NUMERICAL METHODS FOR THE RADIOSITY EQUATION AND RELATED PROBLEMS

by

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An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
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ABSTRACT

In this work, we present numerical methods for the solution of Fredholm integral equations of the second type, for smooth and piecewise smooth surfaces. We use a collocation method based on piecewise polynomial interpolation of the solution. We consider only collocation methods for which the collocation nodes are interior to each triangular face.

In Chapter II we give the general framework for collocation methods based on interpolation. We show that interpolation of degree r of the solution leads to an error in the collocation method of $O(h^{r+1})$, where h is the mesh size of the triangulation, and so collocation methods of any given order can be developed.

In Chapter III we discuss superconvergent methods, as particular cases of the methods introduced in Chapter II. The radiosity equation is introduced, along with some of its properties. Next we discuss two superconvergent collocation methods based on piecewise quadratic interpolation, for the radiosity equation, followed by numerical examples. We conclude this chapter with giving generalized superconvergent methods based on interpolation of any degree r, considering separately the case where r is odd and the case where r is even.

In the following chapter the ideas described earlier are used for finding numerical solutions of the exterior Neumann problem, since in solving this problem we encounter integral equations whose properties are very similar to the ones of the radiosity equation. Considering collocation methods that use only interior nodes is especially useful in solving this problem. We describe a collocation method based on interpolation of the solution, for solving the integral equation derived from the exterior Neumann problem, giving numerical examples for the case of piecewise constant interpolation of the solution (centroid rule).

In the concluding chapter, we draw some important and interesting conclusions as well as discuss some possible ideas for future work in this area.

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CERTIFICATE OF APPROVAL

PH.D. THESIS

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CHAPTER I
INTRODUCTION

Integral equations are an important subject within applied mathematics. They are used as mathematical models for many and diverse physical situations. Also, integral equations occur as reformulations of other mathematical problems, such as Laplace's equation.

In this work, numerical methods are presented and analyzed for the solution of Fredholm integral equations of the second kind of the form

$$u(P) - \int_S u(Q)K(P,Q)dS_Q = f(P), \quad P \in S \tag{1.1}$$

for a smooth or piecewise smooth surface S. In operator form

$$(\mathcal{I} - \mathcal{K})u = f \tag{1.2}$$

We investigate a certain type of collocation method based on piecewise polynomial interpolation of the solution. The general idea of the numerical method is the following: Begin by triangulating S and then approximate the unknown function $u(P)$ by functions which are *piecewise polynomial* in a parametrization over the triangulation of S. Then the numerical solution is found by *collocation,* meaning that the approximate form of the solution is substituted into the equation and then the equation is forced to be true at the *collocation node points,* leading to a system of linear equations for determining the approximate solution.

When the surface S is smooth and the operator $\mathcal{K}$ is compact on $C(S)$, it is relatively easy to do an error analysis of collocation. However, in most applications the surface will only be piecewise smooth, and in this case the analysis of collocation is often more difficult. Also, a lack of smoothness of the kernel function $K(P,Q)$ may imply that $\mathcal{K}$ is no longer compact, nor that any power of it is compact.

Another difficulty in the case where S is not smooth arises in the evaluation of the unit normal to the surface at points located on an edge or at a corner of S. Also, there is a problem in defining the normal at the collocation points which are common to more than one triangular face Δ_k, even for smooth surfaces. To avoid all these problems, we consider only collocation methods for which the collocation points are interior to each triangular face. This also greatly simplifies the programming.

For some approximations of the solution, the function space needs to be changed, namely $C(S)$ must be enlarged to include piecewise polynomial approximants. One way of doing this is by using the space $L^\infty(S)$, the set of all essentially bounded and Lebesgue measurable functions on S, with the essential supremum norm $\|\cdot\|_\infty$.

A general framework and error analysis of these methods is given in Chapter II. We recall the basic results in interpolation and collocation theory, which we need to show that interpolation approximations of degree r of the solution lead to an error in the collocation method of $O(h^{r+1})$, where h is the mesh size of the triangulation. In certain cases, which are described in detail, the error can be improved. In the last part of this chapter, we give a procedure for producing a collocation method for the equation (1.1) of any desired order.

In Chapter III within the framework described in Chapter II, we investigate special collocation methods, arising from certain choices of the node points and certain types of triangulation, which lead to superconvergence for some collocation solutions u_n at the collocation nodes. In the second part of this chapter, we describe such methods for the *radiosity equation. Radiosity* is a method of describing illumination based on a detailed analysis of light reflections off diffuse surfaces. It is typically used to render images of the interior of buildings, and it can achieve extremely photo-realistic results for scenes that are comprised of diffuse reflecting surfaces. In computer graphics, the computation of lighting can be done via radiosity. The radiosity equation

$$u(P) - \frac{\rho(P)}{\pi} \int_S u(Q)G(P,Q)V(P,Q)dS_Q = E(P), \quad P \in S \tag{1.3}$$

and some of its properties are discussed in more detail in Section 3.2. An introduction to the use of equation (1.3) in computer graphics is given in Cohen and Wallace[9], along with methods for its numerical solution. In the past, the Galerkin method has been primarily used to obtain a numerical solution of this equation, with piecewise constant functions as the approximations. In Atkinson and Chandler[5] and Atkinson and Chien[7], the authors investigate different collocation methods for this equation, using piecewise constant (in Atkinson and Chien[7]) or piecewise linear (in Atkinson and Chandler[5]) functions. The methods described in Section 3.3 are obtained using the same approach, with collocation based on piecewise quadratic interpolation of the solution. Numerical examples are given for these cases. In the last part of this chapter, we discuss procedures for developing superconvergent methods for the radiosity equation, based on interpolation of any degree r of the solution. Two cases must be differentiated: the case where r is an odd number and the case where r is an even number. The approaches and the results in the error estimates are different for the two cases.

In the next chapter the ideas described earlier are used for finding numerical solutions of boundary integral equation reformulations of Laplace's equation

$$\Delta u = 0 \tag{1.4}$$

on regions in $\mathbb{R}^3$. In particular, the exterior Neumann problem is studied from this perspective. This is of interest because, in solving the interior and exterior Neumann problems using a single layer potential, we are faced with the problem of evaluating the kernel function for field points that are on the edges of the triangular faces, where this kernel is bad behaved at such points. Considering interior collocation nodes solves this problem. The ideas described in this chapter apply very well to the (interior or exterior) Dirichlet problem also, but because of the existing theory for this problem, it is not of such great interest for this case.

In the concluding chapter, we draw some important and interesting conclusions as well as discuss some ideas for future work in this area.

CHAPTER II
PRELIMINARIES FOR COLLOCATION METHODS

2.1 Preliminaries

Consider the Fredholm integral equation of the second kind

$$u(P) - \int_S u(Q)K(P,Q)dS_Q = f(P), \quad P \in S \tag{2.1}$$

with S a bounded set in $\mathbb{R}^3$. The *kernel function* $K(P,Q)$ is assumed to be absolutely integrable, and it is assumed to satisfy other properties which are sufficient to imply the "Fredholm Alternative Theorem" (see Atkinson[4, Theorem 1.3.1]). The problem to be solved is: Given K and f, find the function u satisfying equation (2.1). Other properties that u may need to satisfy are problem and method dependent.

In this chapter we describe the general framework for collocation methods based on piecewise polynomial interpolation of the solution. We consider a certain type of collocation method. Error formulas and rates of convergence are given. In the end we describe a procedure for developing a collocation method of any desired order.

2.1.1 Interpolation Over the Unit Simplex

We begin by giving some background material needed later. These are well-known results and can be found in more detail in Atkinson[4]. Let σ denote the unit simplex, $\sigma = \{(s,t) \mid 0 \leq s, t, s+t \leq 1\}$. Introduce $u = 1 - s - t$. The coordinates (s,t,u) are called *barycentric coordinates* of a point.

Let $g(s,t)$ be a continuous function on σ. We will approximate g by a polynomial interpolant $p(s,t)$ of degree r, for some $r \geq 0$.

$$p_r(s,t) = \sum_{\substack{i,j\geq 0\\ i+j\leq r}} c_{i,j} s^i t^j \tag{2.2}$$

Since p_r has $f_r \equiv (r+1)(r+2)/2$ degrees of freedom, we will determine the coefficients $c_{i,j}$ from f_r interpolation conditions, namely

$$p_r(q_k) = g(q_k), \quad k = 1, ..., f_r \tag{2.3}$$

where the f_r interpolation nodes will be chosen in the following way.

Let α be a given constant with $0 \leq \alpha \leq \frac{1}{3}$. Define the interpolation nodes by

$$q_{i,j} = \left(\frac{i+(r-3i)\alpha}{r}, \frac{j+(r-3j)\alpha}{r}\right), \quad i,j \geq 0, \quad i+j \leq r \tag{2.4}$$

These f_r nodes form a uniform grid over σ (see Figure 1). If $\alpha = 0$, some of these points are on the edges of σ . If $\alpha > 0$, then they are symmetrically placed points in the interior of σ. For reasons described in Chapter I, throughout this paper we want to consider only nodes that are interior to the triangular elements, so we will work with $0 < \alpha \leq \frac{1}{3}$.

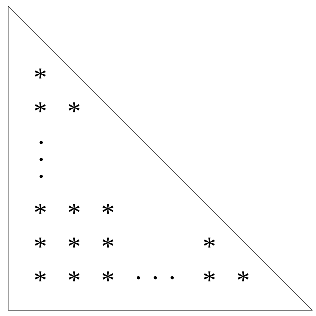

Figure 1: Unit simplex and the interpolation nodes

Denote by $l_{i,j}(s,t)$ the corresponding Lagrange interpolation basis functions. Then for a given $g \in C(\sigma)$, the formula

$$p_r(s,t) = \sum_{0\leq i+j\leq r} g(q_{i,j}) l_{i,j}(s,t) \tag{2.5}$$

is the unique polynomial of degree r that interpolates $g(s,t)$ at the nodes

$$\{q_{i,j} \mid i,j \geq 0, \quad i+j \leq r\}$$

The basis polynomials $l_{i,j}(s,t)$ of degree r are obtained, as usual, from the conditions

$$\begin{aligned} l_{i,j}(q_{i,j}) &= 1 \\ l_{i,j}(q_{l,k}) &= 0, \text{ for } l \neq i \text{ or } k \neq j \end{aligned}$$

So, now we have the interpolation formula

$$g(s,t) \approx \sum_{i+j \leq r} g(q_{i,j}) l_{i,j}(s,t) \tag{2.6}$$

Integrating (2.6) over σ, we obtain the quadrature formula

$$\int_{\sigma} g(s,t) d\sigma \approx \sum_{0 \leq i+j \leq r} \omega_{i,j} g(q_{i,j}) \tag{2.7}$$

where $\omega_{i,j} = \int_{\sigma} l_{i,j}(s,t) d\sigma$. Since the formula (2.6) is exact for all polynomials of degree $\leq r$, formula (2.7) has degree of precision at least r.

To get a better idea of how to obtain the interpolation nodes and the corresponding interpolation polynomials, we construct them explicitly for the cases $r = 0$, 1, 2, and 3. We use a sequential ordering of the nodes $\{q_1, ..., q_{f_r}\}$ to simplify the notation and to lead to formulas more readily adaptable to implementation in computer languages such as *Fortran*.

Example: Constant Interpolation

In this case, $r = 0$, $f_r = 1$. A function $g \in C(\sigma)$ is approximated by its value at the unique interpolation node $q_1 = (\alpha, \alpha)$.

The corresponding basis function is $l_1(s,t) \equiv 1$. We obtain the constant interpolation polynomial

$$p_0(s,t) = g(q_1) \tag{2.8}$$

Formula (2.6) becomes

$$g(s,t) \approx g(q_1) \tag{2.9}$$

Integrating (2.9) over σ yields

$$\int_\sigma g(s,t)d\sigma \approx \frac{1}{2}g(\alpha,\alpha) \tag{2.10}$$

which has degree of precision 0 for any $\alpha \neq \frac{1}{3}$, and degree of precision 1 for $\alpha = \frac{1}{3}$.

A very common choice for the case of constant interpolation is $\alpha = \frac{1}{3}$, meaning q_1 is the centroid of σ. In this case formula (2.10) has degree of precision 1. Later in this paper we will discuss the collocation method based on this type of constant interpolation, called the *centroid rule.*

Example: Linear Interpolation

Now $r = 1$ and there are $f_r = 3$ nodes, denoted $\{q_1, q_2, q_3\}$ (shown in Figure 2), where

$$q_1 = (\alpha,\alpha), \quad q_2 = (\alpha, 1-2\alpha), \quad q_3 = (1-2\alpha,\alpha) \tag{2.11}$$

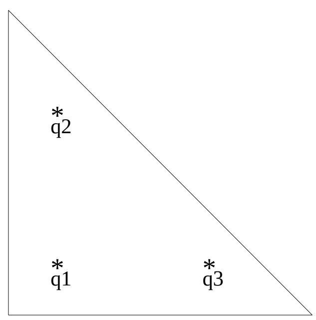

Figure 2: Unit simplex and linear interpolation nodes

The corresponding Lagrange interpolation basis functions are

$$l_1(s,t) = \frac{u-\alpha}{1-3\alpha}, \quad l_2(s,t) = \frac{t-\alpha}{1-3\alpha}, \quad l_3(s,t) = \frac{s-\alpha}{1-3\alpha} \tag{2.12}$$

with $u = 1-s-t$. The approximation

$$g(s,t) \approx \sum_{i=1}^{3} g(q_i)l_i(s,t) \equiv p_1(s,t) \tag{2.13}$$

gives the associated interpolation formula. It leads to the quadrature formula

$$\int_\sigma g(s,t)d\sigma \approx \frac{1}{6}\left[g(\alpha,\alpha)+g(\alpha,1-2\alpha)+g(1-2\alpha,\alpha)\right], \quad 0<\alpha<\frac{1}{3} \tag{2.14}$$

This case is discussed in more detail in Atkinson and Chandler[5].

Example: Quadratic Interpolation

There are 6 nodes now in the earlier grid, and we denote them by $\{q_1, ..., q_6\}$, using (2.11) and

$$q_4=\left(\alpha,\frac{1-\alpha}{2}\right), \quad q_5=\left(\frac{1-\alpha}{2},\frac{1-\alpha}{2}\right), \quad q_6=\left(\frac{1-\alpha}{2},\alpha\right) \tag{2.15}$$

(shown in Figure 3)

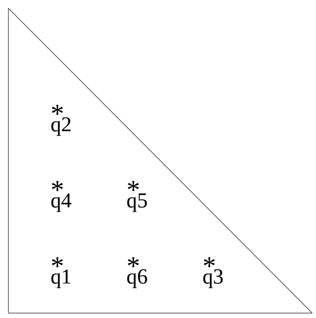

Figure 3: Unit simplex and quadratic interpolation nodes

Introduce the basis functions

$$\begin{aligned}
l_1(s,t) &= \frac{u-\alpha}{1-3\alpha}\left(2\frac{u-\alpha}{1-3\alpha}-1\right)\\
l_2(s,t) &= \frac{t-\alpha}{1-3\alpha}\left(2\frac{t-\alpha}{1-3\alpha}-1\right)\\
l_3(s,t) &= \frac{s-\alpha}{1-3\alpha}\left(2\frac{s-\alpha}{1-3\alpha}-1\right)\\
l_4(s,t) &= 4\frac{t-\alpha}{1-3\alpha}\frac{u-\alpha}{1-3\alpha}\\
l_5(s,t) &= 4\frac{s-\alpha}{1-3\alpha}\frac{t-\alpha}{1-3\alpha}\\
l_6(s,t) &= 4\frac{s-\alpha}{1-3\alpha}\frac{u-\alpha}{1-3\alpha}
\end{aligned} \tag{2.16}$$

The approximation of $g \in C(\sigma)$ is given by

$$g(s,t) \approx \sum_{i=1}^{6} g(q_i) l_i(s,t) \tag{2.17}$$

which after integration yields

$$\begin{aligned} \int_\sigma g(s,t)d\sigma &\approx \frac{\alpha(2-3\alpha)}{6(1-3\alpha)^2}\left[g(\alpha,\alpha)+g(\alpha,1-2\alpha)+g(1-2\alpha,\alpha)\right] \\ &+ \frac{(1-2\alpha)(1-6\alpha)}{6(1-3\alpha)^2}\left[g\left(\alpha,\frac{1-\alpha}{2}\right)+g\left(\frac{1-\alpha}{2},\frac{1-\alpha}{2}\right)\right. \\ &+ \left. g\left(\frac{1-\alpha}{2},\alpha\right)\right] \end{aligned} \tag{2.18}$$

This formula has degree of precision 2 for any $0 < \alpha < \dfrac{1}{3}$.

Example: Cubic Interpolation

In this case the 10 nodes, denoted $\{q_1, ..., q_{10}\}$ are the first 3 nodes given in (2.11) and

$$\begin{array}{lll} q_4 = \left(\alpha, \frac{1}{3}\right) & q_5 = \left(\alpha, \frac{2}{3} - \alpha\right) & q_6 = \left(\frac{1}{3}, \frac{2}{3} - \alpha\right) \\ q_7 = \left(\frac{2}{3} - \alpha, \frac{1}{3}\right) & q_8 = \left(\frac{2}{3} - \alpha, \alpha\right) & q_9 = \left(\frac{1}{3}, \alpha\right) \\ & q_{10} = \left(\frac{1}{3}, \frac{1}{3}\right) & \end{array} \tag{2.19}$$

These are shown in Figure 4.

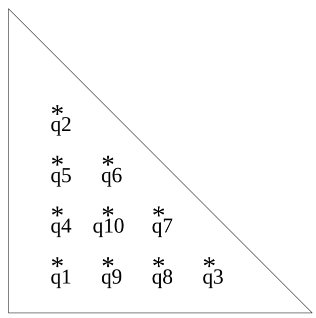

Figure 4: Unit simplex and cubic interpolation nodes

Define the basis functions

$$
\begin{aligned}
l_1(s,t) &= \frac{1}{2}\frac{u-\alpha}{1-3\alpha}(3\frac{u-\alpha}{1-3\alpha}-1)(3\frac{u-\alpha}{1-3\alpha}-2)\\
l_2(s,t) &= \frac{1}{2}\frac{t-\alpha}{1-3\alpha}(3\frac{t-\alpha}{1-3\alpha}-1)(3\frac{t-\alpha}{1-3\alpha}-2)\\
l_3(s,t) &= \frac{1}{2}\frac{s-\alpha}{1-3\alpha}(3\frac{s-\alpha}{1-3\alpha}-1)(3\frac{s-\alpha}{1-3\alpha}-2)\\
l_4(s,t) &= \frac{9}{2}\frac{u-\alpha}{1-3\alpha}\frac{t-\alpha}{1-3\alpha}(3\frac{u-\alpha}{1-3\alpha}-1)\\
l_5(s,t) &= \frac{9}{2}\frac{u-\alpha}{1-3\alpha}\frac{t-\alpha}{1-3\alpha}(3\frac{t-\alpha}{1-3\alpha}-1)\\
l_6(s,t) &= \frac{9}{2}\frac{s-\alpha}{1-3\alpha}\frac{t-\alpha}{1-3\alpha}(3\frac{t-\alpha}{1-3\alpha}-1)\\
l_7(s,t) &= \frac{9}{2}\frac{s-\alpha}{1-3\alpha}\frac{t-\alpha}{1-3\alpha}(3\frac{s-\alpha}{1-3\alpha}-1)\\
l_8(s,t) &= \frac{9}{2}\frac{u-\alpha}{1-3\alpha}\frac{s-\alpha}{1-3\alpha}(3\frac{s-\alpha}{1-3\alpha}-1)\\
l_9(s,t) &= \frac{9}{2}\frac{u-\alpha}{1-3\alpha}\frac{s-\alpha}{1-3\alpha}(3\frac{u-\alpha}{1-3\alpha}-1)\\
l_{10}(s,t) &= 27\frac{u-\alpha}{1-3\alpha}\frac{t-\alpha}{1-3\alpha}\frac{s-\alpha}{1-3\alpha}
\end{aligned}
\tag{2.20}
$$

The polynomial

$$p_3(s,t) = \sum_{i=1}^{10} g(q_i)l_i(s,t) \tag{2.21}$$

is the unique cubic polynomial that interpolates $g(s,t)$ at the nodes $\{q_1, ..., q_{10}\}$.

The interpolation formula (2.6) becomes

$$g(s,t) \approx \sum_{i=1}^{10} g(q_i)l_i(s,t) \tag{2.22}$$

Integrating (2.22) over σ, we obtain

$$
\begin{aligned}
\int_\sigma g(s,t)d\sigma &\approx \frac{1}{60(1-3\alpha)^3}\left[g(\alpha,\alpha)+g(\alpha,1-2\alpha)+g(1-2\alpha,\alpha)\right]\\
&+ \frac{3(1-5\alpha)}{80(1-3\alpha)^3}\left[g\left(\alpha,\frac{1}{3}\right)+g\left(\alpha,\frac{2}{3}-\alpha\right)+g\left(\frac{2}{3}-\alpha,\frac{1}{3}\right)\right.\\
&+ \left. g\left(\frac{2}{3}-\alpha,\alpha\right)+g\left(\frac{1}{3},\alpha\right)\right]\\
&+ \frac{3(-60\alpha^3+60\alpha^2-15\alpha+1)}{40(1-3\alpha)^3}g\left(\frac{1}{3},\frac{1}{3}\right)
\end{aligned}
\tag{2.23}
$$

which has degree of precision 3.

2.1.2 Interpolation Error Formulas Over Triangles

We are extending now the procedures described in the previous section to interpolation over a polygonal region R in the plane $\mathbb{R}^2$.

Let $\mathcal{T}_n = \{\Delta_1, ..., \Delta_n\}$ denote a triangulation of R. For now, we assume that triangles Δ_j and Δ_k can intersect only at vertices or along all of a common edge. Later we will assume additional properties for the triangulation.

Let the vertices of Δ_k be denoted by $\{v_{1,k}, v_{2,k}, v_{3,k}\}$, $\; v_{j,k} \equiv (x_{j,k}, y_{j,k})$, and the vertices of σ by $\{z_1, z_2, z_3\}$, where

$$z_1 = (0,0), \quad z_2 = (0,1), \quad z_3 = (1,0) \tag{2.24}$$

Define $T_k : \sigma \xrightarrow[onto]{1-1} \Delta_k$ by

$$(x,y) = T_k(s,t) = uv_{1,k} + tv_{2,k} + sv_{3,k}, \quad u = 1-s-t, \quad v_{j,k} = T_k(z_j) \tag{2.25}$$

This type of mapping is an *affine mapping*. The inverse of T_k, denoted by $(s,t) = Q_k(x,y)$, is also an affine mapping. It is straight forward to prove that if $p(x,y)$ is a polynomial of degree r in (x,y), then $P(s,t) \equiv p(T_k(s,t))$ is a polynomial of degree r in (s,t). Conversely, if $P(s,t)$ is a polynomial of degree r in (s,t), then $p(x,y) \equiv P(Q_k(x,y))$ is a polynomial of degree r in (x,y).

So, having defined interpolation over σ, we can now use the affine mapping $T_k(s,t)$ to define a corresponding interpolation polynomial over Δ_k, and by extension, over R, similar to formula (2.5) for the unit simplex for q_i defined in (2.4).

For a given $g \in C(R)$, define $\mathcal{P}_n g$ by

$$\mathcal{P}_n g(T_k(s,t)) = \sum_{i=1}^{f_r} g(T_k(q_i)) l_i(s,t), \quad (s,t) \in \sigma, \quad k = 1, ..., n \tag{2.26}$$

The operator norm of $\mathcal{P}_n$, as a mapping from $C(R)$ to $L^\infty(R)$, is given by

$$\|\mathcal{P}_n\| = \max_{(s,t)\in\sigma} \sum_{j=1}^{f_r} |l_j(s,t)| \tag{2.27}$$

In the case $\alpha = 0$, the formula (2.27) defines a projection operator on $C(R)$ and

$$\|\mathcal{P}_n\| = \begin{cases} 1, & \text{for linear interpolation} \\ \frac{5}{3}, & \text{for quadratic interpolation} \end{cases} \tag{2.28}$$

See Atkinson[4, p. 164].

For $0 < \alpha < \frac{1}{3}$, the function $P_n g$ is usually not continuous over R, but it can be regarded as a bounded projection on the larger space $L^\infty(R)$, the set of all essentially bounded and Lebesgue measurable functions on R, with the norm the essential supremum $\|\cdot\|_\infty$. See Atkinson, Graham and Sloan [8] for details on how to extend $\mathcal{P}_n$ from $C(R)$ to $L^\infty(R)$. For this case of α

$$\|\mathcal{P}_n\| = \frac{1+\alpha}{1-3\alpha} \tag{2.29}$$

for linear interpolation, and

$$\|\mathcal{P}_n\| = \begin{cases} \frac{5}{3} & , \text{ if } 0 < \alpha < \frac{15 - 8\sqrt{3}}{33} \\ \frac{1 + 10\alpha - 7\alpha^2}{(1-3\alpha)^2} & , \text{ if } \frac{15-8\sqrt{3}}{33} < \alpha < \frac{1}{3} \end{cases} \tag{2.30}$$

for quadratic interpolation.

The following lemmas give error bounds for the approximation of a function by and interpolatory polynomial. We omit the proofs, as they are relatively straightforward.

Lemma 2.1.1 *Let $\mathcal{T}_n$ be a triangulation of the polygonal region R. Let $g \in C(R)$, let $r \geq 0$ be an integer, and let $\mathcal{P}_n g$ be defined by (2.26). Then*

$$\|g - \mathcal{P}_n g\|_\infty \leq \|\mathcal{P}_n\| \omega(\delta_n, g) \tag{2.31}$$

with $\omega(\delta, g)$ the modulus of continuity of g

$$\omega(\delta, g) = \sup_{\substack{v,w \in R \\ |v-w| \leq \delta}} |g(v) - g(w)|$$

and δ_n the mesh size of the triangulation of R

$$\delta_n = \max_{1 \leq k \leq n} diameter(\Delta_k)$$

Lemma 2.1.2 *Let Δ be a planar triangle, let $r \geq 0$ be an integer, and assume $g \in C^{r+1}(\Delta)$. Then, for the interpolation polynomial $\mathcal{P}_n g(x,y)$ of (2.26)*

$$\max_{(x,y)\in\Delta} |g(x,y) - \mathcal{P}_n g(x,y)| \leq c\delta^{r+1} \max_{\substack{i,j>0 \\ i+j=r+1}} \max_{(\xi,\eta)\in\Delta} \left| \frac{\partial^{r+1} g(\xi,\eta)}{\partial\xi^i \partial\eta^j} \right| \tag{2.32}$$

with $\delta \equiv diameter(\Delta)$. The constant c depends on r, but it is independent of both g and Δ.

See Atkinson[4, p. 158] for the proofs.

2.1.3 Interpolation and Numerical Integration on Surfaces

Recall the integral equation (2.1) that we want to solve

$$u(P) - \int_S u(Q)K(P,Q)dS_Q = f(P), \quad P \in S$$

or, in operator form

$$(\mathcal{I} - \mathcal{K})u = f \tag{2.33}$$

where

$$\mathcal{K}u = \int_S u(Q)K(P,Q)dS_Q \tag{2.34}$$

Before proceeding we need some results on compact operators, which will be needed in studying the solvability of equation (2.1). These results are described in detail in Atkinson[4, Chapter 5], and only the most pertinent points are summarized here.

Definition 2.1 Let X and Y be normed vector spaces, and let $\mathcal{K} : X \longrightarrow Y$ be linear. Then $\mathcal{K}$ is *compact* if the set $\{\mathcal{K}x \mid \|x\|_X \leq 1\}$ has compact closure in Y. Compact operators are also called *completely continuous* operators.

Lemma 2.1.3 *Let X and Y be normed linear spaces with Y complete. Let $\mathcal{K} \in L[X,Y]$, let $\{\mathcal{K}_n\}$ be a sequence of compact operators in $L[X,Y]$, and assume $\mathcal{K}_n \longrightarrow \mathcal{K}$ in $L[X,Y]$, i. e. $\|\mathcal{K}_n - \mathcal{K}\| \longrightarrow 0$. Then $\mathcal{K}$ is compact.*

We assume S is a connected piecewise smooth surface in $\mathbb{R}^3$. By this, we mean

S can be written as

$$S = S_1 \cup S_2 \cup \ldots \cup S_J \tag{2.35}$$

with each S_j the continuous image of a polygonal region in the plane

$$F_j : R_j \xrightarrow[onto]{1-1} S_j, \quad j = 1, \ldots, J \tag{2.36}$$

Generally, we will need to assume that the mappings F_j are several times continuously differentiable.

To create triangulations for S, we first triangulate each R_j and then map this triangulation onto S_j. Let $\{\widehat{\Delta}^j_{n,k} \mid k = 1, \ldots, n_j\}$ be a triangulation of R_j, and then define

$$\Delta^j_{n,k} = F_j(\widehat{\Delta}^j_{n,k})$$

This yields a triangulation of S, which we refer to collectively as $\mathcal{T}_n = \{\Delta_1, \ldots, \Delta_n\}$. We make the following assumptions concerning this triangulation:

T1. The set of all vertices of the surface S is a subset of the set of all vertices of the triangulation $\mathcal{T}_n$.

T2. The union of all edges of S is contained in the union of all edges of all triangles in $\mathcal{T}_n$.

T3. If two triangles in $\mathcal{T}_n$ have a nonempty intersection, then that intersection consists either of (i) a single common vertex, or (ii) all of a common edge.

We call triangulations satisfying T1 - T3 *conforming triangulations.*

Let Δ_k be some element from $\mathcal{T}_n$, and let it correspond to some $\widehat{\Delta}_k$, say $\widehat{\Delta}_k \subset R_j$ and $\Delta_k = F_j(\widehat{\Delta}_k)$. Let $\{\hat{v}_{k,1}, \hat{v}_{k,2}, \hat{v}_{k,3}\}$ denote the vertices of $\widehat{\Delta}_k$. Define $m_k : \sigma \xrightarrow[onto]{1-1} \Delta_k$ by

$$m_k(s,t) = F_j(u\hat{v}_{k,1} + t\hat{v}_{k,2} + s\hat{v}_{k,3}), \quad (s,t) \in \sigma, \quad u = 1 - s - t \tag{2.37}$$

Now we can define interpolation and numerical integration over a triangular surface element Δ by means of a similar formula over σ. Recall the uniform grid over σ defined in (2.4), which we refer to collectively as $\{q_1, ..., q_r\}$. For $g \in C(S)$, restrict g to some $\Delta \in \mathcal{T}_n$ and define

$$(\mathcal{P}_n g)(m_k(s,t)) = \sum_{i=1}^{f_r} g(m_k(q_i))l_i(s,t) \tag{2.38}$$

This will define an interpolation formula over the surface S. The error bounds given in Lemma 2.1.1 and Lemma 2.1.2 can be easily extended to similar results for surfaces. See Atkinson[4, Section 5.3].

Next, we briefly describe the general framework for the collocation and iterated collocation methods. Let X be a Banach space, let $\{X_m \mid m \geq 1\}$ be a sequence of finite dimensional subspaces. Let $\mathcal{P}_m : X \longrightarrow X_m$ be a linear operator with

$$\mathcal{P}_m u = u, \quad u \in X_m \tag{2.39}$$

In attempting to solve the problem (2.33), we will approximate it by solving

$$\mathcal{P}_m(\mathcal{I} - \mathcal{K})u_m = \mathcal{P}_m f, \quad u_m \in X_m \tag{2.40}$$

This is the form in which the method is implemented as it leads directly to equivalent finite linear systems. To make an error analysis, we rewrite (2.40) in the equivalent form

$$(\mathcal{I} - \mathcal{P}_m\mathcal{K})u_m = \mathcal{P}_m f, \quad u_m \in X \tag{2.41}$$

where u_m is the solution of (2.40). This is equivalent to (2.40), since $\mathcal{P}_m u_m = u_m$. We have the following result:

Theorem 2.1.4 *Let X be a Banach space, $\mathcal{K} : X \longrightarrow X$ a bounded operator with $\mathcal{I} - \mathcal{K} : X \xrightarrow[onto]{1-1} X$. Assume that*

$$\|\mathcal{K} - \mathcal{P}_m\mathcal{K}\| \longrightarrow 0 \text{ as } m \longrightarrow \infty \tag{2.42}$$

Then for all sufficiently large m, say $m \geq N$, the operator $(\mathcal{I} - \mathcal{P}_m\mathcal{K})^{-1}$ exists as a *bounded operator from X to X. Moreover, it is uniformly bounded*

$$\sup_{m \geq N} \|(\mathcal{I} - \mathcal{P}_m \mathcal{K})^{-1}\| < \infty \tag{2.43}$$

For the solutions of (2.33) and (2.41)

$$u - u_m = (\mathcal{I} - \mathcal{P}_m \mathcal{K})^{-1}(u - \mathcal{P}_m u) \tag{2.44}$$

$$\frac{1}{\|\mathcal{I} - \mathcal{P}_m \mathcal{K}\|} \|u - \mathcal{P}_m u\| \leq \|u - u_m\| \leq \|(\mathcal{I} - \mathcal{P}_m \mathcal{K})^{-1}\| \cdot \|u - \mathcal{P}_m u\| \tag{2.45}$$

This leads to $\|u - u_m\|$ converging to zero at exactly the same speed as $\|u - \mathcal{P}_m u\|$.

To apply the above theorem, we need to know whether $\|\mathcal{K} - \mathcal{P}_m \mathcal{K}\| \longrightarrow 0$ as $m \longrightarrow \infty$. The following lemma addresses this question.

Lemma 2.1.5 *Let X be a Banach space, and let $\{\mathcal{P}_m\}$ be a family of bounded projections on X with*

$$\mathcal{P}_m u \longrightarrow u \text{ as } m \longrightarrow \infty, u \in X \tag{2.46}$$

Let $\mathcal{K} : X \longrightarrow X$ be compact. Then

$$\|\mathcal{K} - \mathcal{P}_m \mathcal{K}\| \longrightarrow 0 \text{ as } m \longrightarrow \infty \tag{2.47}$$

The proofs of Theorem 2.1.4 and Lemma 2.1.5 are fairly easy and they can be found in Atkinson[4, Section 3.1]. The last lemma includes most cases of interest, but not all. For some approximation processes, $\mathcal{P}_m u \longrightarrow u$ for most $u \in X$, but not all u. In such a case it is necessary to show directly that $\|\mathcal{K} - \mathcal{P}_m \mathcal{K}\| \longrightarrow 0$. Since $u_m \longrightarrow u$ if and only if $\mathcal{P}_m u \longrightarrow u$, such methods are not convergent for some solutions u.

For the iterated collocation method, consider the iteration

$$u^{(k+1)} = f + \mathcal{K} u^{(k)}, \quad k = 0, 1, ... \tag{2.48}$$

If u_m is the solution of the collocation equation (2.41), define the *iterated collocation solution* by

$$\hat{u}_m = f + \mathcal{K} u_m \tag{2.49}$$

Then

$$\mathcal{P}_m \hat{u}_m = \mathcal{P}_m f + \mathcal{P}_m \mathcal{K} u_m = u_m \tag{2.50}$$

and

$$(\mathcal{I} - \mathcal{K}\mathcal{P}_m)\hat{u}_m = f \tag{2.51}$$

Combining (2.41) and (2.51), we obtain

$$u - \hat{u}_m = [f + \mathcal{K}u] - [f + \mathcal{K}u_m] = \mathcal{K}(u - u_m) \tag{2.52}$$

$$\|u - \hat{u}_m\| \leq \|\mathcal{K}\| \cdot \|u - u_m\| \tag{2.53}$$

which proves that the convergence of $\hat{u}_m$ to u is at least as rapid as that of u_m to u.

Also, we see that $(\mathcal{I} - \mathcal{P}_m\mathcal{K})^{-1}$ exists if and only if $(\mathcal{I} - \mathcal{K}\mathcal{P}_m)^{-1}$ exists, since

$$\begin{aligned}(\mathcal{I} - \mathcal{K}\mathcal{P}_m)^{-1} &= \mathcal{I} + \mathcal{K}(\mathcal{I} - \mathcal{P}_m\mathcal{K})^{-1}\mathcal{P}_m \\ (\mathcal{I} - \mathcal{P}_m\mathcal{K})^{-1} &= \mathcal{I} + \mathcal{P}_m(\mathcal{I} - \mathcal{K}\mathcal{P}_m)^{-1}\mathcal{K}\end{aligned} \tag{2.54}$$

We can choose to show the existence of either $(\mathcal{I} - \mathcal{P}_m\mathcal{K})^{-1}$ or $(\mathcal{I} - \mathcal{K}\mathcal{P}_m)^{-1}$, whichever is the more convenient, and the existence of the other inverse will follow immediately. Bounds on one inverse in terms of the other can also be given using (2.54).

2.1.4 Collocation as a Projection Method in $L^\infty(S)$

We want to solve the equation (2.33), using a collocation method based on a piecewise polynomial interpolation operator (2.38). If we choose $X = C(S)$, X_r to be the set of polynomials of degree $\leq r$ and $\mathcal{P}_n$ defined by (2.38), then $\mathcal{P}_n g$ is usually not continuous. If the standard type of collocation error analysis is to be carried out in the context of function spaces, as described in the previous section, then $C(S)$ must be enlarged to include piecewise polynomial approximations $\mathcal{P}_n g$. One way of doing this is by using the space $L^\infty(S)$, the set of all essentially bounded and Lebesgue measurable functions on S, with the norm the essential supremum $\|\cdot\|_\infty$. Here is a brief outline on how $\mathcal{P}_n$ can be extended to a projection operator on $L^\infty(S)$ (for details, see Atkinson, Graham and Sloan[8]):

We call on the mathematical construction of point functions defined and analyzed in Atkinson, Graham and Sloan[8]. Let $C(S)$ denote the subspace of $L^\infty(S)$

consisting of all cosets based on continuous functions

$$\mathcal{C}(S) = \{[g] \mid g \in C(S)\}, \quad g \in [g] \tag{2.55}$$

For a point $P \in S$, define a linear functional on $C(S)$ by

$$l_P([g]) = g(P) \tag{2.56}$$

Then $\|l_P\|$ is bounded with $\|l_P\| = 1$.

Then using the Hahn-Banach Theorem (see Rudin[16]), the functional l_P can be extended (albeit not uniquely) to a linear functional on all of $L^\infty(S)$ with preservation of norm. We continue with the same notation for the extension. Let $[g] \in L^\infty(S)$ and suppose g is continuous at a point P. Then

$$\lim_{Q \to P} l_Q([g]) = l_P([g]) = (\mathcal{P}_n g)(P)$$

Thus the value of $l_P([g])$ possesses the expected value without requiring that $g \in C(S)$. Other properties of the extension are studied in Atkinson, Graham and Sloan[8].

Now, the operator $\mathcal{P}_n$ of (2.38) can be extended to $L^\infty(S)$:

$$(\mathcal{P}_n[g])(P) = \sum_{j=1}^{f_r} l_{m_k(q_i)}([g])\, l_i(s,t), \quad P \in S, \quad [g] \in L^\infty(S) \tag{2.57}$$

where, here, l_P denotes the extension. The range of $\mathcal{P}_n$ is the set of all cosets of functions that are piecewise polynomial (of degree $\leq r$) over the triangulation $\mathcal{T}_n$

With this new definition of $\mathcal{P}_n$, the collocation equation may again be written in the form (2.41), and the rest of the analysis is then entirely analogous to the continuous case. Thus the usual next step is to ensure that (2.42) holds, where $\mathcal{P}_n\mathcal{K}$ is now considered as an operator on $L^\infty(S)$. A sufficient condition for this is that

$$\lim_{n \to \infty} \|\mathcal{P}_n g - g\|_\infty = 0, \quad g \in C(S)$$

with the assumption that $\mathcal{K}$ is compact as an operator from $L^\infty(S)$ to $C(S)$. This follows, using a straightforward modification of the arguments given in the continuous case.

2.2 A Procedure for Developing a Method of Arbitrarily High Order

Given an integral equation of the form (2.1),

$$u(P) - \int_S u(Q)K(P,Q)dS_Q = f(P), P \in S$$

we want to describe a procedure for obtaining a collocation method based on interpolation of degree r for solving the equation (2.1) with an error of order $O(h^{r+1})$.

2.2.1 Error Analysis of the Collocation Method

We consider the same framework as in the previous section. Let S be a smooth surface, satisfying the conditions (2.35) and (2.36), and $\mathcal{T}_n = \{\Delta_1, ..., \Delta k\}$ be a conforming triangulation of S. Let

$$(\mathcal{P}_n g)(m_k(s,t) = \sum_{i=1}^{f_r} g(m_k(q_i))l_i(s,t), g \in C(S), \quad P = m_k(s,t)$$

for q_i and l_i described in (2.4) and (2.5), and m_k the application from (2.37).

We seek solutions of (2.1) of the form

$$u_n(P) = \sum_{j=1}^{f_r} u_n(v_{k,j})l_j(s,t), \quad P = m_k(s,t) \in \Delta_k, \quad v_{k,j} = m_k(q_j), \quad k = 1, ..., n \quad (2.58)$$

Substitute (2.58) into (2.1). To determine the values $\{u_n(v_{k,j})\}$, force the equation resulting from the substitution to be true at the collocation points $\{q_1, ..., q_r\}$ described in (2.4). This leads to the linear system

$$\begin{aligned} u_n(v_i) \quad - \quad & \sum_{k=1}^{n}\sum_{j=1}^{f_r} u_n(v_{k,j}) \int_S K(v_i, m_k(s,t))l_j(s,t) \cdot \\ & |(D_s m_k \times D_t m_k)(s,t)|d\sigma = f(v_i), \quad i = 1, ..., f_r n \qquad (2.59) \end{aligned}$$

which is of order $f_r n$. For the error analysis, the following is true.

Theorem 2.2.1 *Assume S is a smooth surface in $\mathbb{R}^3$ satisfying (2.35) and (2.36) with each $F_j \in C^{r+2}$. Assume that equation (2.1) is uniquely solvable for all functions $f \in C(S)$. Assume $\mathcal{K} : L^\infty(S) \longrightarrow C(S)$ is compact and $u \in C^{r+1}(S)$. Then for all sufficiently large n, say $n \geq n_0$, the operators $\mathcal{I} - \mathcal{P}_n\mathcal{K}$ are invertible on $C(S)$ and have uniformly bounded inverses. Moreover, for the true solution u of (2.1) and the* *solution u_n of (2.41)*

$$\|u - u_n\|_\infty \leq \left\|(\mathcal{I} - \mathcal{P}_n\mathcal{K})^{-1}\right\| \cdot \|u - \mathcal{P}_n u\|_\infty \tag{2.60}$$

Furthermore, if $f \in C^{r+1}(S)$, then

$$\|u - u_n\|_\infty \leq O(h^{r+1}), \quad n \geq n_0 \tag{2.61}$$

Proof: Consider $\mathcal{P}_n$ as a projection operator from $L^\infty(S)$ into itself. It is relatively easy to show that $\mathcal{P}_n u \longrightarrow u$ as $n \longrightarrow \infty$. Since $\mathcal{K}$ is compact, by Lemma 2.1.5 we have that $\|\mathcal{K} - \mathcal{P}_n\mathcal{K}\|_\infty \longrightarrow 0$ as $n \longrightarrow \infty$. From Theorem 2.1.4 and the assumption that the equation (2.1) is solvable, it follows that the operators $(\mathcal{I} - \mathcal{P}_n\mathcal{K})$ are invertible on $C(S)$ and have uniformly bounded inverses for all sufficiently large n, say $n \geq n_0$.

The bound (2.60) follows from the identity

$$u - u_n = (\mathcal{I} - \mathcal{P}_n\mathcal{K})^{-1}(u - \mathcal{P}_n u)$$

The bound (2.61) follows from Lemma 2.1.2. By (2.54), the same bound holds for $\|u - \hat{u}_n\|_\infty$.

□

Although we stated this theorem for the smooth surface case only, we mention that it can be easily generalized to piecewise smooth surfaces.

It is clear now that the accuracy of a collocation method based on piecewise polynomial interpolation depends on the degree of precision of the interpolation formula. Theorem 2.2.1 asserts that an interpolation formula having degree of precision r leads to an error in the collocation method of order at least $O(h^{r+1})$. Since going to high degree polynomials can significantly complicate formulas and computations, a natural question, then, comes to mind: Using an interpolation formula with the degree of precision r, can we do any better than $O(h^{r+1})$ in our error bounds? The answer is sometimes "yes". There are two ways that we can improve the precision of the interpolatory quadrature formula, which in turn, will increase the accuracy of the associated collocation method.

One of them has to do with the triangulation of the surface and the way we refine it. Given a triangulation $\mathcal{T}_n$ of a polygonal region R with grid size δ_n, at each step we refine it to a new triangulation with a smaller grid size. In most finite element methods for solving partial differential equations it does not matter how we do this refinement as long as $\delta_n \longrightarrow 0$ as $n \longrightarrow \infty$; however, when integration is involved, there is an "optimal" type of triangulation that can lead to cancellation of errors.

A simple example may illustrate how that can happen. Let $g(s,t)$ be defined on the unit simplex $\sigma = \{(s,t) \mid 0 \leq s+t \leq 1\}$. Approximate it by a constant polynomial

$$g(s,t) \simeq g(\alpha,\alpha), \quad (s,t) \in \sigma \tag{2.62}$$

with $\alpha \neq \dfrac{1}{3}$. This formula has degree of precision 0. Integrating it over σ, leads to

$$\int\limits_S g(s,t)d\sigma \approx \frac{1}{2}g(\alpha,\alpha) \tag{2.63}$$

which is exact for all polynomials of degree 0. If we extend it to $R = \sigma \cup \bar{\sigma}$, where $\bar{\sigma} = \{(s,t) \mid -1 \leq s+t \leq 0\}$, which is symmetric to σ about the origin (see Figure 5) and consider as a node the reflection of (α,α), which is $(-\alpha,-\alpha)$, we obtain

$$\int\limits_R g(s,t)d\sigma \approx \frac{1}{2}\left[g\left(\alpha,\alpha\right) + g(-\alpha,-\alpha)\right] \tag{2.64}$$

which has degree of precision 1, being exact not only for all constants, but also for s and t. The left-hand side of (2.64) is 0 because $\bar{\sigma}$ is symmetric to σ about the origin, while the right side of (2.64) is 0 by the fact that both s and t are odd functions.

So it appears that by imposing some symmetry in our triangulations $\mathcal{T}_n$, we will sometimes obtain an increase in the degree of precision of a quadrature formula and thus in the rate of convergence of the resulting numerical integration formula. Given a triangle $\Delta \in \mathcal{T}_n$, we will refine it into smaller triangles by using straight line

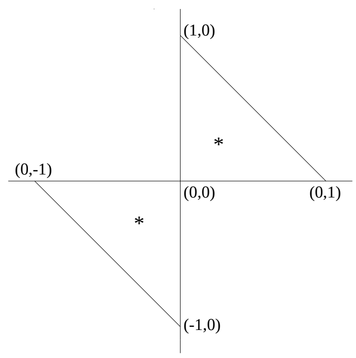

Figure 5: The unit simplex and its symmetric

segments to connect the midpoints of the three sides of Δ. The four new triangles obtained this way will be congruent and similar to the original triangle Δ. After such a refinement of all the triangles in $\mathcal{T}_n$, the new triangulations $\mathcal{T}_{4n}$ will have four times as many triangles as $\mathcal{T}_n$. As for the grid size, we have

$$\delta_{4n} = \frac{1}{2}\delta_n \tag{2.65}$$

We will call such triangulations obtained with this form of refinement *symmetric triangulations.*

If we denote by

$$E_n(g) = \left| \int_\sigma g(s,t)d\sigma - \sum_{k=1}^{n} g(q_i) \sum_{j=1}^{f_r} \int_{\Delta_k} l_{j,k}(s,t)\ d\sigma \right| \tag{2.66}$$

the error for a composite numerical integration of the form (2.7) with $\{q_{j,k}\}$ and $\{l_{j,k}\}$ defined similarly to those defined in (2.4) and (2.5) (only specifically for each triangle Δ_k), and if the integration method has degree of precision d, then the ratio

$$\frac{E_n(g)}{E_{4n}(g)} \tag{2.67}$$

of the errors (2.66) should equal approximately $2^{-(d+1)}$.

If the initial degree of precision d from integrating over σ is an even number, and if we are using a symmetric triangulation scheme, then the degree of precision is increased effectively to $d+1$, as was the case in our previous example, since integrals over symmetric triangles of polynomials of odd degree are 0. In this case it is possible to improve the results in Theorem 2.2.1

Theorem 2.2.2 *Assume all the conditions in Theorem 2.2.1 are satisfied. Furthermore, assume that $\mathcal{T}_n$ is a symmetric triangulation and that the degree of precision r of the interpolation formula is an even number. Assume $u \in C^{r+2}(S)$. Then, for all sufficiently large n, say $n \geq n_0$, the operators $\mathcal{I} - \mathcal{K}\mathcal{P}_n$ are invertible on $C(S)$ and have uniformly bounded inverses. Moreover, if $f \in C^{r+2}(S)$ for the true solution u of (2.1) and the solution $\hat{u}_n$ of (2.49)*

$$\|u - \hat{u}_n\|_\infty \leq O(h^{r+2}), \quad n \geq n_0 \tag{2.68}$$

Proof: Since we use a symmetric triangulation, essentially all the triangles in $\mathcal{T}_n$ can be partitioned into pairs of symmetric triangles (as in Figure 5).

There will be at most $O(\sqrt{n}) = O(h^{-1})$ triangles not included in such pairs of triangles. Proceeding as in the proof of Theorem 2.2.1, the contribution to the errors over the set of all such symmetric pairs is $O(h^{r+2})$, since the degree of precision of the integration formula is now $r+1$. The remaining triangles of number $O(h^{-1})$ will have a composite error of

$$O(h^{-1}) \cdot \text{ Area } (\Delta) O(h^{r+1}) = O(h^{-1}) \cdot O(h^2) \cdot O(h^{r+1}) = O(h^{r+2}) \tag{2.69}$$

Combining the two errors, we have (2.68).

□

Recall the interpolation nodes that we use, given in (2.4). They depend on a parameter α, with $0 \leq \alpha \leq \dfrac{1}{3}$. As long as $\alpha \neq 0$ (so that the nodes are not on the edges of σ) and $\alpha \neq \dfrac{1}{3}$ (so that if $r \geq 1, \alpha \neq 1 - 2\alpha$), we have the liberty to choose any value for α. As it turns out, some particular values for α lead to higher degrees of precision of the quadrature formula (2.7).

As an example, consider the linear interpolation case, which leads to the quadrature formula (2.14)

$$\int_\sigma g(s,t)d\sigma \approx \frac{1}{6}\left[g(\alpha,\alpha)+g(\alpha,1-2\alpha)+g(1-2\alpha,\alpha)\right]$$

As we mentioned before, this formula has degree of precision 1 for any $0<\alpha<\dfrac{1}{3}$.

However, if $\alpha=\dfrac{1}{6}$, formula (2.14) becomes

$$\int_\sigma g(s,t)d\sigma \approx \frac{1}{6}\left[g\left(\frac{1}{6},\frac{1}{6}\right)+g\left(\frac{1}{6},\frac{2}{3}\right)+g\left(\frac{2}{3},\frac{1}{6}\right)\right] \tag{2.70}$$

which has degree of precision 2. The proof is a straightforward computation with the choices $g(s,t)=s^2, st, t^2$. Moreover, if we integrate it over $R=\sigma\cup\bar{\sigma}$, then the formula

$$\begin{aligned}\int_\sigma g(s,t)d\sigma \;\approx\; & \frac{1}{6}\left[g\left(\frac{1}{6},\frac{1}{6}\right)+g\left(\frac{1}{6},\frac{2}{3}\right)+g\left(\frac{2}{3},\frac{1}{6}\right)\right.\\ + \;& \left. g\left(-\frac{1}{6},-\frac{1}{6}\right)+g\left(-\frac{1}{6},-\frac{2}{3}\right)+g\left(-\frac{2}{3},-\frac{1}{6}\right)\right]\end{aligned} \tag{2.71}$$

has degree of precision 3, since this is the case described prior to Theorem 2.2.2. The collocation method using piecewise linear interpolation with $\alpha=\dfrac{1}{6}$, for the radiosity equation is described in great detail in Atkinson and Chandler[5]. We will discuss at length particular choices of α for the quadratic interpolation case in the following chapter.

2.2.2 Determining the Degree of Precision of a Quadrature Formula

What concerns us now is how to find an efficient way of determining the degree of precision of a quadrature formula. To simplify the calculations and the ideas, we will restrict ourselves for the remainder of this chapter to integration formulas over the unit simplex

$$\int_\sigma g(s,t)d\sigma \simeq \sum_{j=1}^{f_r} g(q_j)\int_\sigma l_j(s,t)d\sigma \tag{2.72}$$

for g_j and l_j given in (2.4) and (2.5).

Again, let E_n be the error in formula (2.66)

$$E_n(g) = \int_\sigma g(s,t)d\sigma - \sum_{j=1}^{f_r} g(q_j) \int_\sigma l_j(s,t)d\sigma \qquad (2.73)$$

To show that formula (2.7) has degree of precision r, we must verify that it is exact for all polynomials $s^i t^j$, $0 \le i+j \le r$. We need to find an easier way.

We will follow closely the ideas given in Sobolev[17]. Consider $(s,t) \in \sigma, u = 1-s-t$, and the symmetric group S_3 of permutations in the following context

$$S_3 = \{T_0, T_1, ..., T_5\} \qquad (2.74)$$

where the functions $T_i : \sigma \longrightarrow \sigma$, $i = 0, ..., 5$ are given by

$$\begin{aligned} T_0(s,t) &= (s,t) \\ T_1(s,t) &= (u,t) \\ T_2(s,t) &= (s,u) \\ T_3(s,t) &= (t,s) \\ T_4(s,t) &= (u,s) \\ T_5(s,t) &= (t,u) \end{aligned} \qquad (2.75)$$

Theorem 2.2.3 *For the quadrature formula (2.7) to be exact for all polynomials of a given order r, it is necessary and sufficient that it be exact for all invariant polynomials with respect to S_3, i. e. for those which are unchanged under all mappings $T_0, ..., T_5$.*

Proof: It is straightforward to verify that

$$\begin{aligned} \int_\sigma g(s,t)d\sigma &= \int_\sigma g(T_i(s,t))d\sigma, \quad i = 0, ..., 5 \\ \sum_{j=1}^{f_r} g(q_i) \int_\sigma l_j(s,t)d\sigma &= \sum_{j=1}^{f_r} g\left(T_i(q_i)\right) \int_\sigma l_j\left(T_i(s,t)\right) d\sigma \end{aligned} \qquad (2.76)$$

which means that

$$E_n(g) = E_n(g \circ T_i), \quad i = 0, ..., 5 \qquad (2.77)$$

Then we can write

$$E_n(g) = \frac{1}{6}\sum_{i=0}^{5} E_n(g \circ T_i) = \frac{1}{6}E_n\left(\sum_{i=0}^{5}(g \circ T_i)\right) \tag{2.78}$$

Denote by $\hat{g}$ the *mean* of the function g over the group S_3

$$\hat{g} = \sum_{i=0}^{5} g \circ T_i \tag{2.79}$$

Then, formula (2.78) can be written

$$E_n(g) = \frac{1}{6}E_n(\hat{g}) \tag{2.80}$$

By the properties of S_3, $\hat{g}$ is invariant under all permutations $T_i, i = 0, ..., 5$

$$\hat{g} \circ T_k = \hat{g}, \quad k = 0, ..., 5 \tag{2.81}$$

By (2.80), $E_n(g) = 0$ if and only if $E_n(\hat{g}) = 0$, which means that $E_n(g) = 0$ for all polynomials g of degree $\leq r$ is equivalent to $E_n(\hat{g}) = 0$ for all polynomials $\hat{g}$ of degree $\leq r$ that are invariant under all transformations $T_i, \quad i = 0, ..., 5$.

□

This reduces our task to finding polynomials of a given degree that are invariant under all transformations $T_i, i = 0, ..., 5$, i.e. by (2.75) polynomials of a given degree that are symmetric in s, t, and u. Then, if formula (2.7) is exact for such polynomials, it will also be exact for all polynomials of the given degree.

The following theorem characterizes completely such polynomials that are symmetric in s, t, and u.

Theorem 2.2.4 *Let $\sigma_1 = s + t - s^2 - st - t^2$ and $\sigma_2 = st - s^2t - st^2$. Then $g(s,t)$ is a polynomial symmetric in s, t, and u if and only if g is a polynomial in σ_1 and σ_2.*

Proof: We give only a sketch of the proof. The proof uses field theory and Galois theory, so we do not go into details. The main idea is the following. It is known that the polynomials symmetric in 3 variables, say x_1, x_2, x_3, must be polynomials in

(1) $x_1 + x_2 + x_3$

(2) $x_1x_2 + x_1x_3 + x_2x_3$

(3) $x_1x_2x_3$

Let $x_1 = s, \quad x_2 = t, \quad x_3 = u$. (Since u is not an independent variable, the well-definedness of some embedding mappings must be verified. For details, see Hungerford[13].) Then, we have

(1) $s + t + u = 1$

(2) $st + su + tu = st + (s+t)(1-s-t) = \sigma_1$

(3) $stu = st(1-s-t) = \sigma_2$

which proves our assertion. □

Combining Theorem 2.2.3 and 2.2.4, we obtain that the quadrature formula (2.7) has degree of precision r if and only if it is exact for all polynomials in σ_1 and σ_2 of degree $\leq r$. In the following table we give such polynomials for degrees $0, ..., 6$ ($c_0, c_1, ..., c_6$ denote generic constants).

r	Polynomial
0	c_0
1	c_0
2	$c_0 + c_1\sigma_1$
3	$c_0 + c_1\sigma_1 + c_2\sigma_2$
4	$c_0 + c_1\sigma_1 + c_2\sigma_2 + c_3\sigma_1^2$
5	$c_0 + c_1\sigma_1 + c_2\sigma_2 + c_3\sigma_1^2 + c_4\sigma_1\sigma_2$
6	$c_0 + c_1\sigma_1 + c_2\sigma_2 + c_3\sigma_1^2 + c_4\sigma_1\sigma_2 + c_5\sigma_2^2 + c_6\sigma_1^3$

Table 1: Polynomials in σ_1 and σ_2 of degrees $0, ..., 6$

In conclusion, to develop an integration scheme of a given order, we construct the interpolation polynomial (2.5), integrate it, approximate the integral of a function by the integral of the polynomial in (2.7), and use the error bounds given in Theorems 2.2.1 and 2.2.2. By means described above we may improve the rate of convergence given in Theorems 2.2.1 and 2.2.2.

Finding high order interpolatory formulas comes down to solving a system of equations involving the problem parameter. We developed procedures for interpolation nodes that make use of one parameter, α. Other parameters can be introduced (see the discussion at the beginning of Chapter V).

CHAPTER III
SUPERCONVERGENT METHODS FOR INTEGRATION AND COLLOCATION

In this chapter, we consider some particular cases of the methods described in Chapter II. We begin by discussing some collocation methods. We introduce the radiosity equation, describe its properties and its solvability. We conclude by introducing a particular piecewise quadratic collocation method for determining the numerical solution of the radiosity equation. We show that this method is superconvergent at the collocation nodes and consider numerical examples to illustrate that. We also discuss general superconvergent collocation methods for the radiosity equation based on interpolation of degree r.

3.1 Superconvergent Collocation Methods

Consider the integral equation

$$u(P) - \int_S u(Q)K(P,Q)dS_Q = f(P), \quad P \in S \tag{3.1}$$

for S a smooth surface in $\mathbb{R}^3$, with K and f continuous functions.

Let $\mathcal{T}_n = \{\Delta_1, ..., \Delta_k\}$ be a triangulation of S and $m_k : \sigma \longrightarrow \Delta_k$ be defined as in (2.37). Recall the interpolation formula

$$g(s,t) \approx \sum_{j=1}^{f_r} g(q_j)l_j(s,t), \quad g \in C(S) \tag{3.2}$$

Let

$$\mathcal{P}_n g(m_k(s,t)) = \sum_{j=1}^{f_r} g(m_k(q_j))l_j(s,t), \quad P = m_k(s,t) \in \Delta_k \tag{3.3}$$

with the nodes $\{q_1, ..., q_{f_r}\}$ and $\{l_1, ..., l_{f_r}\}$ given by (2.4) and (2.5).

Define a collocation method using (3.3). Substitute

$$\begin{aligned} u_n(P) &= \sum_{j=1}^{f_r} u_n(v_{k,j}) l_j(s,t), \quad P \in m_k(s,t) \in \Delta_k \\ v_{k,j} &= m_k(q_j), \quad k = 1, \ldots, n \end{aligned} \tag{3.4}$$

into (3.1). This leads to the linear system

$$\begin{aligned} u_n(v_i) \; - \; & \sum_{k=1}^{n} \sum_{j=1}^{f_r} u_n(v_{k,j}) \int_\sigma K(v_i, m_k(s,t)) l_j(s,t) \cdot \\ & |(D_S m_k \times D_t m_k)(s,t)| d\sigma = f(v_i), \quad i = 1, \ldots, n f_r \end{aligned} \tag{3.5}$$

We have shown in Theorem 2.2.1 that under suitable assumptions this method has the error

$$\|u - u_n\|_\infty \le O(h^{r+1}) \tag{3.6}$$

where h is the mesh size of the triangulation $\mathcal{T}_n$. Sometimes at the collocation node points, the collocation method converges more rapidly than over all S, in which case

$$\lim_{n \to \infty} \frac{\max\limits_{1 \le i \le n f_r} |u(v_i) - \hat{u}_n(v_i)|}{\|u - u_n\|_\infty} = 0 \tag{3.7}$$

Such methods are *superconvergent* at the collocation node points.

Let us examine more carefully the terms in (3.7). For simplicity, we work with the solution $\hat{u}_n$ of the iterated collocation equation (2.51). This should cause no problems, since we know that the convergence of $\hat{u}_n$ to u is at least as rapid as that of the solution of the collocation equation (2.41) to u, and the inverses for the collocation equation and iterated collocation equation are related by the identities

$$\begin{aligned} (\mathcal{I} - \mathcal{K}\mathcal{P}_n)^{-1} &= \mathcal{I} + \mathcal{K}(\mathcal{I} - \mathcal{P}_n\mathcal{K})^{-1}\mathcal{P}_n \\ (\mathcal{I} - \mathcal{P}_n\mathcal{K})^{-1} &= \mathcal{I} + \mathcal{P}_n(\mathcal{I} - \mathcal{K}\mathcal{P}_n)^{-1}\mathcal{K} \end{aligned} \tag{3.8}$$

(recall (2.54) for details). Moreover, $\hat{u}(v_i) = u_n(v_i)$ at all collocation nodes.

By looking at the linear system associated with

$$(\mathcal{I} - \mathcal{K}\mathcal{P}_n)(u - \hat{u}_n) = \mathcal{K}(u - \mathcal{P}_n u) \tag{3.9}$$

we have

$$\max_{1 \le i \le nf_r} |u(v_i) - \hat{u}_n(v_i)| \le c \max_{1 \le i \le nf_r} |\mathcal{K}(\mathcal{I} - \mathcal{P}_n)u(v_i)| \tag{3.10}$$

(see Atkinson[4, p. 449]). So, now we can focus on finding errors for $\mathcal{K}(\mathcal{I} - \mathcal{P}_n)u(v_i)$.

First, we need some assumptions for the interpolation over σ. Recall that for $g \in C(\sigma)$, we are considering interpolation of degree r over σ:

$$g(s,t) \approx (\mathcal{L}_\sigma g)(s,t) \equiv \sum_{j=1}^{f_r} g(q_j) l_j(s,t) \tag{3.11}$$

This leads to the numerical integration formula

$$\int_\sigma g(s,t) d\sigma \approx \int_\sigma \mathcal{L}_\sigma g(s,t) d\sigma \tag{3.12}$$

which has a degree of precision of at least r. Assume there is a value $0 < \alpha_0 < \dfrac{1}{3}$ such that for q_j and l_j defined with $\alpha = \alpha_0$, the formula (3.12) is exact for all polynomials in σ_1, σ_2 (introduced in Theorem 2.2.4) of degree $\le r+1$, i. e. has degree of precision $r+1$. For the remainder of this section, we will assume $\alpha = \alpha_0$.

Now, let $\tau \subset \mathbb{R}^2$ be a planar triangle with vertices $\{v_1, v_2, v_3\}$ and define the mapping $m_\tau : \sigma \longrightarrow \tau$ as in (2.37). For $g \in C(\tau)$, define

$$\mathcal{L}_\tau g(x,y) = \sum_{j=1}^{f_r} g(m_\tau(q_j)) l_j(s,t) \tag{3.13}$$

which is a polynomial of degree r in the parametrization variables s and t, interpolating g at the nodes $\{m_\tau(q_1), ..., m_\tau(q_{f_r})\}$.

Define a numerical integration formula over τ by

$$\int_\tau g(x,y) d\tau \approx \int_\tau \mathcal{L}_\tau g(x,y) d\tau \tag{3.14}$$

By our earlier assumption on α_0, this has degree of precision at least $r+1$. In what follows, for differentiable functions g, we will use the notation

$$|D^k g(x,y)| = \max_{0 \le i \le k} \left| \frac{\partial^k g(x,y)}{\partial x^i \partial y^{k-i}} \right| \tag{3.15}$$

We have the following result.

Lemma 3.1.1 *Let τ be a planar right triangle and assume the two sides which form* *the right angle have length h. Assume $\alpha = \alpha_0$. Let $g \in C^{r+2}(\tau), \Phi \in C^1(\tau)$. Then*

$$\left|\int_\tau \Phi(x,y)(\mathcal{I} - \mathcal{L}_\tau)g(x,y)d\tau\right| \le ch^{r+2}\left[\int_\tau (|\Phi| + |D\Phi|)d\tau\right] \cdot \max_\tau \left\{|D^{r+1}g|, |D^{r+2}g|\right\} \tag{3.16}$$

where c denotes a generic constant.

Proof: Let $p_i(x,y)$ denote Taylor expansions of g around a suitable point in τ, of degree i, for $i = r,\ r+1$. Then, since $g \in C^{r+2}(\tau)$, we have that

$$\|g - p_i\|_\infty \le ch^{i+1}\|D^{i+1}g\|_\infty, \quad i = r, r+1 \tag{3.17}$$

with $\|\cdot\|_\infty$ denoting the uniform norm on $C(\tau)$.

From (3.17) it follows that

$$\begin{aligned} \|p_{r+1} - p_r\|_\infty &\le \|g - p_{r+1}\|_\infty + \|g - p_r\|_\infty \\ &\le ch^{r+2}\|D^{r+2}g\|_\infty + ch^{r+1}\|D^k g\|_\infty \\ &= ch^{r+1}\left(h\|D^{r+2}g\|_\infty + \|D^{r+1}g\|_\infty\right) \end{aligned} \tag{3.18}$$

Since $\Phi \in C^1(\tau)$, there is a constant Φ_0 such that

$$\|\Phi - \Phi_0\|_\infty \le ch\,|D\Phi| \tag{3.19}$$

To shorten the notation, let $\mathcal{L}'_\tau = \mathcal{I} - \mathcal{L}_\tau$. We can write

$$\begin{aligned} \int_\tau \Phi\mathcal{L}'_\tau g d\tau &= \int_\tau \Phi\mathcal{L}'_\tau(g - p_{r+1})d\tau \\ &+ \int_\tau (\Phi - \Phi_0)\mathcal{L}'_\tau(p_{r+1} - p_r)d\tau \end{aligned} \tag{3.20}$$

To see why (3.20) is true, note first that

$$\mathcal{L}'_\tau p_r = 0 \tag{3.21}$$

since formula (3.11) has degree of precision r. Also , by our assumption that for $\alpha = \alpha_0$, formula (3.12) has degree of precision $r+1$, we have that

$$\int_\tau \Phi_0\mathcal{L}'_\tau p_{r+1}d\tau = 0 \tag{3.22}$$

Then, using these facts, (3.20) follows from expanding the right side and simplifying. Taking norms in (3.20) and using the bounds in (3.17), (3.18), and (3.19), we have

$$\left|\int_\tau \Phi\mathcal{L}'_\tau g d\tau\right| \leq ch^{r+2}\|\mathcal{L}'_\tau\|\cdot\int_\tau |\Phi|d\tau + ch\|\mathcal{L}'_\tau\|\cdot ch^{r+1}\cdot \Big(h\|D^{r+2}g\|_\infty + \|D^{r+1}g\|_\infty\Big)\cdot\int_\tau |D\Phi|d\tau \tag{3.23}$$

The term on the right of (3.23) is bounded by

$$ch^{r+2}\left[\int_\tau (|\Phi| + |D\Phi|)d\tau\right]\cdot\max_\tau\Big\{|D^{r+1}g|, |D^{r+2}g|\Big\}$$

which proves (3.16).

□

This result can be extended to general triangles, but then the derivatives of g and Φ will involve the mapping m_τ from (2.37). Let $h(\tau)$ denote the diameter of τ and $h^*(\tau)$ the radius of the circle inscribed in τ and tangent to its sides. Define

$$r(\tau) = \frac{h(\tau)}{h^*(\tau)} \tag{3.24}$$

Assume that for our triangulations $\mathcal{T}_n = \{\Delta_{n,k}\}$, $n \geq 1$, we have

$$\sup_n\left[\max_{\Delta_{n,k}\in\mathcal{T}_n} r(\Delta_{n,k})\right] < \infty \tag{3.25}$$

Condition (3.25) prevents the triangles $\Delta_{n,k}$ from having angles which approach 0 as $n \longrightarrow \infty$. Then, Lemma 3.1.1 can be generalized to arbitrary triangles as follows

Corollary 3.1.2 *Let τ be a planar triangle of diameter h, let $g \in C^{r+2}(\tau)$ and $\Phi \in C^1(\tau)$. Assume $\alpha = \alpha_0$. Then*

$$\left|\int_\tau \Phi(x,y)(\mathcal{I}-\mathcal{L}_\tau)g(x,y)d\tau\right| \leq c(r(\tau))h^{r+2}\left[\int_\tau (|\Phi| + |D\Phi|)d\tau\right]\cdot\max_\tau\Big\{|D^{r+1}g|, |D^{r+2}g|\Big\} \tag{3.26}$$

where $c(r(\tau))$ is some function of $r(\tau)$, with $r(\tau)$ from (3.24).

Proof: Let $\bar\tau$ be a right triangle. Then using a mapping of the form (2.37) , $m_{\bar\tau} : \bar\tau \longrightarrow \tau$, we can write

$$\int_\tau \Phi(x,y)(\mathcal{I}-\mathcal{L}_\tau)g(x,y)d\tau = |(D_s m_k \times D_t m_k)|\cdot\int_{\bar\tau}\Phi(m_{\bar\tau}(s,t))(\mathcal{I}-\mathcal{L}_\tau)g(m_{\bar\tau}(s,t))d\bar\tau \tag{3.27}$$

which shows that this case can be reduced to the case where τ is a right triangle whose two sides which form the right angle have length h, keeping in mind that the derivatives of Φ and g will depend on $r(\tau)$. Note that in this case $D_s m_k \times D_t m_k$ is a constant.

$\square$

We want to apply the above results to the individual subintegrals in

$$\begin{aligned} \mathcal{K}u(v_i) &= \sum_{k=1}^{n} \int_{\sigma} K(v_i, m_k(s,t))u(m_k(s,t)) \cdot \\ & \qquad |(D_s m_k \times D_t m_k)(s,t)| \, d\tau \end{aligned} \tag{3.28}$$

Let

$$\begin{aligned} g(x,y) &= u(m_k(s,t)) \, |(D_s m_k \times D_t m_k)(s,t)| \\ \Phi(x,y) &= K(v_i, m_k(s,t)) \end{aligned} \tag{3.29}$$

Then, with the definition of $\mathcal{L}_\tau$ given in (3.13), the term in the right side of (3.10), $|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)u(v_i)|$ can be bounded by

$$\sum_{k=1}^{n} \left| \int_{\Delta_k} \Phi(x,y)(\mathcal{I} - \mathcal{L}_\tau)g(x,y)d\tau \right| \tag{3.30}$$

In the following, by $g \in C^k(S)$ we mean $g \in C(S)$ and $g \in C^k(S_j)$ (i.e $g \circ F_j \in C^k(R_j)$), $j = 1, ..., J$, for R_j and F_j as in (2.35) and (2.36).

Theorem 3.1.3 *Assume the hypotheses of Theorem 2.2.1 with each parametrization function $F_j \in C^{r+3}$, assume $u \in C^{r+2}(S)$ and $K \in C^1(S)$ with respect to Q. Assume the triangulation $\mathcal{T}_n$ of S satisfies (3.25). Then*

$$\max_{1 \le i \le n f_r} |u(v_i) - \hat{u}_n(v_i)| \le c h^{r+2} \tag{3.31}$$

Proof: Following (3.10), we will bound

$$\max_{1 \le i \le n f_r} |\mathcal{K}(\mathcal{I} - \mathcal{P}_n)u(v_i)|$$

using (3.30). On each triangle Δ_k, apply Lemma 3.1.1 or Corollary 3.1.2. ($c(r(\tau))$ of Corollary 3.1.2 will be denoted c to simplify the notation.) Since $u \in C^{r+2}(S)$ and

$K \in C^1(S)$ with respect to Q, we have that

$$|D_Q K|, \left|D^i u\right|, \quad i = r+1, r+2 \tag{3.32}$$

are bounded.

Then, by (3.30)

$$\max_{1 \le i \le nf_r} |\mathcal{K}(\mathcal{I} - \mathcal{P}_n)u(v_i)| \le \sum_{k=1}^{n} ch^{r+2} \int_{\Delta_k} d\tau \tag{3.33}$$

Since there are $n = O(h^{-2})$ triangles, and the integral in the right side of (3.35) is the area of Δ_k, which is $O(h^2)$, (3.35) leads to

$$\max_{1 \le i \le nf_r} |\mathcal{K}(\mathcal{I} - \mathcal{P}_n)u(v_i)| \le ch^{r+2} \tag{3.34}$$

By (3.10), this proves (3.31).

Note that although in this case the result (3.30) can be proven in an easier fashion, we prefer to give this proof, since we want to use it later for other cases.

□

So, for $\alpha = \alpha_0$, the collocation method defined by (3.3) is superconvergent. These results can still be improved, sometimes, using symmetric triangles. This is discussed more in Section 3.3.

3.2 The Radiosity Equation and Its Properties

Radiosity, an important quantity in image synthesis, is defined as being the energy per unit solid angle that leaves a surface. The photometric equivalent is *luminosity*. The *radiosity equation* is a mathematical model for the brightness of a collection of one or more surfaces. The equation is

$$u(P) - \frac{\rho(P)}{\pi} \int_S u(Q) G(P,Q) V(P,Q) dS_Q = E(P), \quad P \in S \tag{3.35}$$

where $u(P)$ is the *radiosity*, or the brightness, at $P \in S$. $E(P)$ is the *emissivity* at $P \in S$, the energy per unit area emitted by the surface.

The function $\rho(P)$ gives the *reflectivity* at $P \in S$, i. e. the bidirectional reflection distribution function. We have that $0 \leq \rho(P) < 1$, with $\rho(P)$ being 0 where there is no reflection at all at P. The radiosity equation is derived from the rendering equation under the *radiosity assumption*: all surfaces in the environment are *Lambertian diffuse reflectors*. What this means is that the reflectivity $\rho(P)$ is independent of the incoming and outgoing directions and, hence, of the angle at which the reflection takes place. Thus, $\rho(P)$ can be taken out from under the integral of a more general formulation (the rendering equation, see Cohen and Wallace[9]), leading to (3.35).

The function G, a geometric term, is given by

$$\begin{aligned} G(P,Q) &= \frac{[(Q-P)\cdot \mathbf{n}_P]\,[(P-Q)\cdot \mathbf{n}_Q]}{|P-Q|^4} \\ &= \frac{\cos\theta_P \cdot \cos\theta_Q}{|P-Q|^2} \end{aligned} \tag{3.36}$$

where $\mathbf{n}_P$ is the inner unit normal to S at P, θ_P is the angle between $\mathbf{n}_P$ and $Q-P$, and $\mathbf{n}_Q$ and θ_Q are defined analogously.

The function $V(P,Q)$ is a *visibility* function. It is 1 if the points P and Q are "mutually visible" (meaning they can "see each other" along a straight line segment which does not intersect S at any other point), and 0 otherwise. Surfaces S for which $V \equiv 1$ on S are called *unoccluded*, and this is the case that we will consider here. More about the radiosity equation can be found in Cohen and Wallace[9].

We can write (3.35) in the form

$$u(P) - \int_S K(P,Q)u(Q)dS_Q = E(P), \quad P \in S \tag{3.37}$$

with

$$K(P,Q) = \frac{\rho(P)}{\pi}G(P,Q)V(P,Q), \quad P,Q \in S \tag{3.38}$$

or, in operator form

$$(\mathcal{I} - \mathcal{K})u = E \tag{3.39}$$

3.2.1 Properties of the Radiosity Equation

We consider only the case that S is a smooth surface, although it need not be connected. The properties of the integral operator $\mathcal{K}$ are not yet fully understood when S is not a smooth surface, but they appear to be similar to the properties of the double layer boundary integral operator on piecewise smooth surfaces from the subject of potential theory.

Assume S has a local representation at each $P_0 \in S$, i.e. there is a plane tangent to S at P_0 with the surface given locally by

$$\zeta = f(\xi, \eta), \quad (\xi, \eta) \text{ in a neighborhood about } P_0 \tag{3.40}$$

We need to assume that each such function f is several times differentiable. Let S be a smooth unoccluded surface in $\mathbb{R}^3$. Decompose S into a finite union

$$S = S_1 \cup \cdots \cup S_J \tag{3.41}$$

with each S_j a smooth surface and intersecting each other along common edges at most. Consider a parametrization function

$$F_j : R_j \xrightarrow[onto]{1-1} S_j \tag{3.42}$$

with R_j a closed simply connected polygon in $\mathbb{R}^2$ and F_j a function having a certain degree of smoothness (later, in Section 3.3 we will impose conditions on the smoothness of F_j). Then having triangulations for the regions R_j, $j = 1, ..., J$ will enable us to produce a triangulation for S, as described earlier in Section 2.1.3, following (2.36).

The function $G(P, Q)$ given in (3.36) has a singularity at $P = Q$ and is smooth otherwise. We also have

$$|G(P, Q)| \le c, \quad P, Q \in S, \quad P \ne Q \tag{3.43}$$

since

$$|\cos\theta_P| \le c\,|P - Q|\,, \quad |\cos\theta_Q| \le \ c\,|P - Q| \tag{3.44}$$

where c denotes a generic constant independent of P and Q. For the proof of (3.44), see Mikhlin[15, pp. 345-349].

If the surface S is smooth and since formula (3.43) holds, it is relatively easy to prove that the integral operator $\mathcal{K}$ of (3.39) is compact as an operator on either $C(S)$ or $L^2(S)$ into itself (see Mikhlin[15, pp. 160-162]).

Next, let us examine the norm of $\mathcal{K}$ when considered as an operator from $C(S)$ to $C(S)$. We have the following, which is proven in Atkinson and Chandler[5].

Lemma 3.2.1 *Let S be the boundary of a convex open set Ω and assume S is a surface to which the Divergence Theorem can be applied. Let $P \in S$, and let S be smooth in an open neighborhood of P. Then*

$$G(P,Q) \geq 0, \quad \text{for} \quad Q \in S \tag{3.45}$$

and

$$\int_S G(P,Q)dS_Q = \pi \tag{3.46}$$

It then follows that

$$K(P,Q) \geq 0, \quad P,Q \in S \tag{3.47}$$

since $V(P,Q)$ and $\rho(P)$ are also nonnegative functions. In the case where S is the unit sphere $x^2+y^2+z^2=1$, a straightforward computation shows that $G(P,Q) \equiv \dfrac{1}{4}$.

3.2.2 Solvability and Regularity of the Radiosity Equation

The solvability theory for the radiosity equation (3.35) is relatively straightforward, being based on the Geometric Series Theorem.

Let S be a smooth unoccluded surface (not necessarily connected). Thus the normal $\mathbf{n}_P$ is to be a continuous function of $P \in S$. In addition to the *radiosity assumption* (discussed at the beginning of 3.2), we will also assume that the reflectivity function $\rho(P) \in C(S)$ and that it satisfies

$$\|\rho\|_\infty < 1 \tag{3.48}$$

From the physical point of view, what (3.48) means is that the surface does not reflect 100% of all the light that it receives, which is a reasonable assumption.

For the regularity of the solution of (3.35), we have

Lemma 3.2.2 *Let $m \geq 0$ be an integer, S a smooth surface satisfying (3.41), with the parametrization functions of (3.42) $F_j \in C^{m+1}(R_j), j = 1, ..., J$. Also, assume the reflectivity function $\rho \in C^{m+1}(S)$. Then*

$$u \in C^m(S) \Rightarrow \mathcal{K}u \in C^{m+1}(S) \tag{3.49}$$

Proof: The proof of this result is based mainly on the fact that

$$\frac{\partial^i G(P,Q)}{\partial P^i} = O\left(\frac{1}{|P-Q|^i}\right) \tag{3.50}$$

which is proven later (see Theorem 3.3.4). In this work, by $\dfrac{\partial F(P)}{\partial P}$ we denote generically the derivatives $\dfrac{\partial F(P)}{\partial x}, \dfrac{\partial F(P)}{\partial y}$, where $P = P(x, y)$.

To get an idea of how the proof goes, consider the case $m = 0$. We have

$$\frac{\partial G(P,Q)}{\partial P} = O\left(\frac{1}{|P-Q|}\right) \tag{3.51}$$

From this we can obtain that

$$\int\limits_S \frac{\partial G(P,Q)}{\partial P} u(Q) dS \in C(S) \tag{3.52}$$

by an argument similar to that of Mikhlin[15, pp. 363-365]. For $m > 0$, one can use an argument similar to that of Günter[11, p. 49] to show the result.

□

It is worth mentioning that using results from potential theory, it is likely that $u \in C^m(S)$ implies something like $\mathcal{K}u \in C^{m+2}(S)$. We do not need such a result, so we do not investigate it further.

Theorem 3.2.3 *Let $m \geq 0$ be an integer. Let $\hat{S}$ be the boundary of a convex open* *set Ω, and assume $\hat{S}$ is a surface to which the Divergence Theorem can be applied. Assume S is a smooth (possibly disconnected) unoccluded surface $S \subset \hat{S}$ that can be represented as in (3.41) with each parametrization function of (3.42) $F_j \in C^{m+2}(R_j)$. Also, assume $\rho, E \in C^m(S)$. Then*

(a) The equation (3.35) is uniquely solvable for each E, with the solution $u(P)$ satisfying

$$\|u\|_\infty \leq \frac{\|E\|_\infty}{1 - \|\mathcal{K}\|} \tag{3.53}$$

(b) The solution $u \in C^m(S)$.

Proof: (a) Since $\rho(P)$ is a continuous function, using Lemma 3.2.1 it follows that $\mathcal{K} : C(S) \longrightarrow C(S)$ is a bounded compact operator with

$$\|\mathcal{K}\| \leq \|\rho\|_\infty \tag{3.54}$$

Then by the assumption (3.48), we have

$$\|\mathcal{K}\| < 1 \tag{3.55}$$

Using the Geometric Series Theorem, we have that the operator $\mathcal{I} - \mathcal{K} : C(S) \longrightarrow C(S)$ is invertible with

$$\left\|(\mathcal{I} - \mathcal{K})^{-1}\right\| \leq \frac{1}{1 - \|\mathcal{K}\|} \tag{3.56}$$

Thus, the equation (3.35) is uniquely solvable for all emissivity functions $E \in C(S)$. Formula (3.53) follows from (3.39) and (3.56).

(b) For $m = 0$, the result follows from part (a). If $m > 0$, write (3.39) in the form

$$u = E + \mathcal{K}u$$

Use induction on m and Lemma 3.2.2 to show that $u \in C^m(S)$.

□

The majority of applications are likely to have surfaces S that are only piecewise smooth. In this case, the function $G(P,Q)$ has singular behavior along all edges and corners, and as a consequence, the operator $\mathcal{K}$ is no longer as well-behaved as for the smooth case. This case is discussed in Atkinson and Chandler[5, Section 4], and Atkinson and Chien[7, Section 3] .

3.3 Superconvergent Collocation Methods for the Radiosity Equation

A superconvergent piecewise linear collocation method for the radiosity equation was developed in Atkinson and Chandler[5]. Following the same ideas we will investigate superconvergent methods based on interpolation of higher degree of the solution of (3.35).

As in Chapter II, consider S a surface satisfying (3.41) and (3.42). Let

$$\left\{ \widehat{\Delta}^j_{n,k} \mid k = 1, ..., n_j \right\} \tag{3.57}$$

be a triangulation of R_j, which will yield a triangulation

$$\left\{ \Delta^j_{n,k} \mid k = 1, ..., n_j \right\}, \quad \Delta^j_{n,k} = F_j \left(\widehat{\Delta}^j_{n,k} \right), \quad k = 1, ..., n_j \tag{3.58}$$

of the subsurface S_j. Then for S as a whole, define

$$\mathcal{T}_n = \bigcup_{j=1}^{J} \left\{ \Delta^j_{n,k} \mid k = 1, ..., n_j \right\} \tag{3.59}$$

Let

$$h \equiv h_n = \max_{1 \le j \le J} \max_{1 \le k \le n_j} diameter \left(\widehat{\Delta}^j_{n,k} \right) \tag{3.60}$$

be the mesh size of this triangulation. (The number of triangles n is to be understood implicitly; from now on, we dispense with it.)

As in Section 3.1 and earlier in Section 2.1, let α be a constant with $0 < \alpha < \frac{1}{3}$ and define the quadratic interpolation nodes in $\sigma = \{(s,t) \mid 0 \le s, t, s+t \le 1\}$, $\{q_1, ..., q_6\}$ as in (2.15). Define corresponding Lagrange interpolation basis functions

$l_1(s,t),...,l_6(s,t)$ as in (2.16). For $g \in C(S)$, define the quadratic interpolating polynomial

$$(\mathcal{P}_n g)(m_k(s,t)) = \sum_{j=1}^{6} g(m_k(q_j))\, l_j(s,t), \quad (s,t) \in \sigma \tag{3.61}$$

with the mapping $m_k : \sigma \xrightarrow[onto]{1-1} \Delta_k^j$ defined in (2.37) and approximate

$$g(P) \approx (\mathcal{P}_n g)(P), \quad P = m_k(s,t) \in \Delta_k \tag{3.62}$$

which leads to

$$\int_{\Delta_k} g(Q) dS_Q \approx \sum_{j=1}^{6} g(m_k(q_j)) \int_{\sigma} l_j(s,t) \left|(D_s m_k \times D_t m_k)(s,t)\right| d\sigma \tag{3.63}$$

After a lengthy calculation, we have

$$\|\mathcal{P}_n\| = \begin{cases} \dfrac{5}{3} & , \text{ if } 0 < \alpha < \dfrac{15 - 8\sqrt{3}}{33} \\ \\ \dfrac{1 + 10\alpha - 7\alpha^2}{(1-3\alpha)^2} & , \text{ if } \dfrac{15 - 8\sqrt{3}}{33} < \alpha < \dfrac{1}{3} \end{cases} \tag{3.64}$$

We define a collocation method with (3.61). Substitute

$$u_n(P) = \sum_{j=1}^{6} u_n(v_{k,j} l_j)(s,t), \quad P = m_k(s,t) \in \Delta_k, \quad k = 1,...,n \tag{3.65}$$

into (3.35), with $V \equiv 1$ for an unoccluded surface. Then determine the values $\{u_n(v_{k,j})\}$ by forcing the equation (3.35) to be true at the collocation nodes, i.e. solve the linear system

$$\begin{aligned} u_n(v_i) \quad - \quad & \frac{\rho(P)}{\pi} \sum_{k=1}^{n} \sum_{j=1}^{6} u_n(v_{k,j}) \int_{\sigma} G(v_i, m_k(s,t))\, l_j(s,t) \\ \cdot \quad & \left|(D_s m_k \times D_t m_k)(s,t)\right| d\sigma = E(v_i), \quad i = 1,...,6n \end{aligned} \tag{3.66}$$

This can be written abstractly as

$$(\mathcal{I} - \mathcal{P}_n \mathcal{K}) u_n = \mathcal{P}_n E \tag{3.67}$$

Also, introduce the iterated collocation solution

$$\hat{u}_n = E + \mathcal{K} u_n \tag{3.68}$$

The collocation solution u_n and the iterated collocation solution $\hat{u}_n$ are related by formulas (2.54). Also,

$$(\mathcal{I} - \mathcal{K}\mathcal{P}_n)\hat{u}_n = E \tag{3.69}$$

The operator

$$\mathcal{K}\mathcal{P}_n : C(S) \longrightarrow C(S) \tag{3.70}$$

is a numerical integral operator based on *product integration*. Thus an error analysis for (3.66) can be based on the general theory for such numerical integral operators (e. g. see Atkinson[4, Section 4.2]). We will give an error analysis based on standard projection operator theory instead. We have

Theorem 3.3.1 *Assume S is a smooth unoccluded surface in $\mathbb{R}^3$, and assume $S \subset \hat{S}$, with $\hat{S}$ the type of surface required in Lemma 3.2.1. Assume S satisfies (3.41) and (3.42) with each $F_j \in C^4$. Then for all sufficiently large n, say $n \geq n_0$, the operators $\mathcal{I} - \mathcal{P}_n\mathcal{K}$ are invertible on $C(S)$ and have uniformly bounded inverses. Moreover, for the true solution u of (3.35) and the solution u_n of (3.67)*

$$\|u - u_n\|_\infty \leq \left\|(\mathcal{I} - \mathcal{P}_n\mathcal{K})^{-1}\right\| \|(u - \mathcal{P}_n u)\|_\infty, \quad n \geq n_0 \tag{3.71}$$

Furthermore, if the emissivity $E \in C^3(S)$, then

$$\|u - u_n\|_\infty \leq O(h^3), \quad n \geq n_0 \tag{3.72}$$

Proof: The proof follows the standard type of collocation error analysis, with $\mathcal{P}_n$ considered as a projection operator from $L^\infty(S)$ onto itself (see Section 2.1.4). Using a standard continuity argument we can show that $u \in L^\infty(S)$ implies $\mathcal{K}u \in C(S)$ and that $\mathcal{K} : L^\infty(S) \longrightarrow C(S)$ is a compact operator. It follows then that

$$\mathcal{P}_n\phi \longrightarrow \phi, \quad \phi \in C(S) \tag{3.73}$$

Then by Lemma 2.1.5 we have that

$$\|\mathcal{P}_n\mathcal{K} - \mathcal{K}\|_\infty \longrightarrow 0, \quad \text{as} \quad n \longrightarrow 0 \tag{3.74}$$

which by Theorem 2.1.4 proves our assertion. The bound (3.72) follows from the fact that we are using quadratic interpolation.

□

3.3.1 Two Superconvergent Piecewise Quadratic Collocation Methods

The first superconvergent method based on quadratic interpolation that we want to discuss is very simple, not requiring a special value for α, but using a symmetric triangulation. First, recall that the interpolation formula

$$g(s,t) \approx \sum_{j=1}^{6} g(q_i) l_j(s,t) \tag{3.75}$$

has degree of precision 2 for any $0 < \alpha < \dfrac{1}{3}$, and so does the quadrature formula

$$\int_\sigma g(s,t)d\sigma \approx \sum_{j=1}^{6} g(q_i) \int_\sigma l_j(s,t)d\sigma \tag{3.76}$$

However, extending formula (3.76) to an integration formula over $U \equiv [0,1] \times [0,1]$ or $R = \sigma \cup \bar{\sigma}$, and considering 6 more nodes (the points symmetric to the nodes about the point $\left(\frac{1}{2}, \frac{1}{2}\right)$, and the origin, respectively), then formula (3.76) has degree of precision 3.

Let

$$\mathcal{L}_\tau g(x,y) = \sum_{j=1}^{6} g\left(m_\tau(q_i)\right) l_j(s,t), \quad (x,y) = m_\tau(s,t) \tag{3.77}$$

for $g \in C(\tau)$, with $m_\tau : \sigma \xrightarrow[onto]{1-1} \tau$ of (2.37). We have the following.

Lemma 3.3.2 *Let τ_1 and τ_2 be planar right triangles that form a square R of length h on a side. Let $g \in C^4(R)$. Let $\Phi \in L^1(R)$ differentiable with first derivatives $D_x\Phi$, $D_y\Phi$ in $L^1(R)$. Then*

$$\left| \int_R \Phi(x,y)(I - \mathcal{L}_\tau)g(x,y)d\tau \right| \le ch^4 \left[\int_R (|\Phi| + |D\Phi|)d\tau \right] \cdot \max_R \left\{ |D^3 g|, |D^4 g| \right\} \tag{3.78}$$

with $\mathcal{L}_\tau g(x,y) \equiv \mathcal{L}_{\tau_i} g(x,y)$, where $(x,y) \in \tau_i$, $i = 1, 2$.

Proof: Let $p_2(x,y)$, $p_3(x,y)$ denote Taylor expansions around a suitable center, of degree 2 and 3 of g over R. As before we have

$$\|g - p_i\|_\infty \le ch^{i+1} \|D^{i+1} g\|_\infty, \quad i = 2, 3 \tag{3.79}$$

From (3.79) it follows that

$$\|p_3 - p_2\|_\infty \leq ch^3 \left(h\|D^4 g\|_\infty + \|D^3 g\|_\infty \right) \tag{3.80}$$

Also, there is a constant Φ_0 such that

$$\|\Phi - \Phi_0\|_1 \leq ch\|D\Phi\|_1 \tag{3.81}$$

Let $\mathcal{L}_\tau^{'} = \mathcal{I} - \mathcal{L}_\tau$. We can write

$$\begin{aligned} \int\limits_R \Phi \mathcal{L}_\tau^{'} g d\tau &= \int\limits_R \Phi \mathcal{L}_\tau^{'} (g - p_3) d\tau \\ &+ \int\limits_R (\Phi - \Phi_0) \mathcal{L}_\tau^{'} (p_3 - p_2) d\tau \end{aligned} \tag{3.82}$$

The reason why (3.82) is true is because

$$\mathcal{L}_\tau^{'} p_2 = 0 \tag{3.83}$$

since formula (3.75) has degree of precision 2. Also, since formula (3.76) has degree of precision 3, and Φ_0 is a constant we have that

$$\int\limits_R \Phi_0 \mathcal{L}_\tau^{'} p_3 d\tau = 0 \tag{3.84}$$

Taking norms in (3.82) and using the bounds in (3.79)-(3.81), we have

$$\begin{aligned} \left| \int\limits_R \Phi \mathcal{L}_\tau^{'} g d\tau \right| &\leq ch^4 \|\mathcal{L}_\tau^{'}\| \cdot \int\limits_R |\Phi| d\tau + ch\|\mathcal{L}_\tau^{'}\| \cdot ch^3 \cdot \\ &\left(h\|D^4 g\|_\infty + \|D^3 g\|_\infty \right) \cdot \int\limits_R |D\Phi| d\tau \end{aligned} \tag{3.85}$$

The term on the right of (3.85) is bounded by

$$ch^4 \left[\int\limits_R (|\Phi| + |D\Phi|) d\tau \right] \cdot \max_R \left\{ |D^3 g|, |D^4 g| \right\}$$

which proves (3.78).

□

If integrating over just one triangle, by a similar argument we can prove the following.

Lemma 3.3.3 *Let τ be a planar right triangle, and assume the two sides which form* *the right angle have length h. Let $g \in C^3(\tau)$ and $\Phi \in L^1(\tau)$. Then*

$$\left| \int_\tau \Phi(x,y)(I - \mathcal{L}_\tau) g(x,y) d\tau \right| \leq ch^3 \left[\int_\tau |\Phi| d\tau \right] \cdot \max_\tau \left\{ |D^3 g| \right\} \tag{3.86}$$

As before, these results can be extended to general triangles, with c replaced by $c(r(\tau))$ or $c(r(R))$, respectively.

Now, we want to apply these results to the individual subintegrals in

$$\begin{aligned} \mathcal{K}u(v_i) &= \frac{\rho(v_i)}{\pi} \sum_{k=1}^{n} \int_\sigma G\left(v_i, m_k(s,t)\right) u\left(m_k(s,t)\right) \\ &\cdot \quad \left|\left(D_s m_k \times D_t m_t\right)(s,t)\right| d\sigma, \quad i = 1, \ldots, 6n \end{aligned} \tag{3.87}$$

with

$$\begin{aligned} g(s,t) &= u\left(m_k(s,t)\right) \left|\left(D_s m_k \times D_t m_t\right)(s,t)\right| \\ \Phi(s,t) &= G\left(v_i, m_k(s,t)\right) \end{aligned} \tag{3.88}$$

For that we need some information about the derivatives of $G(P,Q)$ as $Q \longrightarrow P$. We have the following result.

Theorem 3.3.4 *Let $i \geq 0$ be an integer and let S be a smooth C^{i+1} surface. Then*

$$\left| D_Q^i G(P,Q) \right| \leq \frac{c}{|P-Q|^i}, \quad P \neq Q \tag{3.89}$$

for the function $G(P,Q)$ of (3.35), with c a generic constant independent of P and Q.

Proof: The proof of this theorem is rather long and elaborate, and for this reason we give it in a separate section, Section 3.3.2.

□

Now, we can prove our superconvergence result.

Theorem 3.3.5 *Assume the hypotheses of Theorem 3.3.1, with each parametrization function $F_j \in C^6(S)$. Assume $u \in C^4(S)$. Assume the triangulation $\mathcal{T}_n$ of S satisfies (3.25) and that it is symmetric. For those integrals in (3.87) for which $v_i \in \Delta_k$,* *assume that all such integrals are evaluated with an error of* $O(h^4)$. *Then*

$$\max_{1\le i\le 6n} |u(v_i) - u_n(v_i)| \le ch^4 \tag{3.90}$$

Proof: As in the proof of Theorem 3.1.3, we will bound

$$\max_{1\le i\le 6n} |\mathcal{K}(\mathcal{I} - \mathcal{P}_n)u(v_i))|$$

For a given node point v_i, denote Δ^* the triangle containing it and denote

$$\mathcal{T}_n^* = \mathcal{T}_n - \{\Delta^*\}$$

By our assumption, the error in evaluating the integral of (3.87) over Δ^* will be $O(h^4)$.

Partition $\mathcal{T}_n^*$ into parallelograms to the maximum extent possible. Denote by $\mathcal{T}_n^{(1)}$ the set of all triangles making up such parallelograms and let $\mathcal{T}_n^{(2)}$ contain the remaining triangles. Then

$$\mathcal{T}_n^* = \mathcal{T}_n^{(1)} \cup \mathcal{T}_n^{(2)}$$

It is easy to show that the number of triangles in $\mathcal{T}_n^{(1)}$ is $O(n) = O(h^{-2})$, and the number of triangles in $\mathcal{T}_n^{(2)}$ is $O(\sqrt{n}) = O(h^{-1})$.

It can be shown that all but a finite number of the triangles in $\mathcal{T}_n^{(2)}$, bounded independent of n, will be at a minimum distance from v_i. That means that the triangles in $\mathcal{T}_n^{(2)}$ are "far enough" from v_i, so that the function $G(v_i, Q)$ is uniformly bounded for Q being in a triangle in $\mathcal{T}_n^{(2)}$.

First, consider the contribution to the error coming from the triangles in $\mathcal{T}_n^{(2)}$. By Lemma 3.3.3 the error over each such triangle is $O\left(h^5\|D^5u\|_\infty\right)$, since the area of each triangle is $O(h^2)$ and using our earlier observation. Having $O(h^{-1})$ such triangles in $\mathcal{T}_n^{(2)}$, the total error coming from triangles in $\mathcal{T}_n^{(2)}$ is $O\left(h^4\|D^5u\|_\infty\right)$.

Next, consider the contribution to the error coming from triangles in $\mathcal{T}_n^{(1)}$. By Lemma 3.3.2, the error will be of size $O(h^4)$ multiplied times the integral over each such parallelogram of the maximum of the first derivatives of $G(v_i, Q)$ with respect to Q. Combining these we will have a bound

$$ch^4 \int\limits_{S-\Delta^*} (|G| + |DG|)\, dS_Q \tag{3.91}$$

By Theorem 3.3.4, the quantity in (3.91) is bounded by

$$ch^4 \int\limits_{S-\Delta^*} \left(1 + \frac{1}{|v_i - Q|}\right) dS_Q \tag{3.92}$$

Using a local representation of the surface and then using polar coordinates, the expression in (3.92) is of order

$$ch^4 \left(h^2 + h\right) + ch^4$$

Thus, the error arising from the triangles in $\mathcal{T}_n^{(1)}$ is $O(h^4)$.

Combining the errors arising from the integrals over Δ^*, $\mathcal{T}_n^{(1)}$, and $\mathcal{T}_n^{(2)}$, we have the bound (3.90).

□

For collocation on piecewise smooth functions, see Atkinson and Chandler[5, Section 4]. Here, we only state without proof a convergence result for the collocation method (3.65).

Theorem 3.3.6 *Assume S is a piecewise smooth unoccluded surface in $\mathbb{R}^3$, and assume $S \subset \hat{S}$, with $\hat{S}$ the type of surface required by Lemma 3.2.1. Assume the surface S satisfies (3.41) and (3.42) with each $F_j \in C^4$. For the interpolation method of (3.65), assume*

$$\|\mathcal{P}_n\|\|\mathcal{K}\| \leq \gamma < 1, \quad n \geq n_0 \tag{3.93}$$

for some constant γ and some $n_0 > 0$. The norm $\|\mathcal{P}_n\|$ is given in (3.64) and a bound for $\|\mathcal{K}\|$ is given in (3.54) and (3.55). Then for sufficiently large n, say $n \geq n_0$, the operator $\mathcal{I} - \mathcal{P}_n\mathcal{K}$ are invertible on X and have uniformly bounded inverses. Moreover, for the true solution u of (3.35) and the solution u_n of (3.67),

$$\|u - u_n\|_\infty \leq \left\|(\mathcal{I} - \mathcal{P}_n\mathcal{K})^{-1}\right\| \|u - \mathcal{P}_n u\|_\infty, \quad n \geq n_0 \tag{3.94}$$

Furthermore, if the emissivity $E \in C^3(S)$, then

$$\|u - u_n\|_\infty \le O(h^3), \quad n \ge n_0 \tag{3.95}$$

Next we will develop another superconvergent collocation method based on piecewise quadratic interpolation of the solution. This time, we will increase the order of the quadrature formula, first, by fixing α. Recall the interpolation formula (3.61)

$$g(s,t) \approx (\mathcal{L}_\sigma g)(s,t) \equiv \sum_{j=1}^{6} g(q_j) l_j(s,t)$$

of degree of precision 2 for any $0 < \alpha < \frac{1}{3}$. The formula

$$\int_\sigma g(s,t)d\sigma \approx \int_\sigma \mathcal{L}_\sigma g(s,t)d\sigma \tag{3.96}$$

also has degree of precision 2 for general α. However, for $\alpha = \alpha_0 \doteq 0.103583400062101$, formula (3.96) has degree of precision 4, since it is exact for σ_1, σ_2, and σ_1^2. Extending it over symmetric triangles, it has then degree of precision 5.

With the same notation as before we now have the following

Lemma 3.3.7 *Let τ_1 and τ_2 be planar right triangles that form a square R of length h on a side. Let $g \in C^6(R)$. Let $\Phi \in L^1(R)$ be three times differentiable with partial derivatives of order 1, 2, and 3 in $L^1(R)$. Assume $\alpha = \alpha_0$. Then*

$$\left| \int_R \Phi(x,y)(I - \mathcal{L}_\tau)g(x,y)d\tau \right| \le ch^6 \left[\int_R \sum_{i=0}^{3} |D^i\Phi| d\tau \right] \cdot \max_{\substack{R \\ i=3,\dots,6}} \left\{ |D^i g| \right\} \tag{3.97}$$

with $\mathcal{L}_\tau g(x,y) \equiv \mathcal{L}_{\tau_i} g(x,y)$, where $(x,y) \in \tau_i$, $i = 1,2$.

Proof: Consider Taylor polynomials $p_i(x,y)$ of degree i, for $i = 2, ...5$ such that

$$\|g - p_i\|_\infty \le ch^{i+1} \|D^{i+1}g\|_\infty, \quad i = 2, ..., 5 \tag{3.98}$$

Then

$$\|p_{k+1} - p_k\|_\infty \le ch^{k+1} \left(h\|D^{k+2}g\|^\infty + \|D^{k+1}g\|_\infty \right) \text{ for } \quad k = 2,3,4 \tag{3.99}$$

In addition, let $\Phi_i(x,y)$ be polynomials of degree i over τ satisfying

$$\|\Phi - \Phi_i\|_1 \le ch^{i+1} \|D^{i+1}\Phi\|_1, \quad i = 0,1,2 \tag{3.100}$$

Write

$$\begin{aligned}
\int\limits_R \Phi \mathcal{L}_\tau' g d\tau &= \int\limits_R \Phi \mathcal{L}_\tau'(g - p_5) d\tau \\
&+ \int\limits_R (\Phi - \Phi_0)\mathcal{L}_\tau'(p_5 - p_4) d\tau \\
&+ \int\limits_R (\Phi - \Phi_1)\mathcal{L}_\tau'(p_4 - p_3) d\tau \\
&+ \int\limits_R (\Phi - \Phi_2)\mathcal{L}_\tau'(p_3 - p_2) d\tau \qquad (3.101) \\
&+ \int\limits_R (\Phi_1 - \Phi_0)\mathcal{L}_\tau' p_4 d\tau \\
&+ \int\limits_R (\Phi_2 - \Phi_1)\mathcal{L}_\tau' p_3 d\tau
\end{aligned}$$

To see why (3.101) is true, multiply out the terms on the right. After a series of cancellations, we get

$$\int\limits_R \Phi \mathcal{L}_\tau' g d\tau - \int\limits_R \Phi_0 \mathcal{L}_\tau' p_5 d\tau - \int\limits_R \Phi \mathcal{L}_\tau' p_2 d\tau + \int\limits_R \Phi_2 \mathcal{L}_\tau' p_2 d\tau \qquad (3.102)$$

The last two terms in (3.102) are 0 because formula (3.61) has degree of precision 2, and the second integral in (3.102) is 0 because Φ_0 is a constant and formula (3.96) has degree of precision 5.

Next we will show that the last two integrals in (3.101) are 0. We have

$$\mathcal{L}_\tau \left[(\Phi_1 - \Phi_0)p_4\right] = \mathcal{L}_\tau \left[(\Phi_1 - \Phi_0)\mathcal{L}_\tau p_4\right] \qquad (3.103)$$

since $[(\Phi_1 - \Phi_0)p_4]$ and $[(\Phi_1 - \Phi_0)\mathcal{L}_\tau p_4]$ agree at the collocation node points $\mu_j, \quad j = 1, ..., 12$ as we can see in the following

$$\begin{aligned}
\left[(\Phi_1 - \Phi_0)\mathcal{L}_\tau p_4\right](\mu_j) &= (\Phi_1 - \Phi_0)(\mu_j)(\mathcal{L}_\tau p_4)(\mu_j) \\
&= (\Phi_1 - \Phi_0)(\mu_j) p_4(\mu_j) \qquad (3.104) \\
&= \left[(\Phi_1 - \Phi_0)p_4\right](\mu_j)
\end{aligned}$$

Now, because the integration formula (3.96) has degree of precision 5 and

$\deg[(\Phi_1 - \Phi_0)\mathcal{L}_\tau p_4] \le 3$, we have

$$\int_R \mathcal{L}'_\tau \left[(\Phi_1 - \Phi_0)\mathcal{L}_\tau p_4\right] d\tau = 0$$

i.e.

$$\int_R (\Phi_1 - \Phi_0)\mathcal{L}_\tau p_4 d\tau = \int_R \mathcal{L}_\tau \left[(\Phi_1 - \Phi_0)\mathcal{L}_\tau p_4\right] d\tau \tag{3.105}$$

Next we can write

$$\begin{aligned}
\int_R (\Phi_1 - \Phi_0)\mathcal{L}'_\tau p_4 d\tau &= \int_R (\Phi_1 - \Phi_0) p_4 d\tau - \int_R (\Phi_1 - \Phi_0)\mathcal{L}_\tau p_4 d\tau \\
&= \int_R (\Phi_1 - \Phi_0) p_4 d\tau - \int_R \mathcal{L}_\tau \left[(\Phi_1 - \Phi_0)\mathcal{L}_\tau p_4\right] d\tau \\
&= \int_R (\Phi_1 - \Phi_0) p_4 d\tau - \int_R \mathcal{L}_\tau \left[(\Phi_1 - \Phi_0) p_4\right] d\tau \qquad (3.106) \\
&= \int_R \mathcal{L}'_\tau \left[(\Phi_1 - \Phi_0) p_4\right] d\tau \\
&= 0
\end{aligned}$$

where the second equality is true by (3.105), the third by (3.103) and the last one holds because $\deg[(\Phi_1 - \Phi_0)p_4] \le 5$.

A similar argument leads to

$$\int_R (\Phi_2 - \Phi_1)\mathcal{L}'_\tau p_3 d\tau = 0 \tag{3.107}$$

Now, take norms in (3.101) and use the bounds (3.98)-(3.100) to get (3.97).

□

If the integration is done over a single triangle, then we have the following estimate (the proof is similar to that just given).

Lemma 3.3.8 *Let τ be a planar right triangle and assume the two sides which form the right angle have length h. Assume $\alpha = \alpha_0$. Let $g \in C^5(\tau), \Phi \in L^1(\tau)$ twice differentiable with derivatives of order 1 and 2 in $L^1(\tau)$. Then*

$$\left|\int_\tau \Phi(x,y)(\mathcal{I} - \mathcal{L}_\tau) g(x,y) d\tau\right| \le ch^5 \left[\int_\tau \sum_{i=0}^{2} |D^i \Phi| d\tau\right] \cdot \max_{\substack{\tau \\ i=3,\dots,5}} \left\{|D^i g|\right\} \tag{3.108}$$

where c denotes a generic constant.

As described before, the last two results can be generalized to arbitrary triangles under the assumption (3.25).

In this case, we have the following superconvergence result.

Theorem 3.3.9 *Assume the hypotheses of Theorem 3.3.1, with each $F_j \in C^6$. Assume $u \in C^6(S)$. Assume the triangulation $\mathcal{T}_n$ of S satisfies (3.25) and that it is symmetric. For those integrals in (3.87) for which $v_i \in \Delta_k$, assume that all such integrals are evaluated with an error of $O(h^5)$. Then*

$$\max_{1\leq i\leq 6n} |u(v_i) - \hat{u}_n(v_i)| \leq ch^5 \tag{3.109}$$

Proof: We give bounds for

$$\max_{1\leq i\leq 10n} |\mathcal{K}(\mathcal{I} - \mathcal{P}_n)u(v_i))|$$

With the previous notations we have that the contribution to the error coming from the triangles in $\mathcal{T}_n^{(2)}$ is $O\left(h^5\|D^5u\|_\infty\right)$

The contribution to the error coming from triangles in $\mathcal{T}_n^{(1)}$ is

$$ch^6 \int_{S-\Delta^*} \sum_{j=0}^{3} \frac{1}{|v_i - Q|^j} dS_Q \tag{3.110}$$

Using a local representation of the surface and then using polar coordinates, the expression in (3.110) is of order

$$ch^6\left(h^2 + h + \log h + \frac{1}{h}\right)$$

Combining the errors we have (3.109).

□

Numerical Examples. As a smooth surface consider a "two-piece surface." Define

$$\begin{aligned} S_1 &= \{(x, y, 0) \mid 0 \leq x, y \leq 1\} \\ S_2^{(2)} &= \left\{(x, y, z) \mid 0 \leq x, y \leq 1,\ z = 2 - x^2\right\} \end{aligned} \tag{3.111}$$

and let $S^{(2)} = S_1 \cup S_{(2)}^2$.

We solve the radiosity equation (3.35) with the emissivity $E(P)$ so chosen that the true solution is

$$u(x,y,z) = \frac{1}{\sqrt{x^2 + y^2 + (z - 0.5)^2}} \tag{3.112}$$

The reflectivity function $\rho(P) \equiv 1$; for the solvability function of $(\mathcal{I} - \mathcal{K})u = E$, this is okay, since $\|\mathcal{K}\| < 1$ due to the surface not being closed. In Table 2, we give

$$\|u - u_n\|_\infty = \max_{1 \le i \le 6n} |u(v_i) - u_n(v_i)|$$

for $\alpha = \alpha_0$ and $\alpha = 0.2$. The results for $\alpha = 0.2$ are consistent with a convergence rate of $O(h^4)$ predicted by Theorem 3.3.5. The results for $\alpha = \alpha_0$ appear to agree with a convergence rate of $O(h^5)$ predicted by Theorem 3.3.9, which illustrates the superconvergence.

	$\alpha = \alpha_0$		$\alpha = 0.2$	
n	$\|u - u_n\|_\infty$	Ratio	$\|u - u_n\|_\infty$	Ratio
4	1.78 E-4		1.15 E-3	
16	1.14 E-5	15.6	9.85 E-5	11.73
64	4.53 E-7	25.1	7.62 E-6	12.92
256	1.73 E-8	26.2	5.02 E-7	15.17

Table 2: Errors in solving radiosity equation on $S^{(2)}$

As a simple piecewise smooth surface, we use the unit cube

$$S = [0,1] \times [0,1] \times [0,1]$$

The emissivity is chosen so that the true solution is

$$u(x,y,z) = \frac{1}{\sqrt{(x - 10)^2 + (y - 1)^2 + (z - 3)^2}} \tag{3.113}$$

The reflectivity function is $\rho \equiv 0.5$. The results for $\|u - u_n\|_\infty$ are shown in Table 3. The ratios approach 8 as n increases, which is consistent with a rate of convergence of $O(h^3)$ as predicted by Theorem 3.3.6. For this case we did not take higher values for n, because the system to be solved has order $6n$.

n	$\|u - u_n\|_\infty$	Ratio
12	6.39 E-8	
48	9.04 E-9	7.00
192	1.26 E-9	7.15

Table 3: Errors in solving radiosity equation on the unit cube

3.3.2 The Proof of Theorem 3.3.4

We want to prove (3.89)

$$\left|D_Q^i G(P,Q)\right| \le \frac{c}{|P-Q|^i}, \quad P \ne Q$$

We have by (3.36)

$$G(P,Q) = \frac{\cos\theta_P \cos\theta_Q}{|P-Q|^2}$$

Denote by

$$\begin{aligned} F^P(P,Q) &= \frac{\cos\theta_P}{|P-Q|} \\ F^Q(P,Q) &= \frac{\cos\theta_Q}{|P-Q|} \end{aligned} \tag{3.114}$$

Then, we can write

$$G(P,Q) = F^P(P,Q) \cdot F^Q(P,Q) \tag{3.115}$$

and we have

$$D_Q^n G = \sum_{k=1}^{n} \binom{n}{k} D_Q^{n-k}(F^P)(P,Q) \cdot D_Q^k(F^Q)(P,Q) \tag{3.116}$$

(This derivative notation is explained in the proof of Lemma 3.2.2).

By (3.44)

$$|\cos\theta_P|, |\cos\theta_Q| \le c|P-Q| \tag{3.117}$$

which leads to

$$|F^P|, |F^Q| \leq c \tag{3.118}$$

Claim:

$$\left|D_Q^i F^P\right|, \left|D_Q^i F^Q\right| \leq \frac{c}{|P-Q|^i} \tag{3.119}$$

Proof of claim: Fix $P \in S$. The proof of (3.119) is very delicate. We will use both a local parametrization of the surface as well as formal reasoning. Assume the surface S can be represented locally by

$$z = f(x, y) \tag{3.120}$$

with $f \in C^{i+2}$. We consider P to be the origin of a coordinate system and Q an arbitrary point in S. Then we have

$$\begin{aligned} P &= (0,0,0) \\ Q &= (x, y, f(x,y)) \\ \mathbf{n}_P &= (0,0,1) \\ \mathbf{n}_Q &= (-f_x(x,y), -f_y(x,y), 1) \end{aligned} \tag{3.121}$$

(Implicitly, we then also have that $f(0,0) = f_x(0,0) = f_y(0,0) = 0$). We can write

$$\begin{aligned} \cos\theta_P = \frac{(Q-P)\cdot \mathbf{n}_P}{|P-Q|\cdot|\mathbf{n}_P|} &= \frac{Q}{|Q|}\cdot N^P \\ \cos\theta_Q = \frac{(Q-P)\cdot \mathbf{n}_Q}{|P-Q|\cdot|\mathbf{n}_Q|} &= \frac{Q}{|Q|}\cdot N^Q \end{aligned} \tag{3.122}$$

where we denoted by $N^P = \mathbf{n}_P$ and by $N^Q = \dfrac{\mathbf{n}_Q}{|\mathbf{n}_Q|}$. Note that by (3.121), N^P is independent of Q (and, hence, of x and y), while N^Q is a function of Q, i.e. of x and y. The inequalities (3.118) can now be written

$$\begin{aligned} \left|\frac{Q}{|Q|^2}\cdot N^P\right| &\leq c \\ \left|\frac{Q}{|Q|^2}\cdot N^Q\right| &\leq c \end{aligned} \tag{3.123}$$

Let's proceed first with the derivative of F^P. In what follows we will use the notation g_x, rather than $\dfrac{\partial g}{\partial x}$, for the derivative of a function g with respect to x. We have

$$
\begin{aligned}
|P-Q|\frac{\partial F^P}{\partial x} &= |Q|\left(\frac{Q}{|Q|^2}\cdot N^P\right)_x \\
&= |Q|\left(\frac{Q_x\cdot N^P}{|Q|^2} - 2\frac{\left(Q\cdot N^P\right)(Q\cdot Q_x)}{|Q|^4}\right) \\
&= \frac{Q_x\cdot N^P}{|Q|} - 2\frac{\left(Q\cdot N^P\right)(Q\cdot Q_x)}{|Q|^3}
\end{aligned} \tag{3.124}
$$

For the first term on the right of (3.124) we have

$$
\begin{aligned}
\frac{Q_x\cdot N^P}{|Q|} &= \frac{(1,0,f_x)\cdot(0,0,1)}{\sqrt{x^2+y^2+(f(x,y))^2}} \\
&= \frac{f_x}{\sqrt{x^2+y^2+(f(x,y))^2}} \\
&= O\left(\frac{|x|+|y|}{\sqrt{x^2+y^2}}\right)
\end{aligned} \tag{3.125}
$$

which is bounded. The second term on the right of (3.124) can be rewritten as

$$
2\left(\frac{Q}{|Q|^2}\cdot N^P\right)\left(\frac{Q}{|Q|}\cdot Q_x\right) \tag{3.126}
$$

The first term in (3.126) is clearly bounded because of (3.123). For the second term in of (3.126), note that by (3.121), $Q_x=(1,0,f_x)$. Then by our assumption on the smoothness of f , $|f_x|$ is bounded, and hence, so is $\left|\frac{Q}{|Q|}\cdot Q_x\right|$.

We have just proved that $\left|\frac{\partial F^P}{\partial x}\right| \leq \frac{c}{|P-Q|}$. An identical argument will lead to the result $\left|\frac{\partial F^P}{\partial y}\right| \leq \frac{c}{|P-Q|}$. So we have that

$$
\left|D_Q F^P\right| \leq \frac{c}{|P-Q|} \tag{3.127}
$$

In a similar way we prove that the claim is also true for F^Q. We have

$$
\begin{aligned}
|P-Q|\frac{\partial F^Q}{\partial x} &= |Q|\left(\frac{Q}{|Q|^2}\cdot N^Q\right)_x \\
&= |Q|\left(\frac{Q_x\cdot N^Q}{|Q|^2} + \frac{Q\cdot N^Q_x}{|Q|^2}\right. \\
&\quad - \left. 2\,\frac{\left(Q\cdot N^Q\right)(Q\cdot Q_x)}{|Q|^4}\right)
\end{aligned} \tag{3.128}
$$

$$= \frac{Q_x \cdot N^Q}{|Q|} + \frac{Q \cdot N_x^Q}{|Q|} - 2\frac{\left(Q \cdot N^Q\right)(Q \cdot Q_x)}{|Q|^3}$$

The first term on the right of (3.128) is obviously 0. The second term on the right of (3.128) is bounded because $\frac{Q}{|Q|}$ is a unit vector (so bounded) and

$$N_x^Q = \frac{1}{|n_Q|}(-f_{xx}, -f_{yx}, 0) + \frac{f_x f_{xx} + f_y f_{yx}}{|n_Q|^3}(f_x, f_y, -1)$$

and we assumed $f \in C^{i+2}$. The third term on the right of (3.128) can be rewritten (similarly with (3.126)) as

$$2\left(\frac{Q}{|Q|} \cdot Q_x\right)\left(\frac{Q}{|Q|^2} \cdot N^Q\right) \tag{3.129}$$

which is bounded by (3.123) and by our earlier discussion following (3.126).

The same argument (with x replacing y) proves that $\left|\frac{\partial F^Q}{\partial y}\right| \leq \frac{c}{|P-Q|}$ and so

$$\left|D_Q F^Q\right| \leq \frac{c}{|P-Q|} \tag{3.130}$$

The computations for higher order derivatives get more complicated, but the idea of the proof is the same. Use the inequalities (3.123) and the fact that the norm of a vector of the form $\frac{Q}{|Q|} \cdot A$ is bounded if the components of A involve f and/or its derivatives (e.g. $\frac{Q}{|Q|} \cdot Q_x$, $\frac{Q}{|Q|} \cdot N^Q$, $\frac{Q}{|Q|} \cdot N_x^Q$, etc.)

This concludes the proof of the claim.

For the derivatives of G we have

$$\begin{aligned} \left|D_Q^i G\right| |P-Q|^n &= \sum_{k=0}^{i} \binom{i}{k} \left|D_Q^{i-k} F^P (P-Q)^{i-k}\right| \left|D_Q^k F^Q (P-Q)^k\right| \\ &\leq c \end{aligned}$$

which proves (3.89).

□

3.3.3 Generalized Superconvergent Collocation Methods for the Radiosity Equation

Following the results given in Atkinson and Chandler[5] and our previous work, we want to develop superconvergent collocation methods based on interpolation of higher degree. To better understand how that works, let us consider first the case of cubic interpolation. Recall the interpolation formula

$$g(s,t) \approx \sum_{j=1}^{10} g(q_j) l_j(s,t) \tag{3.131}$$

for q_j and l_j defined in (2.11), (2.19) and (2.20). Formula (3.131) has degree of precision 3, and so does its associated quadrature formula

$$\int_\sigma g(s,t) d\sigma = \sum_{j=1}^{f_r} g(q_j) \int_\sigma l_j(s,t) d\sigma, \quad (s,t) \in \sigma, \quad g \in C(\sigma) \tag{3.132}$$

But for $\alpha = \alpha_0 \doteq 0.199109$ formula (3.132) has degree of precision 4 (it is also exact for σ_2^2). Again, extending it to a formula over a square (formed by symmetric triangles), we obtain a quadrature formula of degree of precision 5. Let

$$\mathcal{L}_\tau g(x,y) = \sum_{j=1}^{10} g(m_\tau(q_j)) l_j(s,t), \quad (x,y) = m_\tau(s,t) \tag{3.133}$$

We have

Lemma 3.3.10 *Let τ_1 and τ_2 be planar right triangles that form a square R of length h on a side. Let $g \in C^6(R)$. Let $\Phi \in L^1(R)$ two times differentiable with derivatives of order 1 and 2 in $L^1(R)$. Assume $\alpha = \alpha_0$. Then*

$$\left| \int_R \Phi(x,y)(I - \mathcal{L}_\tau) g(x,y) d\tau \right| \leq ch^6 \left[\int_R (|\Phi| + |D\Phi| + |D^2\Phi|) d\tau \right] \cdot \max_{\substack{R \\ i=4,5,6}} \left\{ |D^i g| \right\} \tag{3.134}$$

with $\mathcal{L}_\tau g(x,y) \equiv \mathcal{L}_{\tau_i} g(x,y)$, where $(x,y) \in \tau_i$, $i = 1,2$.

Proof: Let $p_i(x,y)$ denote Taylor expansions around a suitable center, of degree i, of g over τ, for $i = 3,...5$. We have

$$\|g - p_i\|_\infty \leq ch^{i+1} \|D^{i+1} g\|_\infty, \quad i = 3,...,5 \tag{3.135}$$

It then follows that

$$\|p_{k+1} - p_k\|_\infty \leq ch^{k+1} \left(h\|D^{k+2} g\|^\infty + \|D^{k+1} g\|_\infty \right) \text{ for } \quad k = 3,4 \tag{3.136}$$

Also, let $\Phi_i(x,y)$ be polynomials of degree i over τ satisfying

$$\|\Phi - \Phi_i\|_1 \leq ch^{i+1} \|D^{i+1}\Phi\|_1, \quad i = 0,1 \tag{3.137}$$

We can write

$$\begin{aligned}\int_R \Phi\mathcal{L}'_\tau g d\tau &= \int_R \Phi\mathcal{L}'_\tau(g-p_5)d\tau \\ &+ \int_R (\Phi-\Phi_0)\mathcal{L}'_\tau(p_5-p_4)d\tau \\ &+ \int_R (\Phi-\Phi_1)\mathcal{L}'_\tau(p_4-p_3)d\tau \\ &+ \int_R (\Phi_1-\Phi_0)\mathcal{L}'_\tau p_4 d\tau\end{aligned} \tag{3.138}$$

Formula (3.138) is true, because after multiplying out the terms on the right we obtain

$$\int_R \Phi\mathcal{L}'_\tau g d\tau - \int_R \Phi_0\mathcal{L}'_\tau p_5 d\tau - \int_R \Phi\mathcal{L}'_\tau p_3 d\tau + \int_R \Phi_1\mathcal{L}'_\tau p_3 d\tau \tag{3.139}$$

The last three terms in (3.139) are 0 because formulas (3.131) and (3.132) have degrees of precision 3 and 5, respectively. The last integral in (3.138) is 0 and the proof is identical to the one given in Lemma 3.3.7.

Now, take norms in (3.138) and use the bounds (3.135)-(3.137) to get (3.134).

□

For integration over single triangles, as expected, the bound will only be of order $O(h^5)$.

Consider formula (3.87) with $i = 1, ..., 10n$. The superconvergence result that follows is

Theorem 3.3.11 *Assume the hypotheses of Theorem 3.3.1, with each $F_j \in C^6$. Assume $u \in C^6(S)$ and $\alpha = \alpha_0$. Assume the triangulation $\mathcal{T}_n$ of S satisfies (3.25) and that it is symmetric. For those integrals in (3.87) for which $v_i \in \Delta_k$, assume that all such integrals are evaluated with an error of $O(h^6)$. Then*

$$\max_{1\le i\le 10n} |u(v_i) - \hat{u}_n(v_i)| \le ch^6 \log h \tag{3.140}$$

Proof: We give bounds for

$$\max_{1\le i\le 10n} |\mathcal{K}(\mathcal{I}-\mathcal{P}_n)u(v_i))|$$

With the previous notations we have that the contribution to the error coming from the triangles in $\mathcal{T}_n^{(2)}$ is $O\left(h^6\|D^5u\|_\infty\right)$.

The contribution to the error coming from triangles in $\mathcal{T}_n^{(1)}$ is

$$ch^6 \int\limits_{S-\Delta^*} \sum_{j=0}^{2} \frac{1}{|v_i - Q|^j} dS_Q \tag{3.141}$$

Using a local representation of the surface and then using polar coordinates, the expression in (3.141) is of order

$$ch^6\left(h^2 + h + \log h\right)$$

Combining the errors we have (3.140).

□

Now we want to investigate superconvergent collocation methods based on interpolation of any degree r. It is clear from our work so far (for quadratic and cubic interpolation) that we have to distinguish two cases: where r is odd and where r is even.

Interpolation of odd degree. We consider the collocation nodes and the interpolation basis functions of (2.4) and (2.5) for some odd number r. The formula

$$g(s,t) \approx \sum_{j=1}^{f_r} g(q_j) l_j(s,t), \quad (s,t) \in \sigma, \quad g \in C(\sigma) \tag{3.142}$$

has degree of precision at least r for any $0 < \alpha < \dfrac{1}{3}$. Assume r is an odd number. The formula

$$\int\limits_{\sigma} g(s,t) d\sigma \approx \sum_{j=1}^{f_r} g(q_j) \int\limits_{\sigma} l_j(s,t) d\sigma, \quad g \in C(\sigma) \tag{3.143}$$

also has degree of precision r. Suppose we can find a value $0 < \alpha_0 < \dfrac{1}{3}$, such that for $\alpha = \alpha_0$, formula (3.143) has degree of precision $r+1$. Then, as it happened in the cubic case, if we extend it to a rectangle, it will have degree of precision $r+2$. We have the following result.

Lemma 3.3.12 *Let τ_1 and τ_2 be planar right triangles that form a square R of length* *h on a side. Let* $g \in C^{r+3}(R)$*. Let* $\Phi \in L^1(R)$ *two times differentiable with derivatives of order 1 and 2 in* $L^1(R)$*. Assume* $\alpha = \alpha_0$*. Then*

$$\left| \int_R \Phi(x,y)(I - \mathcal{L}_\tau) g(x,y) d\tau \right| \leq ch^{r+3} \left[\int_R (|\Phi| + |D\Phi| + |D^2\Phi|) d\tau \right] \cdot \max_{\substack{R \\ i=r+1,r+2,r+3}} \left\{ |D^i g| \right\} \tag{3.144}$$

with $\mathcal{L}_\tau g(x,y) \equiv \mathcal{L}_{\tau_i} g(x,y)$*, where* $(x,y) \in \tau_i$*,* $i = 1,2$*.*

Proof: Let $p_i(x,y)$ denote Taylor expansions around a suitable center, of degree i, of g over τ, for $i = r, r+1, r+2$. We have

$$\|g - p_i\|_\infty \leq ch^{i+1} \|D^{i+1} g\|_\infty, \quad i = r, r+1, r+2 \tag{3.145}$$

Then

$$\|p_{k+1} - p_k\|_\infty \leq ch^{k+1} \left(h\|D^{k+2} g\|^\infty + \|D^{k+1} g\|_\infty \right) \text{ for } \quad k = r, r+1 \tag{3.146}$$

In addition, let $\Phi_i(x,y)$ polynomials of degree i over τ satisfying

$$\|\Phi - \Phi_i\|_1 \leq ch^{i+1} \|D^{i+1}\Phi\|_1, \quad i = 0,1 \tag{3.147}$$

We can write

$$\begin{aligned} \int_R \Phi \mathcal{L}'_\tau g d\tau &= \int_\tau \Phi \mathcal{L}'_\tau (g - p_{r+2}) d\tau \\ &+ \int_R (\Phi - \Phi_0) \mathcal{L}'_\tau (p_{r+2} - p_{r+1}) d\tau \\ &+ \int_R (\Phi - \Phi_1) \mathcal{L}'_\tau (p_{r+1} - p_r) d\tau \\ &+ \int_R (\Phi_1 - \Phi_0) \mathcal{L}'_\tau p_{r+1} d\tau \end{aligned} \tag{3.148}$$

Formula (3.148) is true, because after multiplying out the terms on the right we obtain

$$\int_R \Phi \mathcal{L}'_\tau g d\tau - \int_R \Phi_0 \mathcal{L}'_\tau p_{r+2} d\tau - \int_R \Phi \mathcal{L}'_\tau p_r d\tau + \int_\tau \Phi_1 \mathcal{L}'_\tau p_r d\tau \tag{3.149}$$

The last three terms in (3.149) are 0 because formulas (3.142) and (3.143) have degrees of precision r and $r+2$, respectively. The last integral in (3.148) is 0 since

$$\mathcal{L}_\tau \left[(\Phi_1 - \Phi_0) p_{r+1} \right] = \mathcal{L}_\tau \left[(\Phi_1 - \Phi_0) \mathcal{L}_\tau p_{r+1} \right] \tag{3.150}$$

and hence

$$\int_R (\Phi_1 - \Phi_0)\mathcal{L}'_\tau p_{r+1} d\tau = \int_R \mathcal{L}'_\tau \left[(\Phi_1 - \Phi_0)p_{r+1}\right] d\tau = 0$$

(just as in the proof of Lemma 3.3.7). Taking bounds in (3.148) and using (3.145)-(3.147) we obtain (3.144).

□

If integrating over a single triangle, the bound is given by

Lemma 3.3.13 *Let τ be a planar right triangle and assume the two sides which form the right angle have length h. Assume $\alpha = \alpha_0$. Let $g \in C^{r+2}(\tau), \Phi \in L^1(\tau)$ differentiable with first derivatives in $L^1(\tau)$. Then*

$$\left| \int_\tau \Phi(x, y)(\mathcal{I} - \mathcal{L}_\tau)g(x, y)d\tau \right| \leq ch^{r+2} \left[\int_\tau (|\Phi| + |D\Phi|)d\tau \right] \cdot \max_\tau \left\{ |D^{r+1}g|, |D^{r+2}g| \right\} \tag{3.151}$$

where c denotes a generic constant.

Similar results hold for arbitrary triangles.

Consider formula (3.87) with $i = 1, ..., nf_r$. Now we can address the question of superconvergence.

Theorem 3.3.14 *Assume the hypotheses of Theorem 3.3.1, with each $F_j \in C^{r+2}$. Assume $u \in C^{r+2}(S)$. Assume the triangulation $\mathcal{T}_n$ of S satisfies (3.25) and that it is symmetric. For those integrals in (3.87) for which $v_i \in \Delta_k$, assume that all such integrals are evaluated with an error of $O(h^{r+3})$. Assume $\alpha = \alpha_0$. Then*

$$\max_{1 \leq i \leq nf_r} |u(v_i) - \hat{u}_n(v_i)| \leq ch^{r+3} \log h \tag{3.152}$$

Proof: We bound

$$\max_{1 \leq i \leq nf_r} |\mathcal{K}(\mathcal{I} - \mathcal{P}_n)u(v_i))|$$

using the previous lemmas. By Lemma 3.3.13 the contribution to the error coming from the triangles in $\mathcal{T}_n^{(2)}$ will be $O\left(h^{r+3}\|D^{r+2}u\|_\infty\right)$.

Using Lemma 3.3.12 we have that the contribution to the error coming from triangles in $\mathcal{T}_n^{(1)}$ is of order

$$ch^{r+3} \int\limits_{S-\Delta^*} \sum_{j=0}^{2} \frac{1}{|v_i - Q|^j} dS_Q \tag{3.153}$$

Using a local representation of the surface and then using polar coordinates, the expression in (3.147) is of order

$$ch^{r+3}\left(h^2 + h + \log h\right) = O(h^{r+3}\log h)$$

Combining the errors arising from the integrals over Δ^*, $\mathcal{T}_n^{(1)}$, and $\mathcal{T}_n^{(2)}$, we have (3.152).

$\square$

Interpolation of even degree. Consider the interpolation formula

$$g(s,t) \approx \sum_{j=1}^{f_r} g(q_j) l_j(s,t), \quad (s,t) \in \sigma, \quad g \in C(\sigma) \tag{3.154}$$

with r an even number. Then the quadrature formula

$$\int\limits_{\sigma} g(s,t) d\sigma \approx \sum_{j=1}^{f_r} g(q_j), \quad g \in C(\sigma) \tag{3.155}$$

has degree of precision at least r. Considered over a rectangle formed by two symmetric triangles, it has degree of precision $r+1$, since r is an even number. Defining a collocation method with (3.154), for the solution of the collocation equation and the true solution of the radiosity equation, we have the error estimate

$$\|u - u_n\| = O\left(h^{r+1}\right) \tag{3.156}$$

For the convergence at the collocation nodes we have

Lemma 3.3.15 *Let τ_1 and τ_2 be planar right triangles that form a square R of length h on a side. Let $g \in C^{r+2}(R)$. Let $\Phi \in L^1(R)$ differentiable with first order derivatives in $L^1(R)$. Then*

$$\left|\int\limits_R \Phi(x,y)(I - \mathcal{L}_\tau) g(x,y) d\tau\right| \le ch^{r+2}\left[\int\limits_\tau (|\Phi| + |D\Phi|) d\tau\right] \cdot \max_{\substack{R \\ i=r+1,r+2}} \left\{|D^i g|\right\} \tag{3.157}$$

with $\mathcal{L}_\tau g(x,y) \equiv \mathcal{L}_{\tau_i} g(x,y)$, where $(x,y) \in \tau_i$, $i = 1,2$.

Proof: As mentioned earlier, we can find polynomials $p_i(x,y)$ of degree i such that

$$\|g - p_i\|_\infty \leq ch^{i+1}\|D^{i+1}g\|_\infty, \quad i = r, r+1 \tag{3.158}$$

which implies

$$\|p_{r+1} - p_r\|_\infty \leq ch^{r+1}\left(h\|D^{r+2}g\|_\infty + \|D^{r+1}g\|_\infty\right) \tag{3.159}$$

We can also find a constant Φ_0 such that

$$\|\Phi - \Phi_0\|_1 \leq ch^{i+1}\|D^{i+1}\Phi\|_1 \tag{3.160}$$

Then we have the equality

$$\begin{aligned} \int_R \Phi\mathcal{L}'_\tau g d\tau &= \int_R \Phi\mathcal{L}'_\tau(g - p_{r+1})d\tau \\ &+ \int_R (\Phi - \Phi_0)\mathcal{L}'_\tau(p_{r+1} - p_r)d\tau \end{aligned} \tag{3.161}$$

since formula (3.154) has degree of precision r and formula (3.155) has degree of precision $r+1$.

Using the previous estimates, we obtain (3.157).

□

For integration over one triangle only, the term in h in (3.157) is only h^{r+1}. We use these results to prove the following superconvergence result.

Theorem 3.3.16 *Assume the hypotheses of Theorem 3.3.1, with each* $F_j \in C^{r+2}$. *Assume* $u \in C^{r+2}(S)$. *Assume the triangulation* $\mathcal{T}_n$ *of* S *satisfies (3.25) and that it is symmetric. For those integrals in (3.87) for which* $v_i \in \Delta_k$, *assume that all such integrals are evaluated with an error of* $O(h^{r+2})$. *Then*

$$\max_{1\leq i\leq nf_r} |u(v_i) - \hat{u}_n(v_i)| \leq ch^{r+2} \tag{3.162}$$

Proof: Again we bound

$$\max_{1\leq i\leq nf_r} |\mathcal{K}(\mathcal{I} - \mathcal{P}_n)u(v_i))|$$

using the previous lemma. With the same notations as before, we have that the contribution to the error coming from the triangles in $\mathcal{T}_n^{(2)}$ will be $O\left(h^{r+2}\|D^{r+1}u\|_\infty\right)$.

The contribution to the error coming from triangles in $\mathcal{T}_n^{(1)}$ is of order

$$ch^{r+2} \int\limits_{S-\Delta^*} \sum_{j=0}^{2} \frac{1}{|v_i - Q|^j} dS_Q. \tag{3.163}$$

Using a local representation of the surface and then using polar coordinates, the expression in (3.163) is

$$ch^{r+2}\left(h^2 + h\right) - ch^{r+2} = O(h^{r+2})$$

Combining the errors arising from the integrals over Δ^*, $\mathcal{T}_n^{(1)}$, and $\mathcal{T}_n^{(2)}$, we have (3.162).

□

Note that this case corresponds to the first type of superconvergent piecewise quadratic collocation method, that we described in Section 3.3.1. The second method presented there is somehow “special”, meaning it is not always possible to develop such a method. Notice that in that case we found a value for α that increased the degree of precision with 2, and then 3, using symmetric triangles. Since when r is even we can always improve the precision by considering symmetric triangulations, it makes no sense to determine a value α_0 that would only increase the degree with 1 (since the symmetry does that, anyway). That's why the approaches and the results are different for the two cases: r odd and r even. As for a generalization of the second superconvergent piecewise quadratic method that we presented, there is not much hope there. That would mean to increase the degree of precision with 2, i.e. have the quadrature formula be exact for all the polynomials in σ_1 and σ_2 of the given degree. That means that one variable, α, has to satisfy a number of equations, which (especially if r is large) may not have a solution.

In fact, even increasing with 1 the degree of precision (what we assumed was possible in order to have a superconvergent method) is sometimes difficult. As an easy example, consider the case $r = 5$. To increase the degree of precision to 6, the corresponding quadrature formula must be exact for both σ_1^3 and σ_2^2.

CHAPTER IV
A COLLOCATION METHOD FOR SOLVING THE EXTERIOR NEUMANN PROBLEM

In this chapter we study the numerical solution of a boundary integral equation reformulation of the exterior Neumann problem. We give an outline of the problem and its solvability. Then, we propose a collocation method based on interpolation and give an error analysis. Numerical examples for the piecewise constant collocation method (centroid rule) conclude this chapter.

4.1 The Exterior Neumann Problem

Let D denote a bounded open simply-connected region in $\mathbb{R}^3$, and let S denote its boundary. Let $\overline{D} = D \cup S$ and denote by $D_e = \mathbb{R}^3 - \overline{D}$ the region complementary to D. Let $\overline{D}_e = D_e \cup S$. At a point $P \in S$, let $\mathbf{n}_P$ denote the unit normal directed into D, provided that such a normal exists. In addition, we will assume that S is a piecewise smooth surface satisfying (3.41) and (3.42).

The Exterior Neumann Problem

Find $u \in C^1(\overline{D}_e) \cap C^2(D_e)$ that satisfies

$$\begin{aligned} \Delta u(P) &= 0, \; P \in D_e \\ \frac{\partial u(P)}{\partial \mathbf{n}_P} &= f(P), \; P \in S \end{aligned} \tag{4.1}$$

$$u(P) = O(P^{-1}), \; \frac{\partial u(P)}{\partial r} = O(|P|^{-2}) \quad \text{, as} \quad r = |P| \to \infty \text{ uniformly in } \frac{P}{|P|}$$

with $f \in C(S)$ a given boundary function.

The boundary value problem (4.1) has been studied extensively (see Mikhlin[15, Ch. 18], Günter[11, Ch. 3], Colton[10, Section 5.3]). Here we only give a very brief look at results on the solvability of the problem (4.1).

The Divergence Theorem (see Atkinson[4, Theorem 7.1.2]) can be used to obtain a representation formula for functions that are harmonic inside the region D_e. Let $u \in C^1(\overline{D}_e) \cap C^2(D_e)$ and assume that $\Delta u(P) = 0$ at all $P \in D_e$. Then

$$\int\limits_S \frac{\partial u(Q)}{\partial n_Q} \frac{dS(Q)}{|P-Q|} \quad - \quad \int\limits_S u(Q) \cdot \frac{\partial}{\partial n_Q}\left[\frac{1}{|P-Q|}\right] dS_Q$$
$$= \begin{cases} [4\pi - \Omega(P)]u(P) & , P \in S \\ 4\pi u(P) & , P \in D_e \end{cases} \tag{4.2}$$

(see Atkinson[2].) In formula (4.2), $\Omega(P)$ denotes the *interior solid angle* at $P \in S$, defined in Atkinson[4, p. 430]. If S is smooth, then $\Omega(P) = 2\pi$. For a cube, the corners have interior solid angle of $\frac{1}{2}\pi$, and the edges have interior solid angles of π.

To study the solvability of (4.1), consider representing its solution as a *single layer potential*

$$u(A) = \int\limits_S \frac{\rho(Q)}{|A-Q|}\, dS_Q, \quad A \in D_e \tag{4.3}$$

The function ρ in (4.3) is called a *single layer density* function. The function $u(A)$ in (4.3) is harmonic for all $A \notin S$. For well-behaved density functions and for $A \notin S$, the integrand in (4.3) is nonsingular. Even though for the case $A = P \in S$, the integrand in (4.3) becomes singular, it is relatively straightforward to show that the integral exists and moreover, if ρ is bounded on S, then

$$\sup_{A \in \mathbb{R}^3} |u(A)| \le c\|\rho\|_\infty \tag{4.4}$$

For a complete description of the properties of the single layer potential, see Günter[11, Chapter 2].

Now for the function u of (4.3), impose the boundary condition from (4.1) to get

$$\lim_{\substack{A \to P \\ A \in D_e}} \mathbf{n}_P \cdot \nabla \left[\int\limits_S \frac{\rho(Q)}{|A-Q|}\, dS_Q\right] = f(P), \quad P \in S \tag{4.5}$$

for all $P \in S$ at which the normal $\mathbf{n}_P$ exists (which implies $\Omega(P) = 2\pi$). Using a limiting argument, we obtain the second kind integral equation

$$2\pi\rho(P) + \int_S \rho(Q) \cdot \frac{\partial}{\partial \mathbf{n}_P}\left[\frac{1}{|P-Q|}\right] dS_Q = f(P), \quad P \in S^* \tag{4.6}$$

The set S^* is to contain all points $P \in S$ at which a normal is defined. If S is a smooth surface, then $S^* = S$; otherwise, $S - S^*$ is a set of measure 0. The kernel function in (4.6) is given by

$$\frac{\partial}{\partial \mathbf{n}_P}\left[\frac{1}{|P-Q|}\right] = \frac{\mathbf{n}_P \cdot (P-Q)}{|P-Q|^3} = \frac{\cos\theta_P}{|P-Q|^2} \tag{4.7}$$

where θ_P denotes the angle between $\mathbf{n}_P$ and $(P-Q)$. Equation (4.6) can now be written as

$$\rho(P) + \frac{1}{2\pi}\int_S \rho(P) \cdot \frac{\cos\theta_P}{|P-Q|^2}\, dS_Q = \hat{f}(P), \quad P \in S \tag{4.8}$$

where $\hat{f}(P) = \frac{1}{2\pi} f(P)$. For simplicity, we will write $f(P)$ instead of $\hat{f}(P)$.

Write the equation (4.8) in operator form:

$$(\mathcal{I} - \mathcal{K})\rho = f \tag{4.9}$$

The properties of the integral operator $\mathcal{K}$ and, implicitly, the solvability of equation (4.1) have been studied intensively in the literature, especially for the case that S is a smooth surface. For S sufficiently smooth, $\mathcal{K}$ is a compact operator from $C(S)$ to $C(S)$ and from $L^2(S)$ to $L^2(S)$. These results are contained in many textbooks, for example see Kress[14, Chapter 6], or Mikhlin[15, Chapters 12 and 16]. We will just state the following solvability result.

Theorem 4.1.1 *Let S be a C^2 surface. Then the equation (4.9) has a unique solution $\rho \in X$ for each given function $f \in X$, with $X = C(S)$ or $X = L^2(S)$.*

This theorem then leads to a solvability result for the Exterior Neumann Problem (4.1)

Theorem 4.1.2 *Let S be a smooth surface with $\overline{D_e}$ a region to which the Divergence Theorem can be applied. Assume the function $f \in C(S)$. Then, the Neumann problem (4.1) has a unique solution $u \in C^\infty(D_e)$.*

For the case when S is only piecewise smooth, the properties of $\mathcal{K}$ and the solvability of (4.8) are not yet fully understood. We will assume that Theorem 4.1.1 is true for the piecewise smooth surfaces that we will consider in our work.

4.2 A Collocation Method

We want to study the numerical solution of (4.8) using a an integral equation reformulation of (4.1) have been used before (see Atkinson and Chien[6] or Atkinson[4, Section 9.2]), but with the collocation nodes on the boundary of each triangular element. As mentioned in Chapter I, there are problems with defining the normal at the collocation points which are common to more than one triangular face, especially if the surface itself is approximated. This in turn means it is difficult to evaluate the kernel function in equation (4.8). For these reasons it makes sense to try collocation methods that use only interior collocation node points, like the ones described in Chapter II.

We will use the same framework that we used for the radiosity equation. Assume the surface S satisfies (3.41) and (3.42) and has a triangulation $\mathcal{T}_n = \{\Delta_{n,k} \mid i \leq k \leq n\}$ with mesh size h. For $g \in C(S)$ define an operator $\mathcal{P}_n$ by

$$\mathcal{P}_n g(P) = \sum_{j=1}^{f_r} g\left(m_k(q_j)\right) l_j(s,t), \quad (s,t) \in \sigma, \quad P = m_k(s,t) \in \Delta_k \tag{4.10}$$

with q_j and l_j defined in (2.4) and (2.5). This interpolates $g(P)$ over each triangular element $\Delta_k \in S$, with the interpolating function polynomial in the parameterization variables s and t. Since $\mathcal{P}_n g$ is not continuous in general, we need to enlarge $C(S)$ to include the piecewise polynomial approximations $\mathcal{P}_n g$. To do this, we consider the equation (4.9) within the framework of the function space $L^\infty(S)$ with the uniform norm $\|\cdot\|_\infty$, as described in Section 2.1.4. Then, $\mathcal{P}_n : L^\infty(S) \longrightarrow L^\infty(S)$ is a bounded projection operator, with $\|\mathcal{P}_n\|$ given by (2.27).

Define a collocation method with (4.10). Denote $v_{k,j} = m_k(q_i)$. Substitute

$$\begin{aligned} \rho_n(P) &= \sum_{j=1}^{f_r} \rho_n\left(v_{k,j}\right) l_j(s,t) \\ P &= m_k(s,t) \in \Delta_k, \quad k = 1,...,n \end{aligned} \tag{4.11}$$

into (4.8). To determine the values $\{\rho_n(v_{k,j})\}$, force the equation resulting from the substitution to be true at the collocation nodes $\{v_1, ..., v_{nf_r}\}$. This leads to the linear system

$$\begin{aligned} \rho_n(v_i) &- \frac{1}{2\pi} \sum_{k=1}^{n} \sum_{j=1}^{f_r} \rho_n(v_{k,j}) \int_\sigma \frac{\cos\theta_{v_i}}{|v_i - m_k(s,t)|^2} \\ &\cdot \quad |(D_s m_k \times D_t m_k)(s,t)|\, d\sigma = f(v_i), \quad i = 1, ..., nf_r \end{aligned} \tag{4.12}$$

which we write abstractly as

$$(\mathcal{I} - \mathcal{P}_n\mathcal{K})\rho_n = \mathcal{P}_n f \tag{4.13}$$

which will be compared to (4.9). We have the following result.

Theorem 4.2.1 *Let S be a C^2 surface that satisfies (3.41) and (3.42) with $F_j \in C^{r+2}$. Then for all sufficiently large n, say $n \geq n_0$, the operators $\mathcal{I} - \mathcal{P}_n\mathcal{K}$ are invertible on $L^\infty(S)$ and have uniformly bounded inverses. For the solution ρ of (4.9) and the solution ρ_n of (4.13)*

$$\|\rho - \rho_n\|_\infty \leq \left\|(\mathcal{I} - \mathcal{P}_n\mathcal{K})^{-1}\right\| \cdot \|\rho - \mathcal{P}_n\rho\|_\infty, \quad n \geq n_0 \tag{4.14}$$

Furthermore, if $f \in C^{r+1}(S)$, then

$$\|\rho - \rho_n\|_\infty = O(h^{r+1}), \quad n \geq n_0 \tag{4.15}$$

Proof: The result follows from the standard theory for projection methods (see, for example, Atkinson[1, pp. 50-62]). Since S is smooth, it is known that $\mathcal{K} : L^\infty(S) \longrightarrow C(S)$ and is compact. We then have

$$\|(\mathcal{I} - \mathcal{P}_n)\mathcal{K}\| \longrightarrow 0 \text{ as } n \longrightarrow \infty \tag{4.16}$$

From (4.16) we have the standard result that since $(\mathcal{I} - \mathcal{K})^{-1}$ exists, then $(\mathcal{I} - \mathcal{P}_n\mathcal{K})^{-1}$ exists and is uniformly bounded for sufficiently large n, say $n \geq n_0$.

Combining (4.9) and (4.13) we have (4.14). The bound (4.15) follows from (4.14) and from the fact that we are using interpolation of degree r.

□

As described for the radiosity equation, superconvergent methods can be developed. Next, we want to explore in more detail the collocation method based on piecewise constant interpolation (the centroid method) and show that it is superconvergent at the collocation points. Define the operator $\mathcal{P}_n$ by

$$\mathcal{P}_n g(P) = g(P_k),\ P \in \Delta_k,\ k = 1, ..., n \tag{4.17}$$

for $g \in C(S)$. Then, $\mathcal{P}_n$ is a bounded operator on $C(S)$ with $\|\mathcal{P}_n\| = 1$. Define a collocation method with (4.17). Substitute

$$\rho_n(P) = \rho_n(P_k),\ P = m_k(s,t) \in \Delta_k,\ k = 1, ..., n \tag{4.18}$$

into (4.8). To determine the values $\{\rho_n(P_k)\}$, force the equation resulting from the substitution to be true at the collocation nodes $\{P_k \mid k = 1, ..., n\}$. This leads to the linear system

$$\begin{aligned} \rho_n(P_i) \quad &+ \quad \frac{1}{2\pi} \sum_{k=1}^{n} \rho_n(P_k) \cdot \int_{\sigma} \frac{\cos\theta_{P_k}}{|P_k - m_k(s,t)|^2} \\ &\cdot \quad |(D_s m_k \times D_t m_k)(s,t)| \ d\sigma = f(P_k),\ i = 1, ..., n \end{aligned} \tag{4.19}$$

which can be rewritten abstractly as

$$(\mathcal{I} + \mathcal{P}_n \mathcal{K})\, \rho_n = \mathcal{P}_n f \tag{4.20}$$

which will be compared to (4.9).

By Theorem 4.2.1, for the true solution ρ of (4.9) and the solution ρ_n of the collocation equation (4.20), we have

$$\|\rho - \rho_n\|_\infty = O(h), \quad n \geq n_0 \tag{4.21}$$

For $g \in C(\sigma)$, consider the interpolation formula (4.17), which has degree of precision

0. Integrating it over σ, we obtain

$$\int_\sigma g(s,t)\ d\sigma \approx \int_\sigma \mathcal{L}_\tau g(s,t)\ d\sigma = \frac{1}{2} g\left(\frac{1}{3}, \frac{1}{3}\right) \tag{4.22}$$

which has degree of precision 1.

For $\tau \subset \mathbb{R}^2$, a planar triangle with vertices $\{v_1,\ v_2,\ v_3\}$, define the mapping m_τ as in (2.37). Then for a function $g \in C(\tau)$, the function

$$\mathcal{L}_\tau g(x,y) = g\left(m_\tau\left(\frac{1}{3}, \frac{1}{3}\right)\right) = g(P_\tau) \tag{4.23}$$

is a constant polynomial interpolating g at the node $m_\tau\left(\frac{1}{3}, \frac{1}{3}\right) = P_\tau$ (the centroid of τ). We have the following.

Lemma 4.2.2 *Let τ be a planar right triangle and assume the two sides which form the right angle have length h. Let $g \in C^2(\tau)$. Let $\Phi \in L^1(\tau)$ be differentiable with the first derivatives $D_x\Phi,\ D_y\Phi \in L^1(\tau)$. Then*

$$\left|\int_\tau \Phi(x,y)\left(\mathcal{I} - \mathcal{L}_\tau\right) g(x,y)\ d\tau\right| \le ch^2 \left[\int_\tau \left(|\Phi| + |D\Phi|\right)\ d\tau\right] \cdot \max_\tau \left\{|Dg|, |D^2|g\right\} \tag{4.24}$$

Proof: The proof is very similar to the proof of Lemma 3.1.1. Let $\mathcal{L}'_\tau = \mathcal{I} - \mathcal{L}_\tau$. We can find polynomials $p_0(x,y),\ p_1(x,y)$ of degrees 0 and 1, respectively, and a constant Φ_0 such that

$$\|g - p_0\| \le ch\|D^1\Phi\|_\infty,\ \ \|g - p_1\|_\infty \le ch^2\|D^2 g\|_\infty,\ \ \|\Phi - \Phi_0\|_1 \le ch\|D^1\Phi\|_1 \tag{4.25}$$

From the first two inequalities in (4.25), it follows that

$$\|p_1 - p_0\|_\infty \le ch\left(h\|D^2 g\|_\infty + \|Dg\|_\infty\right) \tag{4.26}$$

We can write

$$\int_\tau \Phi\mathcal{L}'_\tau g\ = \int_\tau \Phi\mathcal{L}'_\tau(g - p_1)\ d\tau + \int_\tau \left(\Phi - \Phi_0\right)\mathcal{L}'_\tau p_1\ d\tau \tag{4.27}$$

Since formula (4.17) has degree of precision 0, it follows that

$$\mathcal{L}'_\tau p_0 = 0 \tag{4.28}$$

Also, by the fact that formula (4.22) has degree of precision 1, the term

$$\int_\tau \Phi_0 \mathcal{L}'_\tau p_1 \; d\tau = 0 \tag{4.29}$$

Using (4.28) we can write

$$\mathcal{L}'_\tau p_1 = \mathcal{L}'_\tau (p_1 - p_0) \tag{4.30}$$

Taking bounds and using (4.26)

$$\|\mathcal{L}'_\tau p_1\|_\infty \leq ch \left(h\|D^2 g\|_\infty + \|Dg\|_\infty \right) \tag{4.31}$$

Using (4.31) and the third inequality in (4.25), we obtain

$$\left| \int_\tau (\Phi - \Phi_0)\mathcal{L}'_\tau p_1 \; d\tau \right| \leq ch^2 \left[\int_\tau |\Phi| + |D\Phi| \; d\tau \right] \cdot \max_\tau \left\{ \|Dg\|_\infty, \; \|D^2 g\|_\infty \right\} \tag{4.32}$$

For the first integral in (4.27), using (4.25) we have the bound

$$\left| \int_\tau \Phi \mathcal{L}'_\tau (g - p_1) \; d\tau \right| \leq ch^2 \left[\int_\tau |\Phi| \; d\tau \right] \cdot \|D^2 g\|_\infty \tag{4.33}$$

Combining (4.32), and (4.33), we have (4.24).

□

As in Section 3.1 (following Lemma 3.1.1), this result can be extended to general triangles, provided

$$\sup_n \left[\max_{\Delta_{n,k} \in \mathcal{T}_n} r(\Delta_{n,k}) \right] < \infty \tag{4.34}$$

where

$$r(\tau) = \frac{h(\tau)}{h^*(\tau)} \tag{4.35}$$

with $h(\tau)$ and $h^*(\tau)$ denoting the diameter of τ and the radius of the circle inscribed in τ, respectively.

Corollary 4.2.3 *Let τ be a planar triangle of diameter h, let $g \in C^2(\tau)$, and let $\Phi \in L^1(\tau)$ with both first derivatives in $L^1(\tau)$. Then*

$$\begin{aligned} \left| \int_\tau \Phi(x,y)(\mathcal{I} - \mathcal{L}_\tau) g(x,y) \right| \quad &\leq \quad c\,(r(\tau))\, h^2 \left[\int_\tau (|\Phi| + |D\Phi|) \; d\tau \right] \\ &\cdot \quad \max_\tau \left\{ \|Dg\|_\infty, \|D^2 g\|_\infty \right\} \end{aligned} \tag{4.36}$$

where $c\,(r(\tau))$ is some multiple of $r(\tau)$ of (4.35).

Since formula (4.22) has degree of precision 1 (odd) over σ, extending it to a square would not improve the degree of precision, which means the same error bound as in Lemma 4.2.2 is true for a parallelogram formed by two symmetric triangles.

We want to apply the above results to the individual subintegrals in

$$\begin{aligned}\mathcal{K}g(P_i) &= \frac{1}{2\pi}\sum_{k=1}^{n}\int_\sigma \frac{\cos\theta_{P_k}}{\left|P_k - m_k(s,t)\right|^2}\rho\left(m_k(s,t)\right) \\ &\qquad \cdot \left|\left(D_s m_k \times D_t m_k\right)(s,t)\right| \; d\sigma \end{aligned} \tag{4.37}$$

with the role of g played by $\rho\left(m_k(s,t)\right)\left|\left(D_s m_k \times D_t m_k\right)(s,t)\right|$, and the role of Φ played by $\dfrac{\cos\theta_{P_k}}{\left|P_k - m_k(s,t)\right|^2}$. For the derivatives of this last function, we have

Theorem 4.2.4 *Let i be an integer and S be a smooth C^{i+1} surface. Then*

$$\left|D_Q^i\left(\frac{\cos\theta_P}{|P-Q|^2}\right)\right| \leq \frac{c}{|P-Q|^{i+1}}, \quad P \neq Q \tag{4.38}$$

with c a generic constant independent of P and Q.

Proof: The proof uses

$$|\cos\theta_P| \leq c|P-Q| \tag{4.39}$$

and the same type of argument that was used to prove Theorem 3.3.5.

□

For the error at the collocation node points, we have the following.

Theorem 4.2.5 *Assume the hypotheses of Theorem 4.2.1, with each $F_j \in C^2$. Assume $\rho \in C^2$. Assume the triangulation $\mathcal{T}_n$ of S satisfies (4.34) and is symmetric. For those integrals in (4.37) for which $P_i \in \Delta_k$, assume that all such integrals are evaluated with an error of $O(h^2)$. Then*

$$\max_{1\leq i\leq n}|\rho(P_i) - \widehat{\rho}_n(P_i)| \leq ch^2\log h \tag{4.40}$$

Proof: As in the proof of 3.1.3, we will bound

$$\max_{1\leq i\leq n}|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)u(v_i))|$$

For a given node point v_i, denote Δ^* the triangle containing it and denote:

$$\mathcal{T}_n^* = \mathcal{T}_n - \{\Delta^*\}$$

By our assumption, the error in evaluating the integral of (4.37) over Δ^* will be $O(h^2)$.

Partition $\mathcal{T}_n^*$ into parallelograms to the maximum extent possible. Denote by $\mathcal{T}_n^{(1)}$ the set of all triangles making up such parallelograms and let $\mathcal{T}_n^{(2)}$ contain the remaining triangles. Then

$$\mathcal{T}_n^* = \mathcal{T}_n^{(1)} \cup \mathcal{T}_n^{(2)}.$$

It is easy to show that the number of triangles in $\mathcal{T}_n^{(1)}$ is $O(n) = O(h^{-2})$, and the number of triangles in $\mathcal{T}_n^{(2)}$ is $O(\sqrt{n}) = O(h^{-1})$.

It can be shown that all but a finite number of the triangles in $\mathcal{T}_n^{(2)}$, bounded independent of n, will be at a minimum distance from v_i. That means that the triangles in $\mathcal{T}_n^{(2)}$ are "far enough" from v_i, so that the function $G(v_i, Q)$ is uniformly bounded for Q being in a triangle in $\mathcal{T}_n^{(2)}$ (where we denote by $G(P, Q) = \dfrac{\cos\theta_P}{|P-Q|^2}$).

First, consider the contribution to the error coming from the triangles in $\mathcal{T}_n^{(2)}$. By Lemma 4.2.2 the error over each such triangle is $O\left(h^2 \|D^2 g\|_\infty\right)$, since the area of each triangle is $O(h^2)$ and using our earlier observation. Having $O(h^{-1})$ such triangles in $\mathcal{T}_n^{(2)}$, the total error coming from triangles in $\mathcal{T}_n^{(2)}$ is $O\left(h^3 \|D^2 g\|_\infty\right)$.

Next, consider the contribution to the error coming from triangles in $\mathcal{T}_n^{(1)}$. By Lemma 4.2.2, the error will be of size $O(h^2)$ multiplied times the integral over each such parallelogram of the maximum of the first derivatives of $G(v_i, Q)$ with respect to Q. Combining these we will have a bound

$$ch^2 \int\limits_{S-\Delta^*} \left(|G| + |DG|\right) dS_Q \tag{4.41}$$

By Theorem 4.2.4, the quantity in (4.41) is bounded by

$$ch^2 \int\limits_{S-\Delta^*} \left(\frac{1}{|P-Q|} + \frac{1}{|P-Q|^2} \right) dS_Q \tag{4.42}$$

Using a local representation of the surface and then using polar coordinates, the expression in (4.42) is of order

$$ch^2 \left(h + \log h\right)$$

Thus, the error arising from the triangles in $\mathcal{T}_n^{(1)}$ is $O(h^2 \log h)$. Combining the error arising from the integrals over Δ^*, $\mathcal{T}_n^{(1)}$, and $\mathcal{T}_n^{(2)}$, we have (4.40).

□

Numerical Examples. As a smooth surface consider the ellipsoid

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1 \tag{4.43}$$

with $(a, b, c) = (1, 1, 1)$ (the surface $E1$), and $(a, b, c) = (2, 3, 5)$ (the surface $E2$) We solve the equation (4.1) with the function $f(P)$ so chosen that the true solution is

$$u = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \tag{4.44}$$

In Tables 4 and 5 we give

$$|u(P) - u_n(P)| \tag{4.45}$$

where $P = P_{ij} = \tau_i \left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}} \right) \in D_e(E_j)$ (the exterior of E_j), where $\tau_1 = 1.1$, $\tau_2 = 2$, and $\tau_3 = 10$ (points situated further and further away from the boundary of the ellipsoid). The results are consistent with a convergence rate of $O(h^2 \log h)$ predicted by Theorem 4.2.5 which illustrates the superconvergence.

As a simple piecewise smooth surface, we use again the unit cube

$$S = [0, 1] \times [0, 1] \times [0, 1] \tag{4.46}$$

$P = P_{11}$			$P = P_{21}$		$P = P_{31}$	
n	$\|u(P) - u_n(P)\|$	Ratio	$\|u(P) - u_n(P)\|$	Ratio	$\|u(P) - u_n(P)\|$	Ratio
4	8.52 E-1		5.05 E-1		1.02 E-1	
16	9.29 E-2	9.16	6.05 E-2	8.35	1.20 E-2	8.53
64	1.10 E-2	8.44	8.32 E-3	7.27	1.63 E-3	7.36
256	2.67 E-3	4.12	1.88 E-3	4.40	3.71 E-4	4.39

Table 4: Errors in solving the Neumann Problem on $E1$

$P = P_{12}$			$P = P_{22}$		$P = P_{32}$	
n	$\|u(P) - u_n(P)\|$	Ratio	$\|u(P) - u_n(P)\|$	Ratio	$\|u(P) - u_n(P)\|$	Ratio
4	2.87 E-1		1.56 E-1		2.70 E-2	
16	5.94 E-2	4.84	2.91 E-2	5.36	5.09 E-3	5.30
64	1.24 E-2	4.77	5.85 E-3	4.98	9.99 E-4	5.10
256	3.02 E-3	4.12	1.29 E-3	4.53	2.07 E-4	4.82

Table 5: Errors in solving the Neumann Problem on $E2$

The function f is chosen so that the true solution is

$$u = \frac{1}{\sqrt{(x-0.5)^2 + (y-0.5)^2 + (z-0.5)^2}} \tag{4.47}$$

In Table 6 we give the results for $|u(P) - u_n(P)|$ for $P = P_i = (\tau_i, \tau_i, \tau_i) \in D_e(S)$, $i = 1, 2, 3$. The ratios approach 2 as n increases, which is consistent with a rate of convergence of $O(h)$ as predicted by Theorem 4.2.1 (with $r = 0$). As shown in the table, the further away from the boundary of S the point P is, the better the approximation.

We conclude this chapter by noting that the ideas used in this section to study

	$P = P_1$		$P = P_2$		$P = P_3$	
n	$\lvert u(P) - u_n(P)\rvert$	Ratio	$\lvert u(P) - u_n(P)\rvert$	Ratio	$\lvert u(P) - u_n(P)\rvert$	Ratio
12	8.98 E-1		2.92 E-1		3.62 E-2	
48	4.11 E-1	2.17	9.18 E-2	3.18	3.01 E-3	12.01
192	1.96 E-1	2.02	3.61 E-2	2.54	3.61 E-4	8.33
768	9.89 E-2	1.98	1.68 E-2	2.14	1.18 E-4	3.05

Table 6: Errors in solving the Neumann Problem on the unit cube

the numerical solution of the exterior Neumann problem (4.1) apply very well to studying the numerical solutions of the interior Neumann problem and the (interior or exterior) Dirichlet problem as well. For the interior Neumann problem (analogous to (4.1), only with D instead of D_e), an auxiliary condition on $f(P)$ is needed for solvability (namely, $\int_S f(Q)\, dS = 0$). Also, this problem does not have a unique solution in the sense that two solutions differ by a constant, and the integral equation corresponding to (4.8) is no longer uniquely solvable.

The interior Dirichlet problem is defined as follows. Find $u \in C(\overline{D}) \cap C^2(D)$ that satisfies

$$\begin{aligned} \Delta u(P) &= 0, \quad P \in D \\ u(P) &= f(P), \quad P \in S \end{aligned} \tag{4.48}$$

with $f \in C(S)$ a given boundary function. The approach is similar to the one used to solve (4.1). Represent the solution of (4.48) as a double layer potential

$$w(A) = \int_S \Psi(Q) \frac{\partial}{\partial \mathbf{n}_Q} \left[\frac{1}{|A - Q|} \right] dS_Q, \; A \in D \tag{4.49}$$

and determine the density function Ψ by imposing the boundary condition in (4.48)

$$\Psi(P) - \frac{1}{2\pi} \int_S \Psi(Q) \cdot \frac{\cos \theta_Q}{|P - Q|}\, dS_Q = f(P), \; P \in S \tag{4.50}$$

See Atkinson[4, Ch. 9], for details.

The equation (4.50) is similar to equation (4.8). But, the interest in solving it using collocation methods with only interior collocation points is not so great in this case, since the kernel does not involve the normal $\mathbf{n}_P$, but the normal $\mathbf{n}_Q$. The exterior Dirichlet problem (defined analogously with (4.48), only with D_e instead of D) can be transformed into an interior Dirichlet problem using a Kelvin transform (see Atkinson[4, pp. 400-402]).

CHAPTER V
CONCLUSIONS

This paper investigates collocation methods for the solution of Fredholm integral equations of the second kind

$$u(P) - \int_S u(Q)K(P,Q)\, dS_Q = f(P), \quad P \in S \tag{5.1}$$

In particular, we are interested in the radiosity equation

$$u(P) - \frac{\rho(P)}{2\pi} \int_S u(Q)G(P,Q)V(P,Q)\, dS_Q = E(P), \quad P \in S \tag{5.2}$$

and in the integral equation reformulation of the exterior Neumann problem

$$\rho(P) + \frac{1}{2\pi} \int_S \rho(Q) \frac{\cos\theta_P}{|P-Q|^2}\, dS_Q = f(P), \quad P \in S \tag{5.3}$$

Collocation methods based on piecewise polynomial interpolation for the numerical solution of (5.1) have been studied extensively, especially piecewise linear methods (with the collocation nodes being the three vertices of each triangle; see Atkinson[4, Section 3.2]) and piecewise quadratic collocation methods (with the six collocation node points being the vertices and midpoints of the sides of each triangle; see Atkinson and Chien[6]). We considered (following the ideas in Atkinson and Chandler[5]) only collocation methods for which the collocation nodes are interior to each triangular face. We did so to avoid the difficult task of evaluating the unit normal to a surface that is not smooth at points located on an edge or at a corner. Also, in choosing the collocation nodes this way, we cannot have collocation points which are common to more that one triangle (in which case, again, there would be problems in defining the normal at such points).

We described in Chapter II a procedure for obtaining a numerical method of any desired order. Using interpolation of the solution of degree r at the $f_r \equiv \dfrac{(r+1)(r+2)}{2}$ interior nodes defined in (2.4), we obtain an error of order $O(h^{r+1})$. If we use symmetric triangulations or particular choices of the parameter α (used in defining the collocation nodes (2.4)), we might improve the rate of convergence.

We want to have collocation node points that are interior to each triangle and symmetrically placed inside each triangle. The set of nodes in (2.4) is not the only possible choice. We defined them that way because using that pattern it is possible to define interpolation of any degree r. But, in applications we rarely consider interpolation of degree higher that 2. One other way of choosing 6 interpolation (and collocation) nodes would be the following: Consider two constants $0 < \alpha, \beta < \dfrac{1}{2}$, $\alpha \neq \beta$ and define

$$q_1 = (\alpha, \alpha),\ q_2 = (\alpha, 1-2\alpha),\ q_3 = (1-2\alpha, \alpha)$$
$$q_4 = (\beta, \beta),\ q_5 = (\beta, 1-2\beta),\ q_6 = (1-2\beta, \beta) \tag{5.4}$$

which is actually a generalization of the nodes for quadratic interpolation considered in (2.11) and (2.15) (letting $\beta = \dfrac{1-\alpha}{2}$, we obtain the nodes in (2.11) and (2.15)). A quadrature formula derived based on quadratic interpolation at the nodes (5.4) has degree of precision 2 for any $0 < \alpha, \beta < \dfrac{1}{2}$, and degree of precision 4 for $\alpha = \alpha_0 \doteq 0.103583$ and $\beta = \beta_0 \doteq 0.448208$. Also, for these values, if the integration is done over symmetric triangles, the degree of precision is 5.

Another set of interpolation nodes for the quadratic case could be

$$q_1 = (\alpha, \beta),\ q_2 = (\alpha, \gamma),\ q_3 = (\beta, \alpha),\ q_4 = (\beta, \gamma),\ q_5 = (\gamma, \alpha),\ q_6 = (\gamma, \beta) \tag{5.5}$$

for constants $0 < \alpha, \beta, \gamma < \dfrac{1}{2}$, $0 \leq \alpha + \beta, \alpha + \gamma, \beta + \gamma \leq 1$. The definition of the six basis functions $l_1, ..., l_6$ and the computation of the norm $\|\mathcal{P}_n\|$ of the corresponding interpolation polynomial would be significantly more difficult, but having more "degrees of freedom" in choosing values for α, β (and γ in (5.5)) may lead to higher degrees of precision of the approximation. For the cubic interpolation case, one might consider the set of nodes in (5.4) and

$$q_7 = (\gamma, \gamma),\ q_8 = (\gamma, 1-2\gamma),\ q_9 = (1-2\gamma, \gamma),\ q_{10} = (\frac{1}{3}, \frac{1}{3}) \tag{5.6}$$

In Chapter III we used piecewise quadratic collocation methods for finding the numerical solution of the radiosity equation (5.2). Under certain smoothness assumptions the equation (5.2) is uniquely solvable for each function E. We proved that in this case, the rate of convergence is $O(h^3)$. At the collocation node points, we obtained superconvergence, $O(h^4)$ when we used only a symmetric triangulation, and $O(h^5)$ when in addition to the symmetry we considered a certain value for the parameter α, which increased the degree of precision of the quadrature formula derived from the interpolation formula. The error analysis is based on the collocation solution and it uses the space $L^\infty(S)$, requiring $\mathcal{P}_n$ to be a projection operator.

We also described procedures for obtaining superconvergent collocation methods using interpolation of higher degree. For the case of cubic interpolation with a special value for the parameter α we obtained a rate of convergence of $O(h^6 \log h)$. To obtain superconvergent methods based on interpolation of the solution of degree r, we must consider separately the cases of r being an odd or an even number. If r is even , we can obtain superconvergence, $O(h^{r+2})$, by simply considering symmetric triangulations of the surface S. Superconvergent methods with r being odd are more difficult to obtain, since they require the existence of a number α_0 for which the quadrature formula has degree of precision $r+1$. The value of such an α_0 is determined by solving a system of equations in only one variable, and that system may not have any solution. However, if such an α_0 exists, then the rate of convergence of the collocation method is proven to be $O(h^{r+3} \log h)$. For the numerical examples in Section 3.3 in implementing the method, we used the boundary element package described in Atkinson[3].

We only considered the simplest of cases, that of unoccluded surfaces ($V \equiv 1$). For computer graphics applications the more interesting case is that of occluded surfaces. In this case there are problems along the "lines of discontinuity" of V in dealing with the integrals over triangles Δ_k where $V(v_i, Q) \equiv 1$ is not true for $v_i \in \Delta_k$. In Atkinson and Chien[7] the authors describe a way of solving this problem in the case of piecewise constant interpolation. Unfortunately, for higher order methods this approach is not good enough.

Also, the surfaces considered are smooth. However, in most applications, the surfaces are likely to be only piecewise smooth. In this case the radiosity kernel is less well-behaved than for the smooth case, being no longer compact. We do not obtain superconvergence anymore, as the examples in Section 3.3 show.

Another factor that might slow down the speed of convergence is the approximation of the surface S. If the boundary S is curved rather than polyhedral, then it is convenient to approximate S by interpolation, obtaining an approximate boundary $\widehat{S}$ (see Atkinson and Chien[6] or Atkinson[4, Sections 9.3.1 and 5.3.3]). The interpolatory surface is then used in the approximate calculation of the Jacobian $|(D_s m_k \times D_t m_k(s,t)|$ and the approximate calculation of the unit normals $\mathbf{n}_P$ and $\mathbf{n}_Q$. In the boundary element package Atkinson[3], we use a quadratic interpolation of a curved surface. The decrease in the rate of convergence due to interpolation of the surface was not shown in our examples in Section 3.3, since for both surfaces the approximation was exact.

In Chapter IV we use a piecewise polynomial collocation method for the numerical solution of equation (5.3). Equation (5.3) arises in solving the exterior Neumann problem. Representing the solution of the Neumann problem as a single layer potential (which is always harmonic)

$$u(A) = \int_S \frac{\rho(Q)}{|A - Q|}\, dS_Q, \quad A \in D_e \tag{5.7}$$

the density function ρ must satisfy equation (5.3). Since the kernels of equations (5.2) and (5.3) are similar, the approaches used in finding numerical solutions for (5.2) apply very well to (5.3). Collocation methods with nodes interior to the triangles (and surface) are especially useful for equation (5.2), since in evaluating the kernel at the collocation points, there are problems with the normal $\mathbf{n}_P$ at either points common to more that one triangular face, or at points that are on a corner or edge of the surface S. In Section 4.2 we describe and give an error analysis for such collocation methods based on interpolation of the solution at the interior collocation nodes (2.4). Numerical examples are given for the case of piecewise constant interpolation (the centroid rule) of the solution. The examples illustrate the superconvergence ($O(h^2 \log h)$) of the centroid rule in the case of a smooth surface (the ellipsoid). For the case of piecewise smooth surface (the unit cube), the rate of convergence obtained is only $O(h)$, the method being no longer superconvergent.

The ideas described in this chapter apply well to studying the numerical solutions of the interior Neumann problem and the (interior or exterior) Dirichlet Problem as well. For the interior Neumann problem (analogous to the exterior Neumann problem, only with D instead of D_e), an auxiliary condition on $f(P)$ is needed for solvability (namely, $\int_S f(Q)\, dS = 0$). Also, this problem does not have a unique solution in the sense that two solutions differ by a constant.

In this work we studied in detail only the case of piecewise constant interpolation of the solution for the Neumann problem. But, as we did in Chapter III for the radiosity equation, superconvergent collocation methods based on interpolation of higher degree can be developed for equation (5.3).

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