The Numerical Solution of Boundary Integral Equations

Kendall E. Atkinson

Dept. of Mathematics, The University of Iowa

[1]

Abstract

Much of the research on the numerical analysis of Fredholm type integral equations during the past ten years has centered on the solution of boundary integral equations (BIE). A great deal of this research has been on the numerical solution of BIE on simple closed boundary curves S for planar regions. When a BIE is defined on a smooth curve S, there are many numerical methods for solving the equation. The numerical analysis of most such problems is now well-understood, for both BIE of the first and second kind, with many people having contributed to the area. For the case with the BIE defined on a curve S which is only piecewise smooth, new numerical methods have been developed during the past decade. Such methods for BIE of the second kind were developed in the mid to late 80s; and more recently, high order collocation methods have been given and analyzed for BIE of the first kind. The numerical analysis of BIE on surfaces S in $\mathbb{R}^3$ has become more active during the past decade, and we review some of the important results. The convergence theory for Galerkin methods for BIE is well-understood in the case that S is a smooth surface, for BIE of both the first and second kind. For BIE of the second kind on piecewise smooth surfaces, important analyses have been given more recently for both Galerkin and collocation methods. In contrast, almost nothing is understood about collocation methods for solving BIE of the first kind, regardless of the smoothness of S. Numerical methods for BIE on surfaces S in $\mathbb{R}^3$ lead to computationally expensive procedures, and a great deal of the research for such BIE has looked at the efficient numerical evaluation of integrals, the use of iterative methods for solving the associated linear systems, and the use of "fast matrix-vector calculations" for use in iteration procedures.

1 Introduction

Boundary integral equations (BIE) are reformulations of boundary value problems for partial differential equations. They have been present in the mathematics literature for almost two centuries, and engineers have made much use of them for solving a wide variety of real-world problems. As a few examples of the

use of BIE, we cite Hess [43] and Newman [58] for potential flow calculations, Rizzo et al. [70] and Rudolphi et al. [72] for crack problems in elasticity, and Jaswon and Symm [44] for electrostatic and elastostatic calculations.

In spite of the widespread use of BIE, much of their numerical analysis has not been well-understood until much more recently, say from 1975 onwards. One of the major contributions to the subject was the recognition that most BIE can be regarded as strongly elliptic pseudo-differential operator equations on suitably chosen Sobolev spaces, and further, that finite element methods can then be applied to their solution. This theory is well-surveyed in the article [87] of W. Wendland, given during the last meeting in 1986 on the *State of the Art in Numerical Analysis,* and in the papers of Costabel [23], Costabel and Wendland [26], and Wendland [88]. For this reason, we omit further mention of it in this survey.

The theory of finite element methods for strongly elliptic pseudo-differential operator equations is most useful when the boundary S is a smooth curve or surface and when Galerkin methods are being used. During the past decade, there has been much research on numerical methods for BIE on piecewise smooth boundaries and on the study of collocation methods for all types of BIE. In addition, more attention has been given to the practical problems associated with solving discretizations of BIE, including numerical integration, iteration methods for solving the associated linear systems, and the use of "fast matrix-vector calculations".

In this survey, we concentrate on the topics mentioned in the last paragraph. We also restrict the survey to BIE for Laplace's equation $\Delta u = 0$. In part, this this is due to space limitations; but also, most of the major aspects of the theory are adequately understood by considering this case alone.

2 Boundary Integral Equations

To begin, we summarize the most popular BIE reformulations of Laplace's equation; and most such BIE are obtained as straightforward applications of the the divergence theorem. We omit BIE for planar problems that are derived as a consequence of applications of Cauchy's formula from functions of a complex variable.

2.1 Green's representation formulas

Let D be a planar simply-connected bounded region with a boundary S to which the divergence theorem can be applied. For example, let S be piecewise smooth, meaning that S is composed of a finite number of smooth sections, each of which has a C^∞ parameterization. Assume $u \in C^1(\overline{D}) \cap C^2(D)$ and $\Delta u \equiv 0$ on D.

Then

$$\int_S \left\{ \frac{\partial u(Q)}{\partial \mathbf{n}_Q} \log|P-Q| - u(Q) \frac{\partial}{\partial \mathbf{n}_Q} [\log|P-Q|] \right\} dS_Q = \begin{cases} 2\pi u(P), & P \in D \\ \Omega(P) u(P), & P \in S \\ 0, & P \in D_e \end{cases} \tag{2.1}$$

In this, $\mathbf{n}_Q$ is the inner normal to S at Q, $\Omega(P)$ is the interior boundary angle at P, and $D_e = \mathbb{R}^2 \backslash \overline{D}$.

An exterior variant is obtained by considering $u \in C^1(\overline{D}_e) \cap C^2(D_e)$ with $\Delta u \equiv 0$ on D_e and

$$\sup_{P \in \overline{D}_e} |u(P)| < \infty$$

Then

$$\begin{aligned} \int_S \left\{ \frac{\partial u(Q)}{\partial \mathbf{n}_Q} \log|P-Q| - u(Q) \frac{\partial}{\partial \mathbf{n}_Q} [\log|P-Q|] \right\} dS_Q \\ = \begin{cases} 2\pi [u(\infty) - u(P)], & P \in D_e \\ 2\pi u(\infty) - [2\pi - \Omega(P)] u(P), & P \in S \\ 2\pi u(\infty), & P \in D \end{cases} \end{aligned} \tag{2.2}$$

2.1.1 Formulas in three dimensions

Let $D \subset \mathbb{R}^3$ be a bounded simply-connected region with a piecewise smooth boundary S. Assume $u \in C^1(\overline{D}) \cap C^2(D)$ and $\Delta u \equiv 0$ on D. Then

$$\int_S \left\{ u(Q) \frac{\partial}{\partial \mathbf{n}_Q} \left[\frac{1}{|P-Q|} \right] - \frac{\partial u(Q)}{\partial \mathbf{n}_Q} \frac{1}{|P-Q|} \right\} dS_Q = \begin{cases} 4\pi u(P), & P \in D \\ \Omega(P) u(P), & P \in S \\ 0, & P \in D_e \end{cases} \tag{2.3}$$

In this, $\mathbf{n}_Q$ is the inner normal to S at Q, $\Omega(P)$ is the interior solid angle at P, and $D_e = \mathbb{R}^3 \backslash \overline{D}$.

For the exterior variant of this formula, assume $u \in C^1(\overline{D}_e) \cap C^2(D_e)$, $\Delta u \equiv 0$ on D_e, and

$$|u(P)| = O\left(|P|^{-1}\right), \quad |\nabla u(P)| = O\left(|P|^{-2}\right) \quad \text{as} \quad |P| \to \infty$$

Then

$$\begin{aligned} \int_S \left\{ u(Q) \frac{\partial}{\partial \mathbf{n}_Q} \left[\frac{1}{|P-Q|} \right] - \frac{\partial u(Q)}{\partial \mathbf{n}_Q} \frac{1}{|P-Q|} \right\} dS_Q \\ = \begin{cases} -4\pi u(P), & P \in D_e \\ -[4\pi - \Omega(P)] u(P), & P \in S \\ 0, & P \in D \end{cases} \end{aligned} \tag{2.4}$$

2.2 Direct boundary integral equations

Direct BIE are obtained directly from the above formulas, either by using one of the given formulas or by using some derivative of them. The unknown function is then the function u or its normal derivative $\partial u/\partial \mathbf{n}$. One of the reasons that direct BIE are sometimes preferred is because the unknown function has immediate physical significance, as compared to the unknown in an indirect BIE. We give some examples of direct BIE.

Solve the *exterior planar Neumann problem*

$$\begin{aligned} \Delta u(P) &= 0, \qquad P \in D_e \\ \frac{\partial u(p)}{\partial \mathbf{n}_P} &= f(P), \quad P \in S \end{aligned} \tag{2.5}$$

and assume S is smooth. For uniqueness, we impose

$$|u(P)| = O\left(|P|^{-1}\right), \quad |\nabla u(P)| = O\left(|P|^{-2}\right)$$

as $|P| \to \infty$. Then solve the BIE

$$-\pi u(P) + \int_S u(Q) \frac{\partial}{\partial \mathbf{n}_Q} \left[\log |P-Q|\right] dS_Q = \int_S f(Q) \log |P-Q| \, dS_Q, \quad P \in S \tag{2.6}$$

This is a classical BIE of the second kind; and it is well known that it is uniquely solvable within $C(S)$ for any given right hand function in $C(S)$, not just those written in the form of the given integral on the right side of (2.6).

Solve the *interior planar Dirichlet problem*

$$\begin{aligned} \Delta u(P) &= 0, \qquad P \in D \\ u(P) &= f(P), \quad P \in S \end{aligned}$$

Do this by solving

$$\int_S \psi(Q) \log |P-Q| \, dS_Q = \varphi(P), \quad P \in S \tag{2.7}$$

$$\varphi(P) = \int_S f(Q) \frac{\partial}{\partial \mathbf{n}_Q} \left[\log |P-Q|\right] dS_Q + \Omega(P) f(P)$$

with the unknown

$$\psi(Q) \equiv \frac{\partial u(Q)}{\partial \mathbf{n}_Q}$$

The above BIE of the first kind has been studied intensively in the past decade; and today it is very well understood for the case that S is a smooth curve.

For other direct BIE reformulations of Laplace's equation, see Blue [16] and Jaswon and Symm [44].

2.3 Indirect boundary integral equations

We will consider some *indirect BIE* in $\mathbb{R}^3$, and analogous formulas can be given in $\mathbb{R}^2$. We begin by deriving some consequences of formulas (2.3) and (2.4).

Let $u_i \in C^1(\overline{D}) \cap C^2(D)$ and $\Delta u_i \equiv 0$ on D; and let $u_e \in C^1(\overline{D}_e) \cap C^2(D_e)$, $\Delta u_e \equiv 0$ on D_e, with

$$|u_e(P)| = O\left(|P|^{-1}\right), \quad |\nabla u_e(P)| = O\left(|P|^{-2}\right) \quad \text{as} \quad |P| \to \infty$$

For points $P \in S$, introduce

$$\begin{aligned} [u(P)] &= u_i(p) - u_e(P) \\ \left[\frac{\partial u(P)}{\partial \mathbf{n}_P}\right] &= \frac{\partial u_i(P)}{\partial \mathbf{n}_P} - \frac{\partial u_e(P)}{\partial \mathbf{n}_P} \end{aligned} \tag{2.8}$$

Subtract the exterior version of Green's representation formula (2.4) from the interior version (2.3). This yields

$$\begin{aligned} &\int_S \left\{ [u(Q)] \frac{\partial}{\partial \mathbf{n}_Q} \left[\frac{1}{|P-Q|}\right] - \left[\frac{\partial u(Q)}{\partial \mathbf{n}_Q}\right] \frac{1}{|P-Q|} \right\} dS_Q \\ &\qquad = \begin{cases} 4\pi u_i(P), & P \in D \\ \Omega(P) u_i(P) + [4\pi - \Omega(P)]\, u_e(P), & P \in S \\ 4\pi u_e(P), & P \in D_e \end{cases} \end{aligned} \tag{2.9}$$

We illustrate the use of (2.9) by deriving a *single layer representation* for the solution of Laplace's equation. Given a function $u_i \in C^1(\overline{D}) \cap C^2(D)$ and $\Delta u_i \equiv 0$ on D, choose a function $u_e \in C^1(\overline{D}_e) \cap C^2(D_e)$, $\Delta u_e \equiv 0$ on D_e, with

$$|u_e(P)| = O\left(|P|^{-1}\right), \quad |\nabla u_e(P)| = O\left(|P|^{-2}\right)$$

and with

$$u_e(P) = u_i(P), \quad P \in S \tag{2.10}$$

Then u_i has the representation

$$u_i(P) = \int_S \frac{\psi(Q)}{|P-Q|} dS_Q, \quad P \in D \tag{2.11}$$

with

$$\psi(Q) = -\frac{1}{4\pi} \left[\frac{\partial u_i(Q)}{\partial \mathbf{n}_Q} - \frac{\partial u_e(Q)}{\partial \mathbf{n}_Q}\right], \quad Q \in S \tag{2.12}$$

This in turn leads to both BIE of the first kind and second kind, depending on the type of boundary condition being given.

We can repeat the above type of argument to obtain the *double layer representation*

$$u_i(P) = \int_S \rho(Q) \frac{\partial}{\partial \mathbf{n}_Q} \left[\frac{1}{|P-Q|}\right] dS_Q, \quad P \in D \tag{2.13}$$

In this case, we replace (2.10) with the matching of the normal derivatives on S. The function ρ has the interpretation

$$\rho(Q) = \frac{1}{4\pi}\left[u_i(Q) - u_e(Q)\right] \tag{2.14}$$

The formulas (2.12) and (2.14) can be used to obtain regularity results for the respective functions ψ and ρ, by using regularity results on the solutions of Laplace's equation. For an example of this, see Petersdorf and Stephan [62].

As an example of the use of (2.13), consider solving the interior Dirichlet problem

$$\begin{aligned} \Delta u(P) &= 0, \qquad P \in D \\ u(P) &= f(P), \quad P \in S \end{aligned} \tag{2.15}$$

as the double layer potential of (2.13). This leads to the classic BIE of the second kind

$$(2\pi + \mathcal{K})\,\rho = f \tag{2.16}$$

$$\mathcal{K}\rho(P) = \int_S \rho(Q)\frac{\partial}{\partial \mathbf{n}_Q}\left[\frac{1}{|P-Q|}\right] dS_Q + [2\pi - \Omega(P)]\,\rho(P), \quad P \in S \tag{2.17}$$

In the case that S is smooth, this is one of the equations studied by Ivar Fredholm in order to show the existence of solutions to the Dirichlet problem for $\Delta u = 0$; and it is also the classic BIE example given in most textbooks on partial differential equations. When the double layer representation is used to solve the interior Neumann problem, one obtains a *hypersingular* BIE. For examples of the solution of this equation in $\mathbb{R}^3$, see Giroire and Nedelec [36] and Petersdorf and Stephan [63].

3 The Numerical Solution of Planar Problems

Until the early 1970s, the most commonly studied BIE in numerical analysis was the classic second kind equation (2.16) with a smooth planar boundary S in $\mathbb{R}^2$. Solving the interior Dirichlet problem using a double layer representation for $u(P)$ leads to a second kind equation $(-\pi + \mathcal{K})\,\rho = f$,

$$-\pi\rho(t) + \int_0^{2\pi} K(t,s)\rho(s)\,ds = f(t)$$

in which K is smooth and periodic. Essentially any numerical method will work well, and Nyström methods with the trapezoidal rule probably work best.

$$-\pi\rho_n(t) + h\sum_{j=1}^{n} K(t,jh)\rho_n(jh) = f(t), \quad 0 \le t \le 2\pi$$

$$\|\rho - \rho_n\|_\infty = O\left(\|\mathcal{K}\rho - \mathcal{K}_n\rho\|_\infty\right)$$

For a complete discussion of this method, see [10]§7.2. Most of the work of the past two decades has looked at other planar BIE problems, especially those of the first kind on smooth boundaries and those of both the first and second kind for piecewise smooth boundaries.

3.1 BIE of the first kind on a smooth boundary

A large literature has developed in the past 10 to 15 years for BIE of the first kind. The most studied of such BIE has been

$$\mathcal{L}x(P) \equiv \int_S x(P) \log|P-Q|\, dS_Q = f(P), \quad P \in S \tag{3.1}$$

often called *Symm's equation*. The curve S is finite, and it can be either open or closed. Unique solvability for all $f \in C(S)$ follows if S is not a Γ-contour; and this is guaranteed by assuming that S has a diameter less than 1 (which can always be attained when solving Laplace's equation by a re-scaling of the variables). For a complete development of the theory of $\mathcal{L}x = f$, see Yan and Sloan [90]. In particular, for S simple, closed, and C^∞,

$$\mathcal{L} : H^r(S) \underset{onto}{\overset{1-1}{\to}} H^{r+1}(S), \quad r \geq 0 \tag{3.2}$$

The operator $\mathcal{L}$ can also be extended to $H^r(S)$ for $r < 0$, and then (3.2) will be true for all real r.

The key to understanding the behaviour of $\mathcal{L}$ and to analyzing numerical methods for solving (3.1) is to decompose $\mathcal{L}$ as follows. Let S have a parameterization $\mathbf{r}(t)$, $0 \leq t \leq 2\pi$; and then write

$$\mathcal{L}x(\mathbf{r}(t)) = \int_0^{2\pi} [\log|\mathbf{r}(t) - \mathbf{r}(s)|]\, |\mathbf{r}'(s)|\, x(\mathbf{r}(s))ds, \quad 0 \leq t \leq 2\pi$$

Introduce the equivalent operator

$$\mathcal{K}\varphi(t) = \int_0^{2\pi} \varphi(s) \log|\mathbf{r}(t) - \mathbf{r}(s)|\, ds, \quad 0 \leq t \leq 2\pi \tag{3.3}$$

The equation $\mathcal{L}x = f$ becomes

$$\mathcal{K}\varphi = g$$

with

$$g(t) = f(\mathbf{r}(t)), \quad \varphi(s) = |\mathbf{r}'(s)|\, x(\mathbf{r}(s))$$

We can write

$$\mathcal{K}\varphi = -\pi\mathcal{A}\varphi + \mathcal{B}\varphi \tag{3.4}$$

$$\mathcal{A}\varphi(t) = \int_0^{2\pi} \varphi(s) \log\left|2e^{-\frac{1}{2}} \sin\left(\frac{t-s}{2}\right)\right| ds \tag{3.5}$$

$$\mathcal{B}\varphi(t) = \int_0^{2\pi} B(t,x)\varphi(s)\,ds$$

$$B(t,s) = \begin{cases} \log \dfrac{|\sqrt{e}\,[\mathbf{r}(t)-\mathbf{r}(s)]|}{\left|2\sin\left(\dfrac{t-s}{2}\right)\right|}, & t-s \neq 2m\pi \\ \log|\sqrt{e}\mathbf{r}'(t)|\,, & t-s = 2m\pi \end{cases}$$

If $\mathbf{r} \in C_p^m(2\pi)$, then $B(t,s)$ is in $C_p^{m-1}(2\pi)$ with respect to both s and t. Consequently, $\mathcal{B}$ is a compact operator on most spaces. For example, $\mathcal{B}$ is compact from H^r to H^{r+1} for $r \geq 0$, provided $m > 2$. Then $\mathcal{K}\varphi = g$ is equivalent to

$$-\pi\varphi + \mathcal{A}^{-1}\mathcal{B}\varphi = \mathcal{A}^{-1}g \tag{3.6}$$

This is a second kind equation, with a compact integral operator; and as noted earlier, much is known about such equations.

It can be shown that

$$\mathcal{A}\varphi(t) = \frac{1}{\sqrt{2\pi}}\left[\widehat{\varphi}(0) + \sum_{|m|>0} \frac{\widehat{\varphi}(m)}{|m|} e^{ims}\right] \tag{3.7}$$

where

$$\varphi(s) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \widehat{\varphi}(m) e^{ims}$$

In essence, $\mathcal{A}$ is a slight modification of $\mathcal{K}$ with S the unit circle; and (3.7) shows

$$\mathcal{A} : H^r \underset{onto}{\overset{1-1}{\to}} H^{r+1}, \quad r \geq 0$$

The decomposition (3.4) is the major tool used in analyzing many numerical methods for solving $\mathcal{K}\varphi = g$.

3.1.1 Numerical methods

For an excellent survey of numerical methods for solving $\mathcal{L}x = f$, see Sloan [79]. To derive Galerkin methods, regard $\mathcal{L}$ as a strongly elliptic operator from $H^{-\frac{1}{2}}(S)$ to $H^{\frac{1}{2}}(S)$:

$$(\mathcal{L}x, x) \geq c\,\|x\|^2_{-\frac{1}{2}}\,, \quad x \in H^{-\frac{1}{2}} \tag{3.8}$$

and then extend finite element methods and their analysis to this case. Define suitable finite element subspaces $\mathcal{X}_n$, and then find x_n from the Galerkin conditions

$$(\mathcal{L}x_n, v) = (f, v), \quad v \in \mathcal{X}_n$$

The left side uses the extension to $H^{\frac{1}{2}} \times H^{-\frac{1}{2}}$ of the standard inner product on $H^0 = L^2$. Then it follows from Cea's Lemma that x_n exists and

$$\|x - x_n\|_{-\frac{1}{2}} \leq c \inf_{v \in \mathcal{X}_n} \|x - v\|_{-\frac{1}{2}} \tag{3.9}$$

With x_n piecewise polynomial of degree $r \geq 0$,

$$\|x - x_n\|_0 \leq ch^{r+1} \|x\|_{r+1}, \quad x \in H^{r+1} \tag{3.10}$$

Collocation methods for $\mathcal{L}x = f$ divide into two classes, depending on the form of the approximating functions. With piecewise polynomial approximants, we obtain *boundary element methods;* and with trigonometric polynomial approximants, we obtain *spectral methods.* For boundary element methods, we cite Arnold and Wendland [4] and Saranen [73]. Again, the entire literature is well-surveyed in [79].

3.1.2 Spectral methods

The spectral methods define $\mathcal{X}_n$ as the trig polynomials of degree $\leq n$. For a *Galerkin method* to solve $\mathcal{K}\varphi = g$, let $\mathcal{P}_n$ denote the orthogonal projection of $L^2(0, 2\pi)$ onto $\mathcal{X}_n$. Define $\varphi_n \in \mathcal{X}_n$ as the solution of

$$\mathcal{P}_n \mathcal{K} \varphi_n = \mathcal{P}_n g$$

Recalling (3.4) and using the special properties of $\mathcal{A}$, especially

$$\mathcal{P}_n \mathcal{A} \psi = \mathcal{A} \mathcal{P}_n \psi, \quad \psi \in \mathcal{X}_n$$

we have the equivalent equation

$$\varphi_n + \mathcal{P}_n \mathcal{A}^{-1} \mathcal{B} \varphi_n = \mathcal{P}_n \mathcal{A}^{-1} g$$

A complete error analysis can be based on comparing this with

$$-\pi\varphi + \mathcal{A}^{-1}\mathcal{B}\varphi = \mathcal{A}^{-1} g$$

For example, see McLean [51]. A spectral discrete Galerkin-collocation method is given in [7] for Laplace's equation with S a smooth simple closed curve, and this is extended in [15] to smooth open arcs. Extensions to a more general framework and to other BIE are given in McLean [52], McLean et al. [53], and Saranen and Vainikko [74].

3.2 Hypersingular BIE on smooth boundaries

Use a double layer potential

$$u(P) = \int_S \rho(Q) \frac{\partial}{\partial \mathbf{n}_Q} \left[\log |P - Q|\right] dS_Q, \quad P \in D$$

to solve the interior Neumann problem for $\Delta u = 0$ on D. We find ρ by solving

$$\mathcal{H}\rho = f \tag{3.11}$$

$$\begin{aligned}\mathcal{H}\rho(P) &\equiv \lim_{A\to P} \mathbf{n}_P \cdot \nabla_A \int_S \rho(Q) \frac{\partial}{\partial \mathbf{n}_Q} \left[\log|A-Q|\right] dS_Q \\ &\equiv \frac{\partial}{\partial \mathbf{n}_P} \int_S \rho(Q) \frac{\partial}{\partial \mathbf{n}_Q} \left[\log|P-Q|\right] dS_Q\end{aligned}$$

For S = unit circle, let $\mathcal{H}$ be denoted by $\mathcal{H}_u$. Then

$$\mathcal{H}_u : e^{int} \to \pi\,|n|\, e^{int}, \quad n = 0, \pm 1, , ... \tag{3.12}$$

Let $\mathcal{D}$ denote the derivative operator, and let $\mathcal{C}_u$ denote the Cauchy singular integral operator on the unit circle:

$$\mathcal{C}_u\varphi(z) = \frac{1}{\pi i} \int\limits_{|\zeta|=1} \frac{\varphi(\zeta)\, d\zeta}{\zeta - z}$$

Then

$$\begin{aligned}&\mathcal{H}_u = -\pi i \mathcal{D}\mathcal{C}_u = -\pi i \mathcal{C}_u \mathcal{D} \\ &\mathcal{H}_u : H^r \to H^{r-1}, \quad r \geq 1\end{aligned} \tag{3.13}$$

For the general equation $\mathcal{H}\rho = f$, with a parameterization of S proportional to arc-length and L the length of S,

$$\mathcal{H}\rho(\theta) = -\frac{2\pi}{L}\mathcal{H}_u\rho(\theta) + \mathcal{B}\mathcal{D}\rho(\theta) \tag{3.14}$$

with $\mathcal{B}$ a smoothing operator. This can be used to develop spectral methods of solution. Numerical methods for solving $\mathcal{H}\rho = f$ are given in Rathsfeld et al. [69], Kress [50], Chien and Atkinson [22], and Amini and Maines [2].

3.3 Second kind BIE on piecewise smooth boundaries

This is a topic on which enormous progress has been made since the early 1980s, essentially solving the problem. As a particular example, consider solving the interior Dirichlet problem with boundary data f by using a double layer potential representation:

$$u(P) = \int_S \rho(Q) \frac{\partial}{\partial \mathbf{n}_Q} \left[\log|P-Q|\right] dS_Q, \quad P \in D \tag{3.15}$$

Then ρ satisfies

$$(-\pi + \mathcal{K})\,\rho = f \tag{3.16}$$

$$\mathcal{K}\rho(P) = \int_S \rho(Q) \frac{\partial}{\partial \mathbf{n}_Q} \left[\log|P-Q|\right] dS_Q - \left[\pi - \Omega(P)\right] \rho(P), \quad P \in S \tag{3.17}$$

The quantity $\Omega(P)$ denotes the interior angle at $P \in S$. It can be shown that

$$\mathcal{K} : C(S) \to C(S)$$

is bounded. The behaviour of a solution ρ can be studied by using

$$\rho(P) = u(P) - u_e(P)$$

with $u_e(P)$ a solution of an exterior Neumann problem with normal derivative matching that of $u(P)$ on S, as in (2.13)-(2.14).

Let S have a corner at the origin with the interior angle

$$\phi = (1 - \chi)\,\pi, \quad -1 < \chi < 1$$

It follows from regularity results for Laplace's equation that

$$\rho(P) = O\left(|P|^{\beta}\right), \quad \beta = \frac{1}{1 + |\chi|} \tag{3.18}$$

Therefore, ρ has an algebraic singularity with

$$\frac{1}{2} < \beta < 1$$

In addition to the lack of smoothness in ρ, the operator $\mathcal{K}$ is no longer compact on $C(S)$. To gain some insight, look at the special case of a wedge boundary, as in Figure 1. Solve the Dirichlet problem with boundary data f as a double layer potential (3.15), which means solving

$$(-\pi + \mathcal{K})\,\rho = f \tag{3.19}$$

This is equivalent to the system

$$\begin{aligned} -\pi\rho_1(x) - \int_0^1 \frac{x \sin(\chi\pi)\,\rho_2(y)\,dy}{x^2 + 2xy\cos(\chi\pi) + y^2} &= f_1(x) \\ -\pi\rho_2(x) - \int_0^1 \frac{x \sin(\chi\pi)\,\rho_1(y)\,dy}{x^2 + 2xy\cos(\chi\pi) + y^2} &= f_2(x) \end{aligned}$$

for $0 < x \leq 1$. The function ρ_1 and ρ_2 are the restrictions of ρ to the bottom branch and upper branch of S, respectively. This system can be uncoupled as

$$-\pi\psi_{\pm}(x) \pm \int_0^1 \frac{x \sin(\chi\pi)\,\psi_{\pm}(y)\,dy}{x^2 + 2xy\cos(\chi\pi) + y^2} = F_{\pm}(x) \tag{3.20}$$

and each equation can be reduced to a more well-known type of equation.

One approach is to use the change of variables $x = e^{-t}$, $y = e^{-\tau}$. Then (3.20) becomes the Wiener-Hopf equation

$$-\pi\widehat{\psi}(t) \pm \int_0^{\infty} \frac{\sin(\chi\pi)\,\widehat{\psi}(\tau)}{e^{t-\tau} - 2\cos(\chi\pi) + e^{\tau-t}} = \widehat{F}(t), \quad 0 \leq t < \infty \tag{3.21}$$

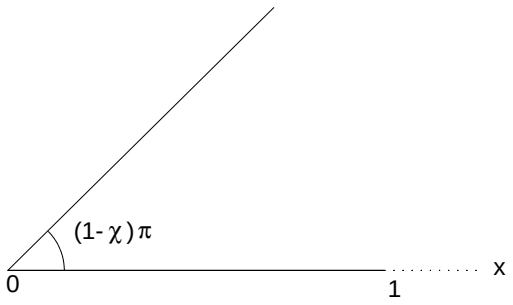

Figure 1. A wedge boundary

From this perspective, we obtain that the spectrum of $\mathcal{K}$ equals $[-\chi\pi, \chi\pi]$, showing clearly $\mathcal{K}$ is not compact.

A second approach is to rewrite (3.20) as the *Mellin convolution equation*

$$-\pi\psi_{\pm}(x) \pm \int_0^1 \kappa\left(\frac{x}{y}\right)\psi_{\pm}(y)\frac{dy}{y} = F_{\pm}(x) \tag{3.22}$$

$$\kappa(u) = \frac{u\sin(\chi\pi)}{u^2 + 2u\cos(\chi\pi) + 1}, \quad 0 \le u < \infty$$

This has become the preferred way to study the wedge equation and its numerical solution.

3.3.1 Galerkin methods

Regard the wedge operator

$$\mathcal{L}\psi(x) \equiv \int_0^1 \frac{x\sin(\chi\pi)\,\psi(y)\,dy}{x^2 + 2xy\cos(\chi\pi) + y^2}, \quad 0 < x \le 1$$

from (3.20) as an operator on $L^2(0,1)$. Then

$$\|\mathcal{L}\| \leq \pi \left|\sin\left(\frac{\chi\pi}{2}\right)\right| < \pi$$

and therefore

$$(-\pi \pm \mathcal{L})\,\psi = F \tag{3.23}$$

is uniquely solvable. Let $\mathcal{X}_n$ be a subspace of piecewise polynomial functions, and let $\mathcal{P}_n$ be the orthogonal projection of $L^2(0,1)$ onto $\mathcal{X}_n$. Then

$$(-\pi \pm \mathcal{P}_n\mathcal{L})\,\psi_n = \mathcal{P}_n F$$

is uniquely solvable for $\psi_n \in \mathcal{X}_n$, the method is stable, and

$$\|\psi - \psi_n\|_2 \leq \pi \left\|(-\pi \pm \mathcal{P}_n\mathcal{L})^{-1}\right\| \|\psi - \mathcal{P}_n\psi\|_2 \tag{3.24}$$

By using a graded mesh that compensates for the behaviour in (3.18), we can obtain an optimal order of convergence. Define a graded mesh using

$$x_i = \left(\frac{i}{n}\right)^q, \quad i = 0, 1, ..., n, \quad q \geq 1 \tag{3.25}$$

Let $\mathcal{X}_n$ consist of all piecewise polynomial functions of degree $\leq r$. Recall the order of convergence of the solution about the origin is $O(x^\beta)$ with $\beta = (1 + |\chi|)^{-1}$. Then choose

$$q > \frac{r+1}{\beta + \frac{1}{2}}$$

It follows that

$$\|\psi - \psi_n\|_2 \leq O\left(n^{-(r+1)}\right) \tag{3.26}$$

Choosing $q \geq r + 1$ will ensure this for all angles. These generalize to results using $\|\cdot\|_\infty$, with a suitably larger choice for q. The above theory was first given in Chandler [17].

3.3.2 Collocation methods

Let $\mathcal{X}_n$ denote all piecewise polynomials of degree $\leq r$ on our graded mesh, with no continuity restrictions. Introduce

$$0 \leq \xi_0 < \xi_1 < \cdots < \xi_r \leq 1$$

and define collocation nodes

$$x_{i,j} = x_{i-1} + \xi_j\,(x_i - x_{i-1}), \quad j = 0, ..., r$$

for $i = 1, ..., n$. [With $\xi_0 = 0$ and $\xi_r = 1$, there are continuity restrictions on $\mathcal{X}_n$.] Define $\mathcal{P}_n$ as the piecewise interpolatory projection of $C[0,1]$ onto $\mathcal{X}_n$ [actually we must use a larger space than $C[0,1]$ to have a well-defined theory, as in [13]].

For the wedge equation (3.23), the resulting collocation method can be written abstractly as

$$(-\pi \pm \mathcal{P}_n \mathcal{L})\, \psi_n = \mathcal{P}_n F \tag{3.27}$$

It is more difficult to analyze than Galerkin's method; and just having $\mathcal{P}_n x \to x$ for all $x \in C[0,1]$, with a suitably graded mesh, is no longer sufficient for convergence (due to an example in [18]). For the numerical method, the approximating subspace $\mathcal{X}_n$ must be modified about the origin, with a corresponding change in $\mathcal{P}_n$. On some neighborhood $[0, x_{i_*}]$, we must define the approximations as piecewise constant, with corresponding collocation nodes the centroids of the subintervals.

Assume

$$\int_0^1 \left(\prod_{i=0}^{r} (\xi - \xi_i)\, g(\xi) d\xi \right) = 0, \quad \deg(g) \leq r'$$

Then for some $i_* \geq 0$ chosen sufficiently large, the inverses $(-\pi \pm \mathcal{P}_n \mathcal{L})^{-1}$ exist and are uniformly bounded for $n \geq N$. Moreover, if

$$q > \frac{r + r' + 2}{\beta}$$

then

$$\max_{i,j} |\psi(x_{i,j}) - \psi_n(x_{i,j})| \leq O\left(n^{-(r+r'+2)} \right) \tag{3.28}$$

This is uniform in χ if

$$q \geq 2\,(r + r' + 2)$$

A crucial role in the analysis is played by the problem of inverting "finite section" equations. In the Mellin convolution formulation, we need to show the stability over ϵ of the approximation

$$-\pi \psi_\epsilon(x) \pm \int_\epsilon^1 \kappa\left(\frac{x}{y}\right) \psi_\epsilon(y) \frac{dy}{y} = F(x), \quad 0 < x \leq 1 \tag{3.29}$$

and its relation to the full equation

$$-\pi \psi(x) \pm \int_0^1 \kappa\left(\frac{x}{y}\right) \psi(y)\, \frac{dy}{y} = F(x), \quad 0 < x \leq 1$$

This is also the problem of analyzing the approximation of the Wiener-Hopf equation

$$-\pi \widehat{\psi}(t) \pm \int_0^\infty k\,(t - \tau)\, \widehat{\psi}(\tau)\, dy = \widehat{F}(t), \quad t \geq 0$$

by the finite section equation

$$-\pi\widehat{\psi}_\gamma(t) \pm \int_0^\gamma k\,(t-\tau)\,\widehat{\psi}_\gamma(\tau)\,dy = \widehat{F}(t), \quad t \geq 0 \tag{3.30}$$

and in this case, the problem was first solved by Anselone and Sloan [3].

The general solvability theory for $(-\pi + \mathcal{K})\,\rho = f$ goes back to Radon [65] in 1919. His methods are akin to the present methods: the problem is divided into two parts, one on a small 'wedge shaped' section about the corner point and the other on the remaining part of the boundary. This is also the basis for studying the corresponding problems on surfaces in $\mathbb{R}^3$.

For Nyström methods for solving $(-\pi + \mathcal{K})\,\rho = f$, see Graham and Chandler [35]. For a general approach to all of these methods and generalizations of them, see Elschner [29]. For some other references to the approximate solution of $(-\pi + \mathcal{K})\,\rho = f$, see Costabel and Stephan [24], Atkinson and de Hoog [14], Kress [49], Rathsfeld [67], Atkinson and Graham [12], and Jeon [45].

3.4 First kind BIE on piecewise smooth boundaries

For S a piecewise smooth boundary, the choice of study for first kind BIE is again

$$\mathcal{L}x(P) \equiv \int_S x(P)\log|P-Q|\,dS_Q = f(P), \quad P \in S$$

Preliminary results on collocation methods for this equation were obtained by Yan [89] and Costabel and Stephan [25]. When compared with the earlier work in §3.1 for S a smooth boundary, the difficulty is that the earlier split

$$\mathcal{L} = -\pi\mathcal{A} + \mathcal{B}$$

is no longer of much use, in that $\mathcal{B}$ is no longer a compact operator.

The first general collocation method was obtained only recently, by Elschner and Graham [32]. The main idea in their work is to use a carefully constructed parameterization of the polygonal boundary, one which will improve the behaviour of the equation in the vicinity of corners of S. For simplicity, consider only a single corner, at the origin. Then introduce a parameterization

$$\gamma : [-\pi, \pi] \to S$$

with $\gamma(0) = \mathbf{0}$, and consider here only the corner at the origin. Then $\mathcal{L}x = f$ becomes

$$\mathcal{K}w(\sigma) \equiv \int_{-\pi}^{\pi} \log|\gamma(s) - \gamma(\sigma)|\,w(\sigma) = g(s) \tag{3.31}$$

$$w(\sigma) = |\gamma'(\sigma)|\,x(\gamma(\sigma)), \quad g(s) = f(\gamma(s))$$

The function γ is so chosen that all low order derivatives vanish around $s = 0$. The form of γ is

$$\gamma(s) = O\,(s^q)$$

for s near to 0, and with q chosen sufficiently large.

The resulting solution w is relatively smooth around 0; and the operator $\mathcal{A}$ of (3.5) can again be used in approximating $\mathcal{K}$. We transform $\mathcal{K}w = g$ to

$$\left[I + \mathcal{A}^{-1}(\mathcal{K} - \mathcal{A})\right] w = \mathcal{A}^{-1} g \tag{3.32}$$

It can be shown that

$$\mathcal{A}^{-1}(\mathcal{K} - \mathcal{A}) = -\mathcal{H}\mathcal{D}(\mathcal{K} - \mathcal{A}) + \mathcal{E}$$

with $\mathcal{E}$ compact, $\mathcal{D}\varphi = \varphi'$, and $\mathcal{H}$ the Hilbert singular integral operator. The operator

$$\mathcal{B} = -\mathcal{H}\mathcal{D}(\mathcal{K} - \mathcal{A})$$

can be shown to be equivalent about the corner at the origin to a Mellin convolution operator plus a compact perturbation. Earlier types of results for such operators can then be brought to bear to analyze the second kind reformulation (3.32) of $\mathcal{K}w = g$.

A collocation method with splines is defined for solving (3.31), although modifications depending on some unknown index $i^* \geq 0$ must be introduced. For the error in solving for w, it can be shown that if q is chosen sufficiently large, and if i_* is chosen sufficiently large, then a collocation method with splines of degree k and a uniform mesh h will have an error in L^2 of $O(h^{k+1})$. Generalizations of this work are being given; for example, see Elschner and Stephan [33].

4 BIE Problems on Smooth Surfaces

The study of BIE on smooth surfaces in $\mathbb{R}^3$ is a well-developed area of classical mathematics, and there are many references to it; for example, see Kress [48]. The general theory of numerical methods for solving BIE of the second kind with a compact operator on smooth surfaces is basically well-understood. But significant practical problems remain, especially concerning numerical integration. For BIE of the first kind, only Galerkin methods are well-understood; and virtually nothing is understood about collocation methods for such equations.

Recall the single layer and double layer potential operators introduced in §2.3:

$$\mathcal{S}\psi(P) = \int_S \frac{\psi(Q)}{|P - Q|} dS_Q, \quad P \in S \tag{4.1}$$

$$\mathcal{K}\rho(P) = \int_S \rho(Q) \frac{\partial}{\partial \mathbf{n}_Q} \left[\frac{1}{|P - Q|}\right] dS_Q, \quad P \in S \tag{4.2}$$

For S the smooth boundary of a closed bounded simply connected region in $\mathbb{R}^3$,

$$\mathcal{S}, \mathcal{K} : H^r(S) \to H^{r+1}(S)$$

are bounded operators. For $S = U$ the unit sphere, $\mathcal{S} = 2\mathcal{K}$. Using a representation of the function $\rho \in L^2(U)$ in terms of its Laplace series expansion in spherical harmonics, the mapping properties of $\mathcal{S}$ are transparent from the result

$$\mathcal{S} : Y_n^m \to \frac{4\pi}{2n+1} Y_n^m$$

for spherical harmonics Y_n^m of degree n. An extensive development of the mapping properties of $\mathcal{S}$ and $\mathcal{K}$ is given in [39]. In particular, both $\mathcal{K}$ and $\mathcal{S}$ are compact on $\mathcal{X}$ to $\mathcal{X}$, with $\mathcal{X}$ equal to either $C(S)$ or $L^2(S)$.

We begin with BIE of the second kind, returning later with some brief remarks on BIE of the first kind. For the interior Dirichlet problem (2.15), we represent the solution u as a double layer potential,

$$u(P) = \int_S \rho(Q) \frac{\partial}{\partial \mathbf{n}_Q} \left[\frac{1}{|P-Q|} \right] dS_Q, \quad P \in D$$

and we find ρ by solving the second kind BIE of (2.16),

$$(2\pi + \mathcal{K})\, \rho = f \tag{4.3}$$

We can also obtain other BIE of the second kind, by other applications of the various identities of §2.

4.1 Numerical methods

It is straightforward to speak of both Galerkin and collocation methods for solving (4.3). The convergence theory follows easily from the compactness of $\mathcal{K}$ and $\mathcal{S}$. The Galerkin and collocation methods are simply

$$(2\pi + \mathcal{P}_n \mathcal{K})\, \rho_n = \mathcal{P}_n f \tag{4.4}$$

with appropriate choices of projection operators $\mathcal{P}_n$. If we assume $\mathcal{P}_n x \to x$ for all $x \in \mathcal{X}$, then

$$\|(I - \mathcal{P}_n)\, \mathcal{K}\| \to 0 \tag{4.5}$$

In turn, this implies the existence and uniform boundedness of $(2\pi + \mathcal{P}_n \mathcal{K})^{-1}$ for all $n \geq N$; and moreover,

$$\|\rho - \rho_n\| \leq 2\pi \left\| (2\pi + \mathcal{P}_n \mathcal{K})^{-1} \right\| \|(I - \mathcal{P}_n)\, \rho\| \tag{4.6}$$

The methods of most interest have been those of *boundary element* type defined using collocation, for which we now describe a framework.

Let $\mathcal{T}_n = \left\{ \Delta_k^{(n)} \mid k = 1, ..., n \right\}$ denote a sequence of triangulations of S; and let

$$h \equiv h_n = \max_k \operatorname{diam} \left(\Delta_k^{(n)} \right)$$

To refine a triangulation $\mathcal{T}_n$, we usually subdivide each element of $\mathcal{T}_n$ by connecting the midpoints of the sides of $\Delta_k^{(n)}$, in order to get the next triangulation $\mathcal{T}_{4n}$. We usually have a parameterization of each element $\Delta_k^{(n)}$, say

$$m_k : \widehat{\Delta}_k^{(n)} \subset \mathbb{R}^2 \underset{onto}{\overset{1-1}{\to}} \Delta_k^{(n)}$$

Let $p(x, y)$ be a polynomial over $\widehat{\Delta}_k^{(n)}$. Then consider

$$P(m_k(x, y)) = p(x, y)$$

as a 'polynomial' over $\Delta_k^{(n)}$.

For some degree $r \geq 0$, we define $\mathcal{X}_n$ as the set of all functions that are piecewise polynomial over the triangulation $\mathcal{T}_n$ of S. In our own work, we often restrict the functions of $\mathcal{X}_n$ to be continuous; but this is sometimes too restrictive. As a particular example for which I have created a public domain package BIEPACK [8], we take $r = 2$ and we base the collocation method on interpolation at the vertices and midpoints of the sides of the elements of $\mathcal{T}_n$. On the boundary of bounded simply connected region, this leads to

$$n_v = 2n + 2$$

interpolation nodes. These are denoted collectively by $\mathcal{V}_n = \left\{v_i^{(n)}\right\}$, and in terms of the element $\Delta_k^{(n)}$ to which they belong by

$$\{v_{k,1}, ..., v_{k,6}\}$$

For interpolation over $\Delta_k^{(n)}$, introduce 'Lagrange basis polynomials' $\{L_{k,1}, ..., L_{k,6}\}$ and write

$$\mathcal{P}_n f(Q) = \sum_{j=1}^{6} f(v_{k,j}) L_{k,j}(Q), \quad Q \in \Delta_k^{(n)}$$

with $\mathcal{P}_n$ the projection of $C(S)$ onto $\mathcal{X}_n$. Formulas (4.4)-(4.6) are valid with this definition of $\mathcal{P}_n$, and it is discussed at greater length in [11].

Also, we often approximate the surface S by interpolation, of the same or greater degree. For a 'piecewise polynomial surface' approximation of degree $q \geq 1$, interpolate the parameterization function $m_k(x, y)$ by a polynomial of degree q over $\widehat{\Delta}_k^{(n)}$. In BIEPACK, we use quadratic interpolation. The *panel method* uses piecewise constant interpolation of functions combined with piecewise planar approximations of the surface; and it is widely used in the aircraft industry.

With piecewise quadratic interpolation defining the quadrature, and with the refinement of $\mathcal{T}_n$ to $\mathcal{T}_{4n}$ based on subdividing each element into 4 new elements based on connecting the midpoints of the sides, we have

$$\max_{v_i} |\rho(v_i) - \rho_n(v_i)| = O\left(h^4 \log h\right)$$

whereas

$$\|\rho - \rho_n\|_\infty = O\left(h^3\right)$$

This appears to generalize to collocation with $\mathcal{X}_n$ composed of piecewise polynomials of degree$\leq r$ for some even r.

When approximating the surface S with a polynomial of degree q, with piecewise polynomial collocation of degree r, the best we have been able to prove is

$$\|\rho - \rho_n\|_\infty = O\left(h^{\min\{r+1,q\}}\right)$$

although empirical results at the node points are better than this. For $r = q = 2$, we obtain something consistent with

$$\max_{v_i} |\rho(v_i) - \rho_n(v_i)| = O\left(h^4\right)$$

As references, see Wendland [86], Chien [20], [21], Atkinson [8], and Atkinson and Chien [11]. For extensions of the package BIEPACK to a distributed processor parallel computer, see Natarajan and Krishnaswamy [56].

4.1.1 The linear system

Consider the linear system associated with

$$(2\pi + \mathcal{P}_n\mathcal{K})\,\rho_n = \mathcal{P}_n f$$

We write

$$\rho_n(Q) = \sum_{j=1}^{6} \rho_n\left(v_{k,j}\right) L_{k,j}(Q), \quad Q \in \Delta_k^{(n)}, \quad k = 1, ..., n$$

and solve

$$2\pi\rho_n\left(v_i^{(n)}\right) + \sum_{k=1}^{n}\sum_{j=1}^{6} \rho_n\left(v_{k,j}\right) \int_{\Delta_k} L_{k,j}(Q) \frac{\partial}{\partial \mathbf{n}_Q}\left[\frac{1}{\left|v_i^{(n)} - Q\right|}\right] dS_Q = f\left(v_i^{(n)}\right), \quad i = 1, ..., n_v \tag{4.7}$$

The most time-consuming part of this method is the setup of the coefficient matrix, which involves the accurate and efficient numerical integration of the integrals over the elements. The cost of this is often an order of magnitude more than that of the solution of the linear system. We consider this further in §6.

4.2 Spectral methods

For the equation $(2\pi + \mathcal{K})\,\rho = f$,

$$2\pi\rho(P) + \int_S K(P,Q)\rho(Q)\, dS_Q = f(P), \quad P \in S \tag{4.8}$$

transform the integration region to the unit sphere U. Let S be given by a smooth mapping

$$M : U \underset{onto}{\overset{1-1}{\to}} S$$

with a smooth inverse M^{-1}. Then our BIE becomes

$$2\pi\widehat{\rho}(P) + \int_S \widehat{K}(P,Q)\widehat{\rho}(Q)\, dS_Q = \widehat{f}(P), \quad P \in U$$

$$\widehat{\rho}(P) \equiv \rho(M(P)), \quad \widehat{f}(P) \equiv f(M(P))$$

$$\widehat{K}(P,Q) = K(M(P), M(Q))J_M(Q), \quad Q \in U$$

with Jacobian $J_M(Q)$. Denote the new integral equation by

$$\left(2\pi + \widehat{\mathcal{K}}\right)\widehat{\rho} = \widehat{f} \tag{4.9}$$

Let $\mathcal{X}_N$ denote all spherical polynomials of degree$\leq N$, which has dimension $d_N \equiv (N+1)^2$. The standard orthogonal basis is the set of spherical harmonics

$$\{Y_n^m(\theta,\phi) \mid 0 \leq n \leq N,\ -n \leq m \leq n\}$$

which we assume here are of norm 1. Also denote these by

$$\{\Phi_k \mid 1 \leq k \leq d_N\}$$

Let $\mathcal{P}_N$ denote the orthogonal projection of $L^2(U)$ onto $\mathcal{X}_N$. The Galerkin method is simply

$$\left(2\pi + \mathcal{P}_N\widehat{\mathcal{K}}\right)\widehat{\rho}_N = \mathcal{P}_N\widehat{f}$$

Write the numerical solution as

$$\widehat{\rho}_N(P) = \sum_{k=1}^{d_N} \alpha_k \Phi_k$$

It is obtained by solving

$$2\pi\alpha_i + \sum_{k=1}^{d_N} \alpha_k \left(\widehat{\mathcal{K}}\Phi_k, \Phi_i\right) = (f, \Phi_i), \quad i = 1, ..., d_N$$

The coefficients $\left(\widehat{\mathcal{K}}\Phi_k, \Phi_i\right)$ are double surface integrals; and as such, they must be evaluated carefully to have a Galerkin method of acceptable cost.

For the speed of convergence, say on $C(U)$,

$$\|\rho - \rho_N\|_\infty \leq 2\pi \left\|(2\pi + \mathcal{P}_N\mathcal{K})^{-1}\right\| \|(I - \mathcal{P}_N)\rho\|_\infty$$

For $\rho \in C^{[\ell,\gamma]}(U)$, a Hölder space of ℓ-times differentiable functions with index γ, and for S sufficiently smooth, we have

$$\|\rho - \rho_N\|_\infty \leq O\left(N^{-(\ell+\gamma-0.5)}\right) \tag{4.10}$$

For this, see [5], [6]. For discrete versions of this Galerkin method, see Ganesh et al. [34].

4.3 First kind BIE

There is a well-developed theory of Galerkin methods for those BIE which can be regarded as strongly elliptic pseudo-differential equations on suitable Sobolev spaces. The most studied of such equations is

$$\mathcal{S}\psi(P) \equiv \int_S \frac{\psi(Q)}{|P-Q|} dS_Q = f(P), \quad P \in S$$

In this, we usually assume S is the boundary of a bounded simply-connected $D \subset \mathbb{R}^3$, and we assume S is as smooth as needed. The basic paper in the study of this equation is that of Nedelec [57].

Nedelec shows that $\mathcal{S} : H^{-\frac{1}{2}} \underset{onto}{\overset{1-1}{\to}} H^{\frac{1}{2}}$ and

$$(\mathcal{S}\varphi, \varphi) \geq c \|\varphi\|^2_{-\frac{1}{2}}, \quad \varphi \in H^{-\frac{1}{2}} \tag{4.11}$$

with $(\cdot, \cdot)$ denoting the extension of the usual inner product on $L^2(S)$ to $H^{\frac{1}{2}} \times H^{-\frac{1}{2}}$. Then he applies the standard theory of finite element methods to obtain a convergent finite element Galerkin method.

Using a triangulation $\mathcal{T}_n = \left\{\Delta_k^{(n)}\right\}$ of S, let $\mathcal{X}_n$ denote all functions which are piecewise polynomial of degree $\leq r$ in the parameterization variables. The Galerkin method is defined by

$$(\mathcal{S}\psi_n, \varphi) = (f, \varphi), \quad \varphi \in \mathcal{X}_n$$

Then using Cea's Lemma, the existence and uniqueness of ψ_n can be shown, together with

$$\|\psi - \psi_n\|_{-\frac{1}{2}} \leq c \inf_{\varphi \in \mathcal{X}_n} \|\psi - \varphi\|_{-\frac{1}{2}}$$

By standard arguments, this can then be extended to the more standard error bound

$$\|\psi - \psi_n\|_0 \leq ch^{r+1} \|\psi\|_{r+1}, \quad \psi \in H^{r+1} \tag{4.12}$$

Nedelec also considers the approximation of the boundary by interpolation, of some degree $k \geq 1$. Denoting the resulting solution by $\widehat{\psi}_n$, the convergence results become

$$\left\| \psi - \widehat{\psi}_n \circ \Phi_n^{-1} \right\|_0 \leq c \left[h^{r+1} \|\psi\|_{r+1} + h^k \|\psi\|_0 \right] \tag{4.13}$$

for $\psi \in H^{r+1}(S)$. In this, Φ_n is a special map from the interpolating surface to the original surface S.

These results generalize to more general strongly elliptic pseudo-differential operator equations. For example, see Wendland [87], [88]. For Galerkin methods based on using spherical polynomials as the approximants, see [19].

Unfortunately, almost nothing is understood of collocation methods for solving BIE of the first kind on smooth surfaces. In [10]§9.2.3, we give some numerical experiments for some collocation methods for such BIE.

5 BIE Problems on Piecewise Smooth Surfaces

When S is only piecewise smooth, the most studied BIE is $(2\pi + \mathcal{K})\,\rho = f$,

$$\mathcal{K}\rho(P) = \int_S \rho(Q)\frac{\partial}{\partial \mathbf{n}_Q}\left[\frac{1}{|P-Q|}\right] dS_Q + [2\pi - \Omega(P)]\,\rho(P), \quad P \in S \tag{5.1}$$

which can arise from either the interior Dirichlet problem or the exterior Neumann problem. The first important paper in the study of the numerical analysis of this equation is Wendland [85]. In it, he gives detailed analyses on the mapping properties of single and double layer operators, including showing that $\mathcal{K}$ is a bounded map of $C(S)$ to $C(S)$. The collocation methods studied were those based on piecewise constant and piecewise linear interpolation. More recently, major extensions have been given by J. Elschner, for Galerkin methods, and A. Rathsfeld, for collocation methods. A related discussion is given in Vainikko [83]. We begin with a description of the ideas of Wendland, and then we describe briefly some of the ideas of Elschner and Rathsfeld.

The properties Wendland assumed for the surface are still those used for most error analyses. In essence, S is assumed to be piecewise smooth,

$$S = S_1 \cup \cdots \cup S_J$$

with each S_j the smooth image of a planar polygonal region. In all cases, the interior solid angle $\Omega(P)$ is assumed to satisfy

$$0 < \Omega(P) < 4\pi \tag{5.2}$$

Also, each face S_j is assumed to have a piecewise smooth boundary with no cusps. A more precise description of the properties of each S_j is given in [85]. The surface S is triangulated as in the case of a smooth surface S,

$$S = \bigcup_{k=1}^{n} \Delta_k^{(n)}$$

with the triangulation $\mathcal{T}_n = \left\{\Delta_k^{(n)}\right\}$ 'respecting the edges and corners' of S.

We let $\mathcal{X}_n$ be a space of piecewise polynomial functions of some degree$\leq r$, as in §4. These functions may be restricted to be continuous; but if they are only piecewise continuous, the space for the error analysis must be enlarged from $C(S)$, much as is discussed in [13]. Let $\mathcal{P}_n$ denote the interpolatory projection of $C(S)$ onto $\mathcal{X}_n$, and then the collocation method for solving $(2\pi + \mathcal{K})\,\rho = f$ is given by

$$(2\pi + \mathcal{P}_n\mathcal{K})\,\rho_n = \mathcal{P}_n f \tag{5.3}$$

The analysis of Wendland [85] is based on a critical pair of assumptions:

1. At each point $P \in S$, the tangent cone is convex or its complement is convex;

2. The surface angles and interpolating projections must satisfy

$$\sup_{P\in S} |2\pi - \Omega(P)| \sup_{n\geq 1} \|\mathcal{P}_n\| < 2\pi$$

With piecewise constant interpolation and piecewise linear interpolation (at the vertices of a triangular element), this last assumption is always true. A convergence theory can then be based on the geometric series theorem for $(2\pi + \mathcal{P}_n\mathcal{K})^{-1}$, localized to corners and edges of S. This work also assumes well-behaved solution functions, so that uniform meshes work well in creating accurate approximations of the unknown function ρ. For some extensions of this work to higher order methods, see [11].

5.1 Galerkin methods

When S is a polyhedral Lipschitz boundary, Elschner [30] gives additional theory on the properties of $\mathcal{K}$. He localizes the operator $\mathcal{K}$ to the vertices of S and then shows the localized operator can be regarded as a Mellin convolution operator with an operator valued kernel function. He then uses results for such operators to look at the solvability properties of $(2\pi + \mathcal{K})\,\rho = f$, including obtaining regularity results for the solution ρ. In [31], he gives a general theory for Galerkin boundary element methods. This includes using a graded mesh to compensate for singular behaviour in the solution ρ in the vicinity of edges and corners of S.

Elschner requires that the localized version of $\mathcal{K}$ on each infinite cone generated at each corner of the surface S satisfy a *finite section assumption*. This amounts to something akin to the invertibility of the finite section equation in the planar case, as discussed earlier in §3.1 for (3.29)-(3.30). This finite section assumption is known to be true in either of two cases (assuming the surface is Lipschitz), and these are cases also covered by the analyses of Wendland and his co-authors.

1. Either the interior or the exterior solid tangent cone is convex at each point of the surface; or
2. The surface S is made up of faces that are parallel to one of the three coordinate planes in $\mathbb{R}^3$.

With the finite section assumption, the author constructs a mesh which is graded towards each edge of S, and it is doubly-graded towards the vertices of S. The subspace $\mathcal{X}_n$ is defined as the set of piecewise polynomial functions degree$\leq r$; and there is no continuity restriction on the approximants in $\mathcal{X}_n$. With a properly graded mesh, Elschner is able to prove stability of a Galerkin method and to show an optimal order of convergence:

$$\|\rho - \rho_n\|_{L^2(S)} = O\left(m^{-(r+1)}\right), \quad n = O\left(m^2\right) \tag{5.4}$$

The grading is of the same algebraic type as used for planar problems and described by (3.25).

5.2 Collocation methods

When S is a polyhedral Lipschitz boundary, A. Rathsfeld [66] gives a general theory for collocation boundary element methods for solving $(2\pi + \mathcal{K})\,\rho = f$. This also includes using a graded mesh to compensate for singular behaviour in the solution ρ in the vicinity of edges and corners of S. As with the analysis of Elschner for Galerkin methods, Rathsfeld makes a "finite section" assumption on the localizations of $\mathcal{K}$.

Rathsfeld uses a graded mesh, similar to that in Elschner's Galerkin methods. For some $r \geq 0$, define $\mathcal{X}_n$ as the functions of degree$\leq r$ in the parameterization variables on each face of S. But as in the planar case, this is too general to allow a successful error analysis for collocation methods. In the vicinity of edges and corners of S, special modifications of the approximating space $\mathcal{X}_n$ are needed; and the collocation nodes are chosen with some care as well. With this modified approximating subspace $\mathcal{X}_n$, Rathsfeld proves convergence and stability of the collocation scheme, with

$$\|\rho - \rho_n\|_\infty = O\left(m^{-(r+1)}\right), \quad n = O(m^2) \tag{5.5}$$

In the work of Elschner and Rathsfeld, the authors study the double layer integral operator $\mathcal{K}$ of (5.1) by localizing it to each corner of the polyhedral surface S. Denote by Γ the infinite cone determined by a generic corner P_0, and consider $\mathcal{K} \equiv \mathcal{W}$ defined over Γ. Let Γ intersect a unit sphere centered at P_0, and call this intersection γ. Parameterize γ by $\omega(s)$, $0 \leq s \leq L$. Write a typical point on Γ as $P = r\omega(s)$. For $g \in C_0(\Gamma)$,

$$\mathcal{W}g(r\omega(s)) = \int_0^\infty \left(\mathcal{B}\left(\frac{r}{\tau}\right) g\left(\tau\omega(\cdot)\right)\right)(\omega(s))\frac{d\tau}{\tau} \tag{5.6}$$

with $\mathcal{B}(t)$ an operator on $C(\gamma)$ to $C(\gamma)$:

$$\left(\mathcal{B}(t)h\right)(\omega(s)) = \int_0^L h(\omega(\sigma))\frac{t\mathbf{n}_{\omega(\sigma)} \cdot \omega(s)}{|t\omega(s) - \omega(\sigma)|}d\sigma, \quad t \geq 0$$

The equation $(2\pi + \mathcal{W})\,\rho = f$ is a Mellin convolution equation using the operator-valued Mellin convolution kernel $\mathcal{B}$. The authors use properties for such operators to obtain convergence and stability results for the numerical solution of boundary element methods for solving $(2\pi + \mathcal{W})\,\rho = f$. This is then used in obtaining a solvability theory for the original integral equation $(2\pi + \mathcal{K})\,\rho = f$, including regularity results for the solution ρ.

5.2.1 An alternative framework

Using results on the solvability of the associated boundary value problems for $\Delta u = 0$, combined with the Green's representation formulas, one can obtain results on solvability of some BIE, including regularity results on their solutions.

For such an approach, see von Petersdorff and Stephan [62]. For an introduction to other graded mesh methods and to *hp*-versions of BIE methods, see Stephan [80] and Stephan and Suri [81].

6 BIE on Surfaces - Other Aspects

There are aspects of the numerical solution of BIE on surfaces which become a source of difficulty because of the rapid increase in cost associated with the surface S being two dimensional. These difficulties are related primarily to the iterative solution of the associated linear system and the calculation of the surface integrals in that system.

The linear system associated with the quadratic collocation method for (5.3) is given by

$$\begin{aligned} 2\pi\rho_n\left(v_i^{(n)}\right) + \sum_{k=1}^{n}\sum_{j=1}^{6}\rho_n\left(v_{k,j}\right)\int_{\Delta_k} L_{k,j}(Q)\frac{\partial}{\partial \mathbf{n}_Q}\left[\frac{1}{\left|v_i^{(n)}-Q\right|}\right]dS_Q \\ + \left[2\pi - \Omega(v_i^{(n)})\right]\rho_n\left(v_i^{(n)}\right) = f\left(v_i^{(n)}\right), \quad i=1,...,n_v \end{aligned} \tag{6.1}$$

and we denote it by

$$(2\pi + K_n)\,\mathbf{v}_n = \mathbf{y}_n \tag{6.2}$$

There remains the problem of estimating $\Omega(v_i^{(n)})$. Note that when $\rho \equiv 1$, it follows that $(2\pi + \mathcal{K})\,\rho = 4\pi$. Using this, the solid angle can be approximated by

$$\Omega(v_i^{(n)}) \approx \sum_{k=1}^{n}\sum_{j=1}^{6}\int_{\Delta_k} L_{k,j}(Q)\frac{\partial}{\partial \mathbf{n}_Q}\left[\frac{1}{\left|v_i^{(n)}-Q\right|}\right]dS_Q \tag{6.3}$$

Some discussion of this is given in [11].

Two-grid iteration methods for solving the system (6.2) are examined in [9] , and multigrid methods are examined in Hackbusch [40]. A general examination and comparison of these and other iteration methods is given in Rathsfeld [68]. For problems over smooth surfaces S, there is no difficulty in applying any of these methods; and their error analysis is relatively straightforward. For piecewise smooth surfaces, one must "precondition" the system. To do this, we can "solve exactly" the portion of the linear system corresponding to the nodes on the surface that are at an edge, a corner, or are nearby to such. Examples of this are given in [9] and [68].

For other discussions of iteration methods for solving discretizations of BIE, see Schippers [76], Hebeker [42], Atkinson and Graham [12], Vavasis [84], and Petersdorf and Stephan [64], [63], Kelley [46], Kelley and Xue [47]. An interesting discussion is given in Edelman [28] of the significance of BIE within the numerical linear algebra of dense linear systems.

6.1 Numerical integration

In (6.1), the integrals

$$\int_{\Delta_k} L_{k,j}(Q) \frac{\partial}{\partial \mathbf{n}_Q} \left[\frac{1}{\left| v_i^{(n)} - Q \right|} \right] dS_Q \tag{6.4}$$

must be evaluated numerically in all but the simplest cases. The functions $L_{k,j}(Q)$ are "Lagrange basis functions" for quadratic interpolation over Δ_k. [In the case of piecewise constant interpolation over polyhedral surfaces S, these integrals can be done analytically.] These are the most time consuming part of most boundary element codes. We divide the evaluation of them into two cases.

Case (a). Let $v_i^{(n)} \in \Delta_k$. Then the integrand in (6.4) is singular. This appears to be a difficult integral, but there is a simple fix, called the "Duffy transformation". Consider the integral

$$I = \int_\sigma g(s,t)\, d\sigma$$

with σ the unit simplex in the plane:

$$\sigma = \{(s,t) \mid 0 \le s, t, s+t \le 1\}$$

Further assume the function has a point singularity at the origin. Introduce the change of variables wh

$$s = (1-y)\, x, \quad t = yx, \quad 0 \le x, y \le 1$$

$$I = \int_0^1 \int_0^1 xg\left((1-y)\, x, yx\right)\, dx\, dy$$

When the original integral arises from a collocation integral of either single or double layer type, then the new integrand contains no singularity, including in its derivatives. Then simply apply Gaussian quadrature to both integrals, usually with only a few nodes (e.g. ≈ 5) in each direction. For a complete discussion and extension to other singular integrals, see Schwab and Wendland [78].

Case (b). Let $v_i^{(n)} \notin \Delta_k$. Then the integrals (6.4) are nonsingular. However, they vary from being almost singular, when $v_i^{(n)}$ is near Δ_k, to having very well-behaved integrands, when $v_i^{(n)}$ is distant from Δ_k. Note that of the $O(n^2)$ integrals in the collocation matrix, only $O(n)$ are of singular type; and therefore, most of the cost of setting up the matrix for (6.1) is in these nonsingular integrals.

An efficient quadrature method with an exponential rate of convergence is defined and analyzed in Schwab [77]. We have used the following somewhat simpler schema in [8], and it is reasonably efficient in most cases. Introduce a parameter μ to indicate a number of level of subdivisions of Δ_k for a composite quadrature scheme. For the basic quadrature scheme, we use the 7-point rule T2:5-1, of degree of precision 5, from Stroud [82].

Table 1. Max errors in ρ_n

n	$\mu = 0$	$\mu = 1$	$\mu = 2$	$\mu = 3$
8	3.63E−2	3.30E−2	3.31E−2	3.31E−2
32	3.78E−3	3.79E−3	3.78E−3	3.78E−3
128	5.49E−4	2.92E−4	2.91E−4	2.91E−4
512	2.62E−4	1.88E−5	1.87E−5	1.87E−5

Table 2. Timings for "graded" mesh quadrature (sec)

n	$\mu = 0$	$\mu = 1$	$\mu = 2$	$\mu = 3$	*Iteration*
8	0.85	0.85	1.9	4.0	
32	3.9	6.1	22.0	59.5	< 0.01
128	31.2	42.7	123	439	0.13
512	388	424	754	2240	2.05

1. If $0 < \text{dist}\,(v_i, \Delta_k) \leq h$, use μ levels of subdivision of Δ_k and apply a given quadrature scheme to each of the resulting 4^μ subelements.

2. If $h < \text{dist}\,(v_i, \Delta_k) \leq 2h$, use $\mu - 1$ levels of subdivision of Δ_k and apply a given quadrature scheme to each of the resulting $4^{\mu-1}$ subelements.

3. Continue this until no subdivision of Δ_k takes place, so that the quadrature scheme is applied directly to Δ_k.

As n increases to $4n$ to $16n$, it seems best to increase μ to $\mu + 1$.

We give an example of solving $(2\pi + \mathcal{K})\,\rho = f$ for an exterior Neumann problem on an ellipsoid. Table 1 contains the errors for varying μ, Table 2 contains timings for the composite scheme described above, and Table 3 contains timings when μ levels of subdivisions are used over all triangular elements. The errors in Table 1 are valid for both Tables 2 and 3; but the timings are far larger for Table 3. The timings are for a Hewlett-Packard HP-720 workstation.

For other discussions of numerical integration over boundary elements for solving integral equations on surfaces, see Chien [21] and Sauter and Schwab [75].

Table 3. Timings using a uniform μ levels of subdivision (sec)

n	$\mu = 0$	$\mu = 1$	$\mu = 2$	$\mu = 3$
8	0.85	0.90	1.6	4.3
32	3.9	7.2	20.8	76.0
128	31.2	89.3	317	1230
512	388	1350	5030	19800

6.2 Fast matrix-vector calculations

Consider a discretization of the single layer operator:

$$\mathcal{S}\rho\left(v_i^{(n)}\right) \approx \sum_{k=1}^{n}\sum_{j=1}^{6}\rho\left(v_{k,j}^{(n)}\right)\int_{\Delta_k}\frac{L_{k,j}(Q)}{\left|v_i^{(n)}-Q\right|}dS_Q \tag{6.5}$$

for $i = 1, ..., n_v$, using a piecewise quadratic interpolation of ρ. Denote the vector of values $\rho\left(v_i^{(n)}\right)$ by $\boldsymbol{\rho}_n$, and let $A\boldsymbol{\rho}_n$ denote the matrix multiplication inherent in the above approximation. Then all iteration methods for solving a linear system involving A, say $A\mathbf{v} = \mathbf{y}$, will involve repeated multiplications $A\mathbf{w}$ for various vectors $\mathbf{w}$. To evaluate A explicitly requires $O\left(n^2\right)$ operations or more, and to evaluate $A\mathbf{w}$ requires $2n^2$ operations. Reducing the cost of computing $A\mathbf{w}$ is of little consequence if we must still evaluate A explicitly.

In the past ten years, several approaches have been used to speed the calculation of $A\mathbf{w}$, reducing it from $O(n^2)$ to $O\left(n\log^d n\right)$ operations, for some small integer d. The three main types of methods are as follows.

- *Fast multipole methods*
- *Clustering methods*
- *Wavelet compression methods*

With all of these methods, we produce a vector

$$\mathbf{u} \approx A\mathbf{w} \tag{6.6}$$

within an acceptably small error and at a cost of $O\left(n\log^d n\right)$ operations.

6.2.1 Fast multipole methods

These were developed by L. Greengard and V. Rokhlin in a series of papers, and they have become widely used in the past few years. For papers of Greengard and Rokhlin, see [37], [38], and [71]; and for a paper dealing more explicitly with BIE approximations, see Nabors et al. [55].

The problem as originally posed by Greengard and Rokhlin is to calculate the potential at points $\Upsilon_1, ..., \Upsilon_p$ as determined by charges of strengths $q_1, ..., q_k$ at k points $Q_1, ..., Q_k$:

$$\Phi(\Upsilon_i) = \sum_{j=1}^{k}\frac{q_j}{|\Upsilon_i - Q_j|}, \quad i = 1, ..., p \tag{6.7}$$

Often $\{\Upsilon_i\} = \{Q_j\}$, with the proviso of omitting the term in the sum for $j = i$. The size of k can be quite large, say $k = 10,000$. The methods used depend on

the use of spherical harmonics expansions and on clusterings of the points $\{Q_j\}$ according to their distance from Υ_i.

Writing points using their spherical coordinates, let $\Upsilon = (r, \theta, \phi)$ and $Q = (\rho, \alpha, \beta)$. Then for $\rho < r$,

$$\frac{1}{|\Upsilon - Q|} = \sum_{n=0}^{\infty} P_n(\cos\gamma) \left(\frac{\rho}{r}\right)^n \tag{6.8}$$

with γ the angle between Υ and Q, and with $P_n(u)$ the Legendre polynomial of degree n. To give a flavor to the method's techniques, we give one theorem from the development of the method.

Theorem. *Suppose that k charges of strengths $q_1, ..., q_k$ are located at the points $Q_i = (\rho_i, \alpha_i, \beta_i)$, $i = 1, ..., k$, and assume that $\rho_i < a$, a given number. Then for any $\Upsilon = (r, \theta, \phi)$ with $r > a$, the potential $\Phi(\Upsilon)$ is given by*

$$\Phi(\Upsilon) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{M_n^m}{r^{n+1}} Y_n^m(\theta, \phi) \tag{6.9}$$

$$M_n^m = \sum_{i=1}^{k} q_i \rho_i^n Y_n^{-m}(\alpha_i, \beta_i)$$

For any $\ell \geq 1$,

$$\left| \Phi(\Upsilon) - \sum_{n=0}^{\ell} \sum_{m=-n}^{n} \frac{M_n^m}{r^{n+1}} Y_n^m(\theta, \phi) \right| \leq \frac{A}{r - a} \left(\frac{a}{r}\right)^{\ell+1} \tag{6.10}$$

$$A = \sum_{i=1}^{k} |q_i|$$

The functions $\{Y_n^m(\theta, \phi)\}$ are spherical harmonics of degree n and order m. The formula (6.9) is called the *multipole expansion* of $\Phi(\Upsilon)$. The *fast multipole method* is based on creating graded subdivisions of space and to combine this with approximations as in (6.10) so as to calculate an estimate of $\Phi(\Upsilon)$ at p points Υ in much fewer than $O(pk)$ operations. An excellent introduction to this is Greengard [37]Chap. 3.

6.2.2 Clustering methods

Assume the creation of a sequence of triangulations of S, much as in §4, §5; and let each triangulation be a refinement of the preceding one. Call these $\mathcal{T}_0$, $\mathcal{T}_1$, $\mathcal{T}_2$, ...,$\mathcal{T}_I$, with $\mathcal{T}_0 = \{\Delta_k \mid 1 \leq k \leq n\}$ the finest mesh, in keeping with the notation in Hackbusch and Nowak [41]. For simplicity here, also assume S is polyhedral. We restrict our interest to the single layer potential approximation of (6.5).

Note that as the distance from v_i is increased, the integrand becomes smoother. For each element τ in some $\mathcal{T}_l$, with $P \notin \tau$, we expand the simple layer in a Taylor series about the "centroid" Q_τ of τ:

$$\frac{1}{|P-Q|} \approx \sum_{l \in I_m} \kappa_l(P;Q_\tau)\Phi_l(Q)$$

The degree of the Taylor series is m, and I_m denotes an index set for the expansion. The accuracy of this expansion will depend on the distance of P from Q_τ, the size of m, and the size of τ. We call τ a "cluster" of elements from the finest level $\mathcal{T}_0$.

Note that $\Phi_l(Q)$ is independent of P and Q_τ, which is critically important in obtaining a low operations count. The functions $\Phi_l(Q)$ are to be simple monomials in the components of Q, thus making them simple to integrate over elements Δ_k of $\mathcal{T}_0$ and thence over clusters from the coarser triangulations $\mathcal{T}_l$, some $l > 0$.

Let $\rho_n(Q)$ denote the piecewise quadratic

$$\rho_n(Q) = \sum_{j=1}^{6} \rho_n(v_{k,j})\, L_{k,j}(Q), \quad Q \in \Delta_k, \quad 1 \le k \le n$$

Let the cluster τ consist of a set of elements from $\mathcal{T}_0$, say $\Delta_1, ..., \Delta_p$. Then

$$\int_\tau \frac{\rho_n(Q)\, dS_Q}{|P-Q|} \approx \sum_{j=1}^{p} \kappa_l(P;Q_\tau) \sum_{l \in I_m} \int_{\Delta_j} \rho_n(Q)\, \Phi_l(Q)\, dS_Q$$

With preprocessing, we can evaluate all of the integrals

$$\int_{\Delta_j} \rho_n(Q)\, \Phi_l(Q)\, dS_Q$$

in $O(n)$ operations, allowing both l and j to vary. Because both $\rho_n(Q)$ and $\Phi_l(Q)$ are polynomials in Q, these integrals are straightforward to evaluate.

For each point $P = v_i$, we do an initial preprocessing to produce the needed clustering of elements of $\mathcal{T}_0$, to have

$$S = [\tau_1 \cup \cdots \cup \tau_c] \cup \left[\Delta_{i_1} \cup \cdots \cup \Delta_{i_q}\right]$$

with $\{\tau_j\}$ elements from various $\mathcal{T}_k$ with $k > 0$. Then we have

$$\int_S \frac{\rho_n(Q)\, dS_Q}{|P-Q|} = \sum_{j=1}^{c} \int_{\tau_j} \frac{\rho_n(Q)\, dS_Q}{|P-Q|} + \sum_{j=1}^{q} \int_{\Delta_{i_j}} \frac{\rho_n(Q)\, dS_Q}{|P-Q|}$$

With a careful attention to detail and with preprocessing where possible, it is possible to evaluate an approximation of $[\mathcal{S}\rho_n(v_i)]$ in $O\left(n \log^3 n\right)$ operations. The main reference to such *clustering methods* is Hackbusch and Nowak [41].

6.2.3 Wavelet compression methods

The main idea is to use a wavelet basis such that the most of the elements of the collocation and Galerkin matrix will be small enough to be neglected. Then the cost of evaluating $[\mathcal{S}\rho_n(v_i)]$ will be only slightly larger than $O(n)$ operations. For some papers in this very recent topic, see Alpert et al. [1], Dahmen et al. [27], Petersdorff and Schwab [60], [61], and Micchelli and Xu [54].

Bibliography

1. B. Alpert, G. Beylkin, R. Coifman, and V. Rokhlin (1993) Wavelets for the fast solution of second-kind integral equations, *SIAM J. Sci. Comp.* **14**, pp. 159-184.

2. S. Amini and N. Maines (1995) Regularisation of strongly singular integrals in boundary integral equations, Tech. Rep. MCS-95-7, Univ. of Salford, United Kingdom.

3. P. Anselone and I. Sloan (1985) Integral equations on the half line, *J. Integral Equations* **9** (Suppl), pp. 3-23.

4. D. Arnold and W. Wendland (1983) On the asymptotic convergence of collocation methods, *Numer. Math.* **41**, pp. 349-381.

5. K. Atkinson (1982) The numerical solution of Laplace's equation in three dimensions, *SIAM J. Num. Anal.* **19**, pp. 263-274.

6. K. Atkinson (1985) Algorithm 629: An integral equation program for Laplace's equation in three dimensions, *ACM Trans. on Math. Soft.* **11**, pp. 85-96.

7. K. Atkinson (1988) A discrete Galerkin method for first kind integral equations with a logarithmic kernel, *Journal of Integral Eqns & Applic.* **1**, pp. 343-363.

8. K. Atkinson (1993) User's Guide to a Boundary Element Package for Solving Integral Equations on Piecewise Smooth Surfaces, *Reports on Computational Mathematics* #43, Dept of Mathematics, University of Iowa, Iowa City.

9. K. Atkinson (1994) Two-grid iteration methods for linear integral equations of the second kind on piecewise smooth surfaces in $\mathbf{R}^3$, *SIAM J. Scientific Computing* **15**, pp. 1083-1104.

10. K. Atkinson (1997) *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge Univ. Press, to appear.

11. K. Atkinson and D. Chien (1995) Piecewise polynomial collocation for boundary integral equations, *SIAM J. Scientific Computing* **16,** pp. 651-681.

12. K. Atkinson and I. Graham (1990) Iterative variants of the Nyström method for second kind boundary integral equations, *SIAM J. Scientific and Stat. Comp.* **13,** pp. 694-722.

13. K. Atkinson, I. Graham, and I. Sloan (1983) Piecewise continuous collocation for integral equations, *SIAM J. Num. Anal.* **20**, pp. 172-186.

14. K. Atkinson and F. de Hoog (1984) The numerical solution of Laplace's equation on a wedge, *JIMA: Numerical Analysis* **4**, pp. 19-41.

15. K. Atkinson and Ian Sloan (1991) The numerical solution of first kind logarithmic-kernel integral equations on smooth open arcs, *Math. of Computation* **56**, pp. 119-139.

16. J. Blue (1978) Boundary integral solutions of Laplace's equation, *The Bell System Technical Journal* **57**, pp. 2797-2822.

17. G. Chandler (1984) Galerkin's method for boundary integral equations on polygonal domains, *J. Australian Math. Soc., Series B,* **26**, pp. 1-13.

18. G. Chandler and I. Graham (1988) Product integration-collocation methods for non-compact integral operator equations, *Math. Comp.* **50**, pp. 125-138.

19. Y. Chen and K. Atkinson (1994) Solving a single layer integral equation on surfaces in $\mathbf{R}^3$, *SIAM J. Num. Anal., Reports on Computational Mathematics* #51, Dept of Mathematics, University of Iowa.

20. D. Chien (1992) Piecewise polynomial collocation for integral equations with a smooth kernel on surfaces in three dimensions, *J. of Int. Eqns. and Applic.* **5**, pp. 315-344.

21. D. Chien (1995) Numerical evaluation of surface integrals in three dimensions, *Mathematics of Comp.* **64**, pp. 727-743.

22. D. Chien and K. Atkinson (1995) The numerical solution of a hypersingular planar boundary integral equation, *IMA J. Num. Anal.*, to appear.

23. M. Costabel (1987) Principles of boundary element methods, *Computer Physics Reports* **6**, pp. 243-274.

24. M. Costabel and E. Stephan (1985) Boundary integral equations for mixed boundary value problems in polygonal domains and Galerkin approximation, *Mathematical Models and Methods in Mechanics, Banach Center Publications* **15**, pp. 175-251.

25. M. Costabel and E. Stephan (1987) On the convergence of collocation methods for boundary integral equations on polygons, *Math. Comp.* **49**, pp. 461-478.

26. M. Costabel and W. Wendland (1986) Strong ellipticity of boundary integral operators, *J. Reine Angewandte Math.* **372**, pp. 34-63.

27. W. Dahmen, B. Kleemann, S. Prößdorf, and R. Schneider (1994) A multiscale method for the double layer potential equation on a polyhedron, in *Advances in Computational Mathematics*, ed. by H. Djkshit and C. Micchelli, World Scientific Pub. Co., pp. 15-57.

28. A. Edelman (1993) Large dense numerical linear algebra in 1993: The parallel computing influence, *J. Supercomputing Applic.* **7**, pp. 113-118.

29. J. Elschner (1990) On spline approximation for a class of non-compact integral equations, *Math. Nachr.* **146**, pp. 271-321.

30. J. Elschner (1992) The double layer potential operator over polyhedral domains I: Solvability in weighted Sobolev spaces, *Applic. Analysis* **45**, pp. 117-134.

31. J. Elschner (1992) The double layer potential operator over polyhedral domains II: Spline Galerkin methods. *Math. Methods Appl. Sci.* **15,** pp. 23-37.

32. J. Elschner and I. Graham (1995) An optimal order collocation method for first kind boundary integral equations on polygons, *Numerische Math.* **70**, pp. 1-31.

33. J. Elschner and E. Stephan (1995) A discrete collocation method for Symm's integral equation on curves with corners, *J. Comp. Appl. Math.*, to appear.

34. M. Ganesh, I. Graham, and J. Sivaloganathan (1993) A pseudospectral 3D boundary integral method applied to a nonlinear model problem from finite elasticity, *SIAM J. Numerical Analysis* **31**, pp. 1378-1414.

35. I. Graham and G. Chandler (1988) High order methods for linear functionals of solutions of second kind integral equations, *SIAM J. Numer. Anal.* **25**, pp. 1118-1137.

36. J. Giroire and J. Nedelec (1978) Numerical solution of an exterior Neumann problem using a double layer potential, *Math. Comp.* **32**, pp. 973-990.

37. L. Greengard (1988) *The Rapid Evaluation of Potential Fields in Particle Systems*, The MIT Press, Cambridge, Massachusetts.

38. L. Greengard and V. Rokhlin (1987) A fast algorithm for particle simulation, *J. Comput. Phys.* **73**, pp. 325-348.

39. N. Günter (1967) *Potential Theory*, Ungar Pub., New York.

40. W. Hackbusch (1994) *Integral Equations: Theory and Numerical Treatment*, Birkhäuser Verlag, Basel.

41. W. Hackbusch and Z. Nowak (1989) On the fast matrix multiplication in the boundary element method by panel clustering, *Numer. Math.* **54**, pp. 463-491.

42. F. Hebeker (1988) On the numerical treatment of viscous flows against bodies with corners and edges by boundary element and multi-grid methods, *Numer. Math.* **52**, pp. 81-99.

43. J. Hess (1990) Panel methods in computational fluid dynamics, *Annual Rev. Fluid Mechanics* **22**, pp. 255-274.

44. M. Jaswon and G. Symm (1977) *Integral Equation Methods in Potential Theory and Elastostatics*, Academic Press.

45. Y.-M. Jeon (1993) A Nyström method for boundary integral equations on domains with a piecewise smooth boundary, *J. Int. Eqns. & Applic.* **5**, pp. 221-242.

46. C. T. Kelley (1995) A fast multilevel algorithm for integral equations, *SIAM J. on Numer. Anal.* **32**, pp. 501-513.

47. C. T. Kelley and Z. Xue (1996) GMRES and integral operators, *SIAM J. Sci. Comp.* **17**, pp. 217-226.

48. R. Kress(1989) *Linear Integral Equations*, Springer-Verlag.

49. R. Kress(1990) A Nyström method for boundary integral equations in domains with corners, *Numer. Math.* **58,** pp. 145-161.

50. R. Kress (1995) On the numerical solution of a hypersingular integral equation in scattering theory, *J. Comp. & Appl. Maths.* **61**, pp. 345-360.

51. W. McLean (1986) A spectral Galerkin method for boundary integral equations, *Math. Comp.* **47**, pp. 597-607.

52. W. McLean (1995) Fully-discrete collocation methods for an integral equation of the first kind, *J. Int. Eqns. & Applic.* **6**, pp. 537-572.

53. W. McLean, S. Prößdorf, and W. Wendland (1993) A fully-discrete trigonometric collocation method, *J. Int. Eqns. & Applic.* **5**, pp. 103-129.

54. C. Micchelli and Y. Xu (1995) Weakly singular Fredholm integral equations I: Singularity preserving wavelet-Galerkin methods, *Approximation Theory VIII*, ed. by C. Chui and L. Schumaker, World Sci. Pub.

55. K. Nabors, F. Korsmeyer, F. Leighton, and J. White (1994) Preconditioned, adaptive, multipole-accelerated iterative methods for three-dimensional first-kind integral equations of potential theory, *SIAM J. Sci. Comp.* **15**, pp. 713-735.

56. R. Natarajan and D. Krishnaswamy (1995) A case study in parallel scientific computing: The boundary element method on a distributed memory multicomputer, *Proceedings of Supercomputing 95*, Assoc. of Comp. Mach.

57. J. Nedelec (1976) Curved finite element methods for the solution of singular integral equations on surfaces in $\mathbf{R}^3$, *Computer Methods in Appl. Mechanics and Engin.* **8**, pp. 61-80.

58. J. Newman (1992) Panel methods in marine hydrodynamics, *Proc. Conf. Eleventh Australasian Fluid Mechanics - 1992.*

59. B. Noble (1977) The numerical solution of integral equations, in *The State of the Art in Numerical Analysis*, ed. by D. Jacobs, Academic Press, 1977, pp. 915-966.

60. T. von Petersdorff and C. Schwab (1994) Wavelet approximations for first kind boundary integral equations on polygons, *Numer. Math.*, to appear.

61. T. von Petersdorff and C. Schwab (1995) Fully discrete multiscale Galerkin BEM, in *Multiresolution Analysis and Partial Differential Equations*, ed. by W. Dahmen, P. Kurdila, and P. Oswald, Academic Press, New York.

62. T. von Petersdorff and E. Stephan (1990) Regularity of mixed boundary value problems in $\mathbf{R}^3$ and boundary element methods on graded meshes, *Math. Methods in the Appl. Sci.* **12**, pp. 229-249.

63. T. von Petersdorff and E. Stephan (1990) On the convergence of the multigrid method for a hypersingular integral equation of the first kind, *Numer. Math.* **57,** pp. 379-391.

64. T. von Petersdorff and E. Stephan (1992) Multigrid solvers and preconditioners for first kind integral equations, *Numerical Methods for Partial Diff. Eqns.* **8**, pp. 443-450.

65. J. Radon (1919) Über die Randwertaufgaben beim logarithmischen Potential, *Sitzungsberichte der Akademie der Wissenschafter Wien* **128 Abt. IIa**, pp. 1123-1167.

66. A. Rathsfeld (1992) The invertibility of the double layer potential operator in the space of continuous functions defined on a polyhedron: The panel method, *Appl. Analysis* **45**, pp. 135-177.

67. A. Rathsfeld (1992) Iterative solution of linear systems arising from the Nyström method for the double layer potential equation over curves with corners, *Math. Methods Appl. Sci.* **15**, pp. 443-455.

68. A. Rathsfeld (1995) Nyström's method and iterative solvers for the solution of the double layer potential equation over polyhedral boundaries, *SIAM J. Num. Anal.* **32**, pp. 924-951.

69. A. Rathsfeld, R. Kieser, and B. Kleemann (1992) On a full discretization scheme for a hypersingular boundary integral equation over smooth curves, *Z. Anal. Anwendungen* **11**, pp. 385-396.

70. F. Rizzo, D. Shippy, and M. Rezayat (1985) A boundary integral equation method for radiation and scattering of elastic waves in three dimensions, *International Journal for Numerical Methods in Engineering* **21**, pp. 115-129.

71. V. Rokhlin (1983) Rapid solution of integral equations of classical potential theory, *J. Comput.Phys.* **60**, pp. 187-207.

72. T. Rudolphi, G. Krishnasamy, L. Schmerr, and F. Rizzo (1988) On the use of strongly singular integral equations for crack problems, in *Proc. 10^{th} International Conference on Boundary Elements,* Southampton, 1988.

73. J. Saranen (1988) The convergence of even degree spline collocation solution for potential problems in the plane, *Numer. Math.* **53,** pp. 499-512.

74. J. Saranen and G. Vainikko (1994) Trigonometric collocation methods with product-integration for boundary integral equations on closed curves, *SIAM J. Num. Anal.* **33**, pp. 1577-1596.

75. S. Sauter and C. Schwab (1996) Quadrature for *hp*-Galerkin BEM in $\mathbf{R}^3$, Res. Rep. #96-02, Seminar für Angew. Math., Eid. Tech. Hoch., Zurich.

76. H. Schippers (1985) Multi-grid methods for boundary integral equations, *Numer. Math.* **46**, pp.351-363.

77. C. Schwab (1994) Variable order composite quadrature of singular and nearly singular integrals, *Computing* **53**, pp. 173-194.

78. C. Schwab and W. Wendland (1992) On numerical cubatures of singular surface integrals in boundary element methods, *Numer. Math.* **62**, pp. 343-369.

79. I. Sloan (1992) Error analysis of boundary integral methods, *Acta Numerica* **1**, pp. 287-339.

80. E. Stephan (1994) The h-p boundary element method for solving 2- and 3-dimensional problems - especially in the presence of singularities and adaptive approaches, preprint.

81. E. Stephan and M. Suri (1989) On the convergence of the *p*-version of the boundary element Galerkin method, *Math. Comp.* **52,** pp. 32-48.

82. A. Stroud (1971) *Approximate Calculation of Multiple Integrals*, Prentice-Hall, New Jersey.

83. G. Vainikko (1993) *Multidimensional Weakly Singular Integral Equations,* Lecture Notes in Mathematics #1549, Springer-Verlag, Berlin.

84. S. Vavasis (1992) Preconditioning for boundary integral equations, *SIAM J. Matrix Analysis* **13**, pp. 905-925.

85. W. Wendland (1968) Die Behandlung von Randwertaufgaben im $\mathbf{R}^3$ mit Hilfe von Einfach und Doppelschichtpotentialen, *Numerische Math.* **11**, pp. 380-404.

86. W. Wendland (1985) Asymptotic accuracy and convergence for point collocation methods, in *Topics in Boundary Element Research, Vol. II: Time-Dependent and Vibration Problems*, Springer-Verlag, Berlin, pp. 230-257.

87. W. Wendland (1987) Strongly elliptic boundary integral equations, in *The State of the Art in Numerical Analysis*, ed. by A. Iserles and J. Powell, Clarendon Press, 1987, pp. 511-562.

88. W. Wendland (1990) Boundary element methods for elliptic problems, in *Mathematical Theory of Finite and Boundary Element Methods*, by A. Schatz, V. Thomée, and W. Wendland, Birkhäuser, Boston, pp. 219-276.

89. Y. Yan (1990) The collocation method for first kind boundary integral equations on polygonal domains, *Math. of Comp.* **54**, pp. 139-154.

90. Y. Yan and I. Sloan (1988) On integral equations of the first kind with logarithmic kernels, *J. Integral Eqns & Applics,* **1**, pp. 517-548.