{K}$ and $\tilde{\mathscr{K}}_n$ on $X(\lambda_0)$ to $X(\lambda_0)$. Let $\{\varphi_1, \cdots, \varphi_m\}$ be a basis for $X(\lambda_0)$. For $x \in X(\lambda_0)$,

$$x = \sum_1^m \alpha_i \varphi_i,$$

$$\|\mathscr{K}x - \tilde{\mathscr{K}}_n x\| \leqq \left(\sum_1^m |\alpha_i|\right) \max_i \|(\mathscr{K} - \tilde{\mathscr{K}}_n)\varphi_i\|.$$

Since

$$\|x\|_* \equiv \sum_1^m |\alpha_i|, \qquad x \in X(\lambda_0),$$

is a norm on $X(\lambda_0)$, and since all norms on a finite-dimensional space are equivalent [9, p. 7], there is $c > 0$ with

$$\|x\|_* \leqq c\|x\|.$$

Thus

$$\|\mathscr{K}x - \tilde{\mathscr{K}}_n x\| \leqq c\|x\| \max_i \|\mathscr{K}\varphi_i - \tilde{\mathscr{K}}_n \varphi_i\|. \tag{3.4}$$

For each $z \in X(\lambda_0)$,

$$\|\mathscr{K}z - \tilde{\mathscr{K}}_n z\| \leqq \|E(\sigma_n, \mathscr{K}_n)^{-1}\| \, \|E(\sigma_n, \mathscr{K}_n)\mathscr{K}z - \mathscr{K}_n E(\sigma_n, \mathscr{K}_n)z\|.$$

Since $E(\sigma_n, \mathscr{K}_n)$ and $\mathscr{K}_n$ commute on X, we can obtain

$$\|\mathscr{K}z - \tilde{\mathscr{K}}_n z\| \leqq \|E(\sigma_n, \mathscr{K}_n)^{-1}\| \, \|E(\sigma_n, \mathscr{K}_n)\| \, \|\mathscr{K}z - \mathscr{K}_n z\|.$$

Apply this to (3.4) to get

$$\|\mathscr{K}x - \tilde{\mathscr{K}}_n x\| \leqq c_1 \|x\| \max_i \|\mathscr{K}\varphi_i - \mathscr{K}_n \varphi_i\|, \qquad x \in X(\lambda_0).$$

for all sufficiently large n. With respect to $X(\lambda_0)$,

$$\|\mathscr{K} - \tilde{\mathscr{K}}_n\| \leqq c_1 \max_i \|\mathscr{K}\varphi_i - \mathscr{K}_n \varphi_i\|. \tag{3.5}$$

To complete the proof of the theorem, take a basis for $X(\lambda_0)$ and reduce the restrictions to $X(\lambda_0)$ of $\mathscr{K}$ and $\tilde{\mathscr{K}}_n$ to matrix equivalents A and A_n, respectively, of order m. It is straightforward that

$$\|A - A_n\| \leqq c_2 \|\mathscr{K} - \tilde{\mathscr{K}}_n\|,$$

which can be combined with (3.5) to bound $\|A - A_n\|$. Invoke Lemma 2 to complete the proof.

The bound in (1.6) or (3.5) can be replaced by one involving $\|(\mathscr{K} - \mathscr{K}_n)\mathscr{K}\|$, although the rate of convergence may not be as great. To see this, let $z \in X(\lambda_0)$. Then

$$(\lambda_0 - \mathscr{K})^\nu z = 0,$$

and z can be written as $z = \mathscr{K}\mathscr{L}z$, with $\mathscr{L}$ bounded. Then

$$\|(\mathscr{K} - \mathscr{K}_n)z\| \leqq \|(\mathscr{K} - \mathscr{K}_n)\mathscr{K}\| \, \|\mathscr{L}\| \, \|z\|.$$

Thus (1.6) becomes

$$\max_{\lambda \in \sigma_n} |\lambda_0 - \lambda| \leqq c_3 \|(\mathscr{K} - \mathscr{K}_n)\mathscr{K}\|^{1/\nu} \tag{3.6}$$

for all large n and an appropriate constant c_3.

To apply these results to integral operators, consider first the case where D is a closed, bounded subset of R^q, $q \geqq 1$, and $K(s, t)$ is a continuous function for $s, t \in D$. Define $\mathscr{K}$ by (1.1). Suppose

$$\int_0 f(t)\, dt \approx \sum_1^n w_{j,n} f(t_{j,n}) \tag{3.7}$$

is a convergent numerical integration method for all $f \in C(D)$. Define $\mathscr{K}_n$, $n \geqq 1$, by

$$\mathscr{K}_n x(s) = \sum_1^n w_{j,n} K(s, t_{j,n}) x(t_{j,n}), \qquad x \in C(D).$$

Then A1–A3 will be satisfied (e.g., see [1], [5]), and Theorem 1 can be applied to the eigenvalues of $\mathscr{K}$ and $\mathscr{K}_n$. For the rate of convergence (1.6),

$$\|\mathscr{K}\varphi_i - \mathscr{K}_n\varphi_i\| = \max_s \left| \int_D K(s,t)\varphi_i(t)\,dt - \sum_1^n w_{j,n}K(s,t_{j,n})\varphi_i(t_{j,n}) \right|. \tag{3.8}$$

The rates of convergence given in earlier papers for self-adjoint and normal operators follow easily by noting that $\nu = 1$ for such operators. The above result is slightly more general for such operators since the weights $w_{j,n}$ are not restricted to being positive as in all earlier results.

For self-adjoint operators whose kernels have a weak singularity, e.g., an algebraic or logarithmic singularity,

$$\log \|s - t\| \quad \text{or} \quad \frac{1}{\|s - t\|^{\alpha}}, \qquad s, t \in \mathbb{R}^q, \quad \alpha < q,$$

earlier results no longer apply. For such cases, product integration must be used to treat the singularity in order to obtain a good approximation to $\mathscr{K}x$. But in such a case, the equivalent linear system (1.4) can no longer be converted by a similarity transformation to a symmetric system in any obvious way, and this was essential to earlier work. By (1.6), the rate of convergence will still depend linearly on the quadrature error since $\nu = 1$; formula (3.8) will be replaced by the error formula for product integration.

Although we are mainly interested in the case with $\mathscr{K}_n$ defined by numerical integration, the analysis applies equally well to cases where (i) $\mathscr{K}_n$ is compact and of finite rank, $n \geqq 1$, and (ii) $\|\mathscr{K} - \mathscr{K}_n\| \to 0$ as $n \to \infty$. It then follows fairly easily that A1–A3 are then satisfied. The main applications are (i) defining $\mathscr{K}_n$ by using a degenerate kernel approximation $K_n(s, t)$ to $K(s, t)$, and (ii) projection methods, e.g., Galerkin's method and the collocation method. See [5] for the associated theory for the approximate solution of nonhomogeneous Fredholm equations. In such cases, the bounds in (1.6) and (3.6) can be replaced by $\|\mathscr{K} - \mathscr{K}_n\|$, although (1.6) may still give a better result.

By specializing $\mathscr{K}$ and $\mathscr{K}_n$ to matrices on $\mathbb{R}^q$ for some $q > 1$, we obtain another interesting corollary. Let A be a matrix of order q, with eigenvalues $\lambda_1, \cdots, \lambda_r$ of index $\nu_1, \cdots, \nu_r$, respectively. Let A_n be a sequence of matrices for which

$$\|A - A_n\| \to 0.$$

Pick $\varepsilon > 0$ small enough to make the circles of radius ε about $\lambda_1, \cdots, \lambda_r$ pairwise disjoint. Let $\sigma_{n,j}$ be the eigenvalues of A_n within ε of λ_j, $j = 1, \cdots, r$. Then for all sufficiently large n, the number of eigenvalues in $\sigma_{n,j}$, counted according to their multiplicity, will equal the multiplicity of λ_j. Moreover, there is a $c > 0$ such that

$$\max_{\lambda \in \sigma_{n,j}} |\lambda_j - \lambda| \leqq c\|A - A_n\|^{1/\nu_j}, \qquad j = 1, 2, \cdots, r. \tag{3.9}$$

The proof is immediate from the theorem, as long as $\lambda = 0$ is not an eigenvalue of A.

If it is, then use the perturbed matrices

$$\alpha I + A, \qquad \alpha I + A_n$$

with $\alpha > \|A\|$. The differences of the eigenvalues will remain unchanged, and zero will no longer be an eigenvalue.

To see that (3.9), and thus (1.6), is best possible, use

$$A = \begin{bmatrix} 1 & 1 & 0 & \cdots & & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \\ \vdots & & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 1 \end{bmatrix}, \qquad A_n = \begin{bmatrix} 1 & 1 & 0 & \cdots & & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \\ \vdots & & & & \ddots & 1 \\ 1/n & 0 & \cdots & \cdots & \cdots & 1 \end{bmatrix}$$

in which A and A_n are order $q \times q$. Then

$$\|A - A_n\| = 1/n, \qquad \nu = q,$$

and the characteristic equation is

$$(\lambda - 1)^q = 1/n.$$

Thus

$$\max_{\lambda \in \sigma_n} |\lambda_0 - \lambda| = (1/n)^{1/q} = \|A - A_n\|^{1/\nu}.$$

Note added in proof. Following submission of this paper, the author became aware of the two related papers [15] and [16].

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