PROJECTION METHODS WITH GLOBALLY SMOOTH APPROXIMATIONS FOR HAMMERSTEIN INTEGRAL EQUATIONS ON SURFACES

by

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CHAPTER I

INTRODUCTION

There are extensions of the familiar trigonometric series in one dimension to some closed surfaces in three space. For example functions on the unit sphere may be associated with a series of spherical harmonics called the Laplace series. Functions on a torus may be associated with a bivariate trigonometric series. Similiarly the approximation properties of trigonometric polynomials have analogues for spherical harmonics and bivariate trigonometric polynomials. We explore using these globally smooth approximations to develop projection methods for nonlinear integral equations on these surfaces. And by a change of variable, these methods may be extended to solving integral equations on other smooth but less regularly shaped surfaces.

The particular nonlinear integral equations we examine are called mixed-type Hammerstein integral equations. Let $\mathcal{X}$ be an appropriate space of functions defined on the smooth surface $\partial\Omega$ of a simply-connected open region $\Omega \subset \mathbf{R}^3$. Define a nonlinear integral operator H on some open subset $D \subseteq \mathcal{X}$ by

$$H(x)(P) := y(P) + \int_{\partial\Omega} \sum_{i=1}^{M} k_i(P,Q) g_i(Q, x(Q))\, dQ. \tag{1.1}$$

This is a Hammerstein integral operator of mixed type. We discuss finding the fixed points of H, i.e. $u \in \mathcal{X}$ satisfying,

$$u = H(u). \tag{1.2}$$

Projection methods generally fall into two categories, Galerkin and collocation methods. These are also known as spectral and pseudospectral methods. They have been used and analyzed extensively for one dimensional and multidimensional problems. But the vast majority of this work has assumed the projection methods to be based on approximations which are not globally smooth. These approximations are defined by local considerations based upon some partition of the domain in question. Typically the surface is divided into sections or elements each of which is the image of a smooth map of a polygon. Then an approximation is the sum of local functions. Such a local function, for example, is a low degree multivariate polynomial on one element of the surface and is identically zero on the remainder of the surface. Then the precision of the approximation is increased by subdividing the elements leading to a larger set of such local functions. The most popular method which takes such an approach to solving operator equations on surfaces is called the boundary element method (BEM).

The BEM is very adaptable to different shapes of surfaces and is almost mandatory for nonsmooth surfaces. But for surfaces which are quite smooth, the global methods we consider are often a reasonable alternative. For smooth solutions they are rapidly convergent. Therefore although the linear systems they lead to will involve global integration, they may be much smaller systems. One cost of using these rapidly converging approximations is that they will not converge for some continuous functions, and their rate of convergence is less regular than for the BEM. Also computing the coefficients of the linear system may be more expensive than for the BEM.

As a model problem on which to apply these projection methods, consider the following Laplace equation with nonlinear boundary conditions. This problem has applications in potential theory. Let Ω be an open domain in three space with a smooth connected boundary $\partial\Omega$. The problem is to find u defined on the closure of

Ω such that

$$\begin{aligned} \Delta u(P) &= 0 && \text{in } \Omega \\ \frac{\partial u}{\partial n}(P) &= g(P, u(P)) - f(P) && \text{on } \partial\Omega. \end{aligned} \tag{1.3}$$

A boundary integral reformulation of this problem yields a mixed-type Hammerstein integral equation on the surface $\partial\Omega$.

The integral operators in our model problem are completely continuous on the standard function spaces of interest although they have singular kernels. Therefore we have based our convergence analysis on the concept of a collectively compact sequence of operators [Anselone (1971).] This complements the general convergence theory provided by Krasnolselskii (1964). Also see Krasnolselskii and Zabreiko (1984). The groundwork for applying this theory to projection methods for nonlinear integral equations has been laid by Atkinson (1973) and Atkinson and Potra (1987). The present work extends the results of the latter paper by not assuming the boundedness of the operator norms of the sequence of projections.

Other have studied projection methods based on globally smooth approximations for the types of surfaces we consider. Our work is an extension of the work by Atkinson (1982). He solves a Dirichlet problem for Laplace's equation in three dimensions using a Galerkin scheme based on spherical harmonics. We are extending the application of his general approach to nonlinear integral equations and to collocation on a torus. Lin (1982) and Wienert (1990) have used spherical harmonics for Galerkin methods solving the Helmholtz equation in three space. Ganesh et. al. (1992) describe a pseudospectral method based on spherical harmonics for a nonlinear boundary integral equation from finite elasticity. Saavedra (1988) examined a collocation scheme using bivariate trigonometric polynomials on the surface of a torus.

Recent work has been done on the solution of our model problem for regions in two space. Using the BEM approach, Ruotsalainen and Wendland (1988) solve it with a spline Galerkin method. Hamina, Ruotsalainen, and Saranen (1992) suggest a collocation approach which uses an orthogonal projection to approximate the nonlinearity. Instead of a projection method, Atkinson and Chandler (1990) propose a Nystrom method based on trigonometric interpolation.

One approach we explore to solving Hammerstein integral equations was introduced by Kumar and Sloan (1987). They developed a new collocation method and applied it to one-dimensional Hammerstein integral equations. Their collocation approximations came from piecewise polynomial function spaces. Kumar (1987) discussed its superconvergence. This approach was extended to mixed-type Hammerstein integral equations by Ganesh and Joshi (1989).

In Chapter 2 we define some terminology and notation which will be used in subsequent chapters. This includes a more thorough discussion of projection methods and the change of variables we use to extend those methods to other surfaces. We introduce two variations on the usual projection method in Chapter 3. We then do a general convergence analysis for the projection method and its variants for the mixed-type Hammerstein integral equation. We apply the methods and the convergence analysis of Chapter 3 to our model problem in Chapter 4. In Chapter 5 we look at practical aspects of implementing these methods including complexity analysis and numerical integration. In Chapter 6 we look at some numerical examples.

CHAPTER II

PRELIMINARIES

In this chapter we define terms and establish notation which will be used in subsequent chapters. Since the methods we are exploring are based on globally smooth functions, they will usually behave poorly unless the solution is a smooth function on a smooth surfaces. First we define a Lyapunov surface, a special kind of smooth surface. Next we will use the concept of Hölder continuity to define classes of smooth functions and also classes of smooth surfaces. We examine the approximating subspaces based on globally smooth functions and corresponding projections which are used to define our methods. Finally we consider extending these methods by a change of variable between surfaces. The change of variable allows us to use the more conveniently defined globally smooth approximations of one surface for approximating functions on another surface. This provides a more general setting for discussing these methods in the following chapters.

2.1 Lyapunov Conditions

We will assume the reader is familiar with the concepts of C^k functions and C^k surfaces; although we will measure smoothness of functions and surfaces even more precisely. The usual approach to existence results for problems in potential theory is to assume the boundary surface is Lyapunov, that is, it meets the three Lyapunov conditions. Therefore for our purposes, we will require the surfaces to meet these conditions. Let Γ be a closed surface in $\mathbf{R}^3$ for which the following are true:

1. At each point of Γ there is a well-defined normal and tangent plane.

2. There exists constants $E > 0$ and $0 < \lambda \leq 1$, such that if θ is the angle between the normals at the points P and Q on Γ, then

$$\theta \leq E \mid P - Q \mid^{\lambda} .$$

3. There exists a constant $d > 0$, such that for any point P on Γ, the portion of Γ inside a sphere of radius d centered at P intersects lines parallel to the normal at P in at most one point.

Condition 3 may be deduced from 1 and 2. A Lyapunov surface is C^1; and a C^2 surface is Lyapunov.

2.2 Hölder Continuity

Related to the measure of smoothness, given by E and λ in condition 2, is the concept of Hölder continuity. For more on the following definitions, see Günter (1967).

If a function x defined on Γ satisfies

$$\mid x(P) - x(Q) \mid \leq A \mid P - Q \mid^{\lambda}, \quad 0 < \lambda \leq 1$$

for all P, $Q \in \Gamma$, then we shall say x is Hölder continuous in Γ with Hölder constant A and Hölder exponent λ. A Hölder continuous function is uniformly continuous on Γ. A continuously differentiable function is Hölder continuous with exponent 1. If x also belongs to $C^l(\Gamma)$ and x and all its derivatives up to order l are bounded by A and the derivatives of order l are Hölder continuous with Hölder constant A and Hölder exponent λ, then we shall say x belongs to the class $H(l, A, \lambda)$ in Γ.

Since there is a defined normal, at each point $P \in \Gamma$, we may choose a rectangular coordinate system (ξ, η, ζ) with P as the origin and the positive ζ axis along the outward normal at P. Suppose there is a circle Λ in the (ξ, η) plane (of fixed radius for all points P) centered at P such that there is a function F with a section of the surface Γ containing P parametrized by $(\xi, \eta, F(\xi, \eta))$ for (ξ, η) inside of Λ. We shall say that Γ belongs to the class $L_k(B, \lambda)$ of surfaces if $F \in H(k, B, \lambda)$ in Λ and the numbers B and λ are independent of the choice of the point P. A Lyapunov surface is of the class L_1 for some B and λ.

These measures of smoothness for Γ and for functions defined on Γ, will be useful later for examining rates of convergence for our numerical methods.

2.3 Function Spaces

In the main, we will be concerned with the two function spaces $C(\Gamma)$ and $L_2(\Gamma)$. $C(\Gamma)$ is the space of (real) functions which are defined and continuous on Γ, with the norm

$$\| x \|_\infty := \sup_{P \in \Gamma} | x(P) | .$$

We will assume the Lebesgue definition of an integral. We may define an inner product for functions defined on Γ by,

$$(x, y)_\Gamma := \int_\Gamma x \, y \, dA.$$

Suppose there is a numerical integration scheme approximating $\int_\Gamma x \, dA$, for $x \in C(\Gamma)$, defined by

$$I_m(x) := \sum_{l=1}^{m} w_l \, x(Q_l).$$

We may define a "discrete" inner product by

$$(x,\, y)_{\Gamma,m} := I_m(x\, y) = \sum_{l=1}^{m} w_l x(Q_l) y(Q_l).$$

The Hilbert space, $L^2(\Gamma)$, is the set of functions, x, defined on Γ such that

$$\int_\Gamma x^2 \, dA < \infty.$$

It has the norm

$$\| x \|_2 := (x, x)_\Gamma^{\frac{1}{2}} = \left[\int_\Gamma x^2 \, dA \right]^{\frac{1}{2}}.$$

Both $C(\Gamma)$ and $L_2(\Gamma)$ are inner product spaces with the inner product $(x, y)_\Gamma$. $L_2(\Gamma)$ is complete with respect to this inner product while $C(\Gamma)$ is not.

When the discussion applies to both $L^2(\Gamma)$ and $C(\Gamma)$, we will use the notation $\mathcal{X}(\Gamma)$ and $\| x \|_{\mathcal{X}(\Gamma)}$, or simply $\mathcal{X}$ and $\| x \|_{\mathcal{X}}$ for the designation of the function space and its norm.

An open ball in $\mathcal{X}$ of radius $R > 0$ is defined by

$$B_R(x) := \{ y \in \mathcal{X} \mid \| x - y \|_{\mathcal{X}} < R \}.$$

When x is the origin, (the function identically zero on Γ), we may simply write B_R.

2.4 Operators

Let L denote an operator from some subset $\mathcal{D}$ of $\mathcal{X}$ to a Banach space $\mathcal{Y}$. Since our problem concerns fixed points of operators, $\mathcal{Y}$ will usually be $\mathcal{X}$. For the image of $x \in \mathcal{X}$ under L, we will write Lx if L is linear, and $L(x)$ if nonlinear. In case L is linear, the norm of L is written as $\| L \|$, with the context making clear which space this is in relation to. For a sequence of operators $\{L_N\}$, we will write $L_N(x) \to L(x)$ when

$$\| L_N(x) - L(x) \|_{\mathcal{Y}} \to 0, \quad \text{as } N \to \infty.$$

$\{L_N\}$ is pointwise convergent in $\mathcal{D}$, if $L_N(x) \to L(x)$ for all $x \in \mathcal{D}$. If the L_N and L are linear, we may speak of norm convergence, i.e.

$$\| L_N - L \| \to 0, \quad \text{as } N \to \infty.$$

Norm convergence implies pointwise convergence on $\mathcal{X}$. For nonlinear L_N, a similar concept is uniform convergence on bounded sets.

Let $\{L_N\}$ be a sequence of operators defined on $\mathcal{D} \subseteq \mathcal{X}$. The sequence is said to be collectively compact if for each bounded set $\mathcal{B} \subseteq \mathcal{D}$, the set $\cup_{N=1}^{\infty} L_N(\mathcal{B})$ is relatively compact. For Banach spaces, relative compactness is equivalent to sequential compactness and to total boundedness.

The Fréchet derivative of an operator L at x is the bounded linear operator $L'(x) : \mathcal{X} \to \mathcal{Y}$ satisfying

$$\lim_{\| s \|_{\mathcal{X}} \to 0} \frac{\| L(x+s) - L(x) - L'(x)s \|_{\mathcal{Y}}}{\| s \|_{\mathcal{X}}} = 0.$$

The Fréchet derivative of a linear operator is itself. L is continuous at x if it is Fréchet differentiable there. We will say L is continuously Fréchet differentiable at x, if

$$\| L'(y) - L'(x) \| \to 0, \quad \text{as } y \to x.$$

2.5 Approximating Subspaces

A sequence of finite dimensional subspaces $\{\mathcal{X}_N\}$ of $\mathcal{X}$ will be called a sequence of approximating subspaces if $\bigcup_{N=1}^{\infty} \mathcal{X}_N$ is dense in $\mathcal{X}$. Let $\{h_k\}_{k=1}^{\infty}$ be an orthogonal set of functions which is dense in $\mathcal{X}$. Often the h_k will be trigonometric or spherical polynomials. Let $\mathcal{X}_N$ denote the span of $h_1, \ldots, h_{d_N}$. We will assume that $h_1, \ldots, h_{d_N}$ satisfy some property in relation to N, such as being the subset of $\{h_k\}_{k=1}^{\infty}$ of degree less than or equal to N. A projection $\mathbf{P}_N$ is a linear operator mapping $\mathcal{X}$ onto $\mathcal{X}_N$ and satisfying $\mathbf{P}_N x = x$ for all $x \in \mathcal{X}_N$. From the fact that $\mathbf{P}_N^2 = \mathbf{P}_N$, we have $\| \mathbf{P}_N \| \leq \| \mathbf{P}_N \|^2$, or $\| \mathbf{P}_N \| \geq 1$. Thus we have a sequence of projections $\{\mathbf{P}_N\}_{N=1}^{\infty}$ corresponding to the sequence of approximating subspaces $\{\mathcal{X}_N\}_{N=1}^{\infty}$.

Since $\cup_{N=1}^{\infty} \mathcal{X}_N$ is dense in $\mathcal{X}$, it may be expected that $\mathbf{P}_N x \to x$ for all $x \in \mathcal{X}$. It is certainly true that $\mathbf{P}_N x \to x$ for all x in the dense set $\cup_{N=1}^{\infty} \mathcal{X}_N$. Since this is true, the principle of uniform boundedness implies

Proposition 1 $\mathbf{P}_N x \to x$ *for all* $x \in \mathcal{X}$, *if and only if there exists a positive constant* c, *such that* $\| \mathbf{P}_N \| \leq c$ *for all* N.

There are two broad categories on how the functions $\{h_k\}$ are chosen. The first approach divides Γ into elements, Δ_j, with

$$\Gamma = \bigcup_{j=1}^{J} \Delta_j.$$

Polynomials which are of a fixed low degree on each element make up $\mathcal{X}_N$. Increasing N means subdividing the elements, but not increasing the degree of the polynomials. This approach corresponds to using, in the one dimensional case, piecewise constant or piecewise linear or piecewise quadratic, etc. functions. The desire for some degree of smoothness across the edges of the Δ_j has motivated using spline functions as a modification. For this approach, functions are constructed according to local considerations imposed by the elements. These local methods are often called boundary element methods (BEMs).

The second approach defines $\mathcal{X}_N$ as polynomials which are smooth over the whole surface Γ. The one dimensional analogue is approximation by Chebyshev or trigonometric polynomials. This global approach corresponds to spectral methods and is the focus of our investigation. For our model problems, the spectral methods often lead to cases where $\| \mathbf{P}_N \| \to \infty$ and thus $\mathbf{P}_N x \not\to x$ for some $x \in \mathcal{X}$.

Suppose we have $\{l_k\}$, $k = 1, \ldots, d_N$, a set of linear functionals on $\mathcal{X}_N$, with the matrix $[l_k(h_j)]_{d_N X\, d_N}$ equal to the identity matrix I. For example, $l_k(x)$ may equal $(h_k, x)_\Gamma$ or $(h_k, x)_{\Gamma,m}$. Using such functionals, we may write

$$\mathbf{P}_N x = \sum_{k=1}^{d_N} l_k(x) h_k.$$

Then $\mathbf{P}_N x = 0$ if and only if $l_k(x) = 0$ for $k = 1, \ldots, d_N$.

If the set $\{h_k\}$ constitutes an orthonormal basis for $\mathcal{X}$, then we may choose $l_k(x) := (h_k, x)_\Gamma$. The operator $\mathbf{P}_N$ will be called an orthogonal projection since $(x - \mathbf{P}_N x,\, h_k)_\Gamma = 0$ for $k = 1, \ldots, d_N$. This means $x - \mathbf{P}_N x$ is orthogonal to $\mathcal{X}_N$. In the L_2 norm, $\mathbf{P}_N x$ is the closest element to x from within the subspace $\mathcal{X}_N$.

If there are d_N distinct points on Γ, call them $\{Q_{l,N}\}$, such that $\mathbf{P}_N x(Q_{l,N}) = x(Q_{l,N})$, for $l = 1, \ldots, d_N$, then $\mathbf{P}_N$ is called an interpolatory projection. This often corresponds to the linear functionals being discrete inner products

$$l_k(x) := (h_k,\, x)_{\Gamma,m} = \sum_{l=1}^{m} w_l\, h_k(Q_{l,N}) x(Q_{l,N}),$$

and $m = d_N$. This displays an interesting similarity between orthogonal and interpolatory projections.

2.6 Transforming the Problem

For some surfaces in $\mathbf{R^3}$, there are well-known globally smooth approximations for continuous or square integrable functions defined on the surface. In the case of the unit sphere, U, the spherical harmonics are a natural choice. Whether $\mathcal{X}(U)$ is the space of continuous functions or of square integrable functions, we may form approximating subspaces, $\mathcal{X}_N(U)$, by defining $\mathcal{X}_N(U)$ to be the space of spherical polynomials of degree less than or equal to N. The other example we use is a torus, T, parameterized by $0 \le \phi < 2\pi$ and $0 \le \theta < 2\pi$. We let $\mathcal{X}_N(T)$ be the bivariate trigonometric polynomials of degree less than or equal to N.

The surface in our problem, $\partial\Omega$, may "borrow" the approximating functions of another surface, Θ, if $\partial\Omega$ is a smooth deformation of Θ. This is done via a standard change of variables. We will define a problem in $\mathcal{X}(\Theta)$ equivalent to our problem situated in $\mathcal{X}(\partial\Omega)$. To do this we now define an appropriate map.

Theorem 1 *Let $\mathcal{M} : \mathcal{X}(\partial\Omega) \to \mathcal{X}(\Theta)$ be a bounded, linear bijection. For an operator $A : \mathcal{D} \subseteq \mathcal{X}(\partial\Omega) \to \mathcal{X}(\partial\Omega)$, let $\hat{A} := \mathcal{M} A \mathcal{M}^{-1}$. Then $\hat{A}$ is a map from $\mathcal{M}(\mathcal{D}) \subseteq \mathcal{X}(\Theta) \to \mathcal{X}(\Theta)$, for which the following is true:*

i) If A is bounded and continuous, then so is $\hat{A}$.

ii) If A is compact, then so is $\hat{A}$.

iii) If A is Fréchet differentiable at x, then $\hat{A}$ is Fréchet differentiable at $\mathcal{M}x$.

iv) If x is an isolated fixed point of A, then $\mathcal{M}x$ is an isolated fixed point of $\hat{A}$.

Proof: ¿From the Open Mapping Theorem, we know that $\mathcal{M}^{-1}$ is bounded. Then i), ii), and iii) are well known results about the composition of operators. We should note that

$$\hat{A}'(x) = \mathcal{M}A'(\mathcal{M}^{-1}x).$$

For iv), let $\mathcal{O}$ be an open neighborhood of x such that x is the only fixed point of A in $\mathcal{O}$. Then $\mathcal{M}(\mathcal{O})$ is an open neighborhood of $\mathcal{M}x$ by the Open Mapping Theorem. Now

$$\hat{A}\mathcal{M}x = \mathcal{M}A\,x = \mathcal{M}x.$$

So $\mathcal{M}x$ is a fixed point of $\hat{A}$. Suppose $y \in \mathcal{M}(\mathcal{O})$ is a fixed point of $\hat{A}$. But $\mathcal{M}A\mathcal{M}^{-1}y = y$ implies $A\mathcal{M}^{-1}y = \mathcal{M}^{-1}y$. Thus $\mathcal{M}^{-1}y \in \mathcal{O}$, is a fixed point of A. $\mathcal{M}^{-1}y$ must be x and $y = \mathcal{M}x$. $\square$

We may find such an $\mathcal{M}$ if we have an appropriate map between the underlying surfaces, $\partial\Omega$ and Θ. What we have in mind is inducing $\mathcal{M} : \mathcal{X}(\partial\Omega) \to \mathcal{X}(\Theta)$ from a map $\mu : \Theta \to \partial\Omega$, by defining $\mathcal{M}x := x \circ \mu$. Such a map supplies the standard change of variables. It is immediate that $\mathcal{M}$ would be linear: $\mathcal{M}(a\,x + b\,y) = a\mathcal{M}x + b\mathcal{M}y$. The next two propositions will indicate what properties we desire for μ.

Proposition 2 *If μ is a continuous bijection, then $\mathcal{M}$ is a bounded linear bijection from $C(\partial\Omega)$ to $C(\Theta)$.*

Proof: $\mathcal{M}x = \mathcal{M}y$ if and only if $x(\mu Q) = y(\mu Q)$ for all $Q \in \Theta$. Since μ is one-to-one and onto, this is true if and only if $x(P) = y(P)$ for all $P \in \partial\Omega$. Thus $\mathcal{M}$ is bijective.

For all $x \in \mathcal{X}$,

$$\frac{\| \mathcal{M}x \|_{C(\Theta)}}{\| x \|_{C(\partial\Omega)}} = 1.$$

So $\| \mathcal{M} \| = \infty$. $\mathcal{M}$ is bounded. $\square$

For the transformation given by μ, there is also a function $J(Q)$, the determinant of the Jacobian matrix, such that

$$\int_{\Theta} x(\mu(Q'))J(Q')\, dA_{Q'} = \int_{\partial\Omega} x(Q)\, dA_Q$$

for every $x \in \mathcal{X}(\partial\Omega)$. Also a form of the Mean Value Theorem will apply. If $\mid J(Q') \mid \leq c$ for all $Q' \in \Theta$, then

$$\mid \mu(Q'_1) - \mu(Q'_2) \mid \leq c \mid Q'_1 - Q'_2 \mid,$$

for all $Q'_1,\, Q'_2 \in \Theta$.

Proposition 3 *If $0 < b \leq J(Q')$ for all $Q' \in \Theta$, then $\mathcal{M} : L^2(\partial\Omega) \to L^2(\Theta)$ is a bounded, linear bijection.*

Proof: The argument in Proposition 2 may again be used to show $\mathcal{M}$ is a bijection by adding the qualification "almost everywhere" to the equalities.

For $x \in \mathcal{X}(\partial\Omega)$,

$$b \int_{\Theta} (\mathcal{M}x)^2(Q')\, dA_{Q'} \leq \int_{\partial\Omega} x^2(Q)\, dA_Q.$$

Therefore

$$\frac{\| \mathcal{M}x \|_{L_2(\Theta)}}{\| x \|_{L_2(\partial\Omega)}} \leq \frac{1}{b}.$$

$\mathcal{M}$ is bounded. $\square$

We will later need to know the effect of $\mathcal{M}$ on "smoothing" operators.

Proposition 4 *If $0 < J(Q') \leq c$ for all $Q' \in \Theta$. Suppose the operator A maps the ball $B_a \subset C(\partial\Omega)$ into $H(0,\, b,\, \lambda)$ on $\partial\Omega$. Then $\hat{A}$ maps $B_a \subset C(\Theta)$ into $H(0,\, c^\lambda\, b,\, \lambda)$ on Θ.*

Proof: Let $y \in B_a \subset C(\Theta)$. Then $\mathcal{M}^{-1}y \in B_a \subset C(\partial\Omega)$. Then

$$\begin{aligned} \mid \hat{A}y(P) - \hat{A}y(Q) \mid &= \mid A\mathcal{M}^{-1}y(\mu(P)) - A\mathcal{M}^{-1}y(\mu(Q)) \mid \\ &\leq b \mid \mu(P) - \mu(Q) \mid^\lambda \\ &\leq b(c \mid P - Q \mid)^\lambda = b\, c^\lambda \mid P - Q \mid^\lambda . \end{aligned}$$

We used the generalization of the Mean Value Theorem. Thus $\hat{A}y \in H(0,\, c^\lambda b,\, \lambda)$ on

Θ. $\square$

2.7 The New Operators

Consider again our fixed point problem, $u = H(u)$, with H an operator mapping some subset $\mathcal{D} \subseteq \mathcal{X}(\partial\Omega) \to \mathcal{X}(\partial\Omega)$. To transform the problem onto Θ, we define the operator $\hat{H}$ by

$$\hat{H} := \mathcal{M} H \mathcal{M}^{-1}.$$

By Theorem 1, $\hat{H}$ is completely continuous. $\hat{H}$ is defined on the set $\mathcal{M}(\mathcal{D}) \subseteq \mathcal{X}(\Theta)$. The function $\hat{u}^* := \mathcal{M}u^*$ satisfies the equation

$$\hat{u}^* = \hat{H}(\hat{u}^*). \tag{2.1}$$

By our change in variables, we now have a fixed point operator equation in the space $\mathcal{X}(\Theta)$. From its solution $\hat{u}^*$, we may recover the solution u^* of our original problem.

In terms of μ,

$$\hat{H}(x)(P') := y \circ \mu^{-1}(\mu(P')) + \int_{\partial\Omega} \sum_{i=1}^{M} k_i(\mu(P'), Q) g_i(Q, x(\mu^{-1}(Q)))\, dA_Q. \tag{2.2}$$

for $x \in \mathcal{X}(\Theta)$. [Here again we are using the prime notation to indicate that a point is on Θ, while points on $\partial\Omega$ will have no prime. Also we will assume $\mu(Q') = Q$ and $\mu(P') = P$.] We may similarily define the new operators $\hat{K}_i$ and $\hat{G}_i$. For $x \in \mathcal{X}(\Theta)$,

$$\begin{aligned} \hat{K}_i x(P') &= \mathcal{M} K_i (x \circ \mu^{-1})(P') \\ &= \int_{\partial\Omega} k_i(\mu(P'), Q)\, x \circ \mu^{-1}(Q)\, dA_Q \\ &= \int_{\Theta} k_i(\mu(P'), \mu(Q'))\, x(Q')\, J(Q')\, dA_{Q'} \end{aligned}$$

So for convenience $\hat{K}_i$ may be viewed as an integral operator on $\mathcal{X}(\Theta)$ with kernel, $k_i(\mu(P'), \mu(Q'))\, J(Q')$. For $x \in \mathcal{M}(\mathcal{D})$,

$$\begin{aligned} \hat{G}_i(x)(Q') &= g_i(\mu(Q'), x \circ \mu^{-1}(Q)) \\ &= g_i(\mu(Q'),\, x(Q')) \end{aligned}$$

With these new operators on $\mathcal{X}(\Theta)$, we may characterize $\hat{H}$ as a Hammerstein integral operator of mixed type with the integration taking place on Θ. That is

$$\hat{H}(x) = \hat{y} + \sum_{i=1}^{M} \hat{K}_i \hat{G}_i(x).$$

$\mathcal{M}$ has allowed us to transform the problem without losing any of its characteristics.

CHAPTER III

CONVERGENCE ANALYSIS

In this chapter we define the projection method for the mixed Hammerstein integral equation and also define some variations of the projection method. For these methods we analyze the existence, uniqueness, convergence and rate of convergence of their approximate solutions. Although we assume that the projections are based on globally smooth functions and the context is the space of continuous functions on Θ or the space of square integrable functions on Θ, much of this analysis would generalize to local methods and other function spaces. The results are applied to the model problem in Chapter 4.

3.1 The Projection Method and Variations

Let $\{\mathcal{X}_N\}$ be a sequence of approximating finite dimensional subspaces of $\mathcal{X} = C(\Theta)$ or $L^2(\Theta)$. Let $\mathbf{P}_N$ denote a linear projection from $\mathcal{X}$ onto $\mathcal{X}_N$. We develop some methods for approximating the fixed points of Hammerstein integral operators of mixed-type defined on $\mathcal{X}$,

$$\hat{u} = \hat{H}(\hat{u}) := \hat{y} + \sum_{i=1}^{M} \hat{K}_i \hat{G}_i(\hat{u}). \tag{3.1}$$

We first summarize a standard projection method. Let $\hat{u}^*$ denote the fixed point of the Hammerstein integral equation of mixed-type (3.1). Approximate $\hat{u}^*$ with $\hat{u}_N^{(1)} \in \mathcal{X}_N$ which satisfies the fixed point equation

$$\hat{u}_N^{(1)} = \hat{H}_N^{(1)}(\hat{u}_N^{(1)}) := \mathbf{P}_N \hat{H}(\hat{u}_N^{(1)}) := \mathbf{P}_N \left(\hat{y} + \sum_{i=1}^{M} \hat{K}_i \hat{G}_i(\hat{u}_N^{(1)}) \right) := \mathbf{P}_N \hat{y} + \sum_{i=1}^{M} \hat{K}_{i,N}^{(1)} \hat{G}_i(\hat{u}_N^{(1)}). \tag{3.2}$$

(The superscript here indicates this is one of the three methods we will discuss, labeled 1, 2, and 3.) Note that (3.2) is the same as requiring the "residual" $\hat{u}_N^{(1)} - \hat{H}(\hat{u}_N^{(1)})$ to be mapped to zero by the projection operator $\mathbf{P}_N$. Methods similar to projection methods are often distinguished by the way in which the residual is minimized.

A variation of this was suggested by Kumar and Sloan (1987), and it was extended to mixed Hammerstein equations by Ganesh and Joshi (1989). The variation was developed as a way of reducing the amount of numerical computation. Begin by considering the functions

$$z_j^* := \hat{G}_j(\hat{u}^*). \tag{3.3}$$

They satisfy the equations

$$z_j = \hat{G}_j(\hat{y} + \sum_{i=1}^{M} \hat{K}_i z_i), \tag{3.4}$$

for $j = 1, \ldots, M$. We approximate the z_j^* with a projection method similar to that given above. We find $z_{j,N} \in \mathcal{X}_N$ satisfying

$$z_{j,N} = \mathbf{P}_N \hat{G}_j(\hat{y} + \sum_{i=1}^{M} \hat{K}_i z_{i,N}). \tag{3.5}$$

Finally an approximation to $\hat{u}^*$ is given by $\hat{u}_N^{(2)}$ where

$$\hat{u}_N^{(2)} = \hat{y} + \sum_{i=1}^{M} \hat{K}_i z_{i,N}. \tag{3.6}$$

So $\hat{u}_N^{(2)}$ in turn satisfies the fixed point equation

$$\hat{u}_N^{(2)} = \hat{H}_N^{(2)}(\hat{u}_N^{(2)}) := \hat{y} + \sum_{i=1}^{M} \hat{K}_i \mathbf{P}_N \hat{G}_i(\hat{u}_N^{(2)}) := \hat{y} + \sum_{i=1}^{M} \hat{K}_{i,N}^{(2)} \hat{G}_i(\hat{u}_N^{(2)}). \tag{3.7}$$

The other variation on projection methods which we will consider is called an *iterated projection method.* Sloan (1976) introduced this technique for linear integral equations. Using $\hat{u}_N^{(1)}$, which is the solution of the projection method (3.2), we form the approximation $\hat{u}_N^{(3)}$ where

$$\hat{u}_N^{(3)} = \hat{H}(\hat{u}_N^{(1)}). \tag{3.8}$$

Since $\mathbf{P}_N \hat{u}_N^{(3)} = \hat{u}_N^{(1)}$, the Sloan iterate satisfies our third fixed point approximate equation

$$\hat{u}_N^{(3)} = \hat{H}_N^{(3)}(\hat{u}_N^{(3)}) := \hat{H}(\mathbf{P}_N \hat{u}_N^{(3)}) := \hat{y} + \sum_{i=1}^{M} \hat{K}_i \hat{G}_i(\mathbf{P}_N \hat{u}_N^{(3)}). \tag{3.9}$$

We will call $\hat{u}_N^{(1)}$ the projection solution, $\hat{u}_N^{(2)}$ the Kumar solution, and $\hat{u}_N^{(3)}$ the Sloan iterate. In Chapter 5, we will compare practical aspects of their implementations. For now we do a convergence analysis. The analysis for the projection and Kumar methods will proceed together since they have much in common. The existence, convergence, etc. of the Sloan iterate will be based on that of the projection solution. At the end we will examine the possible superconvergence of $\hat{u}_N^{(2)}$ and $\hat{u}_N^{(3)}$ with respect to $\hat{u}_N^{(1)}$.

In the analysis which follows, we will be borrowing heavily from Krasnoselskii (1964), Krasnoselskii and Zabreiko (1984), and Atkinson and Potra (1987). This analysis is based on the theory of completely continuous operators, together with the concept of the rotation of a completely continuous vector field. The innovations present here are the application of this analysis to the Kumar method, and the generalization to projection methods for which $\| \mathbf{P}_N \| \to \infty$. Furthermore, instead of assuming $\hat{H}_N^{(\nu)} \to \hat{H}$ uniformly on bounded sets, we use the more general assumptions of the theory of collectively compact operators (Anselone, 1971). For the combination of collectively compact operator theory with geometrical reasoning based on the rotation of completely continuous vector fields, we have relied on Atkinson (1973).

Dropping the assumption of boundedness for $\| \mathbf{P}_N \|$ is often needed in the context of seeking uniform convergence and using approximating subspaces with globally smooth basis functions. In the place of this assumption, we will assume a particular relationship between the $\mathbf{P}_N$ and the $\hat{K}_i$, making our analysis peculiar to Hammerstein operators. This is the reason for implicitly defining the operators $\hat{K}_{i,N}^{(\nu)}$, for $\nu =$ 1,2 in (3.2) and (3.7). Thus we distinguish between the projection method and Kumar's method by the order of composition of the operators $\hat{K}_i$ with $\mathbf{P}_N$. If $\| \mathbf{P}_N \|$ is unbounded, then there are x for which $\mathbf{P}_N x$ does not converge to x. To see this, use the principle of uniform boundedness, which says $\mathbf{P}_N x \to x$ for all $x \in \mathcal{X}$ if and only if $\mathbf{P}_N x \to x$ on a dense subset of $\mathcal{X}$ and the operator norms $\| \mathbf{P}_N \|$ are uniformly bounded. Since $\bigcup_{N=1}^{\infty} \mathcal{X}_N$ is dense in $\mathcal{X}$, $\mathbf{P}_N x \to x$ on a dense set. Thus this shows that if $\| \mathbf{P}_N \|$ is unbounded, then $\mathbf{P}_N x$ does not converge to x for some $x \in \mathcal{X}$.

3.2 Some Assumptions and Some Useful Facts

As mentioned, we use the concept of the rotation of a completely continuous vector field. If A is a completely continuous operator, then $I - A$ is a completely continuous vector field. Suppose $\mathcal{O}$ is a bounded open set, and A is completely continuous on closure($\mathcal{O}$) with no fixed points on $\partial\mathcal{O}$. Then there is an integer, which we denote by $\gamma(I - A, \mathcal{O})$, called the rotation of $I - A$ on $\partial\mathcal{O}$. If x_0 is an isolated fixed point of A, there is similarly an integer called the index of x_0 as a fixed point of A and denoted by $ind(x_0, I - A)$. We are not interested in the precise definitions of these topological concepts. We simply wish to make use of the following properties. (For these properties it is assumed that A, A_1, and A_2 are completely continuous on $closure(\mathcal{O})$, and that they have no fixed points on $\partial\mathcal{O}$.)

R.1 If A has fixed points $x_1, \ldots, x_n$ in $\mathcal{O}$, then

$$\gamma(I - A, \mathcal{O}) = ind(x_1, I - A) + \cdots + ind(x_n, I - A).$$

R.2 If $\gamma(I - A, \mathcal{O}) \neq 0$, A has at least one fixed point in $\mathcal{O}$.

R.3 If $\| A_1 x - A_2 x \|_{\mathcal{X}} < \| x - A_1 x \|_{\mathcal{X}}$ on $\partial\mathcal{O}$, then $\gamma(I - A_2, \mathcal{O}) = \gamma(I - A_1, \mathcal{O})$.

R.4 If $A(x_0) = x_0$ and $A'(x_0)$ exists and does not have 1 as an eigenvalue, then x_0 is an isolated fixed point of A, and for arbitrary r,

$$ind(x_0, I - A) = \gamma(I - A'(x_0), B_r) \neq 0,$$

where B_r is the open ball of radius r centered at the origin.

Now we list the assumptions which we will use repeatedly. Of course other assumptions could be substituted for which the following analysis would still be valid. But one goal we have is to minimize the assumptions (at this point) about the nonlinear operators $\hat{G}_i$. We will later add assumptions about Fréchet differentiability where needed. The following assumption **A.5** will have two versions, (1) and (2), corresponding to whether the projection solution $\hat{u}_N^{(1)}$ or the Kumar solution $\hat{u}_N^{(2)}$ is being considered. In the analysis which follows, the index ν will be used to label other assumptions and quantities, giving two versions of each one, corresponding to the two versions of **A.5**. In equations (1.1) and (1.6), we already defined the operators $\hat{K}_{i,N}^{(1)}$ and $\hat{K}_{i,N}^{(2)}$ according to this scheme. Also note that **A.4** has an addendum in the case $\nu = 1$. We will use these assumptions for the remainder of this chapter.

A.1 $\hat{u}^*$ is an isolated fixed point of $\hat{H}$. In particular, there is an open, bounded subset $\mathcal{D}$ in $\mathcal{X}$, such that $\hat{u}^* \in \mathcal{D}$, and $\hat{u}^*$ is the only fixed point of $\hat{H}$ on the closure of $\mathcal{D}$.

A.2 The $\hat{K}_i$ are compact on $\mathcal{X}$.

A.3 The $\hat{G}_i$ are bounded and continuous on $\mathcal{D}$, (the bounded, open set mentioned in **A.1.**) For a subset $\mathcal{S} \subseteq \mathcal{D}$, we call $m(\mathcal{S})$ the least bound such that

$$\max_i \sup_{x \in \mathcal{S}} \| \hat{G}_i(x) \|_{\mathcal{X}} \leq m(\mathcal{S}).$$

A.4 $\hat{y} \in \mathcal{X}$ and for $\nu = 1$, $\mathbf{P}_N \hat{y} \to \hat{y}$.

A.5 (ν) For each integer i and each $x \in \mathcal{X}$, $\hat{K}_{i,N}^{(\nu)} x \to \hat{K}_i x$, and $\{\hat{K}_{i,N}^{(\nu)} \mid N \geq 1\}$ is a collectively compact family on $\mathcal{X}$.

Note that, in general, neither **A.5**(1) nor **A.5**(2) implies the other.

We give a lemma which will be useful later.

Lemma 1 *Suppose T_N is a set of linear operators, with $T_N x \to Tx$ for all $x \in \mathcal{X}$. Then for a precompact set $\mathcal{B}$, there is a sequence $p_N(\mathcal{B})$ converging to zero, such that*

$$\sup_{x \in \mathcal{B}} \| (T - T_N)x \|_{\mathcal{X}} \leq p_N(\mathcal{B}). \tag{3.10}$$

Proof: See Anselone (1971, Proposition 1.7, page 7). $\square$

3.3 A Digression Concerning A.5

The crucial assumption for this analysis is **A.5**(ν), $\nu = 1, 2$. Therefore to give an idea of what is involved in these assumptions, we prove some propositions which supply conditions under which **A.5**(1) and **A.5**(2) would be true. Of course these approaches are not exhaustive, but are the ones used for the model problem in Chapter 4.

Our first proposition shows that norm convergence for the $\hat{K}_{i,N}^{(\nu)}$ is a less general assumption than **A.5**(ν).

Proposition 1 *If for each i, $\| \hat{K}_{i,N}^{(\nu)} - \hat{K}_i \| \to 0$, then* **A.5**$(\nu)$ *is true.*

Proof: Trivially, we have the pointwise convergence. For the collective compactness, let $\mathcal{B}$ be a bounded set. We show that $\bigcup_{N=1}^{\infty} \hat{K}_{i,N}^{(\nu)}(\mathcal{B})$ is a totally bounded set.

$\hat{K}_i(\mathcal{B})$ is precompact and therefore totally bounded. Choose $\epsilon > 0$ and cover $\hat{K}_i(\mathcal{B})$ with the open balls, $\{B_{\epsilon/2}(y) \mid y \in \hat{K}_i(\mathcal{B})\}$. There exists an $N_0^{(\nu)}$ such that for $N \geq N_0^{(\nu)}$,

$$\| \hat{K}_{i,N}^{(\nu)}x - \hat{K}_i x \|_{\mathcal{X}} < \frac{\epsilon}{2}, \qquad \text{for all} \quad x \in \mathcal{B}.$$

There also exists a finite subcover $\{B_{\epsilon/2}(y_j) \mid j = 1, \ldots, J\}$, of $\hat{K}_i(\mathcal{B})$. So by doubling the size of the balls we have

$$\bigcup_{N=N_0^{(\nu)}}^{\infty} \hat{K}_{i,N}^{(\nu)}(\mathcal{B}) \subset \bigcup_{j=1}^{J} B_\epsilon(y_j).$$

So the tail is totally bounded, while the head, $\bigcup_{N=1}^{N_0^{(\nu)}-1} \hat{K}_{i,N}^{(\nu)}(\mathcal{B})$, is totally bounded since it is the finite union of totally bounded sets. $\square$

Next, we will see that assuming that $\{\mathbf{P}_N x \mid N \geq 1\}$ converges for all x is also a less general assumption than $\mathbf{A.5}(\nu)$. This assumption may be made for example when using piecewise smooth approximations. So much of the following analysis includes that case as well. It is also true when $\mathbf{P}_N$ is an orthogonal projection in a Hilbert space.

Proposition 2 *Assume that $\mathbf{P}_N x \to x$ for all $x \in \mathcal{X}$. Then $\mathbf{A.5}(\nu)$ is true for $\nu =$ 1 or 2.*

Proof: By the principle of uniform boundedness, $\| \mathbf{P}_N \| \leq c < \infty$ for all N. For the case $\nu = 1$, the operator $\hat{K}_i$ being compact and the sequence $\{\| \mathbf{P}_N \|\}$ being bounded implies $\| \mathbf{P}_N \hat{K}_i - \hat{K}_i \| \to 0$ for each i. See Atkinson (1976, lemma 2, page 53). By Proposition 1, this norm convergence implies $\mathbf{A.5}$.

For the case $\nu = 2$, clearly

$$\| \hat{K}_i \mathbf{P}_N x - \hat{K}_i x \|_{\mathcal{X}} \leq \| \hat{K}_i \| \, \| \mathbf{P}_N x - x \|_{\mathcal{X}} \to 0.$$

So $\hat{K}_{i,N}^{(2)}$ converges for all x. Let $\mathcal{B}$ be a bounded subset of $\mathcal{X}$ with $\| x \|_{\mathcal{X}} \leq b$ for all $x \in \mathcal{B}$. Then $\| y \|_{\mathcal{X}} \leq c\, b$ for all $y \in \bigcup_N \mathbf{P}_N(\mathcal{B})$. Since $\hat{K}_i$ is compact, $\bigcup_N \hat{K}_i \mathbf{P}_N(\mathcal{B})$ is precompact. This shows the sequence $\{\hat{K}_i \mathbf{P}_N\}$ is collectively compact. $\mathbf{A.5}(2)$ is true. $\square$

Note that we have proven that $\mathbf{P}_N \hat{K}_i$ is norm convergent to $\hat{K}_i$ since $\hat{K}_i$ is compact. If in addition $\mathbf{P}_N$ is an orthogonal projection in a Hilbert space,

$$\| \hat{K}_i(\mathbf{P}_N - I) \| = \| (\mathbf{P}_N - I)\hat{K}_i^* \| \to 0,$$

where $\hat{K}_i^*$ is the adjoint of $\hat{K}_i$. This is an argument from Sloan (1976, Theorem 1) using the fact that $(\mathbf{P}_N - I)$ is self-adjoint and that the adjoint of a compact operator is compact.

For the next proposition, we begin by considering the concept of the best approximation. We define the best approximation to x from $\mathcal{X}_N$, call it $T_N(x)$, as being the element of $\mathcal{X}_N$ which satisfies

$$\| T_N(x) - x \|_{\mathcal{X}} = \min_{y \in \mathcal{X}_N} \| y - x \|_{\mathcal{X}} . \tag{3.11}$$

Assume that it exists and is unique. This is true for the applications which we consider. Although $\| \mathbf{P}_N \| \to \infty$, it may be that the best approximation, $T_N(y)$, converges to y when y is in range$(\hat{K}_i)$. That is the point of our next proposition. It applies only to version 1 of **A.5,** i.e. the projection method.

Proposition 3 *Let x be in $\mathcal{X}$. For some i, let $y = \hat{K}_i x$. If, independently of i,*

$$\| \mathbf{P}_N \| \| y - T_N(y) \|_{\mathcal{X}} \leq a_N \| x \|_{\mathcal{X}}$$

with $a_N \to 0$, then **A.5***(1) is true.*

Proof: Since $T_N(y)$ is in $\mathcal{X}_N$, we have

$$\begin{aligned} \| \mathbf{P}_N \hat{K}_i x - \hat{K}_i x \|_{\mathcal{X}} &= \| \mathbf{P}_N y - y \|_{\mathcal{X}} \\ &= \| \mathbf{P}_N (y - T_N(y)) + T_N(y) - y \|_{\mathcal{X}} \\ &\leq (1 + \| \mathbf{P}_N \|) \| y - T_N(y) \|_{\mathcal{X}} . \end{aligned}$$

For $\| x \|_{\mathcal{X}} = 1$, the last term is less than $2a_N$ which goes to zero. So we have the norm convergence of Proposition 1. $\square$

For balance, we now give some results which apply only to the Kumar solution and version 2 of **A.5**. First we consider the case when $\mathbf{P}_N$ is an orthogonal projection. We use the fact that if $\mathcal{X}$ has the inner product

$$(x,\,y)_\Theta := \int_\Theta x(Q)\,y(Q)\,dA_Q,$$

and $\mathbf{P}_N$ is an orthogonal projection, i.e.

$$\mathbf{P}_N x(Q) = \sum_{k=1}^{d_N} (h_k,\,x)_\Theta h_k(Q),$$

then $\mathbf{P}_N$ is self-adjoint,

$$\int_\Theta x(Q)\mathbf{P}_N y(Q)\,dA_Q = \int_\Theta \mathbf{P}_N x(Q)\,y(Q)\,dA_Q. \tag{3.12}$$

Proposition 4 *Suppose* $\mathcal{X} = C(\Theta)$ *and* $\mathbf{P}_N$ *is an orthogonal projection. Let* $\hat{k}_{P,i}(Q) := \hat{k}_i(P,Q)$ *be the kernel of the integral operator* $\hat{K}_i$. *Assume for* $q = 1$ *or 2,*

$$\max_i \sup_{P\in\Theta} \parallel (\mathbf{P}_N - I)\hat{k}_{P,i} \parallel_q \to 0.$$

Then **A.5***(2) is true.*

Proof: Applying Hölder's inequality, for $x \in \mathcal{X}$,

$$\begin{aligned}
\parallel \hat{K}_i\mathbf{P}_N x - \hat{K}_i x \parallel_\infty &= \parallel \int_\Theta \hat{k}_{P,i}(Q)(\mathbf{P}_N - I)x(Q)\,dA_Q \parallel_\infty \\
&= \parallel \int_\Theta (\mathbf{P}_N - I)\hat{k}_{P,i}(Q)\,x(Q)\,dA_Q \parallel_\infty \\
&\le \sup_{P\in\Theta} \int \mid (\mathbf{P}_N - I)\hat{k}_{P,i}(Q)x(Q) \mid \; dA_Q \\
&\le \parallel x \parallel_{q/(q-1)} \sup_{P\in\Theta} \parallel (\mathbf{P}_N - I)\hat{k}_{P,i} \parallel_q .
\end{aligned}$$

where we let $q/(q-1) = \infty$ in case $q = 1$. Since $\parallel x \parallel_{q/(q-1)}$ is bounded whether $q = 1$ or 2, the norm convergence of $\hat{K}_i\mathbf{P}_N$ to $\hat{K}_i$ follows for each i. We get the result when we again apply Proposition 1. □

For the Kumar method we also consider the case that $\mathbf{P}_N$ is an interpolatory projection on $C(\Theta)$. Recall that an interpolatory projection may often be defined in terms of a discrete inner product. When $\mathbf{P}_N$ is an interpolatory projection defined by

$$\mathbf{P}_N x = \sum_{k=1}^{d_N} (h_k, x)_{\Theta,m} h_k,$$

let $\mathbf{O}_N$ be the corresponding orthogonal projection,

$$\mathbf{O}_N x = \sum_{k=1}^{d_N} (h_k, x)_\Theta h_k.$$

Then $\mathbf{P}_N$ and $\mathbf{O}_N$ exhibit the relationship

$$\begin{aligned} (x, \mathbf{P}_N y)_\Theta &= \int_\Theta x(Q)\mathbf{P}_N y(Q)\, dA_Q \\ &= (\mathbf{O}_N x,\, y)_{\Theta,m}. \end{aligned} \tag{3.13}$$

Compare this to the self-adjointness of the orthogonal projection in (3.12).

For the following theorem we will need another assumption and a definition. The parameter m of the discrete inner product in (3.13) is dependent upon N. We will write $m(N)$. Thus we have a sequence of discrete inner products corresponding to the sequence of interpolatory projections. Suppose this sequence of discrete inner products has the property that there exists an absolute constant b such that if $x \in \mathcal{X}_N$, then

$$(1, \mid x \mid)_{\Theta,m(N)} \leq b \int_\Theta \mid x \mid \, dA. \tag{3.14}$$

We say that a function ω defined on the positive reals is a *bounding function* if $\omega(x)$ approaches zero monotonically as $x \downarrow 0$.

Proposition 5 *Let $\mathbf{P}_N$ be an interpolatory projection based on discrete inner products on the space $\mathcal{X} = C(\Theta)$ with a corresponding orthogonal projection $\mathbf{O}_N$. Assume the sequence of inner products for this projection satisfies (3.14). Suppose that for each i and all $P \in \Theta$, there exists bounding functions ω_1 and ω_2 such that*

$$\| \mathbf{O}_N \hat{k}_{P,i} - \hat{k}_{P,i} \|_1 \leq \omega_1 \left(\frac{1}{N} \right),$$

and

$$\| \hat{k}_{P,i} - \hat{k}_{P',i} \|_1 \leq \omega_2(\mid P - P' \mid).$$

Then **A.5***(2) is true.*

Proof: First we show that $\| \hat{K}_i \mathbf{P}_N \|$ is bounded for each i. Let $x \in C(\Theta)$ with

$\| x \|_\infty \leq 1$. Then

$$\begin{aligned} \| \hat{K}_i \mathbf{P}_N x \|_\infty &= \| \int_\Theta \hat{k}_i(P, Q) \mathbf{P}_N x(Q) dA_Q \|_\infty \\ &= \sup_P \| (x, \mathbf{O}_N \hat{k}_{P,i})_{\Theta, m(N)} \|_\infty \\ &\leq \| x \|_\infty \sup_P (1, | \mathbf{O}_N \hat{k}_{P,i} |)_{\Theta, m(N)} \\ &\leq b \sup_P \| \mathbf{O}_N \hat{k}_{P,i} \|_1 . \end{aligned}$$

Since $\| \mathbf{O}_N \hat{k}_{P,i} - \hat{k}_{P,i} \|_1 \to 0$, the last expression is bounded. Also since $\hat{K}_i \mathbf{P}_N x \to \hat{K}_i x$ on a dense set (namely the union of the $\mathcal{X}_N$), this boundedness implies $\hat{K}_i \mathbf{P}_N x \to \hat{K}_i x$ for all x.

Next we show equicontinuity of the family $\mathcal{F} = \{\hat{K}_i \mathbf{P}_N x \mid N \geq 1, \| x \|_\infty \leq 1\}$. If we follow the same steps as in the derivation above

$$| \hat{K}_i \mathbf{P}_N x(P) - \hat{K}_i \mathbf{P}_N x(P') | \leq c \; \| \mathbf{O}_N (\hat{k}_{P,i} - \hat{k}_{P',i}) \|_1 . \tag{3.15}$$

We consider two ways to bound $\| \mathbf{O}_N (\hat{k}_{P,i} - \hat{k}_{P',i}) \|_1$. First

$$\begin{aligned} \| \mathbf{O}_N (\hat{k}_{P,i} - \hat{k}_{P',i}) \|_1 &\leq \| \mathbf{O}_N \hat{k}_{P,i} - \hat{k}_{P,i} \|_1 + \| \hat{k}_{P,i} - \hat{k}_{P',i} \|_1 + \| \hat{k}_{P',i} - \mathbf{O}_N \hat{k}_{P',i} \|_1 \\ &\leq 2\,\omega_1 \left(\frac{1}{N} \right) + \omega_2(| P - P' |) \end{aligned} \tag{3.16}$$

Also

$$\| \mathbf{O}_N (\hat{k}_{P,i} - \hat{k}_{P',i}) \|_1 \leq \| \mathbf{O}_N \| \; \omega_2(| P - P' |). \tag{3.17}$$

Let $\epsilon > 0$. Let N_0 be such that $\omega_1(1/N) < \epsilon/4$ for $N \geq N_0$. Let δ be small enough such that if $| P - P' | < \delta$,

$$\max_{N \leq N_0} \| \mathbf{O}_N \| \, \omega_2(| P - P' |) < \frac{\epsilon}{2}.$$

Then using (3.16) for $N \geq N_0$ (and the fact that $\| \mathbf{O}_N \| \geq 1$),

$$\| \mathbf{O}_N (\hat{k}_{P,i} - \hat{k}_{P',i}) \|_1 < \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

Using (3.17) for $N \leq N_0$

$$\| \mathbf{O}_N (\hat{k}_{P,i} - \hat{k}_{P',i}) \|_1 < \epsilon.$$

So if $| P - P' | < \delta$, the expression in (3.15) is bounded by $c\,\epsilon$ for all $v \in \mathcal{F}$. $\mathcal{F}$ is an equicontinuous family. It is also uniformly bounded by the first part of the proof.

Therefore $\{\hat{K}_i \mathbf{P}_N \mid N \geq 1\}$ is a collectively compact family. $\square$

3.4 Existence and Convergence

We first list some preliminary results, which will make our approach and subsequent proofs clearer.

Theorem 1 *¿From the assumptions* **A.1** *through* $\mathbf{A.5}(\nu)$ *we have these related facts for* ν *=1 or 2:*

i) The operators $\hat{H}_N^{(\nu)}$ *and* $\hat{H}$ *are completely continuous on* $\mathcal{D}$*. Therefore we may apply the properties of completely continuous vector fields.*

ii) $\{\hat{H}_N^{(\nu)} \mid N \geq 1\}$ *is a collectively compact family on* $\mathcal{D}$*.*

iii) $\hat{H}_N^{(\nu)}(x) \to \hat{H}(x)$ *for all* $x \in \mathcal{D}$*.*

iv) The operators $\hat{H}_N^{(\nu)}$ *converge uniformly to* $\hat{H}$ *on precompact subsets of* $\mathcal{D}$*.*

Proof: The composition of a bounded, continuous operator and a compact operator is completely continuous. Therefore for each integer i and each N the operators $\hat{K}_i\hat{G}_i$, $\mathbf{P}_N\hat{K}_i\hat{G}_i$, and $\hat{K}_i\mathbf{P}_N\hat{G}_i$ are completely continuous on $\mathcal{D}$ and so are their sums plus $\hat{y}$ or $\mathbf{P}_N\hat{y}$. This proves (i).

Let $\mathcal{S}$ be a bounded subset of $\mathcal{D}$. Then for each i, $\hat{G}_i(\mathcal{S})$ is a bounded subset of $\mathcal{X}$, and $\bigcup_{N=1}^{\infty} \hat{K}_{i,N}^{(\nu)}\hat{G}_i(\mathcal{S})$ is precompact. The set $\bigcup_{N=1}^{\infty} \hat{H}_N^{(\nu)}(\mathcal{S})$ is a finite union of these sets and the precompact set $\{\mathbf{P}_N\hat{y} \mid N \geq 1\}$ in case $\nu = 1$. It is the translation by $\hat{y}$ of the union of these sets in case $\nu = 2$. In either case it is also precompact. (ii) is proved.

For (iii), let $x \in \mathcal{D}$. For each i,

$$\hat{K}_{i,N}^{(\nu)}\hat{G}_i(x) \to \hat{K}_i\hat{G}_i(x)$$

(and $\mathbf{P}_N\hat{y} \to \hat{y}$ in case $\nu = 1$). Thus $\hat{H}_N^{(\nu)}(x) \to \hat{H}(x)$.

Finally, let $\mathcal{S}$ be a precompact subset of $\mathcal{D}$. Because of continuity, $\hat{G}_i(\mathcal{S})$ is also precompact. The convergence of $\hat{K}_{i,N}^{(\nu)}$ is uniform on precompact sets. [Lemma 1.] So $\hat{K}_{i,N}^{(\nu)}\hat{G}_i$ converges uniformly to $\hat{K}_i\hat{G}_i$ on $\mathcal{S}$. From this, (iv) easily follows. $\square$

Now we may use the results from Atkinson (1973) to prove the following proposition, setting the stage for our use of completely continuous vector field theory.

Theorem 2 *i) For a closed bounded subset $\mathcal{S} \subset \mathcal{D}$, with $\hat{u}^*$ not on $\mathcal{S}$, $\hat{H}_N^{(\nu)}$ does not have fixed points on $\mathcal{S}$, for N sufficiently large.*
ii) Let $\mathcal{O}$ be a bounded open subset of $\mathcal{D}$, such that $\hat{u}^*$ is not on $\partial\mathcal{O}$. Then for N large enough,

$$\gamma(I - \hat{H}_N^{(\nu)}, \mathcal{O}) = \gamma(I - \hat{H}, \mathcal{O}).$$

Proof: In Atkinson (1973), (i) is Theorem 1 and (ii) is Theorem 2. $\square$

Our next theorem is the main convergence result. The proof is borrowed from Krasnoselskii and Zabrieko (1984, p. 326). We repeat it here to show its generalization to the Kumar method under the assumption **A.5**(2) and to clarify the use of completely continuous vector fields.

Theorem 3 *Assume $\hat{u}^*$ has a nonzero index with respect to $\hat{H}$. Let $\mathcal{O}$ be a bounded open subset of $\mathcal{D}$ containing $\hat{u}^*$ and isolating $\hat{u}^*$ from all other fixed points of $\hat{H}$. For each $\epsilon > 0$, there exists an $N_0^{(\nu)}$ such that for $N \geq N_0^{(\nu)}$, there exists at least one fixed point $\hat{u}_N^{(\nu)}$ of $\hat{H}_N^{(\nu)}$ in $\mathcal{O}$, and each such $\hat{u}_N^{(\nu)}$ is in $B_\epsilon(\hat{u}^*) \cap \mathcal{O}$.*

Proof: Since $\hat{u}^*$ is the only fixed point of $\hat{H}$ in $\mathcal{O}$, by **R.1,**

$$\gamma(I - \hat{H}, \mathcal{O}) = ind(\hat{u}^*, I - \hat{H}) \neq 0.$$

By Theorem 2, part (ii), there exists $N_1^{(\nu)}$ such that for $N \geq N_1^{(\nu)}$,

$$\gamma(I - \hat{H}_N^{(\nu)}, \mathcal{O}) = \gamma(I - \hat{H}, \mathcal{O}) \neq 0.$$

Then by **R.2**, $\hat{H}_N^{(\nu)}$ has at least one fixed point in $\mathcal{O}$ for $N \geq N_1^{(\nu)}$.

Let $\epsilon > 0$. Let $\mathcal{S} = \text{closure}(\mathcal{O} - B_\epsilon(\hat{u}^*))$. By Theorem 2, part (i), there exists $N_2^{(\nu)}$ such that for $N \geq N_2^{(\nu)}$, $I - \hat{H}_N^{(\nu)}$ has no fixed points on $\mathcal{S}$. So for $N \geq \max(N_1^{(\nu)}, N_2^{(\nu)})$, $\hat{H}_N^{(\nu)}$ has at least one fixed point in $\mathcal{O} \cap B_\epsilon(\hat{u}^*)$, and none in $\mathcal{O} - \mathcal{B}_\epsilon(\hat{u}^*)$. □

Theorem 3 tells us, that (assuming $\mathcal{D}$ is a bounded domain) the set of fixed points of $\hat{H}_N^{(\nu)}$ in $\mathcal{D}$ converges to $\hat{u}^*$.

3.5 Rate of Convergence

Suppose each $\hat{G}_i$ is Fréchet differentiable at $\hat{u}^*$. ¿From this it follows easily that $\hat{H}$ is also Fréchet differentiable at $\hat{u}^*$. We will denote the linear operator which is the Fréchet derivative of $\hat{G}_i$ at $\hat{u}^*$ by $\hat{G}_i'(\hat{u}^*)$. Explicitly,

$$\hat{G}_i'(\hat{u}^*)x(Q) = \hat{g}_{i,2}(Q,\, \hat{u}^*(Q))\, x(Q),$$

where $\hat{g}_{i,2}$ is the partial derivative of $\hat{g}_i$ with respect to the second variable. Consider the sequences

$$r_N^{(\nu)} := \max_i \frac{\| \hat{G}_i(\hat{u}_N^{(\nu)}) - \hat{G}_i(\hat{u}^*) - \hat{G}_i'(\hat{u}^*)(\hat{u}_N^{(\nu)} - \hat{u}^*) \|_{\mathcal{X}}}{\| \hat{u}_N^{(\nu)} - \hat{u}^* \|_{\mathcal{X}}}.$$

As $N \to \infty$, $\hat{u}_N^{(\nu)} \to \hat{u}^*$ by Theorem 3. So by the definition of Fréchet differentiability, $r_N^{(\nu)} \to 0$. Let $F = \hat{H}'(\hat{u}^*)$. If 1 is not an eigenvalue of F, (as we will assume) then $(I - F)^{-1}$ exists. We should note in passing that this will imply **A.1**. We also have a new family of linear operators,

$$\hat{H}_N^{(\nu)\prime}(\hat{u}^*)x := \sum_{i=1}^{M} \hat{K}_{i,N}^{(\nu)} \hat{G}_i'(\hat{u}^*)x,$$

with properties similar to the nonlinear operators $\hat{H}_N^{(\nu)}$.

Proposition 6 *The following is true about $\hat{H}_N^{(\nu)\prime}(\hat{u}^*)$.*

i) $\{\hat{H}_N^{(\nu)\prime}(\hat{u}^*) \mid N \geq 1\}$ is collectively compact.

ii) $\hat{H}_N^{(\nu)\prime}(\hat{u}^*)x \to Fx$, for all $x \in \mathcal{X}$.

iii) $[I - \hat{H}_N^{(\nu)\prime}(\hat{u}^*)]^{-1}$ exists and is uniformly bounded for N large enough.

iv) For N large enough and for any $R > 0$,

$$\gamma(I - \hat{H}_N^{(\nu)\prime}(\hat{u}^*), B_R) = \gamma(I - F, B_R). \tag{3.18}$$

(This fact will be used for proving uniqueness in Theorem 5.)

v) If for each i, $\| \hat{K}_{i,N}^{(\nu)} - \hat{K}_i \| \to 0$, then $\| \hat{H}_N^{(\nu)\prime}(\hat{u}^*) - F \| \to 0$. (This would also imply (i) through (iv).)

Proof: The proof of (i) and (ii) is exactly as for (ii) and (iii) of Theorem 1, by substituting $\hat{G}_i'(\hat{u}^*)x$ for $\hat{G}_i(x)$ as needed.

(iii) is Theorem 1.6 of Anselone (1971).

(iv) is like the proof of Theorem 2, (ii).

For (v),

$$\| \hat{H}_N^{(\nu)\prime}(\hat{u}^*) - F \| \leq \| \hat{K}_{i,N}^{(\nu)} - \hat{K}_i \| \| \hat{G}_i'(\hat{u}^*) \| \to 0.$$

To see that this implies (i) through (iv), see Proposition 1. □

For the proofs of the following theorems, we introduce some notation, for shorthand. Let $c = \| (I - F)^{-1} \|$, and $a = (\| I - F \|)^{-1}$. Define the bound $b^{(\nu)}$ by

$$b^{(\nu)} := \sup_{i,N} \| \hat{K}_{i,N}^{(\nu)} \| .$$

$b^{(\nu)}$ is finite by the principle of uniform boundedness and **A.5**(ν). Finally let

$$p_N^{(\nu)} := \| \hat{H}_N^{(\nu)\prime}(\hat{u}^*) - F \| .$$

If the $\hat{K}_{i,N}^{(\nu)}$ are norm convergent, $p_N^{(\nu)} \to 0$.

We now give results on the rate of convergence for $\hat{u}_N^{(\nu)}$, and these are possible with the assumption of differentiability for the $\hat{G}_i$ at the point $\hat{u}^*$ and norm convergence of the sequences $\{\hat{K}_{i,N}^{(\nu)}\}$. Our proof is an adaptation of Theorem 2.2 of Atkinson and Potra (1987).

Theorem 4 *Suppose the $\hat{G}_i$ are Fréchet differentiable at $\hat{u}^*$. Suppose 1 is not an eigenvalue of $\hat{H}'(\hat{u}^*)$. Assume the sequences of operators $\{\hat{K}_{i,N}^{(\nu)}\}$ are norm convergent. Then there exist positive sequences, $\alpha_N^{(\nu)}$, $\beta_N^{(\nu)}$ converging to zero, such that for N large* *enough*

$$a(1-\beta_N^{(\nu)}) \parallel \hat{H}_N^{(\nu)}(\hat{u}^*) - \hat{H}(\hat{u}^*) \parallel_{\mathcal{X}} \leq \parallel \hat{u}_N^{(\nu)} - \hat{u}^* \parallel_{\mathcal{X}} \leq c(1+\alpha_N^{(\nu)}) \parallel \hat{H}_N^{(\nu)}(\hat{u}^*) - \hat{H}(\hat{u}^*) \parallel_{\mathcal{X}} .$$

Proof: We derive the following identity:

$$\begin{aligned}
\hat{u}_N^{(\nu)} - \hat{u}^* &= \hat{H}_N^{(\nu)}(\hat{u}_N^{(\nu)}) - \hat{H}(\hat{u}^*) + \hat{H}_N^{(\nu)}(\hat{u}^*) - \hat{H}_N^{(\nu)}(\hat{u}^*) \\
&= [\hat{H}_N^{(\nu)}(\hat{u}_N^{(\nu)}) - \hat{H}_N^{(\nu)}(\hat{u}^*) - \hat{H}_N^{(\nu)\prime}(\hat{u}^*)(\hat{u}_N^{(\nu)} - \hat{u}^*)] \\
&\quad + \hat{H}_N^{(\nu)\prime}(\hat{u}^*)(\hat{u}_N^{(\nu)} - \hat{u}^*) + \hat{H}_N^{(\nu)}(\hat{u}^*) - \hat{H}(\hat{u}^*)
\end{aligned}$$

Rearranging this equation we have

$$\begin{gathered}
(I-F)(\hat{u}_N^{(\nu)} - \hat{u}^*) - (\hat{H}_N^{(\nu)\prime}(\hat{u}^*) - F)(\hat{u}_N^{(\nu)} - \hat{u}^*) \\
-[\hat{H}_N^{(\nu)}(\hat{u}_N^{(\nu)}) - \hat{H}_N^{(\nu)}(\hat{u}^*) - \hat{H}_N^{(\nu)\prime}(\hat{u}^*)(\hat{u}_N^{(\nu)} - \hat{u}^*)] = \hat{H}_N^{(\nu)}(\hat{u}^*) - \hat{H}(\hat{u}^*).
\end{gathered} \tag{3.19}$$

Now taking bounds in (3.19) yields the inequalities

$$[1 - c\, p_N^{(\nu)} - c\, M\, b^{(\nu)}\, r_N^{(\nu)}] \parallel \hat{u}_N^{(\nu)} - \hat{u}^* \parallel_{\mathcal{X}} \leq c \parallel \hat{H}_N^{(\nu)}(\hat{u}^*) - \hat{H}(\hat{u}^*) \parallel_{\mathcal{X}}, \tag{3.20}$$

and

$$[1 + a\, p_N^{(\nu)} + a\, M\, b^{(\nu)}\, r_N^{(\nu)}] \parallel \hat{u}_N^{(\nu)} - \hat{u}^* \parallel_{\mathcal{X}} \geq a \parallel \hat{H}_N^{(\nu)}(\hat{u}^*) - \hat{H}(\hat{u}^*) \parallel_{\mathcal{X}} . \tag{3.21}$$

The assertion is proved with

$$\beta_N^{(\nu)} = \frac{a\, p_N^{(\nu)} + a\, M\, b^{(\nu)}\, r_N^{(\nu)}}{1 + a\, p_N^{(\nu)} + a\, M\, b^{(\nu)}\, r_N^{(\nu)}} \quad \text{and} \quad \alpha_N^{(\nu)} = \frac{c\, p_N^{(\nu)} + c\, M\, b^{(\nu)}\, r_N^{(\nu)}}{1 - c\, p_N^{(\nu)} - c\, M\, b^{(\nu)}\, r_N^{(\nu)}}. \quad \square$$

This theorem states that the rate of convergence of $\hat{u}_N^{(\nu)}$ to $\hat{u}^*$ is the same as the rate of convergence of $\hat{H}_N^{(\nu)}$ to $\hat{H}$ at the point $\hat{u}^*$. By Theorem 1 (iii), we know that this does indeed converge. In the projection method case,

$$\parallel \hat{H}_N^{(1)}(\hat{u}^*) - \hat{H}(\hat{u}^*) \parallel_{\mathcal{X}} = \parallel \mathbf{P}_N \hat{u}^* - \hat{u}^* \parallel_{\mathcal{X}} . \tag{3.22}$$

Thus we may simply say, that for the projection method, the rate of convergence of the approximate solution matches the rate of convergence of the projections at $\hat{u}^*$. For the Kumar method,

$$\parallel \hat{H}_N^{(2)}(\hat{u}^*) - \hat{H}(\hat{u}^*) \parallel_{\mathcal{X}} = \parallel \sum_{i=1}^{M} \hat{K}_i [\mathbf{P}_N \hat{G}_i(\hat{u}^*) - \hat{G}_i(\hat{u}^*)] \parallel_{\mathcal{X}} . \tag{3.23}$$

Later we will compare these quantities, looking for possible superconvergence.

3.6 Uniqueness of the Approximate Solutions

The assumptions made in Theorem 3 are not sufficient to prove the uniqueness of the solutions to our approximate equations. If we assume continuous Fréchet differentiability of the $\hat{G}_i$ around $\hat{u}^*$, we may show the uniqueness of the approximate solutions for N sufficiently large. At the same time, we may drop the assumption that $\hat{H}_N^{(\nu)\prime}(\hat{u}^*)$ is norm convergent to F.

Theorem 5 *Assume the $\hat{G}_i$ are continuously differentiable in $\mathcal{D}$, and that 1 is not an eigenvalue of $F = \hat{H}'(\hat{u}^*)$. Then there exists an $N_0^{(\nu)}$ such that for $N \geq N_0^{(\nu)}$, there exists a unique fixed point $\hat{u}_N^{(\nu)}$ of $\hat{H}_N^{(\nu)}$ in $\mathcal{D}$.*

Proof: Without loss of generality, we will assume that each ball $B_\delta(\hat{u}^*)$ mentioned in this proof is contained in $\mathcal{D}$. For each $\epsilon > 0$, there exists a $\delta(\epsilon)$ such that for $x \in B_\delta(\hat{u}^*)$,

$$\max_i \parallel \hat{G}_i'(x) - \hat{G}_i'(\hat{u}^*) \parallel < \epsilon.$$

This is an operator norm for the difference of the linear operators, $\hat{G}_i'(x)$ and $\hat{G}_i'(\hat{u}^*)$.

By Proposition 6, there exists N_1, $c < \infty$, and $d > 0$, such that for $N \geq N_1$, $(I - \hat{H}_N^{(\nu)\prime}(\hat{u}^*))^{-1}$ exists,

$$\parallel (I - \hat{H}_N^{(\nu)\prime}(\hat{u}^*))^{-1} \parallel \leq c,$$

and

$$\inf_{\|x\|_{\mathcal{X}}=1} \parallel (I - \hat{H}_N^{(\nu)\prime}(\hat{u}^*))x \parallel_{\mathcal{X}} \geq d.$$

Let

$$\alpha = \min\left\{\frac{d}{2\,M\,b^{(\nu)}},\ \frac{1}{2\,M\,b^{(\nu)}\,c}\right\}.$$

Let $x \in B_{\delta(\alpha)}(\hat{u}^*)$ and $N \geq N_1$. Subsequently, we will simply use δ for $\delta(\alpha)$. Then

$$\| \hat{H}_N^{(\nu)\prime}(x) - \hat{H}_N^{(\nu)\prime}(\hat{u}^*) \| < M\, b^{(\nu)}\, \alpha \leq \frac{1}{2\,c} \leq \frac{1}{2 \| (I - \hat{H}_N^{(\nu)\prime}(\hat{u}^*))^{-1} \|}.$$

This implies, by the standard perturbation theorem, that $(I - \hat{H}_N^{(\nu)\prime}(x))^{-1}$ exists and

$$\| (I - \hat{H}_N^{(\nu)\prime}(x))^{-1} \| < 2 \| (I - \hat{H}_N^{(\nu)\prime}(\hat{u}^*))^{-1} \| \leq 2\,c.$$

Also, for $\| y \|_{\mathcal{X}} = 1$, and for the same restrictions on x and N,

$$\| [\hat{H}_N^{(\nu)\prime}(x) - \hat{H}_N^{(\nu)\prime}(\hat{u}^*)] y \|_{\mathcal{X}} < \frac{d}{2} \leq \| y - \hat{H}_N^{(\nu)\prime}(\hat{u}^*) y \| .$$

By **R.3**,

$$\gamma(I - \hat{H}_N^{(\nu)\prime}(x), B_R) = \gamma(I - \hat{H}_N^{(\nu)\prime}(\hat{u}^*), B_R) \qquad \text{for all} \quad R > 0. \tag{3.24}$$

By Theorem 2, part (ii), there exists an $N_2^{(\nu)}$ such that for $N \geq N_2^{(\nu)}$,

$$\gamma(I - \hat{H}_N^{(\nu)}, B_\delta(\hat{u}^*)) = \gamma(I - \hat{H}, B_\delta(\hat{u}^*)). \tag{3.25}$$

By Theorem 3, there is an $N_3^{(\nu)}$ such that for $N \geq N_3^{(\nu)}$, all fixed points of $\hat{H}_N^{(\nu)}$ in $\mathcal{D}$ are in $B_\delta(\hat{u}^*)$. Let $N_0^{(\nu)} = \max\{N_1^{(\nu)}, N_2^{(\nu)}, N_3^{(\nu)}\}$. Let $N \geq N_0^{(\nu)}$. Let $x_1, \ldots, x_n$ be the fixed points of $\hat{H}_N^{(\nu)}$ in $B_\delta(\hat{u}^*)$. We know at least one such fixed point exists. By the existence of $(I - \hat{H}_N^{(\nu)\prime}(x_j))^{-1}$, we know that $\hat{H}_N^{(\nu)\prime}(x_j)$ does not have 1 as an eigenvalue. Then since the x_j are isolated by **R.4**, we may use the precompactness of $\hat{H}_N^{(\nu)}(B_\delta(\hat{u}^*))$ to infer that there is a finite number of them. For each j the following string of implications hold.

$$\begin{aligned} ind(x_j, I - \hat{H}_N^{(\nu)}) &= \gamma(I - \hat{H}_N^{(\nu)\prime}(x_j), B_R) && \mathbf{R.4} \\ &= \gamma(I - \hat{H}_N^{(\nu)\prime}(\hat{u}^*), B_R) && (3.24) \\ &= \gamma(I - F, B_R) && \text{Proposition 6, (iv)} \\ &= ind(\hat{u}^*, I - \hat{H}) && \mathbf{R.4} \\ &= \gamma(I - \hat{H}, B_\delta(\hat{u}^*)) && \mathbf{R.1} \\ &= \gamma(I - \hat{H}_N^{(\nu)}, B_\delta(\hat{u}^*)) && (3.25). \end{aligned}$$

Combining this with **R.1**, we have

$$\gamma(I - \hat{H}_N^{(\nu)}, B_\delta(\hat{u}^*)) = ind(x_1, I - \hat{H}_N^{(\nu)}) + \cdots + ind(x_n, I - \hat{H}_N^{(\nu)}) = n\, \gamma(I - \hat{H}_N^{(\nu)}, B_\delta(\hat{u}^*)).$$

n must be 1. There is a unique fixed point of $\hat{H}_N^{(\nu)}$ in $\mathcal{D}$. $\square$

3.7 Superconvergence and the Iterated Solution

The Kumar method and the iterated projection method are especially attractive when they exhibit the property called superconvergence. For $\nu = 2$ or 3, we say $\hat{u}_N^{(\nu)}$ is superconvergent if

$$s_N^{(\nu)} := \frac{\| \hat{u}_N^{(\nu)} - \hat{u}^* \|_{\mathcal{X}}}{\| \hat{u}_N^{(1)} - \hat{u}^* \|_{\mathcal{X}}} \to 0, \qquad \text{as} \quad N \to \infty.$$

We first consider the superconvergence of the Kumar solution. For the rate of convergence result in Theorem 4, we assumed the sequences $\{\hat{K}_{i,N}^{(\nu)}\}$ are norm convergent which is less general than **A.5**(ν). For notation, let

$$\rho_N^{(\nu)} := \max_i \| \hat{K}_{i,N}^{(\nu)} - \hat{K}_i \| .$$

We also need to know a little more about $\hat{G}_i$. The condition in the next theorem may often be interpreted as saying that $\hat{G}_i$ preserves the smoothness of $\hat{u}^*$.

Theorem 6 *Assume the sequences of operators $\{\hat{K}_{i,N}^{(\nu)}\}$ are norm convergent for $\nu =$ 1 and 2. If for N large enough,*

$$\max_i \frac{\| (\mathbf{P}_N - I)\hat{G}_i(\hat{u}^*) \|_{\mathcal{X}}}{\| (\mathbf{P}_N - I)\hat{u}^* \|_{\mathcal{X}}} \le c < \infty \tag{3.26}$$

then $\hat{u}_N^{(2)}$ is superconvergent.

Proof: By Theorem 4, $s_N^{(2)}$ is of the order of

$$\sigma_N := \frac{\| \hat{H}_N^{(2)}(\hat{u}^*) - \hat{H}(\hat{u}^*) \|_{\mathcal{X}}}{\| \mathbf{P}_N\hat{u}^* - \hat{u}^* \|_{\mathcal{X}}} = \frac{\| \sum_{i=1}^{M} \hat{K}_i(\mathbf{P}_N - I)\hat{G}_i(\hat{u}^*) \|_{\mathcal{X}}}{\| (\mathbf{P}_N - I)\hat{u}^* \|_{\mathcal{X}}}. \tag{3.27}$$

Using the fact that $\mathbf{P}_N - I = (\mathbf{P}_N - I)^2$,

$$\begin{aligned} \| \hat{K}_i(\mathbf{P}_N - I)\hat{G}_i(\hat{u}^*) \|_{\mathcal{X}} &= \| \hat{K}_i(\mathbf{P}_N - I)(\mathbf{P}_N - I)\hat{G}_i(\hat{u}^*) \|_{\mathcal{X}} \\ &\le \rho_N^{(2)} \| (\mathbf{P}_N - I)\hat{G}_i(\hat{u}^*) \|_{\mathcal{X}} . \end{aligned}$$

Thus σ_N of (3.27) is less than or equal to $M\rho_N^{(2)}\, c \to 0$. So $s_N^{(2)}$ goes to zero. $\square$

One consequence of this proof is that $\hat{u}_N^{(2)}$ is superconvergent with respect to

$$\max_i \| (I - \mathbf{P}_N)\hat{G}_i(\hat{u}^*) \|_{\mathcal{X}} .$$

Recalling the definition of z_i in the description of Kumar's method, this quantity is

$$\max_i \| (I - \mathbf{P}_N)z_i \|_{\mathcal{X}} .$$

It may be shown that for each i,

$$\| z_{i,N} - z_i \|_{\mathcal{X}} = O(\| (I - \mathbf{P}_N)z_i \|_{\mathcal{X}}).$$

Thus $\hat{u}_N^{(2)}$ is superconvergent with respect to the worst rate of convergence over i of $z_{i,N}$ to z_i. [Compare this with Kumar (1989).] But for the superconvergence that we are interested in, it seems quite possible for the ratio in the condition (3.26) of Theorem 6 not to be bounded. For example, $\hat{u}^*$ may belong to some $\mathcal{X}_N$, while for some i, $\hat{G}_i(\hat{u}^*)$ does not belong to any $\mathcal{X}_N$ even though it may be quite smooth. Therefore it seems that the Kumar solution $\hat{u}_N^{(2)}$ may sometimes be superconvergent with respect to $\hat{u}_N^{(1)}$, and other times not. Also we should remind ourselves that this result is based on (3.26) and the assumption that $\rho_N^{(2)} \to 0$, which may not be true even when $\rho_N^{(1)} \to 0$.

Now we turn our attention to the Sloan iterate, $\hat{u}_N^{(3)}$. We need not worry that we did not prove the existence of the solution. Its existence follows, by definition, from the existence of $\hat{u}_N^{(1)}$. But to infer its convergence from that of $\hat{u}_N^{(1)}$, we will need the assumption that the $\hat{G}_i$ are Fréchet differentiable at $\hat{u}^*$. If, in addition, $\mathcal{X}$ is a Hilbert space and $\mathbf{P}_N$ is an orthogonal projection, we will see that superconvergence is assured.

Again we adapt a proof from Atkinson and Potra (1987, Theorem 2.4), but we do not, as they do, assume that $\| \mathbf{P}_N \|$, and therefore $\| F(I - \mathbf{P}_N) \|$, is bounded on any closed subspace of $\mathcal{X}$. Let $k := \max_i \| \hat{K}_i \|$. Let $g := \max_i \| \hat{G}_i'(\hat{u}^*) \|$. Let

$f :=\| F \|$. Let

$$c_N := \frac{\| (\mathbf{P}_N - I)\hat{u}^* \|_{\mathcal{X}}}{\| \hat{u}_N^{(1)} - \hat{u}^* \|_{\mathcal{X}}}.$$

By Theorem 3, c_N is bounded away from 0 and ∞.

Theorem 7 *Assume the $\hat{G}_i$ are Fréchet differentiable at $\hat{u}^*$ and that 1 is not an eigenvalue of $F = \hat{H}'(\hat{u}^*)$. Also assume that $\rho_N^{(1)} \to 0$. Then $\hat{u}_N^{(3)}$ is superconvergent if and only if*

$$h_N := \frac{\| F(I - \mathbf{P}_N)\hat{u}^* \|_{\mathcal{X}}}{\| (I - \mathbf{P}_N)\hat{u}^* \|_{\mathcal{X}}} \to 0, \qquad as \quad N \to \infty.$$

Proof: We make the following derivation.

$$\begin{aligned}
(I - F)(\hat{u}_N^{(3)} - \hat{u}^*) &= \hat{H}(\hat{u}_N^{(1)}) - \hat{H}(\hat{u}^*) - F(\hat{H}(\hat{u}_N^{(1)}) - \hat{H}(\hat{u}^*)) \\
&= \hat{H}(\hat{u}_N^{(1)}) - \hat{H}(\hat{u}^*) - F(I - \mathbf{P}_N)(\hat{H}(\hat{u}_N^{(1)}) - \hat{H}(\hat{u}^*)) \\
&\quad -F\mathbf{P}_N(\hat{H}(\hat{u}_N^{(1)}) - \hat{H}(\hat{u}^*)) \\
&= [I - F(I - \mathbf{P}_N)](\hat{H}(\hat{u}_N^{(1)}) - \hat{H}(\hat{u}^*)) - F\mathbf{P}_N(\hat{u}_N^{(1)} - \hat{u}^*) \\
&= [I - F(I - \mathbf{P}_N)](\hat{H}(\hat{u}_N^{(1)}) - \hat{H}(\hat{u}^*) - F(\hat{u}_N^{(1)} - \hat{u}^*)) \\
&\quad -F\mathbf{P}_N(\hat{u}_N^{(1)} - \hat{u}^*) + [I - F(I - \mathbf{P}_N)]F(\hat{u}_N^{(1)} - \hat{u}^*) \\
&= [I - F(I - \mathbf{P}_N)](\hat{H}(\hat{u}_N^{(1)}) - \hat{H}(\hat{u}^*) - F(\hat{u}_N^{(1)} - \hat{u}^*)) \\
&\quad -F(I - \mathbf{P}_N)F(\hat{u}_N^{(1)} - \hat{u}^*) - F(I - \mathbf{P}_N)\hat{u}^* \\
&= [I - F(I - \mathbf{P}_N)]\sum_{i=1}^{M} \hat{K}_i(\hat{G}_i(\hat{u}_N^{(1)}) - \hat{G}_i(\hat{u}^*) - \hat{G}_i'(\hat{u}^*)(\hat{u}_N^{(1)} - \hat{u}^*)) \\
&\quad -F(I - \mathbf{P}_N)\sum_{i=1}^{M} \hat{K}_i\hat{G}_i'(\hat{u}^*)(\hat{u}_N^{(1)} - \hat{u}^*) - F(I - \mathbf{P}_N)\hat{u}^*.
\end{aligned}$$

We may take bounds in two directions. First,

$$\begin{aligned}
\| \hat{u}_N^{(3)} - \hat{u}^* \|_{\mathcal{X}} \geq\ & a \left[\frac{\| F(I - \mathbf{P}_N)\hat{u}^* \|_{\mathcal{X}}}{\| (\mathbf{P}_N - I)\hat{u}^* \|_{\mathcal{X}}} c_N \right. \\
& \left. -(M\,k + f\,\rho_N^{(1)})r_N^{(1)} - f\,\rho_N^{(1)}\,g \right] \| \hat{u}_N^{(1)} - \hat{u}^* \|_{\mathcal{X}} \; .
\end{aligned} \tag{3.28}$$

On the other hand,

$$\| \hat{u}_N^{(3)} - \hat{u}^* \|_{\mathcal{X}} \leq a \left[\frac{\| F(I-\mathbf{P}_N)\hat{u}^* \|_{\mathcal{X}}}{\| (I-\mathbf{P}_N)\hat{u}^* \|_{\mathcal{X}}} c_N \right. \qquad (3.29)$$
$$\left. +(M\,k + f\,\rho_N^{(1)})r_N^{(1)} + f\,\rho_N^{(1)}\,g \right] \| \hat{u}_N^{(1)} - \hat{u}^* \|_{\mathcal{X}} .$$

¿From this we can see that since $r_N^{(1)} \to 0$, and $\rho_N^{(1)} \to 0$, it must be true that $s_N^{(3)} \to 0$ if and only if $h_N \to 0$. □

This theorem leads us to consider the operators F and $F\mathbf{P}_N$. We immediately have the following corollary.

Corollary 1 *If* $\| F(I-\mathbf{P}_N) \| \to 0$, *then* $\hat{u}_N^{(3)}$ *is superconvergent. This is true if* $\mathcal{X}$ *is a Hilbert space and* $\mathbf{P}_N$ *is an orthogonal projection.*

Proof: Using the fact that $(I-\mathbf{P}_N) = (I-\mathbf{P}_N)^2$,

$$h_N \leq \frac{\| F(I-\mathbf{P}_N) \| \, \| (I-\mathbf{P}_N)\hat{u}^* \|_{\mathcal{X}}}{\| (I-\mathbf{P}_N)\hat{u}^* \|_{\mathcal{X}}} = \| F(I-\mathbf{P}_N) \| \to 0.$$

For the second statement, $\hat{H}$ being completely continuous implies that F is compact on $\mathcal{X}$. See Krasnoselskii and Zabrieko (1984, page 77). Using the argument from Sloan (1976, Theorem 1),

$$\| F(I-\mathbf{P}_N) \| = \| (I-\mathbf{P}_N)F^* \| \to 0,$$

where the superscript $(^*)$ denotes the adjoint operator. This uses the fact that $\mathbf{P}_N$ is self adjoint. The right hand side converges since F^* will be compact. Again, see Atkinson (1976, lemma 2, page 53). □

The condition $\| F(I-\mathbf{P}_N) \| \to 0$ of this corollary is closely related to the condition $\rho_N^{(2)} \to 0$. With sufficient smoothness for $\hat{g}_{i,2}(Q, \hat{u}^*(Q))$, these conditions would probably be equivalent. Therefore the propositions in section 3 which apply to Kumar's method, also supply some approaches to showing $F_i\mathbf{P}_N$ satisfies the conditions of Theorem 8. We illustrate this in section 6 of the next chapter. In subsequent chapters, we will attempt to determine the presence of superconvergence in Kumar's method and in the iterated projection method. Any such superconvergence must be weighed against possible additional computational costs entailed.

3.8 Alternative Approach to Kumar's Method

For some applications it may be difficult to show that assumption **A.5**(2) is satisfied. Therefore we present an alternative approach to the convergence theory for Kumar's method. It requires somewhat more restrictions on the $\hat{G}_i$ and the notation of product spaces.

Let $\mathcal{X}^M$ be the product space $\mathcal{X} \times \cdots \times \mathcal{X}$ (M times). We denote an element of $\mathcal{X}^M$ by $\bar{x} =< x_1, x_2, \ldots, x_M >$ where each x_i belongs to $\mathcal{X}$. Let $\mathcal{X}^M$ have the norm

$$||| \bar{x} |||= \max_i \| x_i \|_{\mathcal{X}} .$$

Give similar definitions for $\mathcal{X}_N^M = \mathcal{X}_N \times \cdots \times \mathcal{X}_N$ (M times). Define the projection operator $\bar{\mathbf{P}}_N : \mathcal{X}^M \to \mathcal{X}_N^M$ by

$$\bar{\mathbf{P}}_N(\bar{x}) =< \mathbf{P}_N x_1, \ldots, \mathbf{P}_N x_M > .$$

Keep the same assumptions **A.1** to **A.3** as above. Let $\mathcal{D}$ be as in assumption **A.3**. Define the subset $\mathcal{E}$ of $\mathcal{X}^M$ by

$$\mathcal{E} = \{\bar{x} \in \mathcal{X}^M \mid \hat{y} + \sum_{i=1}^{M} \hat{K}_i x_i \in \mathcal{D}\}.$$

From the continuity of the $\hat{K}_i$, $\mathcal{E}$ is an open subset of $\mathcal{X}^M$. Define $V_i : \mathcal{E} \to \mathcal{X}$ by

$$V_i(\bar{x}) = \hat{G}_i(\hat{y} + \sum_{j=1}^{M} \hat{K}_j x_j).$$

Since the $\hat{K}_j$ are compact and the $\hat{G}_i$ are bounded, it may be seen that each V_i is completely continuous on $\mathcal{E}$. Finally define $V : \mathcal{X}^M \to \mathcal{X}^M$ by

$$V(\bar{x}) =< V_1(\bar{x}), \ldots, V_M(\bar{x}) > .$$

Since $\hat{u}^* \in \mathcal{D}$, $\bar{z}^* =< \hat{G}_1(\hat{u}^*), \ldots, \hat{G}_M(\hat{u}^*) >$ is in $\mathcal{E}$. The coordinates of $\bar{z}^*$ satisfy

$$\hat{u}^* = \hat{y} + \sum_{i=1}^{M} \hat{K}_i z_i^*.$$

Since for each i, $\hat{G}_i(\hat{u}^*) = V_i(\bar{z}^*)$, $\bar{z}^*$ is a fixed point in $\mathcal{E}$ of V. Kumar's method prescribes applying a projection method to approximate the fixed point $\bar{z}^*$ by the solution $\bar{z}_N$ of

$$\bar{z}_N = \bar{\mathbf{P}}_N V(\bar{z}_N).$$

We wish to show that $\bar{z}_N$ exists for N sufficiently large and converges to $\bar{z}^*$. The Kumar solution is acquired by use of $\bar{z}_N$ by calculating

$$\hat{u}_N^{(2)} = \hat{y} + \sum_{i=1}^{M} \hat{K}_i z_{i,N}.$$

The operators V_i are not Hammerstein integral operators. So some aspects of our approach to convergence analysis taken above do not apply here. But the work done by Atkinson (1973) supplies a framework for analyzing projection methods for general nonlinear operators. To take the place of **A.5**(2), we must add some new assumptions

A.5′ For each integer i and each $\bar{x} \in \mathcal{E}$, $\mathbf{P}_N V_i(\bar{x}) \to V_i(\bar{x})$, and $\{\mathbf{P}_N V_i \mid N \geq 1\}$ is a collectively compact family on $\mathcal{E}$.

A.6′ For each integer i and at each point $\bar{x} \in \mathcal{E}$, $\{\mathbf{P}_N V_i\}$ is an equicontinuous family.

¿From these assumptions it is easy to infer the following properties for V : $\bar{\mathbf{P}}_N V$ and V are completely continuous on $\mathcal{E}$; $\{\bar{\mathbf{P}}_N V \mid N \geq 1\}$ is a collectively compact family on $\mathcal{E}$; $\bar{\mathbf{P}}_N V(\bar{x}) \to V(\bar{x})$ for all $\bar{x} \in \mathcal{E}$; and the family of operators $\{\bar{\mathbf{P}}_N V\}$ is equicontinuous at each point $\bar{x} \in \mathcal{E}$. Therefore we have the following convergence result from the assumptions **A.1** through **A.6′**.

Theorem 8 *Assume $\bar{z}^*$ has a nonzero index with respect to V. Let $\mathcal{O}$ be a bounded open subset of $\mathcal{E}$ containing $\bar{z}^*$ and isolating $\bar{z}^*$ from all other fixed points of V. For each $\epsilon > 0$, there exists an N_0 such that for $N \geq N_0$, there exists at least one fixed point $\bar{z}_N$ of $\bar{\mathbf{P}}_N V$ in $\mathcal{O}$, and each such $\bar{z}_N$ is in $B_\epsilon(\bar{z}^*) \cap \mathcal{O}$.*

Proof: See Theorem 3 of Atkinson (1973). □

To satisfy some of the conditions of this theorem it could be shown that $\bar{z}^*$ is an isolated fixed point of V if $\hat{u}^*$ is an isolated fixed point of $\hat{H}$.

Often the less general conditions of the next proposition are satisfied. This provides another way to show convergence.

Proposition 7 *If for each i, $\mathbf{P}_N V_i$ converges uniformly to V_i on bounded sets of $\mathcal{E}$, then* **A.5′** *and* **A.6′** *are satisfied.*

Proof: The same proof that was used for Proposition 1 shows that **A.5′** is satisfied. It remains to show **A.6′**.

Let $\bar{x} \in \mathcal{E}$ with $B_r(\bar{x})$ also in $\mathcal{E}$. Let $\epsilon > 0$. There exists N_0 such that for $N \geq N_0$,

$$\| (\mathbf{P}_N - I)V_i(\bar{y}) \|_{\mathcal{X}} \leq \frac{\epsilon}{4} \tag{3.30}$$

for $\bar{y} \in B_r(\bar{x})$. By the continuity of V_i, we may chose r' such that for $\bar{y} \in B_{r'}(\bar{x})$,

$$\max_{N<N_0} \| \mathbf{P}_N \| \| V_i(\bar{x}) - V_i(\bar{y}) \|_{\mathcal{X}} \leq \epsilon. \tag{3.31}$$

Then we have the following. For $\bar{y} \in B_{r'}(\bar{x})$,

$$\begin{aligned} \| \mathbf{P}_N V_i(\bar{x}) - \mathbf{P}_N V_i(\bar{y}) \|_{\mathcal{X}} &\leq \epsilon \qquad \text{for } N < N_0, \\ \| \mathbf{P}_N V_i(\bar{x}) - \mathbf{P}_N V_i(\bar{y}) \|_{\mathcal{X}} &\leq \| (\mathbf{P}_N - I)[V_i(\bar{x}) - V_i(\bar{y})] \|_{\mathcal{X}} + \| V_i(\bar{x}) - V_i(\bar{y}) \|_{\mathcal{X}} \\ &\leq 2\frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon \qquad \text{for } N \geq N_0. \end{aligned}$$

The first bound is by (3.31) and the second bound is by (3.30) and (3.31) assuming the maximum of $\| \mathbf{P}_N \|$ over $N < N_0$ is greater than or equal to 2. $\{\mathbf{P}_N V_i\}$ is an equicontinuous family at $\bar{x}$. □

If we assume the operators $\hat{G}_i$ are twice Fréchet differentiable on $\mathcal{D}$, this would imply that the operators V_i are twice differentiable on $\mathcal{E}$. For uniqueness and rate of convergence results, we add the new assumption

A.7′ V and the $\hat{\mathbf{P}}_N V$ are twice differentiable and the second derivatives (although perhaps not continuous) are uniformly bounded in a neighborhood of $\bar{z}^*$.

Atkinson (1973) showed that $\mathbf{A.6}'$ follows from $\mathbf{A.7}'$ for $\bar{x} \in B_r(\bar{z}^*)$. He also had the following theorem.

Theorem 9 *Assume* $\mathbf{A.1}$ *through* $\mathbf{A.5}'$*, and* $\mathbf{A.7}'$*. Assume 1 is not an eigenvalue of* $V'(\bar{z}^*)$*. Then there are* $0 < \epsilon \leq r$ *and* $N_0 > 0$ *such that for* $N \geq N_0$*,* $\bar{\mathbf{P}}_N V$ *has a unique fixed point in* $B_\epsilon(\bar{z}^*)$*. Also there exists a constant* c *such that*

$$||| \; \bar{z}^* - \bar{z}_N \; ||| \leq c \; ||| \; \bar{z}^* - \bar{\mathbf{P}}_N \bar{z}^* \; ||| \; .$$

Proof: See Atkinson (1973) Theorem 4. □

¿From this result, we may calculate the following bound on the rate of convergence of $\hat{u}_N^{(2)}$ to $\hat{u}^*$. Let $b = \max_i \| \hat{K}_i \|$. Then

$$\begin{aligned} \| \hat{u}_N^{(2)} - \hat{u}^* \|_{\mathcal{X}} &\leq \max_i \| \hat{K}_i(z_{i,N} - z_i^*) \|_{\mathcal{X}} \\ &\leq b \; ||| \; \bar{z}_N - \bar{z}^* \; ||| \\ &= b\,c \; ||| \; \bar{\mathbf{P}}_N \bar{z}^* - \bar{z}^* \; ||| \\ &= b\,c \max_i \| \mathbf{P}_N \hat{G}_i(\hat{u}^*) - \hat{G}_i(\hat{u}^*) \|_{\mathcal{X}} \end{aligned}$$

This should be contrasted with the discussion after Theorem 6. Our previous approach to Kumar's method had $\| \hat{u}_N^{(2)} - \hat{u}^* \|_{\mathcal{X}}$ superconvergent with respect to this rate.

CHAPTER IV

THE MODEL PROBLEM

In Chapter 3, we have defined the projection method and its variants, the Kumar method and the iterated projection method. We have attempted to extend the applicability of these methods by using a change of variables from smooth but irregularly shaped surfaces to surfaces which have readily computable approximating subspaces of globally smooth functions. We carried out a convergence analysis for each of these methods when it is applied to general mixed-type Hammerstein operator equations. In this chapter, we apply this analysis to a particular boundary value problem in three space, whose reformulation yields a mixed-type Hammerstein integral equation.

Within the framework of this problem, we consider surfaces of two types: those which admit a change of variables to a sphere, and those admitting a change of variables to a torus. This will allow us to show the application of the convergence analysis to situations where the projections are interpolatory as well as to situations where they are orthogonal. When the projections are orthogonal, we will collectively refer to the projection method, the Kumar method, and the iterated projection method as *Galerkin methods.* When the projections are interpolatory, we will call these methods *collocation methods.* We will analyze the Galerkin methods in $L^2(\Theta)$ as well as in $C(\Theta)$. For the collocation methods, only analysis in $C(\Theta)$ is appropriate. We leave until the next chapter the practical question of implementing these methods.

Consider the following Laplace equation with nonlinear Neumann-type boundary conditions.

$$\begin{aligned} \Delta u(P) &= 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n}(P) &= g(P, u(P)) - f(P) & \text{on } \partial\Omega. \end{aligned} \tag{4.1}$$

Ω is an open region in $\mathbf{R}^3$ whose boundary, $\partial\Omega$, is a Lyapunov surface. $\partial u/\partial n$ is the partial derivative of u with respect to the outer normal n to the surface $\partial\Omega$. u is the unknown function. The functions f and g are given. Assumptions concerning f and g are given later. We denote the true solution by u^*. We will apply a boundary integral equation method to reformulate this problem, yielding a fixed point Hammerstein integral equation of mixed-type.

4.1 The Mixed-Type Hammerstein Integral Equation

Because of the assumption of Lyapunov smoothness for $\partial\Omega$, we may apply a boundary integral equation method (BIEM) to the problem (4.1) using Green's second identity. Details of the derivation are in Jaswon and Symm (Chapter 3, 1977). Then u^* satisfies the following integral equation:

$$u(P) = \frac{1}{4\pi} \int_{\partial\Omega} \left[u(Q) \frac{\partial}{\partial n} \left(\frac{1}{\mid P - Q \mid} \right) - \frac{\partial u(Q)}{\partial n} \frac{1}{\mid P - Q \mid} \right] dA_Q \tag{4.2}$$

where $P \in \Omega$ and n is the outward normal to $\partial\Omega$ at the integration variable Q. When P goes to the boundary $\partial\Omega$, the integrands acquire singularities (when the integration variable Q equals the parameter P.) It may be shown that these integrals still exist and are bounded but a jump condition takes place. For P a point on the boundary $\partial\Omega$, u^* satisfies

$$u(P) = \frac{1}{2\pi} \int_{\partial\Omega} \left[u(Q) \frac{\partial}{\partial n} \left(\frac{1}{\mid P - Q \mid} \right) - \frac{\partial u}{\partial n}(Q) \frac{1}{\mid P - Q \mid} \right] dA_Q \tag{4.3}$$

Equation (4.3) differs from (4.2) in that the constant $1/4\pi$ has been replaced by $1/2\pi$. This is the result of the jump condition. The right hand side of (4.2) uses only boundary values of u and $\partial u/\partial n$. In our problem (4.1), $\partial u/\partial n$ on the boundary is determined by u on the boundary. Therefore the values of u^* on the boundary determine u^* inside Ω via equation (4.2). This allows us to concentrate on solving equation (4.3) for the function u^* restricted to $\partial\Omega$.

By substitution of the boundary conditions in (4.1) into equation (4.3), we obtain

$$u(P) = \frac{1}{2\pi}\int_{\partial\Omega} u(Q)\frac{\partial}{\partial n}\left(\frac{1}{\mid P-Q\mid}\right) dA_Q - \frac{1}{2\pi}\int_{\partial\Omega} g(Q,u(Q))\frac{1}{\mid P-Q\mid} dA_Q$$

$$+\frac{1}{2\pi}\int_{\partial\Omega}\frac{f(Q)}{\mid P-Q\mid} dA_Q, \quad P\in\partial\Omega. \tag{4.4}$$

The right hand side of equation (4.4) matches the form of the mixed-type Hammerstein integral operator with the identifications

$$k_1(P,Q) = \frac{1}{2\pi}\frac{\partial}{\partial n}\left(\frac{1}{\mid P-Q\mid}\right), \qquad g_1(P,x(P)) = x(P);$$

$$k_2(P,Q) = \frac{-1}{2\pi}\frac{1}{\mid P-Q\mid}, \qquad g_2(P,x(P)) = g(P,x(P));$$

$$y(P) = K_1 f(P).$$

For $i = 1$ and 2, we define the operators

$$K_i x(P) := \int_{\partial\Omega} k_i(P,Q)x(Q)\,dA_Q,$$

and

$$G_i(x)(P) := g_i(P,x(P)).$$

Therefore equation (4.4) is a fixed point equation for the type of operator we analyzed above. We will rewrite it as

$$u = H(u) = K_2 f + \sum_{i=1}^{2} K_i G_i(u).$$

In the following discussion, we will often simply use $\hat{G}$ for $\hat{G}_2$ since it represents the only nonlinear function g in the problem.

We wish to transform this problem from the surface $\partial\Omega$ to another surface Θ. We assume there exists a continuously differentiable bijection $\mu : \Theta \rightarrow \partial\Omega$. We use μ to induce a map $\mathcal{M}$ from $C(\partial\Omega)$ to $C(\Theta)$ and from $L^2(\partial\Omega)$ to $L^2(\Theta)$. For a function x defined on $\partial\Omega$, $\mathcal{M}x := x \circ \mu$ is defined on Θ. Conjugating by $\mathcal{M}$, we define new operators on $\mathcal{X}(\Theta) = C(\Theta)$ or $L^2(\Theta)$. [See Chapter 2.] Let $\hat{H} := \mathcal{M}H\mathcal{M}^{-1}$. At the same time we may define the operators $\hat{K}_i := \mathcal{M}K_i\mathcal{M}^{-1}$ and $\hat{G}_i := \mathcal{M}G_i\mathcal{M}^{-1}$ for $i = 1$ and 2. Let $\hat{u}^* := \mathcal{M}u^*$. Then $\hat{u}^*$ is the solution of the fixed point equation,

$$\hat{u} = \hat{H}(\hat{u}) = \hat{K}_2 \hat{f} + \sum_{i=1}^{2} \hat{K}_i \hat{G}_i(\hat{u}). \tag{4.5}$$

Having transformed the problem onto Θ, we may attempt to solve (4.5) with a projection method based on the convenient approximating subspaces defined on Θ.

4.2 Applying the Convergence Theory

In order to apply the convergence theory of Chapter 3, we have to satisfy the assumptions **A.1** through **A.5**. We may discuss **A.1** through **A.4** for our problem without reference to the choice of Θ and its approximating subspaces. (**A.4** may seem to be an exception since it requires $\mathbf{P}_N \hat{y} \rightarrow \hat{y}$ for the projection method. But since for our model problem $\hat{y} = \hat{K}_2 \hat{f}$, that will be true automatically if **A.5**(1) is true. So we may ignore this part of **A.4**.) Since **A.5** depends on properties of the projection operator $\mathbf{P}_N$, we postpone the discussion of **A.5** until a particular surface Θ is under consideration. In this section it is shown that **A.1** through **A.4** are satisfied for the problem (4.5), both when $\mathcal{X} = C(\Theta)$ and when $\mathcal{X} = L^2(\Theta)$. These remarks will apply to both Galerkin methods and collocation methods.

Let us begin by listing the assumptions we have concerning f, g, and the map μ.

P.1 $f \in C(\partial\Omega)$.

P.2 $g(P, x)$ is bounded and continuous on $\partial\Omega \times \mathcal{I}$, where $\mathcal{I}$ is an open interval with the range of the solution u^* contained in $\mathcal{I}$.

P.3 $g(P, x)$ is continuously differentiable with respect to x.

P.4 μ is a continuous bijection from Θ to $\partial\Omega$.

P.5 Let $J : \Theta \to \mathbf{R}$ be the determinant of the Jacobian of μ. Then there exists constants d_1 and d_2 such that

$$0 < d_1 \leq J(Q) \leq d_2$$

for all $Q \in \Theta$.

Because of **P.4** and **P.5**, the operators $\hat{K}_i$ and the operators $\hat{G}_i$ on $\mathcal{X}(\Theta)$ inherit properties which the K_i and the G_i have on $\mathcal{X}(\partial\Omega)$. [Theorem 1, Chapter 2]. So we proceed by investigating the properties of the K_i and the G_i on $\mathcal{X}(\partial\Omega) = C(\partial\Omega)$ and $L^2(\partial\Omega)$. But first, if we make the assumption that u^* is an isolated solution of (4.5), **P.4** and **P.5** suffice to prove **A.1** is satisfied via Theorem 1 of Chapter 2. In the sections which follow, we will explicitly make this assumption so that its role remains clear.

The operator G_1 is the identity operator. It is therefore bounded and continuously differentiable. Its derivative is itself. G_2 is the nontrivial nonlinear operator in our Hammerstein integral operator. Let $\mathcal{D}$ be the subset of $\mathcal{X}(\partial\Omega)$ of functions whose range is in $\mathcal{I}$. $\mathcal{D}$ is open in $\mathcal{X}(\partial\Omega)$. G_2 is continuous on $\mathcal{D}$ and takes bounded subsets of $\mathcal{D}$ to bounded sets. See Krasnoselskii and Zabreiko (1984), page 81. Moreover G_2 is continuously differentiable on $\mathcal{D}$ with derivative

$$[G_2'(x)]v(P) = \frac{\partial}{\partial x} g(P, x(P))v(P).$$

By Theorem 1 of Chapter 2, each $\hat{G}_i := \mathcal{M}G_i\mathcal{M}^{-1}$ is bounded, continuous, and continuously differentiable on $\mathcal{M}(\mathcal{D}) \subseteq \mathcal{X}(\Theta)$. Thus **A.3** is satisfied. We will later use the fact that each $\hat{G}_i$ is continuously differentiable on $\mathcal{M}(\mathcal{D})$ when discussing rate of convergence and uniqueness of approximate solutions.

The solution $\hat{u}^*$ of (4.5) is in $\mathcal{M}(\mathcal{D})$. So all the $\hat{G}_i$ are continuously differentiable at $\hat{u}^*$. Trivially it follows that $\hat{H}$ is continuously differentiable at $\hat{u}^*$. The derivative of $\hat{H}$ at $\hat{u}^*$ is

$$[\hat{H}'(\hat{u}^*)]v = \sum_{i=1}^{2} \hat{K}_i[\hat{G}_i'(\hat{u}^*)]v.$$

We consider the integral operators K_1 and K_2. The operator K_1 is given by

$$K_1x(P) = \int_{\partial\Omega} x(Q)\frac{\partial}{\partial n}\left(\frac{1}{\mid P - Q \mid}\right) dA_Q.$$

It is called a double-layer potential with density x. Usually, for the definition of the double-layer, P is allowed to be any point in $\mathbf{R}^3$, but here we are interested only in the value for $P \in \partial\Omega$. Thus we view K_1 as an operator taking one function, x, defined on $\partial\Omega$ to another function K_1x defined on $\partial\Omega$. Similarly, K_2 is a single-layer (or simple-layer) potential. K_2x is given for $P \in \partial\Omega$ by

$$K_2x(P) = \int_{\partial\Omega} x(Q)\frac{1}{\mid P - Q \mid} dA_Q.$$

K_1 and K_2 are in the category of integral operators with weak singularities. Let us define such operators and discuss some of their properties.

Suppose $a(P,Q)$ is a bounded and integrable function on $\partial\Omega \times \partial\Omega$. Let

$$k(P,Q) := \frac{a(P,Q)}{\mid P-Q \mid^{\alpha}}$$

with $0 \leq \alpha < 2$. The integral operator with kernel $k(P,Q)$,

$$Kx(P) = \int_{\partial\Omega} k(P,Q)x(Q)\, dA_Q,$$

is called an operator with a weak singularity. Clearly K_2 is an operator with a weak singularity. It follows from the assumption that $\partial\Omega$ is Lyapunov, that K_1 has a weak singularity. [See Mikhlin (page 356, 1970).] An operator with a weak singularity is a compact operator on $L^2(\partial\Omega) \to L^2(\partial\Omega)$ and on $C(\partial\Omega) \to C(\partial\Omega)$. [Mikhlin (pages 160 and 162).] Therefore assuming **P.4** and **P.5**, the operators $\hat{K}_1$ and $\hat{K}_2$ inherit the property of compactness from K_1 and K_2. [Theorem 1, Chapter 2]. $\hat{K}_1$ and $\hat{K}_2$ are compact on $L^2(\Theta)$ and $C(\Theta)$. This could also be proved by showing that $\hat{K}_1$ and $\hat{K}_2$ are integral operators on Θ with weak singularities. Assumption **A.2** is satisfied. Also $K_2 f$ will belong to $C(\partial\Omega)$ and to $L^2(\partial\Omega)$. So the same is true of $\hat{K}_2\hat{f}$. **A.3** is satisfied.

Based on **P.1** through **P.5** and the assumption that u^* is an isolated solution, we have shown that the problem (4.5) satisfies **A.1** through **A.4** for $\mathcal{X} = C(\Theta)$ and $L^2(\Theta)$. **A.5** remains to be shown during the course of examining particular surfaces Θ and choosing the space $\mathcal{X}$ to be $C(\Theta)$ or $L^2(\Theta)$.

For later use let us also note the following about K_1 and K_2 since this result is independent of the topology of $\partial\Omega$.

Theorem 1 (Günter) *i) Suppose $\partial\Omega$ is Lyapunov with exponent λ and x is bounded and integrable on $\partial\Omega$ with $\mid x \mid \leq a$. Then for any $0 < \theta < 1$ there exists a constant c dependent on $\partial\Omega$ and θ such that $K_1 x \in H(0, c\,a, \theta\lambda)$ on $\partial\Omega$. (Theorem 1, page 106.) ii) Suppose $\partial\Omega$ is Lyapunov and x is bounded and integrable on $\partial\Omega$ with $\mid x \mid \leq a$. Then for any $0 < \theta < 1$ there exists a constant $c = c(\partial\Omega, \theta)$ such that $K_2 x \in H(0, c\,a, \theta)$ on*

$\partial\Omega$. (Theorem 2, page 106.)

iii) If $\partial\Omega \in L_{d+2}(B, \lambda)$ and $x \in H(d, a, \lambda)$ on $\partial\Omega$ $(d \geq 0)$, then for $0 < \theta < \lambda$ there exists $c = c(d,\, B,\, \theta)$ such that $K_1 x \in H(d+1, ca, \theta\lambda)$ on $\partial\Omega$. (Theorem 3, page 106.)

iv) If $\partial\Omega \in L_{d+1}(B, \lambda)$ and $x \in H(d, a, \lambda)$ on $\partial\Omega$ $(d \geq 0)$, then for $0 < \theta < \lambda$ there exists $c = c(d,\, B,\, \theta)$ such that $K_2 x \in H(d+1, ca, \theta\lambda)$ on $\partial\Omega$. (This follows from Theorem 4, page 102 and Lemma 3, page 101.)

Parts (i) and (ii) tell us that if $\partial\Omega$ is Lyapunov with exponent λ, then K_1 maps the ball B_a in $C(\partial\Omega)$ into $H(0,\, c\,a,\, \theta\lambda)$ while K_2 maps B_a into $H(0,\, c\,a,\, \theta)$. By Proposition 4 of Chapter 2, we have the following corollary:

Corollary 1 *Let $\partial\Omega$ be Lyapunov with exponent λ. Assume* **P.4** *and* **P.5**. *For each $a > 0$ and any θ between 0 and 1, $\hat{K}_1$ maps the ball $B_a \subset C(\Theta)$ into $H(0, d_2\, c\, a, \theta\lambda)$ and $\hat{K}_2$ maps B_a into $H(0, d_2\, c\, a, \theta)$ where c depends only on the choice of θ.*

This corollary will be useful for convergence results. To get precise rate of convergence results, we must assume much more about f, g, $\partial\Omega$, and the map $\mathcal{M}$.

First we list some rules concerning Hölder classes on $\partial\Omega$. See [Günter, page 98]. Suppose $f \in H(l, a, \lambda)$ and $g \in H(l, b, \lambda)$. Then

1. $f + g \in H(l, a + b, \lambda)$.
2. For $0 < \lambda' \leq \lambda$, $f \in H(l, a, \lambda')$.
3. For $0 \leq l' < l$, $f \in H(l', \sqrt{2}\, a, 1)$.

Similarly for classes of surfaces, if $\partial\Omega \in L_l(B, \lambda)$ then

4. For $0 < \lambda' \leq \lambda$, $\partial\Omega \in L_l(B, \lambda')$.
5. For $0 < l' < l$, $\partial\Omega \in L_{l'}(cB, 1)$ for some absolute constant c.

¿From the smoothness of the surface $\partial\Omega$ and of the given functions f, g, and μ, we may infer the smoothness of u^* and $\hat{u}^*$. We will subsequently refer to the conditions in the next theorem when giving rate of convergence results.

Theorem 2 *Suppose there exists* $l \geq 0$ *and* $0 < \lambda \leq 1$ *such that the following are true:*

(a) $\partial\Omega \in L_{l+1}(B, \lambda)$ *for some* B.

(b) $K_2 f \in H(l, a, \lambda)$ *on* $\partial\Omega$ *for some* a.

(c) For every c_1 *there exists a* $c_2 = c_2(c_1, k, \lambda')$, *such that for* $0 \leq k \leq l$ *and* $0 < \lambda' \leq \lambda$, G *maps* $H(k, c_1, \lambda') \bigcap D$ *into* $H(k, c_2, \lambda')$.

(d) For $0 \leq k \leq l$ *and* $0 < \lambda' \leq \lambda$, *there exists a constant* $d = d(k, \lambda')$ *such that* $\mathcal{M}$ *maps* $H(k, b, \lambda')$ *on* $\partial\Omega$ *into* $H(k, d\,b, \lambda')$ *on* Θ.

If u *is bounded and integrable on* $\partial\Omega$, *then given* $0 < \lambda' < \lambda$ *there exist constants* a_1 a_2 *such that* $u^* \in H(l, a_1, \lambda')$ *on* $\partial\Omega$, *and* $\hat{u}^* \in H(l, a_2, \lambda')$ *on* Θ.

Proof: Let $0 < \lambda' < \lambda$. Since u^* is bounded and integrable, $G(u^*)$ is bounded and integrable. Let λ_0 be arbitrary satisfying $\lambda' < \lambda_0 < \lambda$. By parts (i) and (ii) of Günter's Theorem and rule 1 above, $K_1 u^* + K_2 G(u^*) \in H(0, c, \lambda_0)$ for some c. Applying (b) and rules 1 and 2, $u^* = K_1 f + K_1 u^* + K_2 G(u^*) \in H(0, c + a_1', \lambda_0)$. This supplies the base case for the following induction.

Inductive claim: Suppose $u^* \in H(l, c_0, \lambda_0)$ for some $\lambda' < \lambda_0 < \lambda$, $l < d$, and for some c_0. Then for any $\lambda' < \lambda_1 < \lambda_0$, there exists c_1 such that $u^* \in H(l+1, c_1, \lambda_1)$. Proof of claim: By (c), $G(u^*) \in H(l, c', \lambda_0)$ for some c'. So for some c'' $K_2 G(u^*) \in H(l+1, c'', \lambda_1$ and $K_1 u^* \in H(l+1, c''', \lambda_1)$. $K_2 f \in H(l+1, a_1', \lambda_1)$ by (b) and rules 2 and 3. So $u^* \in H(l+1,\ c'' + c''' + a_1',\ \lambda_1)$ by rule 1.

The second part of the theorem follows now by using (d). $\square$

4.3 General Remarks on the Convergence Analysis

When we have defined our sequence of projection operators $\{\mathbf{P}_N\}$, we may more closely inquire into the approximate solutions to (4.4) given by the projection method, the Kumar method, and the iterated projection method. We review the definitions of these methods and make some general comments about the convergence analysis which we will apply in the following sections to both Galerkin methods and collocation methods.

Our definitions may be summarized by the following fixed point equations. The projection solution $\hat{u}_N^{(1)}$ satisfies the equation

$$\begin{aligned}\hat{u}_N^{(1)} = \hat{H}_N^{(1)}(\hat{u}_N^{(1)}) := \mathbf{P}_N \hat{H}(\hat{u}_N^{(1)}) &:= \mathbf{P}_N \hat{K}_2 \hat{f} + \sum_{i=1}^{2} \mathbf{P}_N \hat{K}_i \hat{G}_i(\hat{u}_N^{(1)}) \\ &:= \mathbf{P}_N \hat{K}_2 \hat{f} + \sum_{i=1}^{2} \hat{K}_{i,N}^{(1)} \hat{G}_i(\hat{u}_N^{(1)}). \end{aligned} \tag{4.6}$$

The Kumar solution $\hat{u}_N^{(2)}$ satisfies the equation

$$\hat{u}_N^{(2)} = \hat{H}_N^{(2)}(\hat{u}_N^{(2)}) := \hat{K}_2 \hat{f} + \sum_{i=1}^{2} \hat{K}_i \mathbf{P}_N \hat{G}_i(\hat{u}_N^{(2)}) := \hat{K}_2 \hat{f} + \sum_{i=1}^{2} \hat{K}_{i,N}^{(2)} \hat{G}_i(\hat{u}_N^{(2)}). \tag{4.7}$$

Finally the Sloan iterate $\hat{u}_N^{(3)}$ satisfies the equation

$$\hat{u}_N^{(3)} = \hat{H}_N^{(3)}(\hat{u}_N^{(3)}) := \hat{H}(\mathbf{P}_N \hat{u}_N^{(3)}) := \hat{K}_2 \hat{f} + \sum_{i=1}^{2} \hat{K}_i \hat{G}_i(\mathbf{P}_N \hat{u}_N^{(3)}). \tag{4.8}$$

The projection solution $\hat{u}_N^{(1)}$ is an element of $\mathcal{X}_N$ while in general the Kumar solution and the Sloan iterate are not. The Sloan iterate $\hat{u}_N^{(3)}$ has a special relationship with the projection solution $\hat{u}_N^{(1)}$. This relationship is expressed by the fact that $\hat{u}_N^{(3)} = \hat{H}(\hat{u}_N^{(1)})$ while $\hat{u}_N^{(1)} = \mathbf{P}_N \hat{u}_N^{(3)}$. We based much of our analysis on viewing the operators $\hat{K}_{i,N}^{(1)} := \mathbf{P}_N \hat{K}_i$ and $\hat{K}_{i,N}^{(2)} := \hat{K}_i \mathbf{P}_N$ as approximations to $\hat{K}_i$, for $i = 1, \ldots, M$. Therefore we did the convergence analysis for $\hat{u}_N^{(1)}$ and $\hat{u}_N^{(2)}$ in parallel while the convergence analysis for $\hat{u}_N^{(3)}$ follows from that of $\hat{u}_N^{(1)}$ since $\hat{u}_N^{(3)}$ is defined in terms of $\hat{u}_N^{(1)}$.

For the case $\Theta = U$, the unit sphere in $\mathbf{R}^3$, we will examine Galerkin methods based on spherical harmonics. We will first look at convergence and rate of convergence of our Galerkin methods in the space of square integrable functions $L^2(U)$, and then in the space of continuous functions $C(U)$. How to base collocation methods on spherical harmonics is an open question, so we do not have that approach available.

For the case Θ being a standard torus in $\mathbf{R}^3$, we consider only collocation methods based on double Fourier series (although Galerkin methods would be possible.) These methods only make sense in the space of continuous functions. Thus we give convergence and rate of convergence results only for the uniform norm.

Since we have already established **A.1** through **A.4** on **P.1** through **P.5**, we concentrate on showing **A.5** for the projection and Kumar methods. This usually will be based on the fact that $\hat{K}_{i,N}^{(\nu)}$ is norm convergent to $\hat{K}_i$ for $i = 1$ and 2 and $\nu = 1$ and 2. See Proposition 1 of Chapter 3. This will allow us to use the rate of convergence result in Theorem 4 of Chapter 3. When using Theorem 4 for the Kumar method, we get a suggestion of superconvergence. Under the same conditions, when we show

$$\| \hat{u}_N^{(1)} - \hat{u}^* \|_{\mathcal{X}} \leq \frac{c}{N^{l+\lambda'}},$$

we also show

$$\| \hat{u}_N^{(2)} - \hat{u}^* \|_{\mathcal{X}} \leq \frac{\alpha_N}{N^{l+\lambda'}},$$

where $\alpha_N \to 0$.

When considering convergence in $C(\Theta)$ for Θ the unit sphere and the standard torus, **A.5**(2) for the Kumar method has not been easy to prove. In each case we give conjectures which would lead to **A.5**(2). In the collocation case (on the torus), those conjectures are not enough to prove norm convergence of the $\hat{K}_{i,N}^{(2)}$ to the $\hat{K}_i$. So they do not support the rate of convergence result of Theorem 4 of Chapter 3. They simply give us existence, uniqueness, and convergence of the approximate solutions $\hat{u}_N^{(2)}$. In order not to rely on the unproven conjectures and to get rate of convergence results, we follow the alternative approach to convergence analysis for the Kumar method outlined in the last section of Chapter 3.

In all cases we offer computable rate of convergence results. They are computable in the sense that they are based on properties of the surface $\partial\Omega$ and the given functions f and g and the map μ. We will study these rates empirically later in our numerical examples. We will be especially interested in looking for evidence of superconvergence. We have already informally discussed in Chapter 3 when superconvergence may be expected. Only in the first case below will the superconvergence of $\hat{u}_N^{(3)}$ be theoretically established.

Assuming **P.1** through **P.5**, $\hat{H}$ is continuously differentiable in a neighborhood of $\hat{u}^*$ (in particular $\mathcal{M}(\mathcal{D})$). If we further assume 1 is not an eigenvalue of $\hat{H}'(\hat{u}^*)$, then the uniqueness of the approximations $\hat{u}_N^{(\nu)}$ will immediately follow from the conditions for their uniqueness and convergence. Therefore in the cases we look at, we will mostly concern ourselves with proving $\mathbf{A.5}(\nu)$, $\nu = 1$ and 2, and inferring existence and convergence of $\hat{u}_N^{(\nu)}$ from this.

4.4 Galerkin Methods with Surface Spherical Harmonics

In case Θ is the unit sphere U in $\mathbf{R}^3$, we may choose $\mathcal{X}_N$ to be the space of surface spherical harmonics of degree less than or equal to N. This is a proper choice whether $\mathcal{X}(U)$ is $L^2(U)$ or $C(U)$ since the spherical harmonics are dense in both. How to interpolate using spherical harmonics is an open question. Thus we are limited in our choice of projection methods to using an orthogonal projection method. In this case we refer to the projection method and its variants as Galerkin methods. We discuss spherical harmonics and show how the basis functions $\{h_k\}$ may be chosen in this case.

We give U the standard parametrization σ, where

$$\sigma(\phi, \theta) = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$$

for $0 \le \phi < 2\pi$ and $0 \le \theta \le \pi$. But unless more clarity is needed we will identify

$P = \sigma(\phi, \theta)$ with (ϕ, θ). So $x(\phi, \theta)$ will mean $x(\sigma(\phi, \theta))$ for $x \in \mathcal{X}(U)$.

For $x \in \mathcal{X}(U)$

$$\int_U x(Q) dA_Q = \int_0^{2\pi} \int_0^{\pi} x(\sigma(\phi, \theta)) \sin\theta \, d\theta \, d\phi.$$

By using μ we may also transform integrals over $\partial\Omega$. Let $\rho = \mu \circ \sigma$. Then for $v \in \mathcal{X}$

$$\int_{\partial\Omega} v(Q) dA_Q = \int_0^{2\pi} \int_0^{\pi} v(\rho(\phi, \theta)) \mid \rho_\phi \times \rho_\theta \mid d\theta \, d\phi.$$

$\hat{H}$ may also be redefined in terms of itegration over the rectangle $0 \le \phi < 2\pi$ and $0 \le \theta \le \pi$. Let the parameter $P = \sigma(\phi', \theta')$ and the variable of integration $Q = \sigma(\phi, \theta)$. Then $\hat{K}_i \hat{G}_i(x)(P) =$

$$\int_0^{2\pi} \int_0^{\pi} [k_i(\rho(\phi', \theta'), \rho(\phi, \theta)) g_i(\rho(\phi, \theta), x(\sigma(\phi, \theta)))] \mid \rho_\phi \times \rho_\theta \mid \, d\theta \, d\phi$$

$$= \int_U \hat{k}_i(P, Q) \hat{g}_i(Q, x(Q)) \, dA_Q \tag{4.9}$$

where

$$\hat{g}_i(Q, x) = g_i(\mu(Q), x)$$

and the kernels

$$\hat{k}_i(P, Q) = k_i(\mu(P), \mu(Q)) \mid \rho_\phi \times \rho_\theta \mid / \sin\theta.$$

With these new definitions and with a similar treatment of $\hat{K}_2 \hat{f}$, $\hat{H}$ is still a Hammerstein integral operator of mixed type.

A solution h of Laplace's equation $\triangle v = 0$ which is homogeneous of degree N in x, y, and z is called a solid spherical harmonic of degree N. If we restrict such a function to the surface of U, it is called a surface spherical harmonic of degree N. Let $\mathcal{X}_N(U)$ be the subspace of $\mathcal{X}(U)$ of surface spherical harmonics of degree N or less. An orthogonal basis for $\mathcal{X}_N(U)$ is the set consisting of the Legendre polynomials $P_m(\cos\theta)$, $m \le N$, and the functions $P_l^k(\cos\theta)\cos(k\phi)$ and $P_l^k(\cos\theta) sin(k\phi)$ where $P_l^k(x)$ are the associated Legendre functions and $1 \le k \le l \le N$. Thus the dimension of $\mathcal{X}_N(U)$, call it d_N, equals $(N+1)^2$.

The Laplace series corresponding to a function $x(\theta,\phi)$ defined on the unit sphere is

$$\sum_{n=0}^{\infty}\frac{2n+1}{4\pi}\int_U x(\theta',\phi')P_n(\cos\gamma)\,dA'$$

with γ the angle between the two vectors from the center of the sphere to the points (θ,ϕ) and (θ',ϕ'). We keep this notation in what follows. We are interested in the projection onto $\mathcal{X}_N(U)$ given by the truncated Laplace series. A well-known addition theorem [MacRobert (1967)] gives the identity

$$\begin{aligned}P_n(\cos\gamma) &= P_n(\cos\theta')\,P_n(\cos\theta)\\ &\quad +2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}[P_n^m(\cos\theta')\cos m\phi'\,P_n^m(\cos\theta)\cos m\phi\\ &\qquad +P_n^m(\cos\theta')\sin m\phi'\,P_n^m(\cos\theta)\sin m\phi].\end{aligned}$$

Therefore the truncated series is

$$\mathbf{P}_N x(\theta,\phi) = \sum_{n=0}^{N}\frac{2n+1}{4\pi}\int_U x(\theta',\phi')P_n(\cos\gamma)\,dA' \tag{4.10}$$

$$\begin{aligned}&= \sum_{n=0}^{N}\frac{2n+1}{4\pi}\left\{\left(\int_U x(\theta',\phi')P_n(\cos\theta')\,dA'\right)P_n(\cos\theta)\right.\\ &+2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}\left[\left(\int_U x(\theta',\phi')P_n^m(\cos\theta')\cos m\phi'\,dA'\right)P_n^m(\cos\theta)\cos m\phi\right.\\ &\quad\left.\left.+\left(\int_U x(\theta',\phi')P_n^m(\cos\theta')\sin m\phi'\,dA'\right)P_n^m(\cos\theta)\sin m\phi\right]\right\}.\end{aligned}$$

In order to have an orthonormal basis for $\mathcal{X}_N$, we use these basis functions:

$$\sqrt{\frac{2n+1}{4\pi}}P_n(\cos\theta),\qquad \sqrt{\frac{2n+1}{2\pi}\frac{(n-m)!}{(n+m)!}}P_n^m(\cos\theta)\left\{\begin{array}{l}\cos m\phi\\ \sin m\phi\end{array}\right. . \tag{4.11}$$

We give these basis functions some order and call them h_k for $k=1,\ldots,d_N$. then the above projection (4.10) is simplified to the standard form for an orthogonal projection,

$$\mathbf{P}_N x(\theta,\phi) = \sum_{k=1}^{d_N}(h_k,\,x)_{\mathrm{U}}h_k\,(\theta,\phi). \tag{4.12}$$

4.5 Convergence of the Galerkin Methods in $L^2(U)$

We begin by analyzing convergence of $\hat{u}_N^{(\nu)}$ to $\hat{u}^*$, $\nu = 1$ and 2, in $L^2(U)$. The space $L^2(U)$ has the nice property of being a Hilbert space. With respect to its inner product

$$(x, y)_U := \int_U x(Q)y(Q)\,dA_Q,$$

the basis functions (4.11) are orthonormal. This will make the analysis in this case somewhat easier than for $C(U)$, as we shall see.

$\mathbf{P}_N$ is an orthogonal projection from $L^2(U)$ to $\mathcal{X}_N(U)$, and the L^2 operator norm for this projection, $\| \mathbf{P}_N \|$, equals 1 for all N. Recall that since $\bigcup_{N=1}^{\infty} \mathcal{X}_N(U)$ is dense in $L^2(U)$, the boundedness of $\| \mathbf{P}_N \|$ is equivalent to $\mathbf{P}_N x \to x$ for all $x \in L^2(U)$. Of course this convergence is in the L^2 norm. Therefore by Proposition 2 of Chapter 3, we have $\mathbf{A.5}(\nu)$ for both $\nu = 1$ and $\nu = 2$. This sets the groundwork for convergence of both the projection method and Kumar's method.

Theorem 3 *Assume* **P.1** *through* **P.5**. *Let u^* be an isolated solution of (4.4). Then the approximations $\hat{u}_N^{(\nu)}$, $\nu = 1$ and 2, exist for N large enough and they converge to $\hat{u}^*$ in $L^2(U)$. Also if 1 is not an eigenvalue of $\hat{H}'(\hat{u}^*)$, $\hat{u}_N^{(\nu)}$ is unique for N large enough.*

Moreover, since $\mathbf{P}_N$ is an orthogonal projection on a Hilbert space, the sequences $\{\hat{K}_{i,N}^{(\nu)}\}$ are norm convergent for both $\nu = 1$ and 2. Therefore we may apply the rate of convergence result in Theorem 4 of Chapter 3. Let $T_N \in \mathcal{X}_N$ be the best approximation (in the uniform norm) to $\hat{u}^*$. Since $\mathbf{P}_N T_N = T_N$, we may make the following estimate.

$$\begin{aligned} \| \mathbf{P}_N \hat{u}^* - \hat{u}^* \|_2 &= \| \mathbf{P}_N(\hat{u}^* - T_N) + T_N - \hat{u}^* \|_2 \\ &\le (1 + \| \mathbf{P}_N \|) \| \hat{u}^* - T_N \|_2 \qquad (4.13) \\ &\le 4\sqrt{\pi}\, \| \hat{u}^* - T_N \|_\infty . \end{aligned}$$

We have used the fact that for $x \in C(U)$

$$\| x \|_2 = \left[\int_U | x |^2 \, dA_Q \right]^{\frac{1}{2}} \leq \| x \|_\infty \left[\int_U dA_Q \right]^{\frac{1}{2}} = 2\sqrt{\pi} \| x \|_\infty .$$

The last quantity in (4.13) may be estimated by the use of a theorem from Ragozin.

Theorem 4 (Ragozin) *If $x \in H(k, a, \lambda)$ on U, then there exists $T_N \in \mathcal{X}_N$ such that*

$$\| x - T_N \|_\infty \leq \frac{a \, b_k}{N^{k+\lambda}}.$$

b_k is a constant which depends only on k.

Thus we have the following:

Theorem 5 *Assume* **P.1** *through* **P.5** *and that u^* is an isolated solution of (4.4) that is bounded and integrable. Assume there exists l and λ such that $\partial\Omega$, f, g, and μ satisfy the conditions of Theorem 2. Then for $0 < \lambda' < \lambda$, there exists a constant c with*

$$\| \hat{u}_N^{(1)} - \hat{u}^* \|_2 \leq \frac{c}{N^{l+\lambda'}},$$

and a sequence $\alpha_N \to 0$ with

$$\| \hat{u}_N^{(2)} - \hat{u}^* \|_2 \leq \frac{\alpha_N}{N^{l+\lambda'}}.$$

Proof: Let $0 < \lambda' < \lambda$. By Theorem 2 above, $\hat{u}^* \in H(l, c', \lambda')$ for some c'. By (4.13) and Ragozin's Theorem,

$$\| \mathbf{P}_N \hat{u}^* - \hat{u}^* \|_2 \leq \frac{4\sqrt{\pi} \, c' \, b_l}{N^{l+\lambda'}}.$$

Since $\| \hat{u}_N^{(1)} - \hat{u}^* \|_2$ is on the same order as $\| \mathbf{P}_N \hat{u}^* - \hat{u}^* \|_2$ by Theorem 4 of Chapter 2, we may find such a constant c.

For the Kumar case, since $u^* \in H(l, c', \lambda')$ for some c', $\hat{G}(\hat{u}^*) = \mathcal{M}G(u^*) \in H(l, c'', \lambda')$ for some c'', by conditions (c) and (d) of Theorem 2. So also

$$\| \mathbf{P}_N \hat{G}(\hat{u}^*) - \hat{G}(\hat{u}^*) \|_2 \leq \frac{4\sqrt{\pi} \, c'' \, b_l}{N^{l+\lambda'}}.$$

By the comments after Theorem 6 of Chapter 3, $\| \hat{u}_N^{(2)} - \hat{u}^* \|_2$ is superconvergent with respect to this rate. That is, such a sequence $\{\alpha_N\}$ exists. $\square$

The existence, convergence, and uniqueness of the Sloan iterate $\hat{u}_N^{(3)}$ follows from that of $\hat{u}_N^{(1)}$. This comes from the fact that it is defined in terms of $\hat{u}_N^{(1)}$. But in this case its superconvergence is also assured.

Theorem 6 *Assume* **P.1** *through* **P.5** *and that 1 is not an eigenvalue of* $\hat{H}'(\hat{u}^*)$. *Then* $\hat{u}_N^{(3)}$ *is superconvergent with respect to* $\hat{u}_N^{(1)}$ *in* $L^2(U)$.

Proof: See the corollary to Theorem 7 of Chapter 3. □

4.6 Convergence of the Galerkin Methods in $C(U)$

By Theorem 3 of the last section, the projection solution $\hat{u}_N^{(1)}$ and the Kumar solution $\hat{u}_N^{(2)}$ exist for N sufficiently large, and they are unique if 1 is not an eigenvalue of $\hat{H}'(\hat{u}^*)$. Since $C(U)$ is contained in $L^2(U)$, this need not be proved in $C(U)$. But it remains to show the uniform convergence of the solutions, $\hat{u}_N^{(\nu)}$ for $\nu = 1$ and 2. In $C(U)$, $\| \mathbf{P}_N \|$ turns out not to be bounded. Gronwall (1914) shows that as an operator on $C(U)$

$$\lim_{N\to\infty} \frac{\| \mathbf{P}_N \|}{\sqrt{N}} = 2\sqrt{\frac{2}{\pi}}. \tag{4.14}$$

Therefore $\mathbf{P}_N x \not\to x$ for some $x \in C(U)$. Proposition 2 of Chapter 3 does not apply.

We consider using the concept of the best approximation in order to apply Proposition 3 of Chapter 3. Let $x \in C(U)$ with $\| x \|_\infty = a$. Let $y_1 = \hat{K}_1 x$ and $y_2 = \hat{K}_2 x$. Suppose $\partial\Omega$ is Lyapunov with exponent λ. Let d_2 be as in **P.5**. By the corollary to Günter's theorem, $y_1 \in H(0,\, d_2 c\, a,\, \theta\lambda)$ and $y_2 \in H(0,\, d_2 c\, a,\, \theta)$ for any $0 < \theta < 1$, with c depending only on $\partial\Omega$ and the choice of θ. By Ragozin's Theorem, the best approximations $T_N(y_1)$ and $T_N(y_2)$ to y_1 and y_2 satisfy

$$\| y_1 - T_N(y_1) \|_\infty \le \frac{d_2 c\, a\, b_0}{N^{\theta\lambda}},$$

and

$$\| y_2 - T_N(y_2) \|_\infty \le \frac{d_2 c\, a\, b_0}{N^{\theta}}.$$

Let $p := \sup_N \parallel \mathbf{P}_N \parallel / \sqrt{N}$. By (4.14), p is finite. Define two sequences by

$$\alpha_N := p\, d_2 c\, b_0 N^{1/2-\theta\lambda},$$

and

$$\beta_N := p\, d_2 c\, b_0 N^{1/2-\theta}.$$

Then

$$\parallel \mathbf{P}_N \parallel \parallel y_1 - T_N(y_1) \parallel_\infty \leq \alpha_N \parallel x \parallel_\infty$$

and

$$\parallel \mathbf{P}_N \parallel \parallel y_2 - T_N(y_2) \parallel_\infty \leq \beta_N \parallel x \parallel_\infty .$$

Now if the Lyapunov exponent of the surface $\partial\Omega$ is greater than $1/2$, θ may be chosen such that $\theta\lambda$ and θ are both greater than $1/2$. In that case α_N and β_N go to zero.

In light of Proposition 3 of Chapter 3, we have just proved the following theorem.

Theorem 7 *Assume* **P.1** *through* **P.5***. Let u^* be an isolated solution of (4.4). If $\partial\Omega$ is Lyapunov with exponent $\lambda > \frac{1}{2}$, then $\hat{u}_N^{(1)}$ converges in $C(U)$ (i.e. uniformly) to $\hat{u}^*$.*

With the tools we developed in the last section, we also have a rate of convergence result.

Theorem 8 *Assume* **P.1** *through* **P.5** *and that u^* is an isolated solution of (4.4), bounded and integrable. Suppose there exist l and λ satisfying the conditions of Theorem 2. Then for $0 < \lambda' < \lambda$ there exists a constant c such that*

$$\parallel \hat{u}_N^{(1)} - \hat{u}^* \parallel_\infty \leq \frac{c}{N^{l+\lambda'-1/2}}.$$

Proof: By Theorem 2, $\hat{u}^* \in H(l, a, \lambda')$ for some a given $0 < \lambda' < \lambda$. Using an estimate as in (4.13),

$$\begin{aligned} \parallel \mathbf{P}_N \hat{u}^* - \hat{u}^* \parallel_\infty &\leq (1+ \parallel \mathbf{P}_N \parallel) \parallel \hat{u}^* - T_N \parallel_\infty \\ &\leq 2\, p \sqrt{N} \parallel \hat{u}^* - T_N \parallel_\infty \\ &\leq \frac{2\, p\, a\, b_l}{N^{l+\lambda'-1/2}}. \end{aligned}$$

Since $\parallel \hat{u}_N^{(1)} - \hat{u}^* \parallel_\infty$ is on the same order as $\parallel \mathbf{P}_N \hat{u}^* - \hat{u}^* \parallel_\infty$, we may find such a constant c. $\square$

For the Kumar solution, we have a conjecture whose truth would be sufficient for proving the uniform convergence of the $\hat{u}_N^{(2)}$. We hope to prove this conjecture in subsequent research. As in section 4 of Chapter 3, define the one parameter family of functions

$$\hat{k}_{P,i}(Q) := \hat{k}_i(P, Q)$$

for each i. Note that this family of functions is in $L^1(U)$ but not in $L^2(U)$.

Conjecture 1: There exists $k_0 \geq 1$ and $0 < \lambda_0 < 1$, such that the following is true for $\partial\Omega \in L_{k_0}(B, \lambda_0)$ for any $B > 0$. If **P.4** and **P.5** are satisfied for $\mu : U \to \partial\Omega$, then

$$\sup_P \parallel (\mathbf{P}_N - I)\hat{k}_{P,i} \parallel_1 \to 0,$$

for $i = 1$ and 2.

Not much seems to have been written about the L^1 convergence of general Fourier series. This conjecture asserts that the family of functions $\{\hat{k}_{P,i}\}$ are relatively nice as far as functions in $L^1(U) - L^2(U)$ are concerned.

If Conjecture 1 is true, then the following theorems will follow.

Theorem 9 *Assume* **P.1** *through* **P.5** *and that 1 is not an eigenvalue of $\hat{H}'(\hat{u}^*)$. Let $\partial\Omega \in L_k(B, \lambda)$ where $k + \lambda \geq k_0 + \lambda_0$. Then $\hat{u}_N^{(2)}$ exists and is unique for N large enough and converges in $C(U)$ (uniformly) to $\hat{u}^*$.*

Proof: There exists a c such that for any B, $L_k(B, \lambda) \subseteq L_{k_0}(cB, \lambda_0)$. (Rule 5 before Theorem 2.) Apply Proposition 4 of Chapter 3. $\square$

Moreover in this case the $\hat{K}_{i,N}^{(2)}$ are norm convergent. (See the proof of Proposition 4 of Chapter 3.) Therefore we may apply Theorem 4 of Chapter 3 to get a rate of convergence result.

Theorem 10 *Assume the conditions of Theorems 8 and 9. Then there exist a sequence $\alpha_N \to 0$ such that*

$$\| \hat{u}_N^{(2)} - \hat{u}^* \|_\infty \leq \frac{\alpha_N}{N^{k+\lambda'-1/2}}.$$

Proof: As in Theorem 5, $\hat{G}(\hat{u}^*) \in H(l, c, \lambda')$ for some c. Then using the same bounds as in Theorem 8,

$$\| \mathbf{P}_N \hat{G}(\hat{u}^*) - \hat{G}(\hat{u}^*) \|_\infty \leq \frac{c'}{N^{l+\lambda'-1/2}}.$$

As Theorem 6 of Chapter 3 shows, $\| \hat{u}_N^{(2)} - \hat{u}^* \|_\infty$ is superconvergent with respect to that rate. $\square$

Since the conjecture is not yet proved it is desirable to employ the alternative approach to convergence analysis of the Kumar method outlined in the last section of Chapter 3. When considering the Kumar method in $C(\Theta)$, the cases for the torus and the sphere are so similar that we will treat them together after discussing the torus case. See the last section of this chapter.

Slightly strengthening Conjecture 1 gives us the basis for showing the superconvergence of the Sloan iterate $\hat{u}_N^{(3)}$.

Conjecture 1$'$: Under the same conditions as in Conjecture 1, it is also true that

$$\sup_P \| (\mathbf{P}_N - I)\{\hat{k}_{P,2} \hat{G}'(\hat{u}^*)\} \|_1 \to 0.$$

Based on this, we have the theorem,

Theorem 11 *Assume* **P.1** *through* **P.5** *and that 1 is not an eigenvalue of $\hat{H}'(\hat{u}^*)$. Let $\partial\Omega \in L_k(B, \lambda)$ where $k + \lambda > k_0 + \lambda_0$. Then the Sloan iterate $\hat{u}_N^{(3)}$ is superconvergent with respect to the projection solution $\hat{u}_N^{(1)}$.*

Proof: Let $F = \hat{H}'(\hat{u}^*)$ as in Chapter 3. We show that $\| F(\mathbf{P}_N - I) \| \to 0$, and then apply the Corollary to Theorem 7 of Chapter 3 to get the result.

$F(\mathbf{P}_N - I)x = \hat{K}_1(\mathbf{P}_N - I)x + \hat{K}_2[\hat{G}'(\hat{u}^*)](\mathbf{P}_N - I)x$. It was shown in Theorem 9 that $\| \hat{K}_1(\mathbf{P}_N - I) \| \to 0$. It remains to show $\| \hat{K}_2[\hat{G}'(\hat{u}^*)](\mathbf{P}_N - I) \| \to 0$.

Using an argument as in Proposition 4 of Chapter 3,

$$\| \hat{K}_2[\hat{G}'(\hat{u}^*)]x \|_\infty \leq \| x \|_\infty \sup_{P\in\Theta} \| (\mathbf{P}_N - I)\{\hat{k}_{P,2}\hat{G}'(\hat{u}^*)\} \|_1 .$$

By the conjecture we have the norm convergence. $\square$

4.7 Collocation Methods on the Torus

Another surface admitting conveniently computable approximating subspaces of globally smooth functions is a torus. If Θ is a standard torus T, approximating subspaces may be composed of bivariate trigonometric polynomials given the usual bivariate parameterization of T. They are dense in both $C(T)$ and in $L^2(T)$. But we will only investigate collocation methods based on an interpolatory projection. Therefore we will be interested only in $C(T)$.

Suppose the torus T is the revolution about the $z-$axis of a circle in the $(x, z)-$plane with center on the $x-$axis. T is parametrized by

$$\tau(\phi,\theta) = ([a + b\sin\theta]\cos\phi,\ [a + b\sin\theta]\sin\phi,\ b\ \cos\theta)$$

with $a > b > 0$, and (ϕ,θ) from the rectangle R in the $(\phi,\theta)-$plane given by R : $0 \leq \phi < 2\pi,\ 0 \leq \theta < 2\pi$. We use these as our coordinates for a point on T. With a function x defined on T will implicitly be associated the function $x \circ \tau$ defined on the rectangle R.

Functions which are continuous on T, may be naturally associated with a continuous function $2\pi-$periodic in each variable in the $(\phi,\ \theta)-$plane. Thus we may approximate such functions with bivariate trigonometric interpolation. The double Fourier series associated with a function x may be expressed as

$$x(\phi,\theta) \sim \left(\frac{1}{2}a_0 + \sum_{i=1}^{\infty}[a_i\cos i\phi + b_i\sin i\phi]\right)\left(\frac{1}{2}c_0 + \sum_{j=1}^{\infty}[c_j\cos j\theta + d_j\sin j\theta]\right).$$

If each of the sums is truncated to N, a bivariate trigonometric polynomial of degree N results. Let $\mathcal{X}_N$ be the finite dimensional subspace of $C(T)$ of bivariate trigonometric polynomials of degree less than or equal to N. A basis for $\mathcal{X}_N$ is given by 1 and $\cos(i\phi)\cos(j\theta)$, $\cos(i\phi)\sin(j\theta)$, $\sin(i\phi)\cos(j\theta)$, $\sin(i\phi)\sin(j\theta)$, for $1 \le i,\ j \le N$. The subspace $\mathcal{X}_N$ of $\mathcal{X}$ has dimension $(2N+1)^2$. For example if $N = 1$, then we may chose our basis functions as

$$\begin{array}{lll} h_1 = 1 & h_2 = \cos\phi & h_3 = \sin\phi \\ h_4 = \cos\theta & h_5 = \sin\theta & h_6 = \cos\phi\cos\theta \\ h_7 = \cos\phi\sin\theta & h_8 = \sin\phi\cos\theta & h_9 = \sin\phi\sin\theta. \end{array}$$

We choose $(2N+1)^2$ node points according to a rectangular grid of node points evenly spaced on R. For $k = 1, \dots, d_N$, choose the non-negative integers i and j such that

$$k = i(2N+1) + j + 1, \qquad 0 \le j < 2N+1. \tag{4.15}$$

Let

$$\phi_i = \frac{2\pi i}{2N+1} \quad \text{and} \quad \theta_j = \frac{2\pi j}{2N+1}.$$

We then identify the point P_k with the coordinates (ϕ_i, θ_j). (Or if P_k is thought of in terms of x, y, and z, we may say $P_k = \tau(\phi_i, \theta_j)$.)

It is well known that the matrix $[h_k(P_l)]$ is nonsingular. This is an extension of the one-dimensional case. See Zygmund (1959), page 7. Therefore there is a unique function in $\mathcal{X}_N = \text{span}(h_1, \dots, h_{d_N})$ which iterpolates a given function x at the set of points P_k. So we construct an interpolatory projection $\mathbf{P}_N$ defined by

$$\mathbf{P}_N x = \sum_{k=1}^{d_N} \alpha_k h_k$$

such that the α_k satisfy

$$\sum_{k=1}^{d_N} \alpha_k h_k(P_l) = x(P_l) \qquad l = 1, \dots, d_N.$$

The general discussion in Section 4.3 concerning the application of the convergence theory of Chapter 3 to the projection method and its variants applies here as well. We will not address the rate of convergence of the Kumar solution $\hat{u}_N^{(2)}$ or of the Sloan iterate $\hat{u}_N^{(3)}$. Nor will we try to prove their superconvergence with respect to $\hat{u}_N^{(1)}$. We approach these questions empirically in our numerical examples.

4.8 Convergence of the Collocation Methods

In this section we address the uniform convergence of the projection solution $\hat{u}_N^{(1)}$ and the Kumar solution $\hat{u}_N^{(2)}$ to $\hat{u}^*$. As before this will be done showing the satisfaction of the assumption $\mathbf{A.5}(\nu)$ for $\nu = 1$ and 2. We will again be concerned with the quantities $\| \mathbf{P}_N \|$ and will use the concept of best approximation.

First we estimate the operator norm $\| \mathbf{P}_N \|$. Consider the Dirichlet kernel,

$$D_N(u) = \frac{1}{2} + \sum_{j=1}^{N} \cos ju = \frac{\sin(N+\frac{1}{2})u}{2\,\sin\frac{1}{2}u}.$$

It has the property that

$$D_N\left(\frac{2\pi j}{2N+1}\right) = \begin{cases} 0\,, & j \neq 0 \\ N+\frac{1}{2}\,, & j = 0. \end{cases}$$

Therefore, if i and j are chosen as in (4.15), the element of $\mathcal{X}_N$ which interpolates a function x at the points $\{P_k\}$ is

$$\left(\frac{2}{2N+1}\right)^2 \sum_{k=1}^{d_N} x(P_k) D_N(\phi - \phi_i) D_N(\theta - \theta_j).$$

So this is an alternative definition of $\mathbf{P}_N$. It allows us to calculate the norm of $\mathbf{P}_N$.

$$\begin{aligned} \| \mathbf{P}_N \| &= \max_{\phi,\theta \in R} \left(\frac{2}{2N+1}\right)^2 \sum_{k=1}^{d_N} | D_N(\phi-\phi_i) D_N(\theta-\theta_j) | \\ &= \max_{0 \le \phi < 2\pi} \left(\frac{2}{2N+1}\right)^2 \left(\sum_{i=1}^{2N+1} | D_N(\phi-\phi_i) | \right)^2 \\ &= O((\log N)^2). \end{aligned}$$

For the last equality see Zygmund (1959), page 19.

Since $\| \mathbf{P}_N \|$ is unbounded, $\mathbf{P}_N x \not\to x$ for some $x \in C(T)$. Therefore we examine basing the proof of $\mathbf{A.5}(1)$ on the concept of the best approximation. A theorem which parallels that of Ragozin for the sphere is given in Lorentz (page 90, 1966).

Theorem 12 *Let $x \in H(k,a,\lambda)$ on T. Then the best approximation $T_N(x) \in \mathcal{X}_N$ satisfies*

$$\| x - T_N(x) \|_\infty \leq \frac{a\, b_k}{N^{k+\lambda}}.$$

Using the same techniques we have already used in the case of the sphere, we may prove the following.

Theorem 13 *Assume **P.1** through **P.5**, and that 1 is not an eigenvalue of $\hat{H}(\hat{u}^*)$. Then the projection solution $\hat{u}_N^{(1)}$ exists for N large enough, and it converges to $\hat{u}^*$ in $C(T)$ and $\hat{u}_N^{(1)}$ is unique for N large enough.*

Proof: Let

$$p := \sup_{N>1} \frac{\| \mathbf{P}_N \|}{(logN)^2}.$$

Let $x \in C(T)$ with $\| x \|_\infty = a$. Let $y_1 = \hat{K}_1 x$ and $y_2 = \hat{K}_2 x$. Let d_2 be as in **P.5**. By the corollary to Günter's Theorem, $y_1 \in H(0, d_2 c\, a, \theta\lambda)$ and $y_2 \in H(0, d_2 c\, a, \theta)$ where λ is the Lyapunov exponent of the surface $\partial\Omega$, $0 < \theta < 1$, and c is a constant depending only on $\partial\Omega$ and the choice of θ. Define the two sequences

$$\alpha_N := \frac{(\log N)^2 p\, d_2 c\, b_0}{N^{\lambda\theta}}$$

and

$$\beta_N := \frac{(\log N)^2 p\, d_2 c\, b_0}{N^{\theta}}.$$

Then for $N > 1$,

$$\| \mathbf{P}_N \| \| y_1 - T_N(y_1) \|_\infty \leq \alpha_N \| x \|_\infty$$

and

$$\| \mathbf{P}_N \| \| y_2 - T_N(y_2) \|_\infty \leq \beta_N \| x \|_\infty .$$

Both α_N and β_N go to zero. By Proposition 3 of Chapter 3, **A.5**(1) is true. The rest follows from Theorems 3 and 5 of Chapter 3. □

We also have the following rate of convergence result.

Theorem 14 *Assume* **P.1** *through* **P.5** *and that* u^* *is an isolated solution of (4.4) which is bounded and integrable. Suppose there exists* l *and* λ *satisfying the conditions of Theorem 2. Then there exists a constant* c *with*

$$\| \hat{u}_N^{(1)} - \hat{u}^* \|_\infty \leq \frac{c\,(\log N)^2}{N^{k+\lambda}}.$$

Proof: The proof is the same as the proof of Theorem 8 with $(\log N)^2$ substituted for $\sqrt{N}$. $\square$

Again the convergence analysis of the Kumar method offers some problems related to the L^1 convergence of double Fourier series. We will again make some conjectures that will be the subject of further research.

Let $\mathbf{O}_N$ be the orthogonal projection onto $\mathcal{X}_N$. Its relationship with $\mathbf{P}_N$ is described in Chapter 3. Recall the definition of the one parameter families of functions $\{\hat{k}_{P,i}\}$. Also recall the definition of a bounding function before Proposition 5.

Conjecture 2: There exists $k_1 \geq 1$ and $0 < \lambda_1 < 1$, such that the following is true for $\partial\Omega \in L_{k_1}(B, \lambda_1)$. If **P.4** and **P.5** are satisfied for $\mu : T \to \partial\Omega$, then there exists a bounding function ω_1 with

$$\| \mathbf{O}_N \hat{k}_{P,i} - \hat{k}_{P,i} \|_1 \leq \omega_1(\frac{1}{N})$$

for $i = 1$ and 2.

Conjecture 3: There exists $k_2 \geq 1$ and $0 < \lambda_2 < 1$, such that the following is true for $\partial\Omega \in L_{k_2}(B, \lambda_2)$ for any B. If **P.4** and **P.5** are satisfied for $\mu : T \to \partial\Omega$, then there exists a bounding function ω_2 with

$$\| \hat{k}_{P_1,i} - \hat{k}_{P_2,i} \|_1 \leq \omega_2(| P_1 - P_2 |)$$

for $i = 1$ and 2. These conjectures express the fact that if $\partial\Omega$ is sufficiently smooth, then the functions $\hat{k}_{P,i}$ are 'nice' functions in $L^1(T) - L^2(T)$.

A a necessary condition for Proposition 5 of Chapter 3 is that the sequence of discrete inner products associated with the sequence of projection $\{\mathbf{P}_N\}$ must satisfy the condition

$$(1, \mid x \mid)_{\Theta, m(N)} \leq b \int_{\Theta} \mid x \mid \, dA,$$

for some absolute constant b. To show this for trigonometric interpolation in one dimension, see Zygmund (1959), page 28. That result can easily be extended to multiple dimensions.

With Conjectures 2 and 3 we have the following theorem:

Theorem 15 *Assume* **P.1** *through* **P.5** *and that 1 is not an eigenvalue of* $\hat{H}'(\hat{u}^*)$. *Let* $\partial\Omega \in L_k(B, \lambda)$ *with* $k + \lambda \geq \max\{k_1 + \lambda_1,\ k_2 + \lambda_2\}$. *Then* $\hat{u}_N^{(2)}$ *exists and is unique for* N *large enough and converges in* $C(T)$ *to* $\hat{u}^*$.

Proof: See Proposition 5 of Chapter 3. □

Unfortunately these conjectures do not establish the norm convergence of the $\hat{K}_{i,N}^{(\nu)}$ to the $\hat{K}_i$. Therefore this approach does not yield rate of convergence results, since norm convergence is required by Theorem 4 of Chapter 3.

4.9 Convergence for the Kumar Solution

If the conjectures given in sections 6 and 8 are established, the convergence analysis for the Kumar method may be done with few assumptions about g. Since they have not been proven, we employ the alternative approach to convergence analysis of the Kumar method outlined in the last section of Chapter 3. In this section we apply that approach to both the Galerkin method on the sphere and the collocation method on the torus. That analysis was based on **A.1** through **A.4** plus the new assumptions **A.5**$'$ and **A.6**$'$. Again we already have shown **A.1** through **A.4** for our problem. It remains to show **A.5**$'$ and **A.6**$'$. Finally we will present rate of convergence and uniqueness results based on **A.7**$'$ of Chapter 3.

Again let Θ denote either the unit sphere U or a torus. Let $\mathcal{X}^2 = \mathcal{X} \times \mathcal{X}$ where

$\mathcal{X} = C(\Theta)$. The subset $\mathcal{E}$ of $\mathcal{X}^2$ is defined by

$$\mathcal{E} = \{\bar{x} \in \mathcal{X}^2 \mid \hat{K}_1 x_1 + \hat{K}_2 x_2 + \hat{K}_2 \hat{f} \in \mathcal{M}(\mathcal{D})\}.$$

The operators V_1 and V_2 are maps from $\mathcal{E}$ to $\mathcal{X}$ and defined as follows:

$$V_1(\bar{x}) = \hat{K}_1 x_1 + \hat{K}_2 x_2 + \hat{K}_2 \hat{f},$$

and

$$V_2(\bar{x}) = \hat{G}(\hat{K}_1 x_1 + \hat{K}_2 x_2 + \hat{K}_2 \hat{f}).$$

The fixed point $\bar{z}^* \in \mathcal{E}$ of the operator

$$V(\bar{x}) =< V_1(\bar{x}), V_2(\bar{x}) >,$$

satisfies the equation

$$\hat{u}^* = \hat{K}_1 z_1^* + \hat{K}_2 z_2^* + \hat{K}_2 \hat{f}.$$

So we approximate $\bar{z}^*$ with the solution $\bar{z}_N$ of the equation

$$\bar{z}_N = \bar{\mathbf{P}}_N V(\bar{z}_N).$$

Then the Kumar solution is calculated by

$$\hat{u}_N^{(2)} = \hat{K}_1 z_{1,N} + \hat{K}_2 z_{2,N} + \hat{K}_2 \hat{f}.$$

We will need a new assumption about the nonlinear function g of our model problem. Recall the interval $\mathcal{I}$ from **P.2**.

P.6 The function g is Hölder continuous in the P variable. In particular for all $x \in \mathcal{I}$, $g(\cdot, x) \in H(0, e, \gamma)$ with $\gamma > \zeta$. We assume $\zeta = 1/2$ for the sphere case and simply $\zeta = 0$ for the torus case.

We present a couple of lemmas based on this assumption.

Lemma 2 *Assume* **P.6**. *Then there exist constants c_1 and c_2 such that if $x \in H(0, a, \lambda)$ on Θ with $\lambda \leq \gamma$, then $\hat{G}(x) \in H(0, c_1 + c_2 a, \lambda)$ on Θ.*

Proof: With the observation that $\hat{g}(P, x) = g(\mu(P), x)$, we may use the proof of Proposition 4 of Chapter 2 to show that $\hat{g}(\cdot, x) \in H(0, d_2 e, \gamma)$ on Θ. Since $\partial \hat{g}/\partial x$ is a continuous function of the compact set $\Theta \times \mathcal{I}$ (**P.3**), it is bounded there, say by h.

Then for any P and $Q \in \Theta$,

$$\begin{aligned} |\, \hat{g}(P, x(P)) - \hat{g}(Q, x(Q)) \,| &\le |\, \hat{g}(P, x(P)) - \hat{g}(Q, x(P)) \,| \\ &\qquad + |\, \hat{g}(Q, x(P)) - \hat{g}(Q, x(Q)) \,| \\ &\le d_2 e \,|\, P - Q \,|^{\gamma} + h \,|\, x(P) - x(Q) \,| \\ &\le d_2 e \,|\, P - Q \,|^{\gamma} + h\, a \,|\, P - Q \,|^{\lambda} \\ &\le (d_2 e + h\, a) \,|\, P - Q \,|^{\lambda} . \end{aligned}$$

Let $c_1 = d_2 e$ and $c_2 = h$. $\square$

Lemma 3 *Assume* **P.1** *through* **P.6**. *Suppose* $\partial\Omega \in L_1(B, \lambda)$ *with* $\zeta < \lambda \le \gamma$. *Then for* $0 < \theta < 1$, *there exist constants* c_1 *and* c_2 *such that if* $\| x_1 \|_\infty, \| x_2 \|_\infty \le a$, *then* $\hat{G}(\hat{K}_1 x_1 + \hat{K}_2 x_2 + \hat{K}_2 \hat{f}) \in H(0, c_1 + c_2 a, \theta\lambda)$ *on* Θ.

Proof: Let $0 < \theta < 1$. Let $\| \hat{f} \|_\infty$ be bounded by a'. By the corollary to Theorem 1, there exists c such that

$$\hat{K}_1 x_1 \in H(0, d_2 c\, a, \theta\lambda),$$

while

$$\hat{K}_2 (x_2 + \hat{f}) \in H(0, d_2 c\,(a + a'), \lambda).$$

The sum of these would be in $H(0, d_2 c\,(2a + a'), \theta\lambda)$. Using the constants in the proof of Lemma 1 above, $\hat{G}(\hat{K}_1 x_1 + \hat{K}_2 x_2 + \hat{K}_2 \hat{f}) \in H(0, d_2 e + h\, d_2 c\,(a + a'), \theta\lambda)$. Let $c_1 = d_2 e + h\, d_2 c\, a'$ and $c_2 = 2\, h\, d_2 c$. $\square$

Based on these lemmas we have the following convergence result.

Theorem 16 *Assume* **P.1** *throught* **P.6** *and that* u^* *is an isolated solution of (4.4). If* $\partial\Omega \in L_1(B, \lambda)$ *with* $\zeta < \lambda \le \gamma$, *then* $\hat{u}_N^{(2)}$ *exists for* N *sufficiently large and coverges to* $\hat{u}^*$.

Proof: We show that $\mathbf{P}_N V_i$ converges uniformly to V_i on bounded sets of $\mathcal{E}$. Then by Proposition 7 of Chapter 3 **A.5$'$** and **A. 6$'$** are true. Then we have the result by Theorem 8 of Chapter 3.

For V_1, this follows from the norm convergence of $\mathbf{P}_N \hat{K}_i$ for $i = 1$ and 2. Therefore we need consider only V_2. Let $\mathcal{B}$ be a bounded set of $\mathcal{E}$ with all elements of $\mathcal{B}$ bounded by a. Let $\bar{x} =< x_1, x_2 >\in \mathcal{B}$. Using the same constants and steps as in the proof of Theorem 7,

$$\| (I - \mathbf{P}_N)\hat{G}(\hat{K}_1 x_1 + \hat{K}_2 x_2 + \hat{K}_2 \hat{f}) \|_\infty \leq \frac{2\, p\, b_0[c_1 + c_2 a]}{N^{\theta\lambda - 1/2}}.$$

Therefore $\mathbf{P}_N V_2$ converges uniformly to V_2 on bounded subsets of $\mathcal{E}$. $\square$

We also have a uniqueness and rate of convergence result.

Theorem 17 *Assume* **P.1** *through* **P.5** *and that u^* is an isolated solution of (4.4) bounded and integrable. Assume V and the V_N are twice differentiable and the second derivatives are uniformly bounded in a neighborhood of $\hat{z}^*$. Suppose there exists l and λ such that $\partial\Omega$, f, g and μ satisfy the conditions of Theorem 2 with $l + \lambda > 1 + \zeta$. Then the $\hat{u}_N^{(2)}$ are unique for N large enough and for $0 < \lambda' < \lambda$, there exists c such that*

$$\| \hat{u}_N^{(2)} - \hat{u}^* \|_\infty \leq \frac{c\,(\log N)^2}{N^{l+\lambda'}}.$$

Proof: Uniqueness follows directly from Theorem 9 of Chapter 3.

By the discussion after Theorem 9 of Chapter 3, $\| \hat{u}_N^{(2)} - \hat{u}^* \|_\infty$ is of the order of $\max_i \| \mathbf{P}_N \hat{G}_i(\hat{u}^*) - \hat{G}_i(\hat{u}^*) \|_\infty$. Let $0 < \lambda' < \lambda$. As in Theorem 5, $\hat{G}(\hat{u}^*) \in H(l, c', \lambda')$ for some c'. So we have the same kind of bound as in Theorem 14. $\square$

As the discussion in the last section of Chapter 3 notes, here we are not able to show the kind of superconvergence (Theorem 5 and Theorem 10) which we had for Kumar's method in the other approach to convergence analysis.

CHAPTER V

IMPLEMENTING THE PROJECTION METHODS

We have proven the convergence of the projection method and its variants for our model problem on certain types of smooth surfaces. In this chapter we explain more fully how each method finds its approximation to the solution. As an example of using an iterative method to solve the nonlinear operator equations for our problem, we consider Newton's method. We do a complexity analysis of the projection and Kumar methods of Galerkin and collocation types as a way of comparing the methods. We consider ways of reducing the complexity. Finally we explain what numerical integrals were used in our implementations, especially those that are meant to handle the singularity in the integral operators.

5.1 The Projection Operator Equations

In this first section, we motivate the following discussion by showing more clearly how we would define the projection solution, the Kumar solution, and the Sloan iterate for our model problem. We begin by applying a projection method to the problem (4.4). The projection solution $\hat{u}_N^{(1)}$ is an element of $\mathcal{X}_N$ which satisfies

$$\hat{u}_N^{(1)} = \mathbf{P}_N \hat{K}_1 \hat{u}_N^{(1)} + \mathbf{P}_N \hat{K}_2 \hat{G}(\hat{u}_N^{(1)}) + \mathbf{P}_N \hat{K}_2 f. \tag{5.1}$$

In the next section we will see how an iterative scheme and well chosen linear functionals can lead to a sequence of approximations which converge to $\hat{u}_N^{(1)}$ and are the solutions to linear systems.

The Sloan iterate $\hat{u}_N^{(3)}$ uses the projection solution $\hat{u}_N^{(1)}$ and performs one 'iteration." Given $\hat{u}_N^{(1)}$,

$$\hat{u}_N^{(3)} = \hat{K}_1 \hat{u}_N^{(1)} + \hat{K}_2 \hat{G}(\hat{u}_N^{(1)}) + \hat{K}_2 f. \tag{5.2}$$

Thus $\hat{u}_N^{(3)}$ requires additional work to calculate. If $\hat{u}_N^{(3)}$ is superconvergent with respect to $\hat{u}_N^{(1)}$, this additional work may pay off. (There are situations when (5.2) may lead directly to a Nystrom method, perhaps with product integration. We will not consider such situations here.)

The calculation of the Kumar solution $\hat{u}_N^{(2)}$ turns out to involve an iteration like that of the Sloan iterate. First the following vector equation is solved. We find $[z_{1,N},\, z_{2,N}] \in \mathcal{X}_N \times \mathcal{X}_N$ which satisfies the system of equations

$$\begin{aligned} z_{1,N} &= \mathbf{P}_N \hat{K}_1 z_{1,N} + \mathbf{P}_N \hat{K}_2 z_{2,N} + \mathbf{P}_N \hat{K}_2 f \qquad (5.3) \\ z_{2,N} &= \mathbf{P}_N \hat{G}(\hat{K}_1 z_{1,N} + \hat{K}_2 z_{2,N} + \hat{K}_2 f) \end{aligned}$$

Once $z_{1,N}$ and $z_{2,N}$ are found, we perform an iteration to get $\hat{u}_N^{(2)}$. We let

$$\hat{u}_N^{(2)} = \hat{K}_1 z_{1,N} + \hat{K}_2 z_{2,N} + \hat{K}_2 f. \tag{5.4}$$

The iteration in (5.4) and in (5.2) for $\hat{u}_N^{(3)}$ will in general involve numerical integrals to approximate the the integral operators.

5.2 Linearization with Newton's Method

Since $\hat{G}$ introduces nonlinearity into the operator equation (5.1), some iterative method must be used to solve the fixed point problem. The most popular such scheme is Newton's method. Because of assumption **P.3** of Chapter 4, $\hat{H}$ is Fréchet differentiable.

We rewrite equation (5.1) as

$$\hat{u}_N^{(1)} - \mathbf{P}_N \hat{H}(\hat{u}_N^{(1)}) = 0.$$

Newton's method for operator equations will now be applied as a root finding scheme for the operator $I - \mathbf{P}_N \hat{H}$. It produces a sequence $x^{(n)}$ with the intent that $x^{(n)} \to \hat{u}_N^{(1)}$. Given an initial guess $x^{(0)}$, let $x^{(n+1)}$ be defined inductively by

$$x^{(n+1)} = x^{(n)} - [I - \mathbf{P}_N \hat{H}'(x^{(n)})]^{-1}(x^{(n)} - \mathbf{P}_N \hat{H}(x^{(n)})). \tag{5.5}$$

Recall that $\hat{H}'(x^{(n)})$ is a linear operator on $\mathcal{X}(\Theta)$. If (5.5) is rearranged to eliminate the inverse operator and cancellation is used,

$$x^{(n+1)} - \mathbf{P}_N \hat{H}'(x^{(n)}) x^{(n+1)} = \mathbf{P}_N(\hat{H}(x^{(n)}) - \hat{H}'(x^{(n)}) x^{(n)}). \tag{5.6}$$

If $x^{(0)} \in \mathcal{X}_N$, then inductively $x^{(n)} \in \mathcal{X}_N$ for all n. The problem of generating an appropriate initial guess with total confidence that the resulting sequence will converge is theoretically difficult. We do not examine this question, but assume $x^{(0)}$ is chosen sufficiently close to $\hat{u}_N^{(1)}$ so as to ensure convergence.

Now we may discretize equation (5.6) by carefully choosing a set of linear functionals $\{l_k\}$, $k = 1, \ldots, d_N$ with the property that $\mathbf{P}_N x = 0$ if and only if $l_k(x) = 0$ for $k = 1, \ldots, d_N$. Then if $x^{(0)} \in \mathcal{X}_N$, equation (5.6) is equivalent to

$$l_k(x^{(n+1)} - \hat{H}'(x^{(n)}) x^{(n+1)}) = l_k(\hat{H}(x^{(n)}) - \hat{H}'(x^{(n)}) x^{(n)}) \tag{5.7}$$

for $k = 1, \ldots, d_N$. For example, we may use the set of linear functionals which was used to define $\mathbf{P}_N$ in Chapter 2, section 5. If $\mathbf{P}_N$ is orthogonal, we will use that set. Then $\{l_k\}$ is the set of inner products with the basis functions $\{h_k\}$:

$$l_k(x) := (h_k,\, x)_\Theta.$$

But if $\mathbf{P}_N$ is interpolatory, a simpler set of linear functionals can be used. We let

$$l_k(x) := x(P_k),$$

where P_k is one of the interpolation node points. These will be the same as the integration node points for the discrete inner product

$$(x,\, y)_{\Theta, d_N}. := \sum_{l=1}^{d_N} w_l\, x(P_l) y(P_l),$$

when the weights $\{w_l\}$ and the nodes $\{P_l\}$ are chosen such that $(x,\, y)_{\Theta, d_N} = (x,\, y)_\Theta$

for all $x,\, y \in \mathcal{X}_N$.

By substituting

$$x^{(n+1)} = \sum_{k=1}^{d_N} \alpha_k^{(n+1)} h_k$$

into (5.7), we have the linear system

$$\sum_{k=1}^{d_N} \alpha_k^{(n+1)} l_j(h_k - \hat{H}'(x^{(n)})h_k) = l_j(\hat{H}(x^{(n)}) - \hat{H}'(x^{(n)})x^{(n)}) \tag{5.8}$$

for $j = 1, \ldots, d_N$. Equation (5.1) becomes

$$\begin{aligned} \sum_{k=1}^{d_N} \alpha_k^{(n+1)} l_j(h_k - \hat{K}_1 h_k - \hat{K}_2[\hat{G}'(x^{(n)})h_k]) \\ = l_j(\hat{K}_2[\hat{G}'(x^{(n)})x^{(n)}] - \hat{K}_2\hat{G}(x^{(n)}) - \hat{K}_2 f), \end{aligned} \tag{5.9}$$

for $j = 1, \ldots, d_N$. With each iterate $x^{(n)}$ is associated the vector $[\alpha_1^{(n)}, \ldots, \alpha_{d_N}^{(n)}]$. We stop the iterations and use $x^{(n+1)}$ as our approximation when

$$\max_k \mid \alpha_k^{(n+1)} - \alpha_k^{(n)} \mid < \epsilon,$$

for some user supplied tolerance ϵ. (In case the sequence $\{x^{(n)}\}$ is not convergent, the iterations are also stopped after some bound. In our numerical examples, the sequence of iterates always converged.)

Equation (5.9) may be examined to determine the amount of work necessary for each iteration. The $d_N \times d_N$ matrix $[l_j(h_k - \hat{K}_1 h_k)]$ and the vector $[l_1(\hat{K}_2 f), \ldots, l_{d_N}(\hat{K}_2 f)]$ of length d_N may be calculated once and used for each iteration. On the other hand the $d_N \times d_N$ matrix

$$[l_j(\hat{K}_2[\hat{G}'(x^{(n)})h_k])]$$

and the length d_N vector

$$[l_1(\hat{K}_2[\hat{G}'(x^{(n)})x^{(n)} - \hat{G}(x^{(n)})]), \ldots, l_{d_N}(\hat{K}_2[\hat{G}'(x^{(n)})x^{(n)} - \hat{G}(x^{(n)})])]$$

must be updated for each iteration.

Solving for the Kumar solution $\hat{u}_N^{(2)}$ from (5.3) may follow the same steps. Let

$$x_1^{(n)} = \sum_{k=1}^{d_N} \alpha_{1,k}^{(n)} h_k \qquad \text{and} \qquad x_2^{(n)} = \sum_{k=1}^{d_N} \alpha_{2,k}^{(n)} h_k.$$

$x^{(n+1)} = [x_1^{(n+1)}, x_2^{(n+1)}]$ is the solution of the system of equations

$$\sum_{k=1}^{d_N} \alpha_{1,k}^{(n+1)} l_j(h_k - \hat{K}_1 h_k) + \sum_{k=1}^{d_N} \alpha_{2,k}^{(n+1)} l_j(-\hat{K}_2 h_k) = l_j(\hat{K}_2 f) \tag{5.10}$$

$$\sum_{k=1}^{d_N} \alpha_{1,k}^{(n+1)} l_j(-\hat{G}'(\hat{K}_1 x_1^{(n)} + \hat{K}_2 x_2^{(n)} + \hat{K}_2 f)\hat{K}_1 h_k) \\ + \sum_{k=1}^{d_N} \alpha_{2,k}^{(n+1)} l_j(h_k - \hat{G}'(\hat{K}_1 x_1^{(n)} + \hat{K}_2 x_2^{(n)} + \hat{K}_2 f)\hat{K}_2 h_k)$$

$$= l_j(\hat{G}'(\hat{K}_1 x_1^{(n)} + \hat{K}_2 x_2^{(n)} + \hat{K}_2 f)(\hat{K}_1 x_1^{(n)} + \hat{K}_2 x_2^{(n)}) - \hat{G}(\hat{K}_1 x_1^{(n)} + \hat{K}_2 x_2^{(n)} + \hat{K}_2 f))$$

The $d_N \times d_N$ matrices $[l_j(h_k - \hat{K}_1 h_k)]$ and $[l_j(\hat{K}_2 h_k)]$ and the length d_N vector $[l_1(\hat{K}_2 f), \ldots, l_{d_N}(\hat{K}_2 f)]$ may be evaluated once to be used for all iterations. But the quantities which need to be updated when calculating the next iterate are the $d_N \times d_N$ matrices

$$[l_j(\hat{G}'(\hat{K}_1 x_1^{(n)} + \hat{K}_2 x_2^{(n)} + \hat{K}_2 f)\hat{K}_1 h_k)],$$
$$[l_j(\hat{G}'(\hat{K}_1 x_1^{(n)} + \hat{K}_2 x_2^{(n)} + \hat{K}_2 f)\hat{K}_2 h_k)],$$

and the length d_N vector with the jth element,

$$l_j(\hat{G}'(\hat{K}_1 x_1^{(n)} + \hat{K}_2 x_2^{(n)} + \hat{K}_2 f)(\hat{K}_1 x_1^{(n)} + \hat{K}_2 x_2^{(n)}) - \hat{G}(\hat{K}_1 x_1^{(n)} + \hat{K}_2 x_2^{(n)} + \hat{K}_2 f)).$$

5.3 Complexity

In evaluating the complexity of the numerical algorithms that implement the projection methods, we will concentrate on three stages. The first two stages have already been alluded to in the last section. First certain quantities are calculated which are used for each iteration. These quantities correspond to the linear part of the operator $\hat{H}$. This first stage is executed only once. The second stage refers to the quantities which are calculated anew for each iteration. These quantities correspond to the nonlinear part of $\hat{H}$. In our problem this is the operator $\hat{G}$. The third stage is solving the linear system. The algorithm repeats the second and third stages until a sufficiently close approximation is achieved. This consideration leads to the goal of reducing the cost of these stages probably at the expense of making the first stage more costly.

The first stage will consist of evaluating those matrices and vectors which will be used in each iteration. They were listed in the last section for both the projection method and the Kumar method. The second stage is evaluating those matrices and vectors which must be updated for each iteration. Again, these were listed for the projection and Kumar methods. For the third stage we assume standard linear system solvers have been used. Therefore we consider the cost of this final stage as beyond our power to improve. Therefore its complexity is a lower bound for the complexity of the whole algorithm. This suggests the goal of making sure the complexity of the other two parts does not exceed that of the third part.

In the following discussion, we will talk of increasing or decreasing the complexity of some stage by a constant factor. This has been based on some programs which we have written. It is quite possible that a more clever implementation would change these estimates. They are used in the hope that some idea of the tradeoffs between approaches may be conveyed. But caution should be used in taking these estimates as final.

Projection methods with globally smooth approximations lead to linear systems with dense matrices whose entries involve integration over the entire surface. This is one of their disadvantages. This disadvantage is offset by rapid convergence in many cases. That is, we hope to get a good approximation with low N, the degree of the approximation, and thus with relatively small linear systems. In stage three, a linear system with a dense matrix of dimension $d_N \times d_N$ is solved. The complexity of stage three will therefore be $O(d_N^3)$. For the two surfaces we considered in Chapter 4, $d_N = O(N^2)$. Therefore stage three has complexity $O(N^6)$.

First consider the Galerkin methods on the sphere. For degree of approximation N, the dimension d_N of $\mathcal{X}_N$ is $(N+1)^2$. So the matrix in the linear system is of order $(N+1)^2$. The linear functionals are defined by

$$l_j(x) := (h_j, x)_U.$$

In general this inner product (a surface integral) must be approximated numerically. Each entry in the matrix of the linear system represents an inner product of a basis function with the image of an integral operator. Let I_p be the complexity of the numerical approximation to the inner product. Let I_o be the complexity of the numerical approximation to the integral operator evaluated at one point. The complexity of these numerical integrals is of the order of the number of their integration node points. As we shall see later, I_p and I_o should be at least $O(N^2)$. This is related to the number of nodes the discrete inner product needs in order for it to be exact for the inner product of two elements of $\mathcal{X}_N$. The complexity I_o will be less bound by this consideration. Then the complexity of each entry in the matrix is $I_pI_o \geq O(N^4)$. Since there are $O(N^4)$ entries, the total cost of calculating the matrix would seem to be at least $O(N^8)$! This would be true both for the matrices of stage 1 and of stage 2.

Actually very little effort is needed to reduce this complexity. Let P_k, $k = 1, \ldots, m$ be the integration node points of the numerical approximation to the inner product. For each node point P_k and each basis function h_j, $\hat{K}_1 h_j(P_k)$ must first be evaluated. The total complexity to calculate the $m \times d_N$ matrix $[\hat{K}_1 h_j(P_k)]$ would be $m\, d_N I_o = I_p I_o (N+1)^2 \geq O(N^6)$. Then multiplying that matrix by the $d_N \times m$ matrix $[w_l h_k(P_l)]$ produces the $d_N \times d_N$ matrix $[(h_k, \hat{K}_1 h_j)]$. The complexity of these multiplications is $I_o(N+1)^4$. So the complexity of these two steps is $\max\{O(I_p I_o (N+1)^2), O(I_o(N+1)^4)\}$. The idea here is that some quantities are used for each entry of a column say, in the final matrix. Instead of evaluating these quantities again and again for each entry, they are temporarily saved in memory. So the computational savings are at the expense of some memory space. This approach may be used for the matrices in both stage 1 and stage 2. If I_o and I_p are $O(N^2)$, then the complexity of each of the three stages is $O(N^6)$. The goal of being bounded by the complexity of the last stage is achieved.

The Kumar method attempts to reduce the complexity of stage 2 by eliminating the need to evaluate the integral operator for each iteration. But for the Galerkin methods it is still necessary to calculate an inner product for each entry in the matrix at a cost of $O(I_p N^4)$. On the other hand the amount of computation is reduced by a constant factor (perhaps 2 or 3) for stage 2. This may be significant as long as N is a small number. On the other hand, the dimension of the linear system is doubled. Since stage 3 has complexity the cube of the dimension of the linear system, this would increase the amount of computation in stage 3 by a factor of 8.

There is a simpler way to achieve an even better result. This approach is natural in the context of Galerkin's method. The inner products $(\cdot,\cdot)_U$ allow the use of adjoints for the linear integral operator components of the Hammerstein operators. Thus we transform these expressions so that the integral resulting from the integral operator need not be updated for each iterate. This is done, for example, by using the identity

$$(h_k, \hat{K}_2[\hat{G}'(x)y])_U = \int_U h_k(P) \int_U \hat{k}_2(P, Q) \frac{\partial}{\partial x} \hat{g}(Q, x(Q)) y(Q)\, dA_Q\, dA_P \quad (5.11)$$

$$= \int_U \frac{\partial}{\partial x}\hat{g}_i(Q, x(Q))y(Q) \int_U \hat{k}_2(P,Q)h_k(P)\, dA_P\, dA_Q.$$

Similar identities may be used for the quantities of the form $(h_k, \hat{K}_2\hat{G}(x))_U$. The function x represents the entity which must be updated at each step of the iteration. The integrals $\int_U \hat{k}_2(P,Q)h_k(P)\, dA_P$ may be evaluated in stage 1, probably increasing the computation in stage 1 by less than $\frac{1}{2}$. Stage 2 now involves only the evaluation of the surface integrals for the inner products. Stage 2's complexity is reduced by a factor of 4 or 5. This is more efficient than Kumar's method in stage 2 without increasing the computation in stage 3. Also, for situations such as our model problem, the $\hat{k}_i$ will depend only on the surface and not on the boundary data. This allows the storing of the quantities $\int_U \hat{k}_i(P,Q)h_k(P)\, dA_P$ to be used for more than one problem on the same surface.

The collocation methods are attractive in that the linear functionals l_k are so simple to evaluate. But in terms of complexity, it is suprisingly not an immediate improvement over the Galerkin methods. But it is more amenable to improvement by Kumar's method.

Our collocation problem was set on the surface of a torus. For degree of approximation N, the dimension of $\mathcal{X}_N$ was $(2\,N+1)^2$. So again stage 3 will have complexity $O(N^6)$. Stages 1 and 2 require the evaluation of an integral operator for each entry of the matrix of the linear system. Their complexities will be $O(I_oN^4)$. Again we may assume $I_o \geq O(N^2)$. So unless $I_p > O(N^2)$ the complexity of each stage is the same as what we achieved for the Galerkin methods!

But this time if we apply Kumar's method, we need not evaluate the integral operator in stage 2. Each entry for the matrix of the linear system will require only the multiplication and addition of some quantities saved in stage 1. Therefore the complexity of stage 2 is reduced to $O(N^4)$. Recall that stage 2 is crucial since it is repeatedly executed. Thus the improvement in efficiency may be significant. The cost of this improvement is again an increase in the complexity of stage 3 by a factor of 8, and perhaps a doubling of the complexity of stage 1.

The use of Kumar's Galerkin method does not seem to be justified by efficiency considerations. But it may be justified if its solution is superconvergent with respect to the projection solution. But if the desire is for superconvergence, one may more simply use the Sloan iterate. Among collocation methods, the Kumar method will probably be significantly more efficient. If in addition it is superconvergent, it would be a very attractive alternative.

5.4 The Numerical Integrals on the Sphere

Galerkin methods on the unit sphere in general require the numerical approximation of the integral operator and of the inner product. These are both integrals over the surface of U. In this section we discuss the numerical integration used first for the inner product and then for the singular integrals arising from the evaluation of the integral operators.

For surface integrals over the unit sphere U in the inner product we use the product Gaussian quadrature formula, following ideas in Stroud (1971) and Atkinson (1982b) and (1985). For the integral of a function x over U

$$\int_U x\,dA = \int_0^{2\pi}\int_0^{\pi} x(\theta,\phi)\sin\theta\,d\theta\,d\phi = \int_0^{2\pi}\int_{-1}^{1} x(\cos^{-1}t,\phi)\,dt\,d\phi. \tag{5.12}$$

This is approximated by

$$I_m(x) = \frac{\pi}{m}\sum_{j=1}^{2m}\sum_{i=1}^{m} w_i x(\cos^{-1}t_i,\phi_j) \tag{5.13}$$

with the $\{t_i\}$ and $\{w_i\}$ the Gauss-Legendre nodes and weights on $[-1,1]$ and $\phi_j = (j-1/2)\pi/m$ for $j=1,\ldots,m$.

As Stroud (1971) shows, if $p_n(x,y,z)$ is a polynomial of degree $n \le 2m -$

$1, I_m(p_n)$ is exact. We may define the discrete inner product $(\cdot,\cdot)_{U,m}$ by

$$(x,y)_{U,m} = \frac{\pi}{m}\sum_{j=1}^{2m}\sum_{i=1}^{m} w_i x(\theta_i,\phi_j)y(\theta_i,\phi_j).$$

If $m \geq N+1$ then the discrete inner product of two functions in $\mathcal{X}_N$ would satisfy

$$(x,y)_{U,m} = (x,y)_{\mathrm{U}} \quad \text{for } x,y \in \mathcal{X}_N.$$

The kernels of the integral operators are singular. Thus a numerical integration method appropriate for singular integrals must be employed. In particular, ordinary product Gaussian quadrature would not be very accurate. For the kernel $k_i(P,Q)$, $i = 1$ and 2, defined on $\partial\Omega \times \partial\Omega$, the singularity occurs when the integration variable Q equals the parameter P. In each case, the singularity is of the order of $\mid P - Q \mid^{-1}$. After the change in variable via the map μ, the singularity in each $\hat{k}_i$ is of the order $\mid \mu^{-1}(P) - \mu^{-1}(Q) \mid^{-1}$ because of assumption **P.5** of Chapter 4.

If we use spherical coordinates for U, integration over U can be transformed into integration on the rectangle $0 \leq \theta \leq \pi$ by $0 \leq \phi < 2\pi$. If the singularity is at the north pole $N = (0,0,1)$ on U, this corresponds to the singularity being on the line $\theta = 0$ on the rectangle. Both putting the singularity on the boundary of the region of integration and spreading it out on an interval often have a good effect on approximating the integral numerically. Since the singularity may be numerically handled well at the north pole we reorient our spherical coordinates for the surface integral in the integral operator such that its parameter P is the north pole. This must be done for each node point of the quadrature for the inner product (the outer surface integral.) It involves an orthogonal transformation (such as a Householder transformation) of the sphere. [See Atkinson (1982b).] A variation would be to transform the sphere such that P becomes the north or south pole according to which is the closest. But the choice of one destination pole simplified the application of the method chosen below.

At this point integrating in the θ direction involves an endpoint singularity. So we may apply a quadrature scheme equipped to handle endpoint singularities. An excellent candidate is the IMT method. But in numerical experiments, another method performed somewhat better. Also, it is based on the simple concept of the substitution needed to eliminate the singularity.

If $N = (0,0,1)$ i.e. the north pole, then in terms of the spherical coordinates

$$\begin{aligned} \mid N - Q \mid &= \sqrt{(\cos\phi\sin\theta)^2 + (\sin\phi\sin\theta)^2 + (1-\cos\phi)^2} \\ &= \sqrt{2 - 2\cos\theta} = 2\sin\frac{\theta}{2} \end{aligned}$$

Thus if x is an integrand with a singularity at the north pole of order $\mid\ N - Q\ \mid^{-1}$, then

$$\begin{aligned} \int_U x\, dA &= \int_0^{2\pi}\int_0^{\pi} x(\theta,\phi) \mid N - Q(\theta,\phi) \mid \frac{\sin\theta}{2\sin\frac{\theta}{2}}\, d\theta\, d\phi \\ &= \int_0^{2\pi}\int_0^{\pi} x(\theta,\phi) \mid N - Q(\theta,\phi) \mid \cos\frac{\theta}{2}\, d\theta\, d\phi. \end{aligned}$$

Now the integrand $x(\theta,\phi) \mid N - Q(\theta,\phi) \mid$ has a bounded singularity at N. Inside the rectangle $0 \le \phi < 2\pi,\ 0 \le \theta \le \pi$ this kind of singularity does not manifest itself. Product quadrature schemes similar to the product Gaussian method (5.13) will work well.

The simplest approach would be to approximate the integral by

$$I_m(x) = \frac{\pi^2}{2m}\sum_{j=1}^{2m}\sum_{i=1}^{m} w_i x(\frac{\pi}{2}(t_i+1),\phi_j)\sin\frac{\pi}{2}(t_i+1)$$

again with the $\{t_i\}$ and $\{w_i\}$ the Gauss-Legendre nodes and weights on [-1,1] and the ϕ_j chosen as for (5.13). Here we are using the values of θ, not of $\cos\theta$ as in the product Gaussian quadrature, for the interval on which to choose the Gauss-Legendre nodes. As in the IMT method above, this would handle singularities at both the north and south poles.

Another approach which worked somewhat better in numerical experiments is based on the substitution $t = 2\sin\frac{\theta}{2} - 1$ so that $\mid N - Q \mid = t + 1$. In this case

$$\int_U x\,dA = \int_0^{2\pi}\int_{-1}^{1} x(2\sin^{-1}\frac{t+1}{2}, \phi)(t+1)\,dt\,d\phi$$

and this is approximated by

$$I_m(x) = \frac{\pi}{m}\sum_{j=1}^{2m}\sum_{i=1}^{m} w_i x(2\sin^{-1}\frac{t_i+1}{2}, \phi_j)(t_i+1)$$

with the $\{t_i\}$ and $\{\phi_j\}$ chosen as above. This was the scheme which we chose. Note that it assumes that there is a singularity at only the north pole. This may explain its slightly superior performance for the numerical experiments we performed which involved singularities at only the north pole.

5.5 The Numerical Integrals on the Torus

Suppose the torus T is parameterized by the map

$$\tau(\phi, \theta) = (\cos\phi(2+\sin\theta),\ \sin\phi(2+\sin\theta),\ \cos\theta).$$

τ is a one to one map of the rectangle $R: -\pi \le \phi < \pi,\ -\pi \le \theta < \pi$. For the integral of a function x over T,

$$\int_T x\,dA = \int_0^{2\pi}\int_0^{2\pi} x(\tau(\phi,\ \theta))(a + b\sin\theta)\,d\theta\,d\phi. \tag{5.14}$$

Since the function $x \circ \tau$ may be viewed as periodic in both the ϕ variable and in the θ variable, we simply use the trapezoidal rule with respect to each variable to achieve a good approximation. We write the numerical integral as

$$I_m(x) = \frac{4\pi^2}{m^2}\sum_{j=1}^{m}\sum_{i=1}^{m} x(\tau(\phi_j,\ \theta_i)) \tag{5.15}$$

with $\theta_i = (i - 1/2)2\pi/m$ for $i = 1, \ldots, m$, and $\phi_j = (j - 1/2)2\pi/m$ for $j = 1, \ldots, m$.

$I_m(x)$ is a good approximation if x is smooth on T. But for the collocation methods on the torus, the integral operator is the only numerical integral that is calculated. It has a singular kernel. The scheme I_m does not behave well if the integrand has a singularity.

Suppose x is smooth on T except for a singularity at $Q_0 = \tau(0, 0)$. [If the singularity occurs at (ϕ_1, θ_1), we simply parameterize T via the map τ' with $\tau'(\phi, \theta) = \tau(\phi + \phi_0,\, \theta + \theta_0)$. Then the singularity is at $Q_0' = \tau'(0,0)$.] We first divide R into four triangles using the vertices $(0,0)$, $(-\pi,\, -\pi)$, $(-\pi,\, \pi)$, $(\pi,\, -\pi)$, and $(\pi,\, \pi)$. On each triangle, we apply a numerical integration scheme that handles a singularity at $(0,0)$. For example on the triangle with vertices $(0,0)$, $(\pi,\, -\pi)$, and $(\pi,\, \pi)$, which we call Δ_1, we use the change of variables

$$\begin{aligned} \phi &= \pi s \\ \theta &= \pi(2t-1)s. \end{aligned}$$

Others have used this substitution for integrating a singularity at one vertex of a triangle. For example see Fairweather (1979). For a function x defined on T,

$$\begin{aligned} \int_{\Delta_1} x\, dA &= \int_{\Delta_1}\int x(\tau(\phi,\, \theta))(2+\sin\theta)\, d\theta\, d\phi \\ &= 2\pi^2 \int_0^1 \int_0^1 x(\tau(\pi s,\, \pi(2t-1)s))(2+\sin[\pi(2t-1)s])\, s\, ds\, dt. \end{aligned} \tag{5.16}$$

In terms of s and t, the distance $\mid P - Q_0 \mid = O(s)$. Therefore the integrands for the single and double-layer potentials will correspond to smooth integrands by this transformation. Then any standard numerical scheme for integrating over a rectangle may be used to approximate this integral. The integrals over the other three triangles are handled in a similar manner. The sum of these four integrals is the integral over T.

CHAPTER VI

NUMERICAL EXAMPLES

We present a number of numerical examples for various surfaces. We have examples of Galerkin methods when $\partial\Omega$ is the image of a smooth map of the unit sphere U and of collocation methods when $\partial\Omega$ is the image of a smooth map of a torus. We hope to illustrate some of the theoretical results from previous chapters. The examples also show some of the difficulties which may be encountered.

6.1 Example Generation

For our model problem it is required that we produce a function u harmonic in the interior Ω of a region whose boundary is a closed smooth surface $\partial\Omega$. We select a surface of interest $\partial\Omega$ along with a map μ from the surface Θ (the unit sphere or a torus) whose image is $\partial\Omega$. Then we must calculate the inner normal derivative of u on the surface $\partial\Omega$. We then find a way to express that inner normal derivative as a nonlinear function g of the value of u on $\partial\Omega$. Recall the boundary conditions

$$\frac{\partial u}{\partial n} = g(P, u(P)) - f(P).$$

An attempt was made to construct g in a natural way from the true boundary data and not simply choose an arbitrary nonlinear function. For the examples we will indicate how $g(P, x)$ is a function of x without giving the complete definitions of g and f.

During the running of the program, we use the known solution u to calculate the true projection $\mathbf{P}_N \hat{u}^*$. For example, for the Galerkin methods, we calculate the coefficients $(h_k, \hat{u}^*)_U$ for $k = 1, \ldots, d_N$. This true projection serves as the initial guess for the Newton iterations. We also compare the rate of convergence of the sequence of projection solutions with the rate of convergence of the sequence of true projections. Theorem 4 of Chapter 3 says that these rates should be the same.

Since the numerical schemes for integrating on these surfaces reduce to integrating on rectangles, we express the number of nodes used for these double surface integrals as $m \times n$. For simplicity, the Galerkin methods use the same number of nodes for evaluating the integral operators as for the inner products. For the most examples, the number of nodes is kept constant in order that the convergence can be seen to be due to the increase in the degree of approximation. But it is often the case that the same accuracy for the solution could be achieved with fewer node points for the lower degrees and also with fewer nodes for the inner products than for the integral operators. We consider convergence only for the uniform norm.

6.2 The Example Functions

For the first example consider the function

$$u_1(x, y, z) = \frac{z - c_1}{((x - a_1)^2 + (y - b_1)^2 + (z - c_1)^2)^{3/2}}.$$

u_1 is harmonic in any region which does not contain the point (a_1, b_1, c_1). The nonlinear function g in the boundary conditions is

$$g(P, x) = g_1(P)x^{5/3} + g_2(P)x.$$

g has a singularity at $x = 0$. We will examine the effect on our methods of choosing c_1 such that 0 is close to the range of u_1 over Θ.

An interesting harmonic function is given by

$$u_2(x, y, z) = (z - c_1)\sqrt{x - a_1 + \sqrt{(x - a_1)^2 + (y - b_1)^2}}.$$

It is singular on the half plane $y = b_1$ and $x \leq a_1$. First we construct a nonlinear g

with

$$g(P, x) = g_1(P)x + g_2(P)/x.$$

g has a restricted domain in our function space.

The next example uses the function

$$u_3(x, y, z) = \arctan\left(\frac{x + y - a_1}{\sqrt{2}(z - c_1)}\right).$$

This function is singular in the plane $z = c_1$, but harmonic elsewhere. The nonlinearity is

$$g(P, x) = g_1(P)\cos^2(x).$$

The constants, a_1, b_1, and c_1, in the definitions of our example functions will be chosen to avoid singularities being inside the region Ω.

6.3 Ellipsoid

The first surface we consider is an ellipsoid. For simplicity, we assume a map from U to an ellipsoid E may be given by

$$\mu(x,\, y,\, z) = (a\, x,\, b\, y,\, c\, z)$$

with $a,\, b,\, c > 0$. We will denote this ellipsoid as 'ellipsoid (a, b, c).' As the ratio between the maximum and the minimum of a, b, and c becomes greater, the worse the methods will perform.

Consider the example u_1 on the mildly eccentric ellipsoid $(2, \frac{1}{2}, 1)$. (See Figure 1.) We let $a_1 = 0$, $b_1 = 2$, and $c_1 = 2$. We use 16×32 nodes for both the integral operator and the inner product. See Table 1.

This exhibits the expected match between the rate of convergence of the projection solution and the projection of the true solution. Increasing the degree by one seems to be matched by a steady ratio of about 2.2 between the errors. Superconvergence for the Sloan iterate is perhaps indicated. Its error goes from being 50% (for

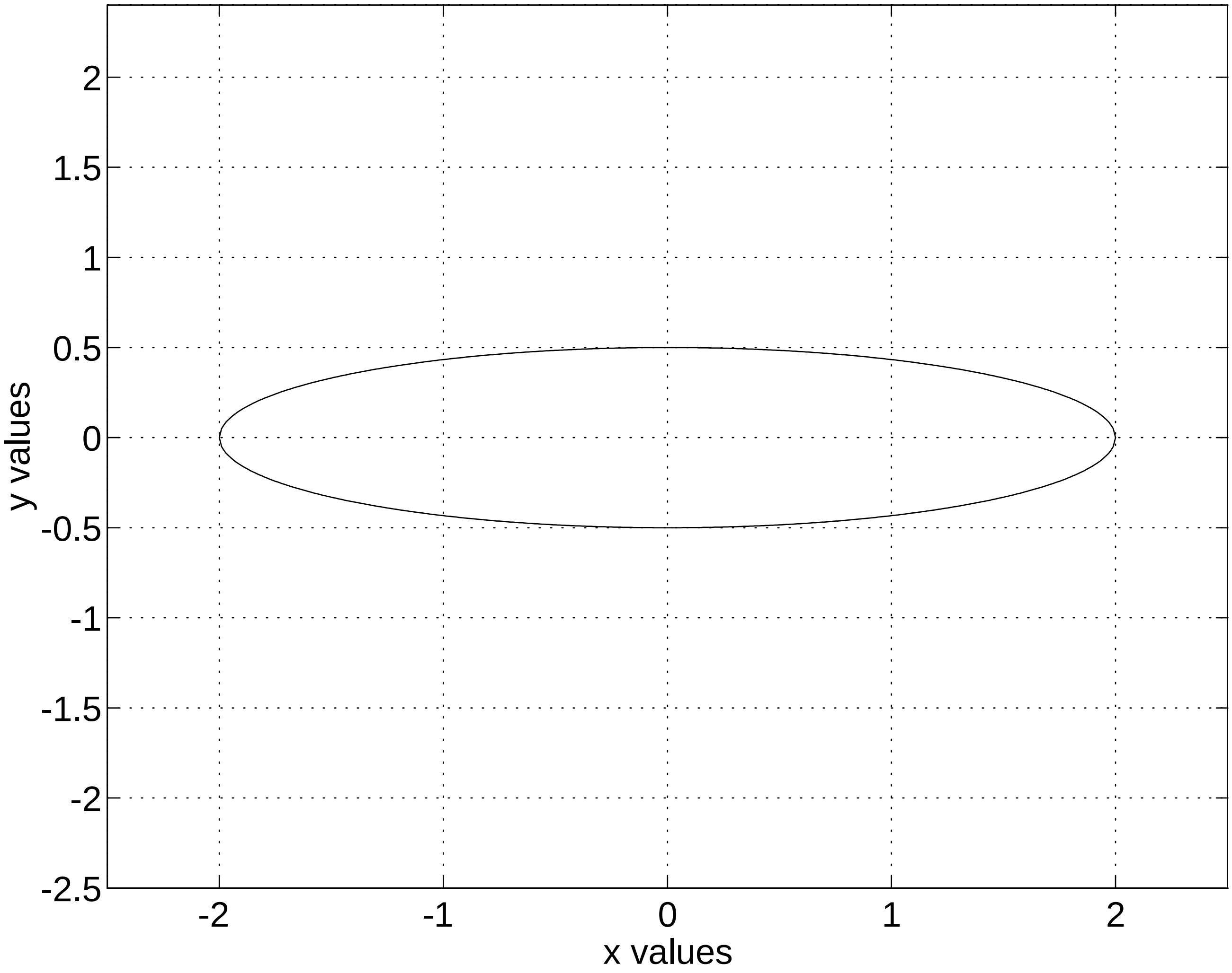

Figure 1: A cross section of the ellipsoid (2,1/2,1).

16×32 Ellipsoid: $(2, \frac{1}{2}, 1)$ $(a_1, b_1, c_1) = (0, 2, 2)$			
Degree	Projection Solution $\parallel \hat{u}_N^{(1)} - \hat{u}^* \parallel_\infty$	Sloan Iterate $\parallel \hat{u}_N^{(3)} - \hat{u}^* \parallel_\infty$	True Projection $\parallel \mathbf{P}_N \hat{u}^* - \hat{u}^* \parallel_\infty$
3	1.03E-02	5.10E-03	1.02E-02
4	5.09E-03	2.21E-03	4.67E-03
5	2.36E-03	9.74E-04	2.15E-03
6	1.07E-03	4.34E-04	9.75E-04
7	4.86E-04	2.51E-04	4.42E-04
8	2.52E-04	2.07E-04	1.99E-04
9	2.13E-04	1.89E-04	8.95E-05
10	2.27E-04	1.97E-04	4.01E-05

Table 1: Example function u_1 on an ellipsoid.

degree 3) to 41% (for degree 6) of the error of the projection solution.

For degrees 7 through 10, the rate of convergence slows down for both solutions. This illustrates what happens when the numerical integration is not accurate enough for a particular solution. This is indicated by the fact that the projection solution's convergence should keep up with the true projection's convergence. Since the Sloan method uses the insufficiently accurate numerical integration to evaluate the iterate, it shows the effect sooner.

For our next example, Ω is again the ellipsoid $(2,\frac{1}{2},1)$. The solution is the example function u_3. We use numerical integration with 32×64 nodes for both the integral operator and the inner product of the Galerkin methods.

At degree 8 a slight slow down in the rate of convergence of the two solutions

Ellipsoid: $(2, \frac{1}{2}, 1)$ $(a_1, c_1) = (6, 2)$				
		Projection Solution	Sloan Iterate	True Projection
Degree	Nodes	$\Vert \hat{u}_N^{(1)} - \hat{u}^* \Vert_\infty$	$\Vert \hat{u}_N^{(3)} - \hat{u}^* \Vert_\infty$	$\Vert \mathbf{P}_N \hat{u}^* - \hat{u}^* \Vert_\infty$
2	16×32	9.53E-03	5.28E-03	8.20E-03
3	32×64	1.85E-03	8.46E-04	1.67E-03
4	32×64	3.54E-04	1.36E-04	3.32E-04
5	32×64	8.10E-05	2.65E-05	7.75E-05
6	32×64	1.84E-05	5.28E-06	1.77E-05
7	32×64	4.11E-06	1.11E-06	3.91E-06
8	32×64	9.59E-07	2.91E-07	9.19E-07
9	32×64	2.77E-07	2.05E-07	1.70E-07
10	32×64	3.28E-07	2.40E-07	4.94E-08

Table 2: Example function u_3 on an ellipsoid.

Ellipsoid: $(2, \frac{1}{2}, 1)$ 16×32 Degree=7 $(a_1, b_1, c_1) = (0, 2, c_1)$			
c_1	Projection Solution $\| (\hat{u}_N^{(1)} - \hat{u}^*)/\hat{u}^* \|_\infty$	Sloan Iterate $\| (\hat{u}_N^{(3)} - \hat{u}^*)/\hat{u}^* \|_\infty$	True Projection $\| (\mathbf{P}_N \hat{u}^* - \hat{u}^*)/\hat{u}^* \|_\infty$
2	7.09E-03	4.22E-03	5.12E-03
1.5	9.11E-03	3.61E-03	8.22E-03
1.3	9.93E-03	3.21E-03	9.34E-03
1.1	1.20E-02	2.89E-03	1.13E-02
1.01	5.43E-02	1.74E-02	5.15E-02

Table 3: Example u_1 on the ellipsoid with $c_1 \to 1$.

takes place. Again, this could be remedied with more accurate numerical integration at the expense of much more computational time. Superconvergence of the Sloan iterate is strongly indicated as its error is 58% of the projection solution error at degree 2 and decreases to 30% of the projection solution error at degree 8.

For example 1, the nonlinear function g has a singularity when $x = 0$. By letting the function parameter c_1 approach 1 from the right, the function u_1 is made to approach 0 for some point on the ellipsoid $(2, \frac{1}{2}, 1)$. For a fixed degree of 7 and numerical integration with 16×32 nodes, we examine the effect on the relative error of the Galerkin solutions as $c_1 \to 1$. See Table 3.

The table shows an increase in the relative error of the projection solution and of the true projection. This may seem to be just a reflection of the smaller values of the true solution, but the absolute error (not shown) was increasing at a similar rate. The effect of the smaller values of the true solution does show up for c_1=1.01. The relative error increases almost five-fold, but the absolute error only increased

32×64 Degree=6 Ellipsoid: $(A,\ 1/A,\ 1)$ $(a_1, b_1, c_1) = (6, 0, 2)$			
A	Projection Solution $\parallel (\hat{u}_N^{(1)} - \hat{u}^*)/\hat{u}^* \parallel_\infty$	Sloan Iterate $\parallel (\hat{u}_N^{(3)} - \hat{u}^*)/\hat{u}^* \parallel_\infty$	True Projection $\parallel (\mathbf{P}_N \hat{u}^* - \hat{u}^*)/\hat{u}^* \parallel_\infty$
2	1.84E-05	5.22E-06	1.80E-05
3	2.35E-03	2.25E-03	1.45E-04
4	1.42E-01	1.39E-01	1.09E-03

Table 4: Example u_3 on the ellipsoid with increasing eccentricity.

about 8%. The relative error in the projection solution closely follows that of the true projection error. This suggests that numerical integration error is not a large factor. Interestingly, the relative error of the Sloan iterate actually decreases for a time as c_1 gets smaller.

The effect of increasing the eccentricity of the ellipsoid may be seen in Table 4. With ellipsoids of the form $(A,\ 1/A,\ 1)$, we increase the eccentricity in the x, y plane by increasing A from 2 to 3 to 4. Again we fix the number of integration nodes and the degree of the projection. The example solution is u_3.

The relative error in the projection solution and the Sloan iterate increase much more rapidly than for the error of the true projection. This strongly suggests that increasing numerical integration error is a major factor.

6.4 Peanut-Shaped Surface

The following smooth map of U may produce a non-convex surface. Consider

$$\mu(x, y, z) = R(x, y, z)(a\,x,\ b\,y,\ c\,z)$$

where given a positive constant α,

$$R(x, y, z) = \sqrt{2z^2 - 1 + \sqrt{\alpha + (2z^2 - 1)^2}}$$

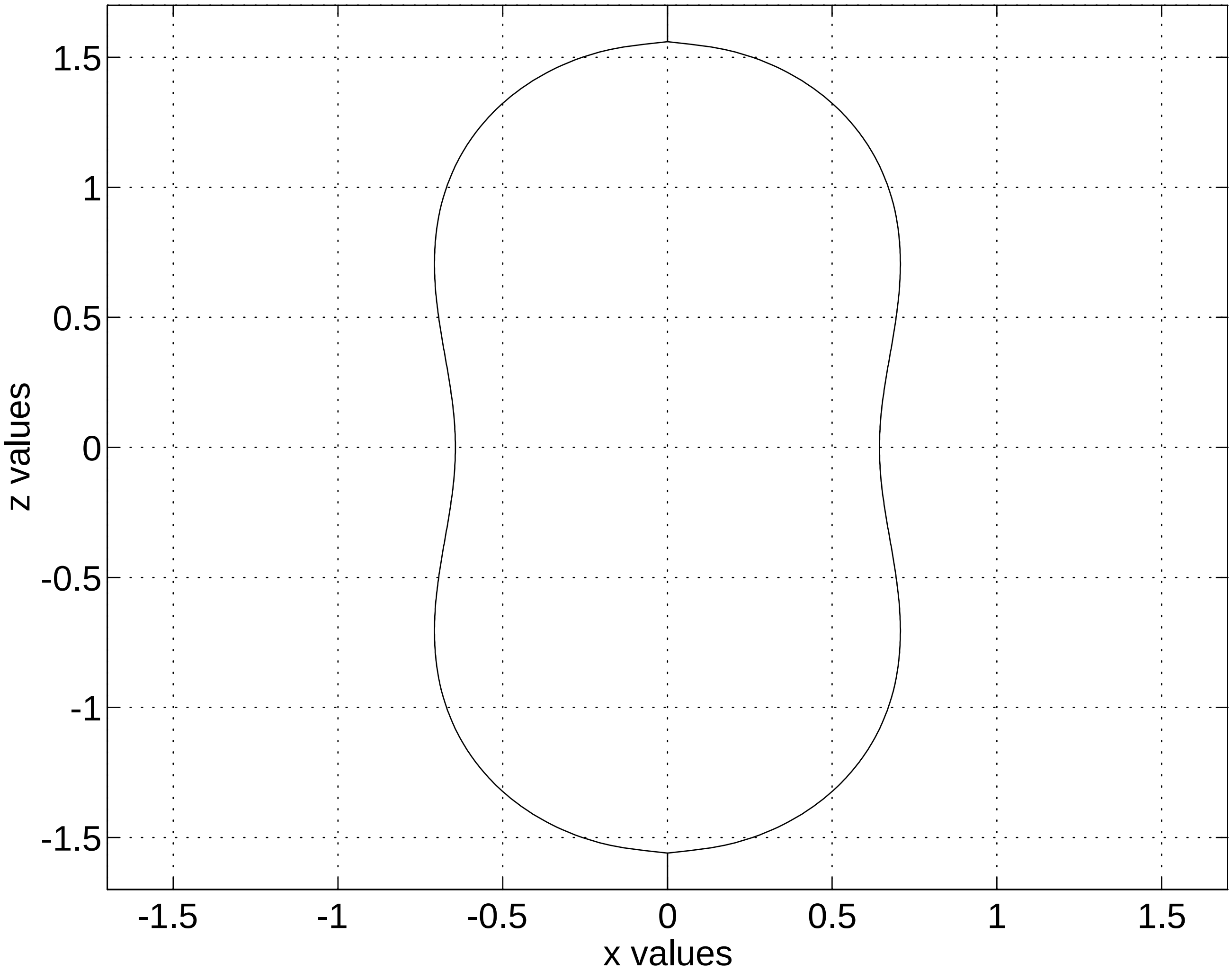

Figure 2: A cross section of the peanut-shaped surface with $\alpha = 1$.

This produces a peanut-shaped region based on the ovals of Cassini. As the parameter $\alpha \to 0$, the surface becomes more constricted along its equator in the plane $z = 0$. For $a = b = c = 1$, Figure 2 shows a cross section for $\alpha = 1$. Figure 3 shows a cross section for $\alpha = 0.2$. Since $a = b$, we get the peanut-shaped surfaces by revolving about the z

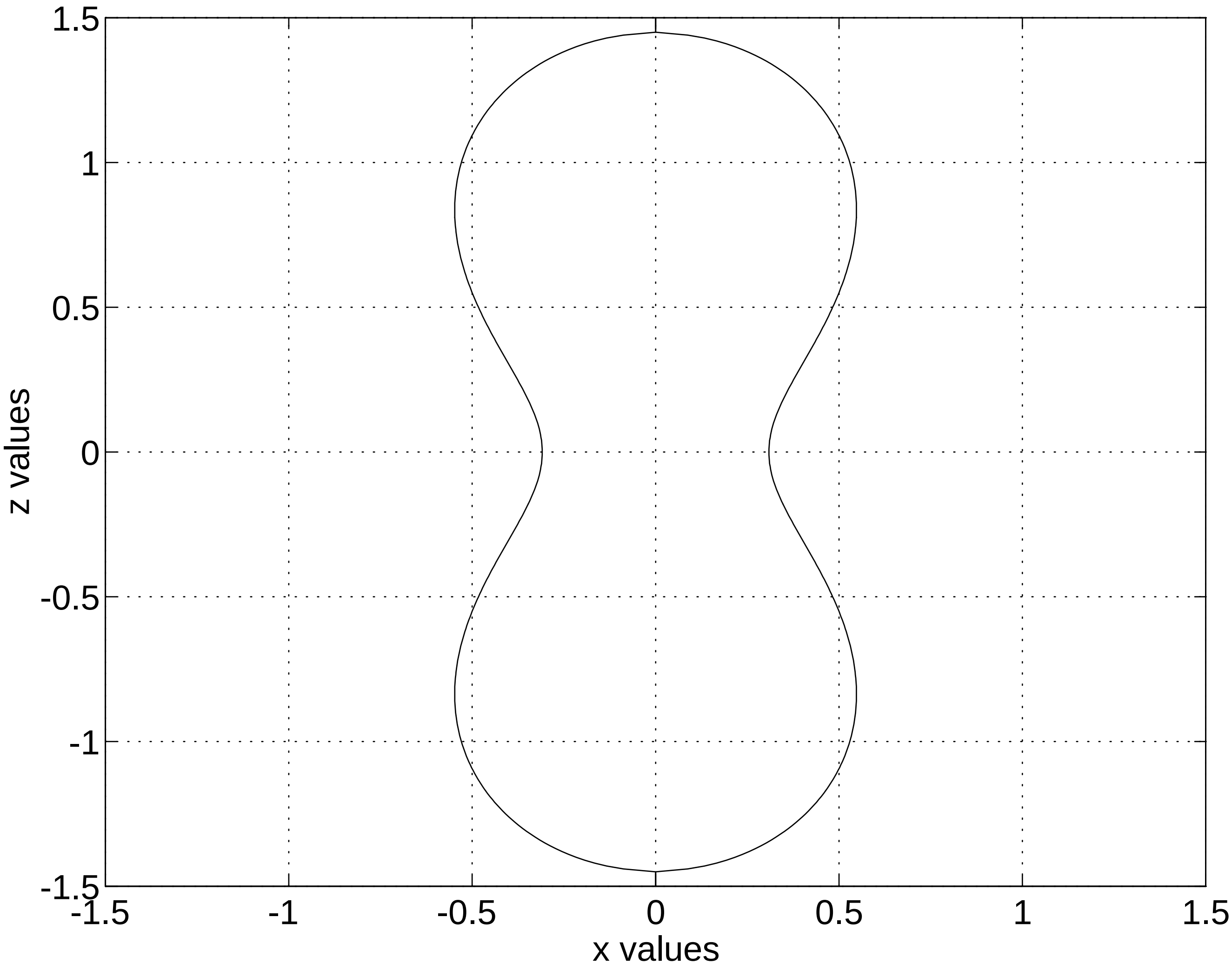

Figure 3: A cross section of the peanut-shaped surface with $\alpha = 0.2$.

axis the cross sectional curves shown.

Results for function u_1 on the peanut-shaped surface are shown in Table 5. Note the very regular rates of convergence exhibited. After degree 3, the projection solution error decreases by a factor of 1.6 or 1.7, the Sloan iterate error decreases by

16×32 Peanut: $(\alpha = 1)$ $(a_1, b_1, c_1) = (0, 2, 2)$			
Degree	Projection Solution $\parallel \hat{u}_N^{(1)} - \hat{u}^* \parallel_\infty$	Sloan Iterate $\parallel \hat{u}_N^{(3)} - \hat{u}^* \parallel_\infty$	True Projection $\parallel \mathbf{P}_N \hat{u}^* - \hat{u}^* \parallel_\infty$
2	9.24E-02	4.86E-02	6.96E-02
3	6.47E-02	3.10E-02	4.92E-02
4	3.99E-02	1.58E-02	3.29E-02
5	2.47E-02	7.87E-03	2.20E-02
6	1.58E-02	4.61E-03	1.46E-02
7	9.89E-03	2.71E-03	9.20E-03
8	6.26E-03	1.71E-03	5.78E-03
9	3.89E-03	1.06E-03	3.57E-03
10	2.33E-03	6.21E-04	2.15E-03

Table 5: Example function u_1 on a peanut-shaped surface.

a factor of 1.6 to 2.0, while the true projection error decreases by a factor of 1.5 to 1.7. Superconvergence of the Sloan iterate is again indicated as the ratio of the Sloan iterate error to the projection solution error is 53% for degree 2 and 27% for degree 10.

An unusual pattern emerges for the example function u_2 on the peanut-shaped surface. From an even to an odd degree the ratio of decrease in error is generally much larger than from an odd to an even degree. In fact, from degree 7 to 8 and from degree 9 to 10 most of the errors increase in magnitude. But from degree 6 to 7 and degree 8 to 9, there is a substantial decrease in error. The projection solution error follows the true projection error quite closely, while the Sloan iterate is superior to

16×32 Peanut: $(\alpha = 1)$ $(a_1, b_1, c_1) = (1, 10, 4)$			
Degree	Projection Solution $\parallel \hat{u}_N^{(1)} - \hat{u}^* \parallel_\infty$	Sloan Iterate $\parallel \hat{u}_N^{(3)} - \hat{u}^* \parallel_\infty$	True Projection $\parallel \mathbf{P}_N \hat{u}^* - \hat{u}^* \parallel_\infty$
2	1.90E+00	9.51E-01	1.34E+00
3	1.75E-01	4.58E-02	1.59E-01
4	1.26E-01	3.62E-02	1.11E-01
5	5.48E-02	8.69E-03	5.38E-02
6	4.46E-02	6.74E-03	4.47E-02
7	6.27E-03	1.22E-03	6.45E-03
8	6.50E-03	1.20E-03	6.70E-03
9	3.75E-03	7.46E-04	4.10E-03
10	3.82E-03	7.68E-04	3.42E-03

Table 6: Example function u_2 on a peanut-shaped surface.

both. There is not strong evidence of superconvergence, however.

Our final peanut example is u_3. It gives evidence of superconvergence of the Sloan iterate. The ratio of the error for the Sloan iterate to the error for the projection solution is 32% for degree 3 and decreases to 18% for degree 10.

6.5 The Torus

For examples on the torus we use collocation methods based on double Fourier series. For our first group of examples, we use a standard torus without distortion. In the next section we will consider what we will call limaçon tori. For the collocation methods, we compare the Kumar solution with the projection solution. We do not consider the Sloan iterate.

16×32 Peanut: $(\alpha = 1)$ $(a_1, c_1) = (3, 2)$			
Degree	Projection Solution $\| \hat{u}_N^{(1)} - \hat{u}^* \|_\infty$	Sloan Iterate $\| \hat{u}_N^{(3)} - \hat{u}^* \|_\infty$	True Projection $\| \mathbf{P}_N \hat{u}^* - \hat{u}^* \|_\infty$
2	2.41E-01	1.50E-01	1.64E-01
3	7.23E-02	2.29E-02	6.99E-02
4	3.46E-02	1.23E-02	2.95E-02
5	1.94E-02	6.79E-03	1.69E-02
6	1.06E-02	3.06E-03	1.05E-02
7	5.08E-03	1.33E-03	4.68E-03
8	2.91E-03	7.50E-04	2.55E-03
9	1.07E-03	2.60E-04	1.08E-03
10	5.78E-04	1.05E-04	6.12E-04

Table 7: Example function u_3 on a peanut-shaped surface.

$16 \times 64 \quad (a_1, b_1, c_1) = (0, 0, 2)$			
Degree	Projection Solution $\parallel \hat{u}_N^{(1)} - \hat{u}^* \parallel_\infty$	Kumar Solution $\parallel \hat{u}_N^{(3)} - \hat{u}^* \parallel_\infty$	True Projection $\parallel \mathbf{P}_N \hat{u}^* - \hat{u}^* \parallel_\infty$
2	5.52E-02	5.04E-02	3.31E-02
3	2.19E-02	2.56E-02	1.58E-02
4	8.11E-03	1.03E-02	4.97E-03
5	2.58E-03	2.68E-03	1.71E-03
6	8.61E-04	6.43E-04	8.15E-04
7	3.45E-04	2.90E-04	3.31E-04
8	8.05E-05	6.87E-05	8.63E-05

Table 8: Example u_1 on the torus.

Consider the example u_1 on standard torus. See Table 8. The ratio between the projection solution error and the Kumar solution error has a generally decreasing trend. This may suggest superconvergence. Suprisingly, there also appears to be superconvergence of both solutions with respect to the true projection. At least numerical integration error is no problem up to degree 8.

The next example shows that numerical integration error can be a problem. Although 16×64 nodes was fine for the last example, here it is apparent by degree 3 or 4 that the accuracy is not enough. The rate of decrease in the error of the solutions is not keeping up with the rate of decrease of the error for the true projection. Therefore we finish the table with an increased number of nodes. Now the rate of decrease for the two solutions' error and the true projection error is about the same. No superconvergence is indicated. The Kumar solution is actually a little bit worse

$(a_1, b_1, c_1) = (1, 10, 4)$				
Degree	Nodes	Projection Solution $\parallel \hat{u}_N^{(1)} - \hat{u}^* \parallel_\infty$	Kumar Solution $\parallel \hat{u}_N^{(3)} - \hat{u}^* \parallel_\infty$	True Projection $\parallel \mathbf{P}_N \hat{u}^* - \hat{u}^* \parallel_\infty$
2	16×64	9.62E-02	9.33E-02	6.53E-02
3	16×64	3.25E-02	3.54E-02	1.02E-02
4	16×64	3.01E-02	3.06E-02	2.91E-03
5	16×64	2.88E-02	2.89E-02	5.35E-04
4	32×128	3.04E-03	3.39E-03	2.91E-03
5	32×128	5.40E-04	6.38E-04	5.35E-04
6	32×128	1.50E-04	1.63E-04	1.46E-04
7	32×128	3.30E-05	3.52E-05	3.18E-05
8	32×128	9.53E-06	1.02E-05	8.38E-06

Table 9: Example u_2 on the torus.

than the projection solution.

Solving for example function u_3 gives quite similar results to the last example. See Table 10. Again, we finish the table after increasing the accuracy of the numerical integration. Also the Kumar solution is again somewhat less accurate and shows no superconvergence.

We once more examine the problem of the nonlinear function g having a singularity by using example function u_1. As $c_1 \to 1$, $u \to 0$ around the top of the torus. The results are similar to those above for u_1 on the ellipsoid. (See Table 3 above.) This time the Kumar solution seems to be less affected than the projection solution. This behavior is similar to that of the Sloan iterate in Table 3.

$(a_1, b_1, c_1) = (4.5, 0, 11)$				
Degree	Nodes	Projection Solution $\parallel \hat{u}_N^{(1)} - \hat{u}^* \parallel_\infty$	Kumar Solution $\parallel \hat{u}_N^{(3)} - \hat{u}^* \parallel_\infty$	True Projection $\parallel \mathbf{P}_N \hat{u}^* - \hat{u}^* \parallel_\infty$
2	16×64	2.64E-03	3.31E-03	1.83E-03
3	16×64	8.25E-04	8.43E-04	2.49E-04
4	16×64	5.87E-04	5.87E-04	8.36E-06
5	16×64	5.87E-04	5.88E-04	3.05E-06
5	32×128	3.09E-06	3.22E-06	3.05E-06
6	32×128	1.79E-07	1.76E-07	1.58E-07
7	32×128	5.27E-08	5.37E-08	3.26E-08

Table 10: Example u_3 on the torus.

16×64 Degree=6 $(a_1, b_1, c_1) = (0, 0, c_1)$			
c_1	Projection Solution $\parallel (\hat{u}_N^{(1)} - \hat{u}^*)/\hat{u}^* \parallel_\infty$	Kumar Solution $\parallel (\hat{u}_N^{(3)} - \hat{u}^*)/\hat{u}^* \parallel_\infty$	True Projection $\parallel (\mathbf{P}_N \hat{u}^* - \hat{u}^*)/\hat{u}^* \parallel_\infty$
2	9.35E-03	7.75E-03	8.44E-03
1.5	1.91E-02	1.66E-02	1.43E-02
1.2	1.21E-01	7.47E-02	1.11E-01
1.1	3.18E-01	1.31E-01	3.28E-01
1.02	2.69E+00	4.55E-01	2.04E+00

Table 11: Example u_1 on the torus as $c_1 \to 1$.

32×128 Degree=5 $(a_1, b_1, c_1) = (1, b_1, 4)$			
b_1	Projection Solution $\| (\hat{u}_N^{(1)} - \hat{u}^*)/\hat{u}^* \|_\infty$	Kumar Solution $\| (\hat{u}_N^{(3)} - \hat{u}^*)/\hat{u}^* \|_\infty$	True Projection $\| (\mathbf{P}_N \hat{u}^* - \hat{u}^*)/\hat{u}^* \|_\infty$
10	5.65E-05	6.76E-05	5.60E-05
6	2.38E-03	3.27E-03	2.17E-03
4	5.91E-02	8.95E-02	5.34E-02
3.5	*	9.20E-01	2.16E-01

Table 12: Example u_2 on the torus as $b_1 \to 3$.

We use u_2 to look at the effect of having a singularity of the harmonic function near the surface. u_2 has the singularity on the plane $y = b_1$ and $x \leq a_1$. The maximum of y on our torus is 3. So we let $b_1 \to 3$ from the right (Table 12). The degree is fixed at 5 and the number of integration nodes at 32×128. In the table, the '*' indicates that the method did not converge within 10 iterations. The relative error of the collocation solutions increase at about the same rate as that of the true projection.

6.6 The Limaçon Torus

Consider the surface parameterized by

$$\begin{aligned} x &= \cos\phi(3 + B\sin\theta + \sin^2\theta) \\ y &= \sin\phi(3 + B\sin\theta + \sin^2\theta) \qquad (6.17) \\ z &= B\cos\theta + \sin\theta\cos\theta. \end{aligned}$$

This surface has the topology of a torus but the cross section given by a plane through the z axis is a pair of limaçons. Limaçons are usually studied in the context of polar coordinates. If $1 < B < 2$, the cross section is a limaçon with a dimple but the surface is smooth. For $B < 1$ the surface is no longer a simple closed surface. If

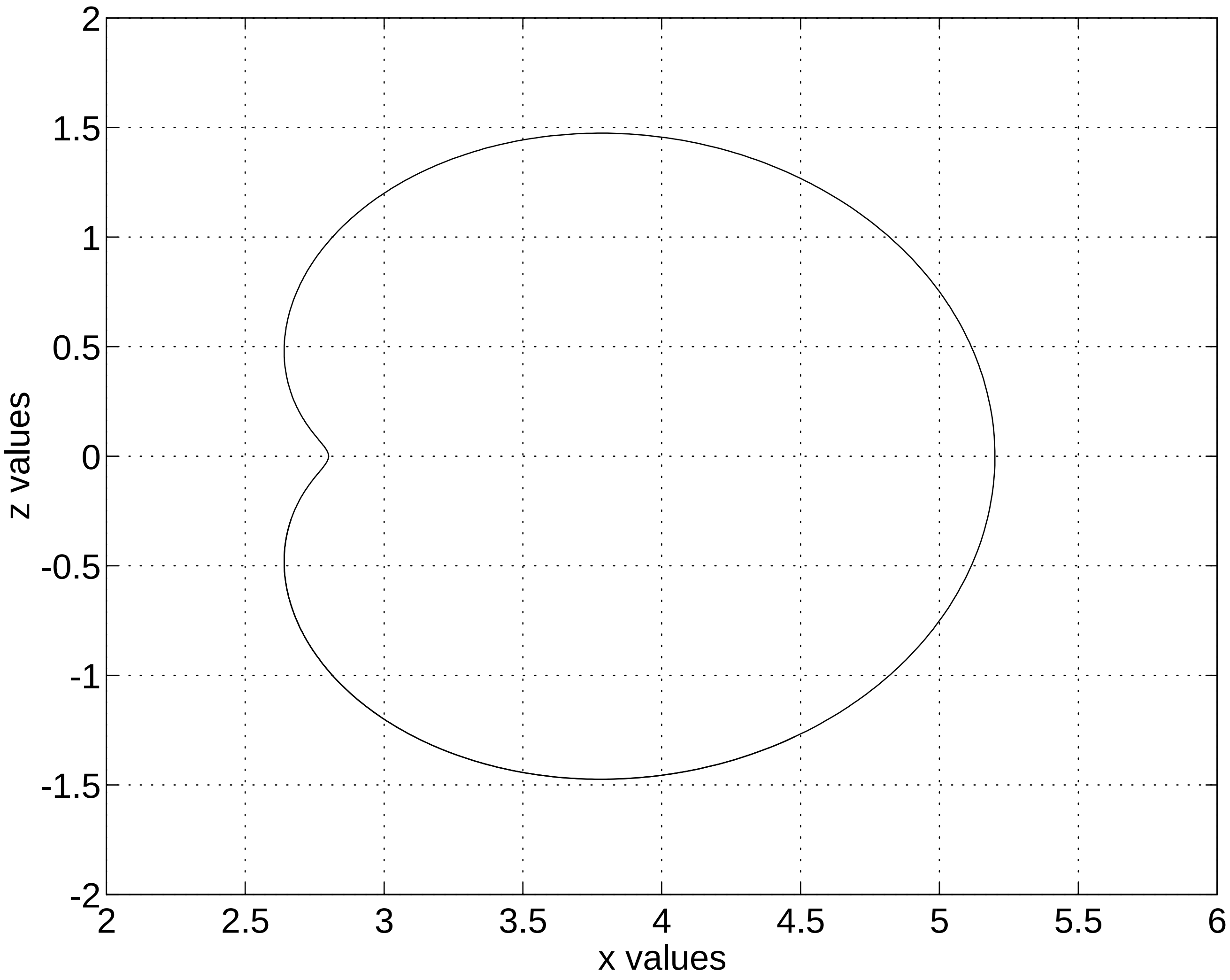

Figure 4: A cross section of the limaçon torus with B = 1.5.

$B = 1$ there is a singularity. The surface has a crease when $\theta = 3\pi/2$ and the surface is not Lyapunov. For $B \geq 2$, the indentation disappears. For our examples we use $1 < B < 2$. See Figure 4 for the shape of this cross section for $B = 1.5$ and Figure 5 for $B = 1.2$.

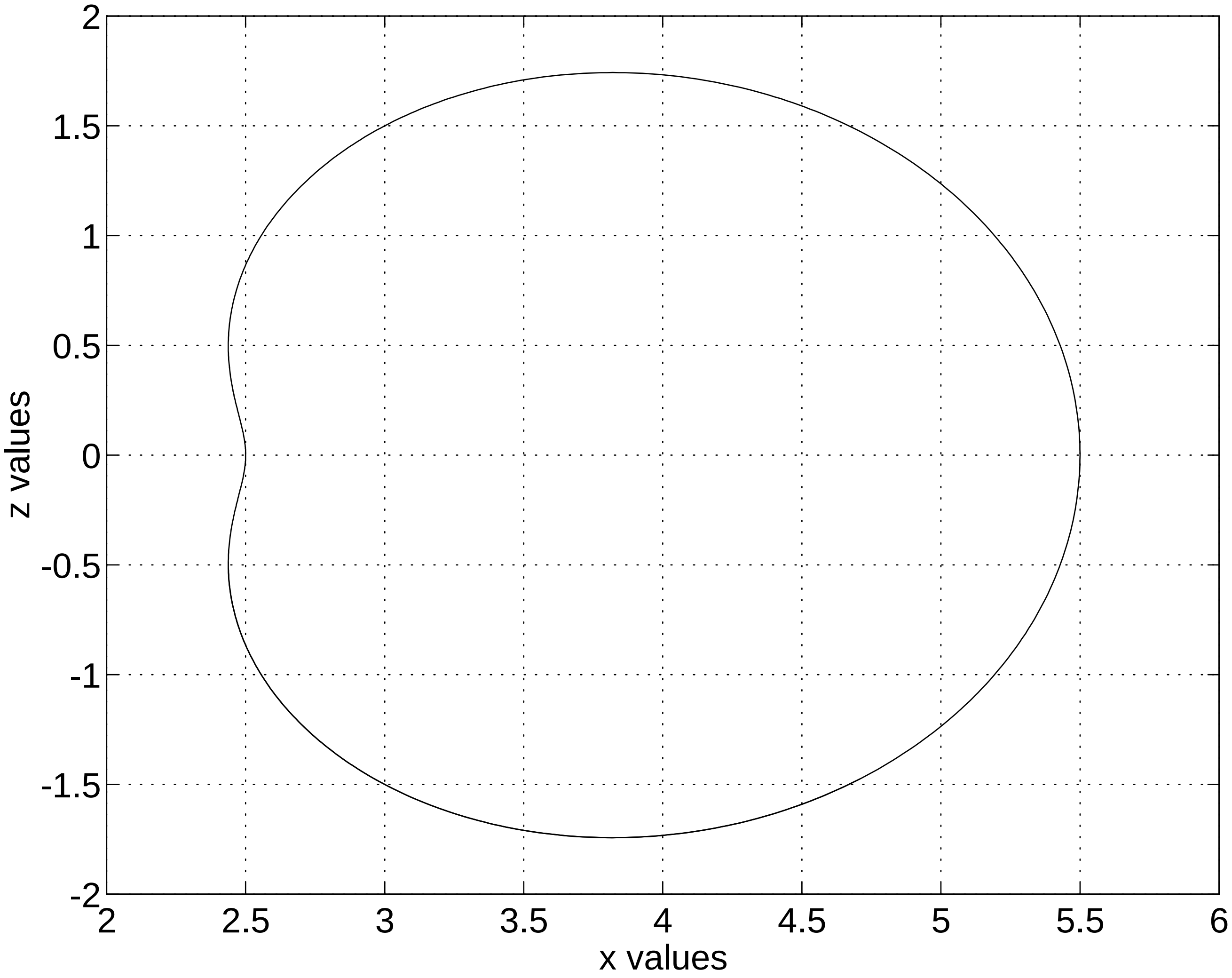

Figure 5: A cross section of the limaçon torus with B = 1.2.

B=1.5 $(a_1, b_1, c_1) = (0, 3, 3)$				
Degree	Nodes	Projection Solution $\parallel \hat{u}_N^{(1)} - \hat{u}^* \parallel_\infty$	Kumar Solution $\parallel \hat{u}_N^{(3)} - \hat{u}^* \parallel_\infty$	True Projection $\parallel \mathbf{P}_N \hat{u}^* - \hat{u}^* \parallel_\infty$
2	16×64	1.39E-03	5.19E-03	5.24E-04
3	16×64	2.89E-04	1.86E-03	1.36E-04
4	16×64	1.01E-04	4.84E-04	3.92E-05
5	16×64	1.74E-05	2.02E-04	8.30E-06
6	16×64	7.48E-06	9.23E-05	2.34E-06
7	32×128	7.21E-07	3.99E-05	7.82E-07
8	32×128	2.49E-07	2.65E-05	2.54E-07

Table 13: Example u_1 on the limaçon torus.

Let $\tau(\phi, \theta)$ be the function which parameterizes the standard torus on the rectangle $R : 1 \leq \phi < 2\pi,\ 0 \leq \theta < 2\pi$, as in Chapter 4. Call the parameterization given above (6.17) for the limaçon torus τ_l. Then the map $\mu = \tau_l \circ \tau^{-1}$, gives us the required map from the torus to the limaçon torus.

With the example function u_1 on the limaçon torus, we have an example in which the Kumar solution performs much worse than the projection solution. Its error starts in degree 2 as about 4 times as bad as that of the projection solution until in degree 8 it is 106 times as bad. For degree 7 and 8 we increase the number of nodes.

Results for u_2 on the limaçon torus is shown in Table 14. Note that the Kumar solution and the projection solution have about the same rate of convergence as the true projections.

B=1.5 $(a_1, b_1, c_1) = (1, 10, 4)$				
Degree	Nodes	Projection Solution $\| \hat{u}_N^{(1)} - \hat{u}^* \|_\infty$	Kumar Solution $\| \hat{u}_N^{(3)} - \hat{u}^* \|_\infty$	True Projection $\| \mathbf{P}_N \hat{u}^* - \hat{u}^* \|_\infty$
2	16×64	5.45E-01	7.31E-01	5.09E-01
3	16×64	1.38E-01	1.83E-01	1.21E-01
4	16×64	7.71E-02	9.96E-02	7.48E-02
5	16×64	2.81E-02	3.32E-02	2.63E-02
6	16×64	2.12E-02	2.48E-02	1.39E-02
7	16×64	1.54E-02	1.56E-02	5.70E-03
6	32×128	1.47E-02	1.65E-02	1.39E-02
7	32×128	6.03E-03	8.31E-03	5.70E-03

Table 14: Example u_2 on the limaçon torus.

For u_3, the Kumar solution behaves much poorer than the projection solution. (Table 15). This is similar to the results above for u_1.

As discussed above, if the parameter B in the map defining the limaçon torus is 1, the resulting surface is not Lyapunov. The next example looks at letting $B \to 1$. We use the example function u_3 fixing the degree and number of integration nodes. See Table 16.

The relative error in the true projection actually decreases somewhat. On the other hand, the relative errors for the collocation solutions get much worse. This indicates that the loss of accuracy for the numerical integration is a large factor.

B=1.5 $(a_1, b_1, c_1) = (9, 0, 11)$				
Degree	Nodes	Projection Solution $\parallel \hat{u}_N^{(1)} - \hat{u}^* \parallel_\infty$	Kumar Solution $\parallel \hat{u}_N^{(3)} - \hat{u}^* \parallel_\infty$	True Projection $\parallel \mathbf{P}_N \hat{u}^* - \hat{u}^* \parallel_\infty$
2	16×64	7.64E-02	2.98E-01	1.37E-02
3	16×64	5.84E-03	4.81E-03	4.11E-03
4	16×64	1.93E-03	2.13E-03	5.33E-04
5	16×64	1.42E-03	1.55E-03	1.48E-04
6	16×64	1.42E-03	1.51E-03	3.53E-05
3	32×128	4.57E-03	3.55E-03	4.11E-03
4	32×128	6.60E-04	8.62E-04	5.33E-04
5	32×128	1.45E-04	2.91E-04	1.48E-04
6	32×128	3.46E-05	1.62E-04	3.53E-05
7	32×128	8.19E-06	6.96E-05	8.26E-06
8	32×128	1.51E-06	4.43E-05	1.38E-06

Table 15: Example u_3 on the limaçon torus.

32×128 Degree=5 $(a_1, c_1) = (9, 11)$			
B	Projection Solution $\parallel (\hat{u}_N^{(1)} - \hat{u}^*)/\hat{u}^* \parallel_\infty$	Kumar Solution $\parallel (\hat{u}_N^{(3)} - \hat{u}^*)/\hat{u}^* \parallel_\infty$	True Projection $\parallel (\mathbf{P}_N \hat{u}^* - \hat{u}^*)/\hat{u}^* \parallel_\infty$
1.5	6.87E-04	1.61E-03	6.80E-04
1.3	9.40E-04	3.90E-03	6.09E-04
1.1	9.68E-02	1.10E-01	5.29E-04
1.02	5.54E-01	6.16E-01	4.97E-04

Table 16: Example u_3 on the limaçon torus as $B \to 1$.

6.7 Conclusions

The rate of convergence of the projection solution always closely matched that of the true projections. For the Galerkin methods the Sloan iterate almost always showed some evidence of superconvergence. But for the collocation examples, the Kumar method displayed no superconvergence and often had a worse rate of convergence.

Maintaining sufficient accuracy for the numerical integrals is a problem as higher and higher degrees of projections are used.

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