A Discrete Galerkin Method for a Hypersingular Boundary Integral Equation

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Abstract

Consider solving the interior Neumann problem

$$\begin{aligned} \Delta u(P) &= 0, & P \in D \\ \frac{\partial u(P)}{\partial \mathbf{n}_P} &= f(P), & P \in S \end{aligned}$$

with D a simply-connected planar region and $S = \partial D$ a smooth curve. A double layer potential is used to represent the solution, and it leads to the problem of solving a hypersingular integral equation. This integral equation is reformulated as a Cauchy singular integral equation. A discrete Galerkin method with trigonometric polynomials is then given for its solution. An error analysis is given; and numerical examples complete the paper.

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1 Introduction

Let D be a bounded open simply-connected region in the plane, and let its boundary S be sufficiently smooth. Consider the Neumann problem: Find $u \in C^1(\overline{D}) \cap C^2(D)$ that satisfies

$$\begin{aligned} \Delta u(P) &= 0, \qquad && P \in D \\ \frac{\partial u(P)}{\partial \mathbf{n}_P} &= f(P), \qquad && P \in S \end{aligned} \tag{1.1}$$

with $f \in C(S)$ a given boundary function.

One way of solving this problem is to express the solution u as a *double layer potential*,

$$u(A) = \int_S \rho(Q) \frac{\partial}{\partial \mathbf{n}_Q} \log \mid A - Q \mid dS_Q, \qquad A \in D \tag{1.2}$$

The function ρ is called a *double layer density function* or a *dipole density function.* Form the derivative of $u(A)$ in the direction $\mathbf{n}_P$, the inner normal to the boundary S at P, and take the limit as $A \to P$, thus obtaining the normal derivative. For the Neumann problem, this leads to

$$f(P) = \frac{\partial u(P)}{\partial \mathbf{n}_P} \tag{1.3}$$

$$= \lim_{A \to P} \mathbf{n}_P \cdot \nabla_A \int_S \rho(Q) \frac{\partial}{\partial \mathbf{n}_Q} \log \mid A - Q \mid dS_Q, \qquad P \in S \tag{1.4}$$

The integral operator is often referred to as *hypersingular*, and we are looking for the density function ρ. For some discussion of this for $S = U$ the unit circle, see Atkinson [5, §7.3.2].

Section 2 gives preliminary information on integral equations for $S = U$ the unit circle; and Section 3 relates the hypersingular integral operator to other potential representations. Section 4 gives a reformulation of the integral equation. Section 5 gives the numerical method and Section 6 gives numerical examples. The numerical method is based on using trigonometric approximations of the unknown density function, and we give what can be regarded as either a discrete Galerkin method or a discrete collocation method.

The general idea of using an approximation scheme using trigonometric approximations is quite old. An early use of this is given in Gabdulhaev [7]. Work from more recent years is given by Amosov [3], Atkinson [4], Atkinson and Sloan [6], Mclean [12], and McLean, Prößdorf, and Wendland [13]. Other approaches to the solution of the hypersingular equation are given in Amini and Maines [1], [2], Giroire and Nedelec [8], Kress [11], and Rathsfeld, Kieser, and Kleemann [15].

2 Preliminaries

In this paper, we consider the Neumann problem given in equation (1.1). Let D be a bounded open simply-connected region in the plane, and assume its boundary S is sufficiently smooth. Thus, S has a parameterization

$$\beta(s) = (\xi(s),\, \eta(s))\,, \qquad 0 \le s \le L \tag{2.1}$$

where s is the arc length coordinate of the point P on S and L is the arc length of S. Assume $\beta(s) \in C^2[0,\, L]$ and $|\beta'(s)| \neq 0$ for every $s \in [0,\, L]$. The normal vector $\mathbf{n}$ at P on S is directed into the interior of the domain D; and we assume the direction of integration on S to be counterclockwise.

Consider the normal derivative of $u(A)$ in the inner direction to S at P:

$$\begin{aligned}
\frac{\partial u(P)}{\partial \mathbf{n}_P} &= \lim_{A \to P} \mathbf{n}_P \cdot \nabla_A \int_S \rho(Q) \frac{\partial}{\partial \mathbf{n}_Q} \log \mid A - Q \mid dS_Q && (2.2)\\
&\equiv \frac{\partial}{\partial \mathbf{n}_P} \int_S \rho(Q) \frac{\partial}{\partial \mathbf{n}_Q} \log \mid P - Q \mid dS_Q && (2.3)\\
&\equiv \mathcal{H}\rho(P), \qquad P \in S && (2.4)
\end{aligned}$$

The resulting integral contains an integrand with a strongly nonintegrable singularity if the integral and derivative operators are interchanged. Such integral operators $\mathcal{H}$ are often referred to as hypersingular, and the integrals do not exist in the usual sense.

The hypersingular integral operator is very closely related to the Cauchy singular integral operator:

$$C\rho(z) = \frac{1}{2\pi i}\int_S \frac{\rho(\zeta)}{\zeta - z}d\zeta\,, \qquad z \in S$$

where S is the boundary of D, as defined before. Properties of Cauchy singular integral operators can be found in Kress [10, p. 82].

For a function $\varphi \in L^2(0, 2\pi)$, we write its Fourier expansion as

$$\varphi(s) = \sum_{m=-\infty}^{\infty} a_m\psi_m(s), \qquad \psi_m(s) = \frac{1}{\sqrt{2\pi}}e^{ims}$$

$$a_m = \int_0^{2\pi}\varphi(s)\overline{\psi_m(s)}ds$$

For any real number $q \geq 0$, define $H^q(2\pi)$ to be the set of all functions $\varphi \in L^2(0, 2\pi)$ for which

$$\|\varphi\|_q \equiv \left[|a_0|^2 + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty}|m|^{2q}|a_m|^2\right]^{\frac{1}{2}} < \infty$$

Consider the case in which S=U, the unit circle. We denote the Cauchy singular integral operator by C_u in this case; and from Henrici [9, p. 109],

$$C_u : e^{ikt} \longrightarrow \text{sign}(k)\cdot\frac{e^{ikt}}{2}, \qquad k = 0, \pm 1, \pm 2, \ldots \tag{2.5}$$

with $\text{sign}(0) = 1$. We can interpret C_u as a operator on $H^q(2\pi)$, and

$$C_u : H^q(2\pi) \xrightarrow[\text{onto}]{1-1} H^q(2\pi), \qquad q \geq 0$$

Consider the same boundary for the hypersingular integral operator, and denote the latter by $\mathcal{H}_u$ in this case. From Atkinson [5, Sec. 7.3], we have

$$\mathcal{H}_u : e^{ikt} \longrightarrow \pi|k|e^{ikt}, \qquad k = 0, \pm 1, \pm 2, \dots \tag{2.6}$$

For $\varphi \in H^1(2\pi)$ with $\varphi = \sum a_m \psi_m$, introduce the derivative operator $\mathcal{D}$:

$$\mathcal{D}\varphi(t) \equiv \frac{d\varphi(t)}{dt} = i \sum_{m \neq 0} m a_m \psi_m(t)$$

Regarding the Cauchy singular integral operator C_u as an operator on $H^q(2\pi)$, and using the mapping properties (2.5) and (2.6), we have

$$\mathcal{H}_u \varphi = -2\pi i \mathcal{D} C_u \varphi = -2\pi i C_u \mathcal{D} \varphi$$

3 Connection With Logarithmic Potential

Consider $\varphi(t)$ as a real function, and assume z does not lie on the boundary S. Introduce

$$\Phi(z) = U(x, y) + iV(x, y) = \frac{1}{2\pi i} \int_S \frac{\varphi(\zeta)\, d\zeta}{\zeta - z} \tag{3.1}$$

Substitute

$$\zeta - z = re^{i\vartheta} \tag{3.2}$$

where $r = \mid \zeta - z \mid$ and $\vartheta = \arg(\zeta - z)$. Taking the logarithmic derivative of (3.2) (for variable ζ and constant z),

$$\frac{d\zeta}{\zeta - z} = d\log r + i d\vartheta = \left(\frac{\partial \log r}{\partial s} + i \frac{\partial \vartheta}{\partial s} \right) ds.$$

By the Cauchy-Riemann equations, applied to $\log(\zeta - z) = \log r + i\vartheta$, we have

$$\frac{\partial \vartheta}{\partial s} = -\frac{\partial \log r}{\partial \mathbf{n}}.$$

Substituting this into (3.1) and separating real and imaginary parts, we obtain

$$U(x,\, y) = \frac{1}{2\pi}\int_S \varphi\, d\vartheta = \frac{1}{2\pi}\int_0^L \varphi \frac{d\vartheta}{ds} ds = \frac{-1}{2\pi}\int_0^L \varphi \frac{\partial}{\partial \mathbf{n}_\zeta} \log r\, ds$$

and

$$V(x,\, y) = \frac{-1}{2\pi}\int_S \varphi\, d \log r \tag{3.3}$$

After an integration by parts (assuming that φ has an integrable derivative with respect to s) equation (3.3) can be written as

$$V(x,\, y) = \frac{1}{2\pi}\int_0^L \frac{d\varphi}{ds} \log r\, ds.$$

These formulae indicate that for real valued densities, the real part of the Cauchy integral coincides with the double layer potential (1.2)

$$u(x,\, y) = \int_0^L \rho(\beta(s)) \frac{\partial}{\partial \mathbf{n}_\zeta} \log r\, ds \qquad (x, y) \in D \tag{3.4}$$

where

$$\rho(\beta(s)) = -\frac{1}{2\pi}\varphi(s).$$

From Kress [10, p. 100], we have the following theorem:

Theorem 1 *The double layer potential u with Hölder continuous density ρ can be extended uniformly Hölder continuously from D into $\overline{D}$.*

Proof: The definition of $C^{0,\alpha}(S)$, the set of all functions which are Hölder continuous, can be found from Kress [10, p. 82]. □

The next theorem gives us the existence and representation of the normal derivative of the double layer potential u on the boundary S.

Theorem 2 *The normal derivative of the double layer potential u with density $\rho \in C^{1,\alpha}(S)$ can be extended uniformly Hölder continuously from D to $\overline{D}$. The normal derivative is given by*

$$\frac{\partial u(P)}{\partial \mathbf{n}_P} = \frac{d}{ds_0} \int_0^L \frac{d\rho}{ds} \log \mid \beta(s) - \beta(s_0) \mid \, ds \qquad \beta(s_0) = P \in S \tag{3.5}$$

Proof: $C^{1,\alpha}(S)$ is the set of all continuously differentiable functions φ such that $\varphi' \in C^{0,\alpha}(S)$; and recall $\beta(s)$ from (2.1), a parameterization of S. See the proof in Kress [10, p. 102] □

Notice that the right-hand side of the equation (3.5) is the tangential derivative of the simple layer potential V; and from Muskhelishvili [14, p. 31], we have

$$\begin{aligned} \frac{\partial u(P)}{\partial \mathbf{n}_P} &= \frac{dV}{ds_0} = \int_0^L \frac{d\rho}{ds} \frac{\partial}{\partial s_0} \log \mid \beta(s) - \beta(s_0) \mid ds \\ &= -\int_0^L \frac{d\rho}{ds} \frac{\beta'(s_0) \cdot (\beta(s) - \beta(s_0))}{\mid \beta(s) - \beta(s_0) \mid^2} ds \end{aligned} \tag{3.6}$$

For the Neumann problem (1.1), the double layer potential

$$u(A) = \int_S \rho(Q) \frac{\partial}{\partial \mathbf{n}_Q} \log \mid A - Q \mid dS_Q, \qquad A \in D \tag{3.7}$$

solves the Neumann problem with boundary condition $\partial u / \partial n = f$ on S provided the density $\rho \in C^{1,\alpha}(S)$ solves the integral equation

$$\frac{\partial}{\partial \mathbf{n}_P} \int_0^L \rho(\beta(s)) \frac{\partial}{\partial \mathbf{n}_{\beta(s)}} \log \mid P - \beta(s) \mid ds = f(P), \qquad P \in S \tag{3.8}$$

Theorem 3 *Let $f \in C^{0,\alpha}(S)$ satisfy the solvability condition*

$$\int_0^L f \, ds = 0.$$

The Neumann problem (1.1) has a solution u of the form (3.7), with $\rho \in C^{1,\alpha}(S)$. Two solutions u can differ only by a constant, as do two solutions ρ.

Proof: See Kress [10, p. 104] □

This establishes the solvability of the integral equation (3.8), and symbolically we write this equation as

$$\mathcal{H}\rho = f.$$

4 Reformulation

With equation (3.6), we have

$$\mathcal{H}\rho(\beta(s_0)) = -\int_0^L \frac{d\rho}{ds}\frac{\beta'(s_0)\cdot(\beta(s)-\beta(s_0))}{\mid \beta(s)-\beta(s_0)\mid^2}ds \tag{4.1}$$

Change from the variable s to θ, with

$$s = \frac{L\theta}{2\pi}, \qquad 0 \leq \theta \leq 2\pi,$$

and do similarly with s_0 and θ_0. Then equation (4.1) becomes

$$\mathcal{H}\rho(\beta(s_0)) = -\frac{L}{2\pi}\int_0^{2\pi} \frac{\beta'(\frac{L\theta_0}{2\pi})\cdot\left(\beta(\frac{L\theta}{2\pi})-\beta(\frac{L\theta_0}{2\pi})\right)}{\left|\beta(\frac{L\theta}{2\pi})-\beta(\frac{L\theta_0}{2\pi})\right|^2}\frac{d\rho}{ds}d\theta \tag{4.2}$$

Introduce a function η defined on $[0, 2\pi]$, and implicitly on the unit circle U, by

$$\eta(\theta) = \rho\left(\beta\left(\frac{L\theta}{2\pi}\right)\right), \qquad \eta_s(\theta) = \frac{d}{ds}\rho\left(\beta\left(\frac{L\theta}{2\pi}\right)\right), \qquad 0 \leq \theta \leq 2\pi$$

The parameterization of the unit circle is

$$\beta_u(\theta) = (\cos(\theta), \sin(\theta)), \qquad 0 \leq \theta \leq 2\pi$$

Using these definitions, write (4.2) as

$$\mathcal{H}\eta(\theta_0) = -\frac{L}{2\pi}\int_0^{2\pi} \frac{\beta'(\frac{L\theta_0}{2\pi})\cdot\left(\beta(\frac{L\theta}{2\pi})-\beta(\frac{L\theta_0}{2\pi})\right)}{\left|\beta(\frac{L\theta}{2\pi})-\beta(\frac{L\theta_0}{2\pi})\right|^2}\eta_s(\theta)\,d\theta$$

$$
\begin{aligned}
&= -\int_0^{2\pi} \frac{\beta_u'(\theta_0)\cdot(\beta_u(\theta)-\beta_u(\theta_0))}{|\beta_u(\theta)-\beta_u(\theta_0)|^2}\cdot \\
&\qquad \left[\frac{|\beta_u(\theta)-\beta_u(\theta_0)|^2}{\beta_u'(\theta_0)\cdot(\beta_u(\theta)-\beta_u(\theta_0))}\frac{\beta'(\frac{L\theta_0}{2\pi})\cdot\left(\beta(\frac{L\theta}{2\pi})-\beta(\frac{L\theta_0}{2\pi})\right)}{\left|\beta(\frac{L\theta}{2\pi})-\beta(\frac{L\theta_0}{2\pi})\right|^2}\right]\eta'(\theta)\,d\theta \\
&= -\int_0^{2\pi} \frac{\sin(\theta-\theta_0)}{2(1-\cos(\theta-\theta_0))}\cdot\left[\frac{2(1-\cos(\theta-\theta_0))}{\sin(\theta-\theta_0)}\cdot\frac{\beta'(\frac{L\theta_0}{2\pi})\cdot\left(\beta(\frac{L\theta}{2\pi})-\beta(\frac{L\theta_0}{2\pi})\right)}{\left|\beta(\frac{L\theta}{2\pi})-\beta(\frac{L\theta_0}{2\pi})\right|^2}\right]\eta'(\theta)\,d\theta \\
&= -\frac{2\pi}{L}\left(\int_0^{2\pi}\frac{\sin(\theta-\theta_0)}{2(1-\cos(\theta-\theta_0))}\eta'(\theta)\,d\theta+\mathcal{B}\mathcal{D}\eta(\theta_0)\right) \\
&= \frac{2\pi}{L}\left(\mathcal{H}_u\eta(\theta_0)+\mathcal{B}\mathcal{D}\eta(\theta_0)\right) \\
&= \frac{2\pi}{L}\left(-2\pi i C_u\mathcal{D}\eta(\theta_0)+\mathcal{B}\mathcal{D}\eta(\theta_0)\right) \qquad (4.3)
\end{aligned}
$$

where the kernel B of the integral operator $\mathcal{B}$ is

$$
B(\theta_0,\,\theta) = -\left(\frac{L}{2\pi}\right)\left[\frac{\beta'(\frac{L\theta_0}{2\pi})\cdot\left(\beta(\frac{L\theta}{2\pi})-\beta(\frac{L\theta_0}{2\pi})\right)}{\left|\beta(\frac{L\theta}{2\pi})-\beta(\frac{L\theta_0}{2\pi})\right|^2}-\frac{\pi}{L}\frac{\sin(\theta-\theta_0)}{(1-\cos(\theta-\theta_0))}\right] \qquad (4.4)
$$

The kernel $B(\theta_0,\theta)$ is continuous, and it has periodicity 2π for both θ and θ_0. It's easy to see B is a periodic function, and we need to show it is continuous when either $\sin(\theta-\theta_0)\to 0$ or $\beta(\frac{L\theta}{2\pi})-\beta(\frac{L\theta_0}{2\pi})\to 0$.

Theorem 4 *Assume $\beta(s)\in C^2[0,\,L]$, then the kernel function $B(\theta_0,\,\theta)$ is continuous over $[0,\,2\pi]\times[0,\,2\pi]$, and it is periodic with respect to both θ and θ_0, with period 2π.*

Proof: It suffices to show three cases:

Case 1: $\theta_0\in(0,\,2\pi)$ and $\theta\to\theta_0$.

Note that we drop the coefficient $-L/2\pi$ in (4.4) for convenience and rewrite it as

$$
B(\theta_0,\,\theta) \;=\; \frac{\beta'(\frac{L\theta_0}{2\pi})\cdot\left(\beta(\frac{L\theta}{2\pi})-\beta(\frac{L\theta_0}{2\pi})\right)}{\left|\beta(\frac{L\theta}{2\pi})-\beta(\frac{L\theta_0}{2\pi})\right|^2}-\left(\frac{\pi}{L}\right)\frac{\sin(\theta-\theta_0)}{1-\cos(\theta-\theta_0)} \qquad (4.5)
$$

$$= \frac{\beta'(\frac{L\theta_0}{2\pi}) \cdot \left(\beta(\frac{L\theta}{2\pi}) - \beta(\frac{L\theta_0}{2\pi})\right)}{\left|\beta(\frac{L\theta}{2\pi}) - \beta(\frac{L\theta_0}{2\pi})\right|^2} - \frac{2\pi}{L(\theta-\theta_0)} \tag{4.6}$$

$$-\frac{\pi}{L}\left(\frac{\sin(\theta-\theta_0)}{1-\cos(\theta-\theta_0)} - \frac{2}{\theta-\theta_0}\right) \tag{4.7}$$

In this proof, we take the advantage of the parameterization β of the boundary S. Since s is the arc coordinate of the point P on S, we have

$$|\beta'(s_0)| = 1 \qquad \text{and} \qquad \beta'(s_0) \cdot \beta''(s_0) = 0 \qquad \forall\, \beta(s_0) \in S$$

The term (4.7) approaches 0 as θ approaches θ_0. For the term (4.6), we first expand β about θ_0:

$$\beta\left(\frac{L\theta}{2\pi}\right) = \beta\left(\frac{L\theta_0}{2\pi}\right) + \frac{L}{2\pi}\beta'\left(\frac{L\theta_0}{2\pi}\right)(\theta-\theta_0) + \left(\frac{L}{2\pi}\right)^2 \beta''\left(\frac{L\theta_1}{2\pi}\right)\frac{(\theta-\theta_0)^2}{2}$$

where θ_1 is between θ and θ_0. Then

$$\beta'\left(\frac{L\theta_0}{2\pi}\right) \cdot \left(\beta(\frac{L\theta}{2\pi}) - \beta(\frac{L\theta_0}{2\pi})\right) = \frac{L}{2\pi}(\theta-\theta_0) + \left(\frac{L}{2\pi}\right)^2 \beta'\left(\frac{L\theta_0}{2\pi}\right) \cdot \beta''\left(\frac{L\theta_1}{2\pi}\right)\frac{(\theta-\theta_0)^2}{2} \tag{4.8}$$

and

$$\left|\beta(\frac{L\theta}{2\pi}) - \beta(\frac{L\theta_0}{2\pi})\right|^2$$

$$= \left(\frac{L}{2\pi}\right)^2 (\theta-\theta_0)^2 + \left(\frac{L}{2\pi}\right)^3 \beta'\left(\frac{L\theta_0}{2\pi}\right) \cdot \beta''\left(\frac{L\theta_1}{2\pi}\right)(\theta-\theta_0)^3 + c_1(\theta-\theta_0)^4 \tag{4.9}$$

where

$$c_1 = \frac{1}{4}\left(\frac{L}{2\pi}\right)^4 \left|\beta''\left(\frac{L\theta_1}{2\pi}\right)\right|^2$$

Substituting (4.8) and (4.9) to (4.6) we have

$$\frac{\beta'(\frac{L\theta_0}{2\pi}) \cdot \left(\beta(\frac{L\theta}{2\pi}) - \beta(\frac{L\theta_0}{2\pi})\right)}{\left|\beta(\frac{L\theta}{2\pi}) - \beta(\frac{L\theta_0}{2\pi})\right|^2} - \frac{2\pi}{L(\theta-\theta_0)}$$

$$= \frac{\frac{L}{2\pi}(\theta-\theta_0)\left(1+\left(\frac{L}{2\pi}\right)\beta'\left(\frac{L\theta_0}{2\pi}\right)\cdot\beta''\left(\frac{L\theta_1}{2\pi}\right)\frac{(\theta-\theta_0)}{2}\right)}{\left(\frac{L}{2\pi}\right)^2(\theta-\theta_0)^2\left(1+\left(\frac{L}{2\pi}\right)\beta'\left(\frac{L\theta_0}{2\pi}\right)\cdot\beta''\left(\frac{L\theta_1}{2\pi}\right)(\theta-\theta_0)+c_2(\theta-\theta_0)^2\right)}$$

$$-\frac{2\pi}{L(\theta-\theta_0)}$$

$$= \frac{2\pi}{L(\theta-\theta_0)}\left(\frac{1+\left(\frac{L}{2\pi}\right)\beta'\left(\frac{L\theta_0}{2\pi}\right)\cdot\beta''\left(\frac{L\theta_1}{2\pi}\right)\frac{(\theta-\theta_0)}{2}}{1+\left(\frac{L}{2\pi}\right)\beta'\left(\frac{L\theta_0}{2\pi}\right)\cdot\beta''\left(\frac{L\theta_1}{2\pi}\right)(\theta-\theta_0)+c_2(\theta-\theta_0)^2}-1\right)$$

$$= \frac{2\pi}{L}\left(\frac{-\left(\frac{L}{2\pi}\right)\beta'\left(\frac{L\theta_0}{2\pi}\right)\cdot\beta''\left(\frac{L\theta_1}{2\pi}\right)\frac{1}{2}-c_2(\theta-\theta_0)}{1+\left(\frac{L}{2\pi}\right)\beta'\left(\frac{L\theta_0}{2\pi}\right)\cdot\beta''\left(\frac{L\theta_1}{2\pi}\right)(\theta-\theta_0)+c_2(\theta-\theta_0)^2}\right) \tag{4.10}$$

Let $\theta\to\theta_0$, (4.10) becomes

$$\lim_{\theta\to\theta_0}\frac{2\pi}{L}\left(\frac{-\left(\frac{L}{2\pi}\right)\beta'\left(\frac{L\theta_0}{2\pi}\right)\cdot\beta''\left(\frac{L\theta_1}{2\pi}\right)\frac{1}{2}-c_2(\theta-\theta_0)}{1+\left(\frac{L}{2\pi}\right)\beta'\left(\frac{L\theta_0}{2\pi}\right)\cdot\beta''\left(\frac{L\theta_1}{2\pi}\right)(\theta-\theta_0)+c_2(\theta-\theta_0)^2}\right)=0$$

since

$$\lim_{\theta\to\theta_0}\beta'\left(\frac{L\theta_0}{2\pi}\right)\cdot\beta''\left(\frac{L\theta_1}{2\pi}\right)=\beta'\left(\frac{L\theta_0}{2\pi}\right)\cdot\beta''\left(\frac{L\theta_0}{2\pi}\right)=0$$

Thus, $B(\theta_0,\,\theta)$ is continuous over $(0,\,2\pi)\times(0,\,2\pi)$, and $B=0$ for $\theta_0=\theta\in(0,\,2\pi)$.

Case 2: $\theta_0=0$, $\theta>0$, and $\theta\to\theta_0$.

The proof of this case is the same as for case 1.

Case 3: $\theta_0=0$, $\theta<2\pi$, and $\theta\to 2\pi$.

Since B has period 2π, $B(0,\,\theta)=B(2\pi,\,\theta)$. Therefore, let $\theta_0=2\pi$ and the proof follows as for the case 1.

This completes the proof that B is continuous over $[0,\,2\pi]\times[0,\,2\pi]$; and $B=0$ for $\theta_0=\theta\in[0,\,2\pi]$. □

Corollary 5 *Assume $\beta(s)\in C^n[0,\,L]$, then the kernel function $B(\theta_0,\,\theta)$ is $n-2$ times continuously differentiable over* $[0,\,2\pi]\times[0,\,2\pi]$.

Proof: B is expressed in terms of (4.6) and (4.7). (4.7) can be checked easily that it is a very smooth function. For (4.6), we examine (4.10) carefully, we can see that the denominator of

(4.10) never equal to zero when θ and θ_0 are close to each other. Therefore, (4.6) is $n-2$ times continuously differentiable if $\beta(s)$ is n times continuously differentiable. □

5 The Numerical scheme

We begin by defining a Galerkin method for solving the hypersingular integral equation (3.8) in the space $L^2(0,\, 2\pi)$. However, instead of solving equation (3.8), we solve the equation (4.3):

$$-2\pi i C_u \mathcal{D}\eta(\theta_0) + \mathcal{B}\mathcal{D}\eta(\theta_0) = g(\theta_0) \tag{5.1}$$

where

$$g(\theta_0) \equiv \frac{L}{2\pi} f(\beta(\frac{L\theta_0}{2\pi})).$$

Let

$$\phi(\theta) \equiv \mathcal{D}\eta(\theta). \tag{5.2}$$

We solve (5.1) for $\phi \in L^2(0,\, 2\pi)$:

$$-2\pi i C_u \phi + \mathcal{B}\phi = g \tag{5.3}$$

From Theorem 3, this is uniquely solvable on $L^2(0,\, 2\pi)$. By making the unknown a derivative, we are decreasing the order of the pseudo-differential operator. Also, the first term of (5.3) is a Cauchy singular integral operator on the unit circle, and therefor, we can compute it easily.

The equation (5.3) is equivalent to

$$\phi - \frac{1}{2\pi i} C_u^{-1} \mathcal{B}\phi = -\frac{1}{2\pi i} C_u^{-1} g \tag{5.4}$$

The right side function $C_u^{-1} g$ is in $L^2(0,\, 2\pi)$. Because $\mathcal{B}$ has a continuous differentiable kernel B, $\mathcal{B}$ is a bounded compact operator from $H^q(2\pi)$ into $H^{q+1}(2\pi)$, and $C_u^{-1}\mathcal{B}$ is a compact mapping from $L^2(0,\,2\pi)$ into $L^2(0,\,2\pi)$. Thus, (5.4) is a Fredholm integral equation of the second kind. By the earlier assumption on the unique solvability of (5.3), we have $\left(I - \frac{1}{2\pi i} C_u^{-1} \mathcal{B}\right)^{-1}$ exists on $L^2(0,\,2\pi)$ to $L^2(0,\,2\pi)$.

Introduce

$$\mathcal{X}_n = \text{span}\,\{\psi_{-n}, \ldots, \psi_0, \ldots, \psi_n\}$$

for a given $n \geq 0$, and let $\mathcal{P}_n$ denote the orthogonal projection of $L^2(0,\,2\pi)$ onto $\mathcal{X}_n$. For $\varphi = \sum a_m \psi_m$, we have

$$\mathcal{P}_n \varphi(\theta) = \sum_{m=-n}^{n} a_m \psi_m(\theta)$$

the truncation of the Fourier series for φ.

Approximate (5.3) by the equation

$$\mathcal{P}_n \left(-2\pi i C_u \phi_n + \mathcal{B}\phi_n\right) = \mathcal{P}_n g, \qquad\qquad \phi_n \in \mathcal{X}_n \tag{5.5}$$

Let

$$\phi_n(\theta) = \sum_{\substack{m=-n \\ m \neq 0}}^{n} a_m^{(n)} \psi_m(\theta)$$

Note that ϕ_n does not have the constant term, i.e., $\phi_n \in \{\phi_n \in \mathcal{X} \mid a_0^{(n)} = 0\}$, because ϕ is the derivative of η (see (5.2)). The equation (5.5) implies that the coefficients $\left\{a_m^{(n)}\right\}$ are determined from the linear system

$$\begin{aligned} -\text{sign}(k) i\pi a_k^{(n)} &+ \sum_{\substack{m=-n \\ m \neq 0}}^{n} a_m^{(n)} \int_0^{2\pi} \int_0^{2\pi} B(\theta_0,\,\theta)\psi_m(\theta)\overline{\psi_k(\theta_0)}\,d\theta\,d\theta_0 \\ = & \int_0^{2\pi} g(\theta)\overline{\psi_k(\theta_0)}\,d\theta, \qquad\qquad k = \pm 1, \ldots, \pm n \end{aligned} \tag{5.6}$$

Using

$$\mathcal{P}_n C_u = C_u \mathcal{P}_n, \qquad\qquad \mathcal{P}_n C_u^{-1} = C_u^{-1} \mathcal{P}_n,$$

the approximating equation (5.5) is equivalent to

$$\phi_n - \frac{1}{2\pi i}\mathcal{P}_n C_u^{-1}\mathcal{B}\phi_n = -\frac{1}{2\pi i}\mathcal{P}_n C_u^{-1} g \tag{5.7}$$

This is simply a standard Galerkin method for solving (5.4).

Since $\mathcal{P}_n\phi \to \phi$, for all $\phi \in L^2(0,\, 2\pi)$, and since $C_u^{-1}\mathcal{B}$ is compact, we have

$$\left\|(I - \mathcal{P}_n)\, C_u^{-1}\mathcal{B}\right\| \longrightarrow 0 \text{ as } n \to \infty$$

Then by standard arguments, the existence of $\left(I - \frac{1}{2\pi i}C_u^{-1}\mathcal{B}\right)^{-1}$ implies that of $\left(I - \frac{1}{2\pi i}\mathcal{P}_n C_u^{-1}\mathcal{B}\right)^{-1}$ exists and is uniformly bounded for all sufficiently large n, and

$$\|\phi - \phi_n\|_0 \le \left\|\left(I - \frac{1}{2\pi i}\mathcal{P}_n C_u^{-1}\mathcal{B}\right)^{-1}\right\| \|\phi - \mathcal{P}_n\phi\|_0$$

where $\|\cdot\|_0$ is the norm for $H^0(2\pi) \equiv L^2(0,\, 2\pi)$. For more detailed bounds on the rate of convergence, see Atkinson [5, §7.3]:

$$\|\phi - \phi_n\|_0 \le \frac{c}{n^q}\|\phi\|_q\,, \qquad\qquad \phi \in H^q(2\pi)$$

for any $q > 0$.

Generally the integrals in (5.6) must be evaluated numerically, and therefore we introduce a *discrete Galerkin method*. We give a numerical method which amounts to using the trapezoidal rule to numerically integrate the integrals in (5.6). Introduce the *discrete inner product*

$$(f,\, g)_n = h\sum_{j=0}^{2n} f(t_j)\overline{g(t_j)}, \qquad\qquad f, g \in C_p(2\pi) \tag{5.8}$$

with $h = 2\pi/(2n+1)$, and $t_j = jh$, $j = 0, 1, \ldots, 2n,$; and note $(\cdot,\, \cdot)_n$ is only semi-definite. This is the trapezoidal rule with $2n+1$ subdivisions of the integration interval $[0,\, 2\pi]$, because the integrand is 2π-periodic; and $(\cdot,\, \cdot)_n$ is a true inner product on the set of all trigonometric polynomials of degree less than or equal to n. Also, approximate the integral operator $\mathcal{B}$ of (4.4) by

$$\mathcal{B}_n\phi(\theta_0) = h\sum_{j=0}^{2n} B(\theta_0,\, t_j)\phi(t_j), \qquad \phi \in C_p(2\pi)$$

We approximate (5.6) using

$$\sigma_n(\theta) = \sum_{\substack{m=-n \\ m\neq 0}}^{n} b_m^{(n)}\psi_m(\theta)$$

with $\left\{b_m^{(n)}\right\}$ determined from the linear system

$$-\operatorname{sign}(k)i\pi b_k^{(n)} + \sum_{\substack{m=-n \\ m\neq 0}}^{n} b_m^{(n)}(\mathcal{B}_n\psi_m,\, \psi_k)_n = (g,\, \psi_k)_n, \qquad k = \pm 1, \pm 2, \ldots, \pm n \tag{5.9}$$

We give the framework of the error analysis of the discrete Galerkin method here, and the proof of the error analysis follows the same pattern as the proof of Theorem 6 in Atkinson and Sloan [6].

Associated with the discrete inner product (5.8) is the discrete orthogonal projection operator $\mathcal{Q}_n$ mapping $\mathcal{X} = C_p(2\pi)$ into $\mathcal{X}_n$; for more details about $\mathcal{Q}_n$ see Atkinson [5, §4.4]. In particular,

$$(\mathcal{Q}_n\varphi,\, \psi)_n = (\varphi,\, \psi)_n, \qquad \forall\psi \in \mathcal{X}_n \tag{5.10}$$

$$\mathcal{Q}_n\varphi = \sum_{m=-n}^{n} (\varphi,\, \psi_m)_n\psi_m \tag{5.11}$$

Using (5.10) and (5.11), equation (5.9) can be written symbolically as

$$\mathcal{Q}_n\left(-2\pi i C_u\sigma_n + \mathcal{B}_n\sigma_n\right) = \mathcal{Q}_n g, \qquad \sigma_n \in \mathcal{X}_n \tag{5.12}$$

This equation is equivalent to the equation

$$-2\pi i C_u\sigma_n + \mathcal{Q}_n\mathcal{B}_n\sigma_n = \mathcal{Q}_n g, \qquad \sigma_n \in \mathcal{X} \tag{5.13}$$

In order to prove the equivalence, we begin by assuming (5.13) is solvable. Then

$$-2\pi i C_u \sigma_n = \mathcal{Q}_n g - \mathcal{Q}_n \mathcal{B}_n \sigma_n \in \mathcal{X}_n.$$

Using (2.5) for C_u, this implies $\sigma_n \in \mathcal{X}_n$ and $\mathcal{Q}_n \sigma_n = \sigma_n$. Using this in (5.13) implies the equation (5.12). A similar argument shows that (5.12) implies (5.13).

Equation (5.13) is equivalent to

$$\sigma_n - \frac{1}{2\pi i} C_u^{-1} \mathcal{Q}_n \mathcal{B}_n \sigma_n = -\frac{1}{2\pi i} C_u^{-1} \mathcal{Q}_n g \tag{5.14}$$

This is an approximation of (5.3). The equation (5.4), which is equivalent to (5.3), and its approximation (5.14)

$$\phi - \frac{1}{2\pi i} C_u^{-1} \mathcal{B} \phi = -\frac{1}{2\pi i} C_u^{-1} g \tag{5.15}$$

$$\sigma_n - \frac{1}{2\pi i} C_u^{-1} \mathcal{Q}_n \mathcal{B}_n \sigma_n = -\frac{1}{2\pi i} C_u^{-1} \mathcal{Q}_n g \tag{5.16}$$

are used for an error analysis of the discrete Galerkin method (5.9).

Then follow the same pattern as the proof for Theorem 6 in Atkinson and Sloan [6], we can show

$$\|\phi - \sigma_n\|_\infty \le \frac{c}{n^{q-0.5-\epsilon}} \tag{5.17}$$

when $g \in H^q(2\pi)$ and $\phi \in C_p(2\pi) \cap H^{q-1}(2\pi)$, for some $q > 0.5$ and any small $\epsilon > 0$.

6 Numerical Examples

We give two numerical examples for the interior Neumann problem (1.1). The domain D for both of the examples is an ellipse and its boundary S is

$$\beta(t) = (a\cos t,\, b\sin t), \qquad 0 \le t \le 2\pi$$

where $a = 0.5$ and $b = 2.5$. Consider the interior Neumann problem

$$\begin{aligned} \Delta u(P) &= 0, & P \in D \\ \frac{\partial u(P)}{\partial \mathbf{n}_P} &= f(P), & P \in S \end{aligned}$$

We represent the solution u as the double layer potential (1.2). The derivative of the

Table 1: Errors in u_n, true solution $= e^x \sin y$

n	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
4	8.28E−3	8.28E−2	2.07E−1	4.17E−1	7.70E−1
8	3.40E−3	3.40E−2	8.56E−2	1.75E−1	3.29E−1
12	1.79E−3	1.79E−2	4.50E−2	9.13E−2	1.73E−1
16	1.01E−3	1.01E−2	2.54E−2	5.16E−2	9.75E−2
20	5.96E−4	5.96E−3	1.50E−2	3.04E−2	5.74E−2
24	3.61E−4	3.61E−3	9.07E−3	1.84E−2	3.49E−2
28	2.23E−4	2.23E−3	5.60E−3	1.14E−2	2.15E−2
32	1.40E−4	1.40E−3	3.51E−3	7.13E−3	1.35E−2
36	8.82E−5	8.82E−4	2.22E−3	4.50E−3	8.51E−3
40	5.66E−5	5.66E−4	1.42E−3	2.89E−3	5.44E−3

unknown density function ρ is obtained by solving (5.7).

Table 2: Errors in u_n, $u(Q) = \log \mid Q - P \mid$, $P = (1,\,2)$

n	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
4	−1.96E−3	−1.91E−2	−4.94E−2	−1.28E−1	−3.57E−1
8	−1.90E−3	−1.97E−2	−5.28E−2	−1.19E−1	−2.65E−1
12	−9.46E−4	−9.87E−3	−2.70E−2	−6.06E−2	−1.36E−1
16	−5.49E−4	−5.74E−3	−1.55E−2	−3.50E−2	−7.76E−2
20	−3.27E−4	−3.42E−3	−9.21E−3	−2.08E−2	−4.52E−2
24	−1.99E−4	−2.08E−3	−5.60E−3	−1.26E−2	−2.83E−2
28	−1.24E−4	−1.29E−3	−3.47E−3	−7.84E−3	−1.71E−2
32	−7.75E−5	−8.10E−4	−2.18E−3	−4.92E−3	−1.08E−2
36	−4.92E−5	−5.14E−4	−1.38E−3	−3.12E−3	−6.91E−3
40	−3.14E−5	−3.28E−4	−8.83E−4	−1.99E−3	−4.34E−3

After solving the equation (5.7) for the approximate solution σ_n, the approximate density function η_n is given by

$$\eta_n(\theta) = \frac{2\pi}{L} \sum_{\substack{m=-n \\ m\neq 0}}^{n} \frac{b_m}{im} \psi_m(\theta) \qquad\qquad \theta \in [0,\, 2\pi]$$

We obtain an approximation u_n by substituting η_n for ρ in equation (1.2) and then integrating it numerically. The integral is evaluated with the trapezoidal rule T_{2m+1} where $m = 256$.

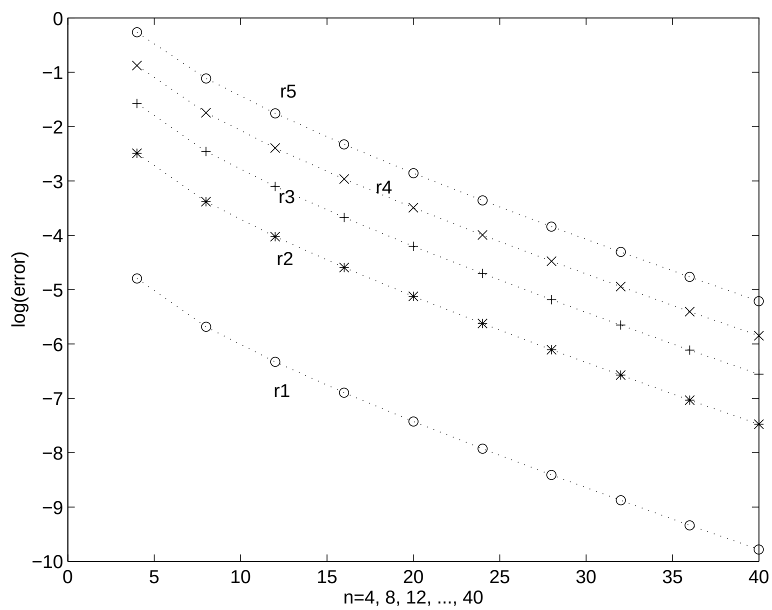

Figure 1: n vs log(error) for $u(x,\, y) = e^x \sin y$

We give the results of this integration at a set of five points inside of D:

$$(x_j,\, y_j) = r_j \left(a\cos(\frac{4}{\pi}),\, b\sin(\frac{4}{\pi}) \right), \qquad\qquad j = 1,\, 2,\, 3,\, 4,\, 5$$

with $r_j = 0.01,\, 0.1,\, 0.25,\, 0.5,\, 0.9$. The point $(x_5,\, y_5)$ is close to the boundary S, making the integrand in (1.2) quite peaked.

Two problems have been solved. The true solution for the first example is

$$u(x,\, y) = e^x \sin y, \qquad\qquad \forall\, (x,\, y) \in D.$$

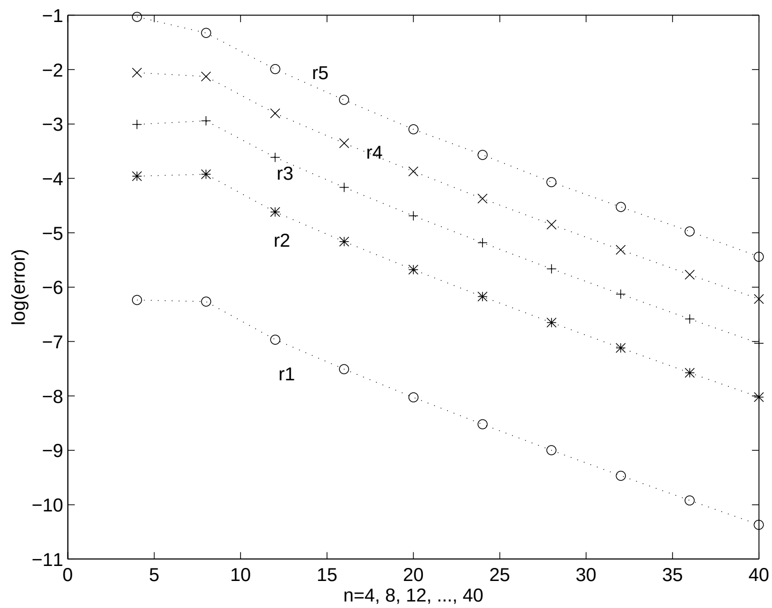

Figure 2: n vs log(error) for $u(Q) = \log \mid Q - P \mid$

The true solution of the second example is

$$u(x,\, y) = \log \mid (x, y) - P \mid, \qquad\qquad \forall\, (x,\, y) \in D$$

where P is a point out side of D, and we arbitrarily choose $P = (1,\, 2)$. Boundary data f for the Neumann problem are computed based on these two true solutions.

Tables 1 and 2 are errors for the true solutions $e^x \sin y$ and $\log \mid Q - P \mid$, respectively. We also plot the errors as Figures 1 and 2. The y-axis of the figures are the natural logarithm of the absolute value of the errors.

From Tables 1 and 2, we have noticed that the closer the points are to the boundary, the larger are the errors. From Figures 1 and 2, it appears that the rate of convergence is exponential:

$$u(A) - u_n(A) = \mathcal{O}(e^{-cn})$$

for some positive number c, which is better than what is proved in (5.17).

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