PIECEWISE POLYNOMIAL COLLOCATION FOR INTEGRAL EQUATIONS ON SURFACES IN THREE DIMENSIONS

by

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<u>An Abstract</u>

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ABSTRACT

We consider solving integral equations on a piecewise smooth surface S in $\mathbf{R}^3$ with a smooth kernel using collocation with isoparametric piecewise polynomial interpolation. Symbolically, we write the integral equation as $(\lambda - \mathcal{K})f = g$. We approximate both the surface S and the unknown solution f by using such interpolation.

P_n is the bounded projection such that $P_n f$ is the piecewise quadratic interpolant of f. For the collocation method,

$$\|f - f_n\|_\infty \le B\,\|f - P_n f\|_\infty$$

for an appropriate B and all sufficiently large n. We can define $\mathcal{K}_n f$ by using piecewise quadratic interpolation of f and the surface S; and by the theory of the Nyström method, we obtain a solution $\tilde{f}_n$, $(\lambda - \mathcal{K}_n)\tilde{f}_n = g$.

The result of the two methods coincides at the node points, and for the Nyström method,

$$\|f - \tilde{f}_n\|_\infty \le B\,\|\mathcal{K}f - \mathcal{K}_n f\|_\infty.$$

We have proved that $\|\mathcal{K}f - \mathcal{K}_n f\|_\infty = O(\widehat{\delta}^4)$ where $\widehat{\delta}$ is the mesh size of the triangulation, and this is faster than $\|f - P_n f\|_\infty$ which is $O(\widehat{\delta}^3)$. The techniques we use generalize to higher degree interpolation of f and S.

Next we consider the boundary integral equation for the Laplace's equation in $\mathbf{R}^3$. For example, the interior Dirichlet problem is to find u such that

$$\Delta u(A) = 0 \quad A \in D; \qquad u(P) = f(P), \quad P \in S\,.$$

We assume u can be represented as a double layer potential:

$$u(A) = \int_S \rho(Q) \frac{\partial}{\partial \nu_Q} [\frac{1}{\mid P - Q \mid}]\, dS_Q\,, \qquad A \in D\,.$$

The density function ρ is determined from the integral equation

$$2\pi\rho(P) + \int_S \rho(Q) \frac{\partial}{\partial \nu_Q} [\frac{1}{\mid P - Q \mid}]\, dS_Q + [2\pi - \Omega(P)]\rho(P) = f(P), \quad P \in S\,,$$

where ν_Q denotes the interior unit normal to S at Q, and $\Omega(P)$ is the inner solid angle of S at $P \in S$. We have investigated theoretically the effect of the numerical integration errors and surface approximation errors that occur in the evaluation of single layer integrals and $\Omega(P)$. The same procedures will be extended in future work to the numerical solution of the full integral equation by collocation.

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CERTIFICATE OF APPROVAL

PH.D. THESIS

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CHAPTER I

INTRODUCTION

Consider the numerical solution of integral equations of the second kind on piecewise smooth surfaces in $\mathbf{R}^3$. Many mathematical problems in physics and engineering can be solved by means of integral equations. For instance, problems in fluid mechanics, hydraulics, geomechanics, and plate bending have been studied through the use of boundary integral equations.

In general, most work has been for the two-dimensional problem; see Brebbia[13, 14, 15, 16, 17]. All aspects of the three-dimensional problem are less developed than for the two-dimensional problem. Several methods have been used to study the three-dimensional problem (see Atkinson[8]), and most numerical methods can be considered to be of collocation or Galerkin type. Atkinson[2, 6, 7] and Wendland[33, 36] used the collocation method, and Costabel and Stephan[22], Ervin and Stephan[23, 24], Giroire and Nedelec[25], and Wendland[35] studied Galerkin's method. Also, Nedelec[29] used a variational formulation and Galerkin's method to study a special case of the first-kind equation. When solving the boundary integral equations, one has to solve a linear system with a dense matrix. Atkinson[3] resolves this problem by using an iterative method; but work is only beginning on the development of iterative methods.

For the collocation method, it is common to use polynomials of degree k to approximate surfaces and polynomials of degree d to approximate functions. Wendland[33] states that for smooth boundaries, the error of the numerical solution is of order n, where n is the minimum of d+1 and k+1. Most practical work has used piecewise constant collocation with piecewise linear approximation of the surface.

In this thesis, we investigate numerical methods for the solution of Fredholm integral equations of the second kind on surfaces in $\mathbf{R}^3$. In Chapter II, we give terminology, notation, and results that are required in defining and solving problems that are presented in the succeeding chapters. Chapter III studies the collocation method for solving the integral equations on a piecewise smooth surface with a smooth kernel function. We also examine the error at nodal points for the collocation method. Following that, we discuss the computation of the single layer integral and the solid angle at points on the surface. In Chapter IV, we introduce the discrete collocation method for solving the integral equations, and we analyze the rate of convergence of the method. Finally, we evaluate the single layer integral and the solid angle numerically, and give an error analysis.

Because of the heavy computation involved in this thesis, we used the symbolic mathematics program Maple to aid in studying the rate of convergence of the numerical method. This thesis only presents the work of using polynomials of degree two to approximate both of the surface and the solution. We also used Maple to do error analysis for other degrees of interpolation of the solution and the surface, and the results are consistent with the kind of results we have obtained for the quadratic case.

We give a preliminary result here, and rigorous definitions and proofs will be given in Chapter III and IV. Consider the integral equation

$$\lambda f(P) - \int_S k(P,Q)\, f(Q)\, dS_Q = g(P), \qquad P \in S$$

where S is a piecewise smooth surface in $\mathbf{R}^3$. Using collocation with isoparametric piecewise quadratic interpolation for both surface S and the unknown function f, we obtain that the error of the numerical solution at node points is $O(\widehat{\delta}^4)$, where $\widehat{\delta}$ is the mesh size of the triangulation. For generalization involving other degrees of polynomial interpolation, see Sections 3.7 and 4.7.

CHAPTER II

PRELIMINARIES

2.1 Introduction

This chapter gives terminology, notation, and results that are required in defining and solving problems that are presented in the succeeding chapters. The presentation is divided into four sections. The first section states the boundary integral equation (BIE) method for Laplace's equation in $\mathbf{R}^3$, with two specific problems that motivate our studies. These are

1. The interior Dirichlet problem, using an indirect BIE formulation.

2. The exterior Neumann problem, using a direct BIE formulation.

The second section gives the definitions, notation and assumptions on the triangulation of the surface S. The third section describes how to interpolate a given function. The last section of this chapter gives general results that are associated with the collocation method and the Nyström method.

2.2 The two BIE problems

Two problems for Laplace's equation and associated boundary integral equations were studied.

P1 The interior Dirichlet problem. Let D be a bounded, open, simply connected region in $\mathbf{R}^3$, and let its boundary S be piecewise smooth. The problem is to find $u \in C(\overline{D}) \cap C^2(D)$ such that

$$\Delta u(A) = 0\,, \qquad A \in D$$

$$u(P) = f(P), \qquad P \in S\,.$$

We assume u can be represented as a double layer potential:

$$u(A) = \int_S \rho(Q) \frac{\partial}{\partial \nu_Q} \left[\frac{1}{\mid A - Q \mid} \right] dS_Q\,, \qquad A \,\in\, D\,. \tag{2.1}$$

The density function ρ is determined from the integral equation

$$2\pi\rho(P) + \int_S \rho(Q) \frac{\partial}{\partial \nu_Q} \left[\frac{1}{\mid P - Q \mid} \right] dS_Q + [2\pi - \Omega(P)]\,\rho(P) = f(P),\ hspace3exP \in S \tag{2.2}$$

in which ν_Q denotes the unit normal to S at Q (if it exists), pointing into D. $\Omega(P)$ is the inner solid angle of S at $P \in S$; and we assume $0 < \Omega(P) < 4\pi$ (see Mikhlin[30, pp. 349–357]).

Symbolically, we write the integral equation (2.2) as

$$(2\pi + \mathcal{K})\rho = f.$$

Under suitable additional assumptions on S,

$$\mathcal{K} : C(S) \longrightarrow C(S)$$

is a bounded linear operator; see Wendland[36].

P2 The exterior Neumann problem. Let D and S be the same as for P1, and let $D_e = \mathbf{R}^3 \setminus \overline{D}$, the region exterior to D and S. The problem is to find $u \in C(\overline{D_e}) \bigcap C^2(D_e)$ such that

$$\Delta u(A) = 0\,, \qquad A \in D_e$$

$$\frac{\partial u(P)}{\partial \nu_P} = f(P)\,, \qquad P \in S$$

$$u(P) = O(\mid P \mid^{-1}), \quad \mid \nabla u(P) \mid = O(\mid P \mid^{-2}) \quad as \ \mid P \mid \rightarrow \infty.$$

It can be shown that such a function u exists (under suitable assumption on S and f) and that Green's third identity can be applied to u:

$$4\pi u(A) = \int_S f(Q) \frac{1}{\mid A - Q \mid} dS_Q - \int_S u(Q) \frac{\partial}{\partial \nu_Q} \left[\frac{1}{\mid A - Q \mid} \right] dS_Q\,, \qquad A \in D_e\,. \tag{2.3}$$

To find u on S, we solve the integral equation

$$2\pi u(P) + \int_S u(Q) \frac{\partial}{\partial \nu_Q} \left[\frac{1}{\mid P - Q \mid} \right] dS_Q + [2\pi - \Omega(P)]\, u(P)$$

$$= \int_S f(Q) \frac{1}{\mid P - Q \mid} dS_Q, \qquad P \in S. \tag{2.4}$$

Then (2.3) gives u on D_e.

The integral equations (2.2) and (2.4) are different only in their right hand inhomogeneous terms.

2.3 The triangulation of S

In this section, we describe the triangulation of the surface S and discuss its refinement to a finer mesh. As discussed in Atkinson[6], we assume

$$S = S_1 \cup S_2 \cup \cdots \cup S_J \tag{2.5}$$

where each S_i is a closed, "smooth" surface in $\mathbf{R}^3$. The only possible intersection of a pair S_i and S_j is to be along a common portion of the edges of these two sub-surfaces. We also assume each S_i has a parametrization in a region of $\mathbf{R}^2$, with the parametrization six times continuously differentiable. In this case, we say S is *piecewise smooth*. By a *smooth surface*, we mean that for each point $P \in S$, there is a neighborhood on S of P, with the neighborhood having a local six times continuously differentiable parametrization in $\mathbf{R}^2$ with its Jacobian determinant not vanishing.

The surface S of (2.5) is then divided into a triangular mesh

$$\{\triangle_{K,N} \mid 1 \le K \le N\} \tag{2.6}$$

for a sequence $N = N_1, N_2, \ldots$. Each S_j is to be broken apart into a set of nonoverlapping triangular shaped elements $\triangle_{K,N_j}$'s. In referring to the element $\triangle_{K,N}$, the reference to N will be omitted, but understood implicitly. Define the mesh size of (2.6) by

$$\delta_N = \max_{1 \le K \le N} \, diam(\triangle_K),$$

$$diam(\triangle_K) = \max_{p,q \in \triangle_K} \mid p - q \mid .$$

Let σ denote the unit simplex in the $st-plane$

$$\sigma = \{(s,t) \mid 0 \leq s, t, s+t \leq 1\} \ ,$$

Let $\rho_1, \ldots, \rho_6$ denote the three vertices and three midpoints of the sides of σ, numbered according to Figure 2.1.

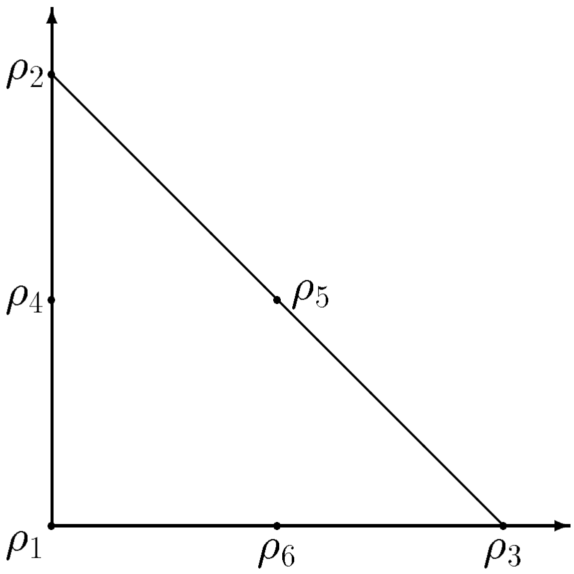

Figure 2.1: The unit simplex

One way of obtaining the triangulation (2.6) and the mappings from σ to each $\triangle_K$ is by means of a parametric representation for the region S_j of (2.5). Assume that for each S_j, there is a mapping

$$F_j : R_j \xrightarrow[\text{onto}]{1\text{–}1} S_j \ , \qquad 1 \leq j \leq J \ , \tag{2.7}$$

where R_j is a polygonal domain in the plane and $F_j \in C^6(R_j)$. Then triangulations of R_j map onto triangulations of S_j . Since the R_j's are polygonal domains and can be written as a union of triangles, without loss of generality, we assume that the R_j's are triangles in this thesis. A paraboloid with top is a good example of an S for the assumption (2.7); but a circular cone is an example of an S for which the assumption (2.7) is not valid, because of the discontinuity of the gradient at the vertex. Let $\widehat{\triangle}_K$ be an element in the triangulation of R_j, and let $\widehat{v}_1$, $\widehat{v}_2$, and $\widehat{v}_3$ be its vertices. Define

$$m_K(s,t) = F_j(u\widehat{v}_1 + t\widehat{v}_2 + s\widehat{v}_3), \qquad u = 1 - s - t\,, \qquad (s,t) \in \sigma \quad (2.8)$$

and let $\triangle_K$ be the image of $\widehat{\triangle}_K$ under this mapping. Most surfaces S of interest can be decomposed as in (2.5), with each S_j representable as in (2.7). Also, the surface S could be smooth, and we would often still want to decompose it as in (2.5).

The mapping (2.8) is used in defining interpolation and numerical integration on $\triangle_K$. Introduce the node points for $\triangle_K$ by

$$v_{j,K} = m_K(\rho_j) \qquad j = 1, \ldots, 6$$

Collectively, the node points of the triangulation $\{\triangle_K\}$ will be denoted by $\{v_i \mid 1 \le i \le M_N\}$, with M_N the number of distinct node points.

The sequence of triangulations (2.6) will usually be obtained by successive refinements. The refinement process is based on connecting the midpoints of the sides of a given element $\widehat{\triangle}_K$. Given $\{\widehat{v}_1, \ldots, \widehat{v}_6\}$, connect $\widehat{v}_4, \widehat{v}_5, \widehat{v}_6$ by lines

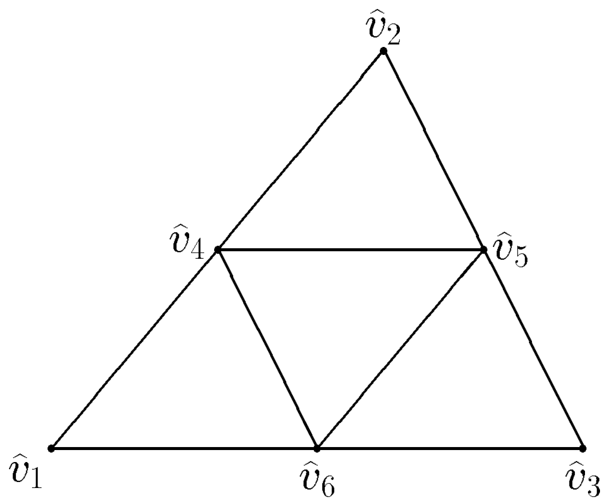

Figure 2.2: Refinement

parallel to the sides of $\widehat{\triangle}_K$, as in Figure 2.2, producing four new triangular elements. The new elements all are congruent, and they are similar to $\widehat{\triangle}_K$. More importantly, any *symmetric pair of triangles*, as shown in Figure 2.3,

have the following property:

$$\hat{v}_1 - \hat{v}_2 = -(\hat{v}_1 - \hat{v}_4)$$

$$\hat{v}_1 - \hat{v}_3 = -(\hat{v}_1 - \hat{v}_5)$$

The assumptions on S and the node points that we made in this section are for the use of quadratic interpolation. There are other degrees of interpolation that can be used, and the assumptions on the smoothness of S and the definition of the nodes will change appropriately. But the general process of refinement will still remain the same, and we still subdivide $\triangle_K$'s in the same way as we do for the quadratic interpolation.

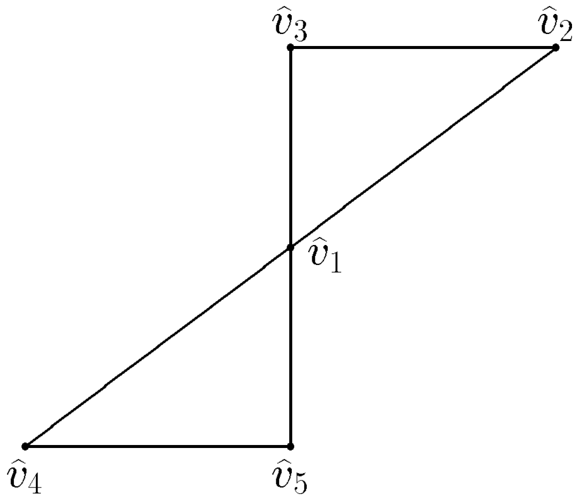

Figure 2.3: A Symmetric pair of triangles

2.4 Interpolation

To define interpolation, introduce the basis functions for quadratic interpolation on σ. Letting $u = 1-(s+t)$, define

$$l_1(s,t) = u(2u-1), \quad l_2(s,t) = t(2t-1), \quad l_3(s,t) = s(2s-1),$$

$$l_4(s,t) = 4tu, \qquad l_5(s,t) = 4st, \qquad l_6(s,t) = 4su.$$

Define a corresponding set of basis functions $\{l_{j,K}(q)\}$ on $\triangle_K$:

$$l_{j,K}(m_K(s,t)) = l_j(s,t)\,, \qquad 1 \le j \le 6\,, \quad 1 \le K \le N\,.$$

Given a function $f \in C(S)$, define

$$\mathcal{P}_N f(q) = \sum_{j=1}^{6} f(v_{j,K}) l_{j,K}(q)\,, \qquad q \in \triangle_K, \tag{2.9}$$

for $K = 1, \ldots, N$. This is called the *piecewise quadratic isoparametric function interpolating* f on the nodes of the mesh $\{\triangle_K\}$ for S.

Other kinds of interpolation can be used, such as piecewise cubic isoparametric function interpolation to f, and in this case, we need ten node points, $\rho_1, \ldots, \rho_{10}$, and ten basis functions for this interpolation on σ.

Lemma 2.1 Let $\{\triangle_K\}$ be a sequence of triangulations for S. Then $\mathcal{P}_N$, as in (2.10), defines a bounded projection operator on $C(S)$ with

$$\|\mathcal{P}_N\| = 5/3\,.$$

For any $f \in C(S)$,

$$\|f - \mathcal{P}_N f\|_\infty \le \frac{5}{3}\,\omega(f, \delta_N) \tag{2.10}$$

with ω the modulus of continuity function.

Proof: See Atkinson[6]. ∎

To improve the order of convergence in (2.10), we use the Taylor error formula. Let $g \in C^3(\sigma)$. Then

$$g(s,t) - T_2(s,t) = \frac{1}{2}\int_0^1 [(s\frac{\partial}{\partial x} + t\frac{\partial}{\partial y})^3 g(x,y)]\,|_{(x,y)=\xi(s,t)}\; d\xi$$

with $T_2(s,t)$ the degree two Taylor polynomial for $g(s,t)$ about $(0,0)$.

Lemma 2.2 Let the triangulation $\{\triangle_{K,N}\}$ be defined by using the mapping (2.8) and a triangulation $\{\widehat{\triangle}_{K,N}\}$ of the polygonal regions R_j in (2.7). Then for $f \in C^3(S)$,

$$\|f - \mathcal{P}_N f\|_\infty \le c\widehat{\delta}_N^3\,.$$

$\widehat{\delta}_N$ is the mesh size of the triangulation $\{\widehat{\triangle}_{K,N}\}$. The constant c depends on f and F_j, but not on N or the triangulation.

Proof: See Atkinson[6]. ■

2.5 Some general results

In this section we state some of the important results used in this paper. First we describe the collocation method which will be applied to solve problems on a surface S in $\mathbf{R}^3$. This material is taken from Atkinson [9, pp. 54–62].

Let $\mathcal{K}$ be a compact operator on the Banach space X, and consider solving the Fredholm integral equation of the second kind

$$(\lambda - \mathcal{K})f = g\,, \qquad g \in X\,.$$

For our work, X is the space of continuous functions on the piecewise smooth surface S, and the integral equation is

$$\lambda f(P) - \int_S k(P,Q)\, f(Q)\, dQ \,=\, g(P), \qquad P \in S\,, \tag{2.11}$$

where $g \in C(S)$ and $k(P,Q)$ is continuous on $S \times S$. Let $\{Q_1, \ldots, Q_n\}$ be a set of distinct *collocation nodes*, and let $\phi_1(P), \ldots, \phi_n(P)$ be continuous functions on S with

$$\det[\phi_i(Q_j)] \neq 0.$$

Define the collocation method as following: for

$$f_n(P) = \sum_{j=1}^{n} \alpha_j \phi_j(P)\,, \tag{2.12}$$

define the residual as

$$r_n(P) = \lambda f_n(P) - \mathcal{K} f_n(P) - g(P)\,.$$

Determine $\alpha_1, \ldots, \alpha_n$ by making

$$r_n(Q_i) = 0 \qquad i = 1, 2, \ldots, n\,; \tag{2.13}$$

and therefore, hopefully, making $r_n(P)$ also small everywhere else in S. The resulting linear system is

$$\sum_{j=1}^{n} \alpha_j [\lambda \phi_j(Q_i) - \int_S k(Q_i, Q) \phi_j(Q) dQ] = g(Q_i) \qquad i = 1, 2, \ldots, n\,.$$

In Atkinson[9, pp. 50–62], it is shown that (2.12) and (2.13) define a projection method. Let X be a Banach space, X_n a finite dimensional subspace, and $\mathcal{P}_n$ a bounded projection operator from X onto X_n. The projection method for solving $(\lambda - \mathcal{K})f = g$ is to solve

$$(\lambda - \mathcal{P}_n \mathcal{K}) f_n = \mathcal{P}_n g\,.$$

Theorem 2.3 Let X be a Banach space, let $\mathcal{K}$ be a bounded operator from X into X, and let $\mathcal{P}_n$ be a bounded projection operator from X onto the subspace X_n. Assume that $(\lambda - \mathcal{K})^{-1}$ exists on X and that

$$\|\mathcal{K} - \mathcal{P}_n \mathcal{K}\| < \frac{1}{\|(\lambda - \mathcal{K})^{-1}\|}\;.$$

Then $(\lambda - \mathcal{P}_n \mathcal{K})^{-1}$ exists on X with

$$\|(\lambda - \mathcal{P}_n \mathcal{K})^{-1}\| \leq \frac{\|(\lambda - \mathcal{K})^{-1}\|}{1 - \|(\lambda - \mathcal{K})^{-1}\|\,\|\mathcal{K} - \mathcal{P}_n \mathcal{K}\|}\;.$$

For $(\lambda - \mathcal{K})f = g$ and $(\lambda - \mathcal{P}_n\mathcal{K})f_n = \mathcal{P}_n g$,

$$\frac{|\lambda|}{\|\lambda - \mathcal{P}_n\mathcal{K}\|}\|f - \mathcal{P}_n f\| \leq \|f - f_n\| \leq |\lambda| \; \|(\lambda - \mathcal{P}_n\mathcal{K})^{-1}\| \, \|f - \mathcal{P}_n f\| \, .$$

From this theorem, we notice that the rate of convergence of f_n to f is exactly the same as the interpolation error $f - \mathcal{P}_n f$.

We describe now the Nyström method, following Atkinson[9, pp. 88–93]. Let

$$\int_S f(P)dP \approx \sum_{j=1}^{n} \omega_{j,n} f(Q_{j,n}) \qquad f \in C(S)\,, \quad n \geq 1 \tag{2.14}$$

denote a numerical method which converges for all continuous functions f on S, and assume that all the node points $Q_{j,n}$ lie in S; generally the second subscript n will be dropped, although implicitly understood. The Nyström method approximates (2.11) by the equation

$$\lambda f_n(P) - \sum_{j=1}^{n} \omega_{j,n} k(P, Q_{j,n}) f_n(Q_{j,n}) = g(P) \quad P \in S \, . \tag{2.15}$$

By letting $P = Q_1, \ldots, Q_n$, we convert the equation (2.15) into the linear system

$$\lambda z_i - \sum_{j=1}^{n} \omega_j k(Q_i, Q_j) z_j = g(Q_i) \qquad i = 1, \ldots, n \, . \tag{2.16}$$

At this point, Nyström found that to each solution $\{z_1, \ldots, z_n\}$ of (2.16), there corresponds a unique solution $z(P)$ of (2.15) with which it agrees at node points $Q_1, \ldots, Q_n$.

Now, introduce the numerical integration operator $\mathcal{K}_n$,

$$\mathcal{K}_n f(P) = \sum_{j=1}^{n} \omega_j k(P, Q_j) f(Q_j), \qquad P \in S\,, \quad f \in C(S)\,. \tag{2.17}$$

$\mathcal{K}_n$ is a compact operator on $C(S)$ to $C(S)$. Symbolically, we can write equations (2.11) and (2.15) as

$$(\lambda - \mathcal{K})f = g\,, \qquad (\lambda - \mathcal{K}_n)f = g\,,$$

respectively.

Lemma 2.4 Let $k(P,Q)$ be continuous and $\mathcal{K}$ be the associated integral operator on $C(S)$. Let the numerical scheme (2.14) converge for all continuous functions, and let $\mathcal{K}_n$ be defined by (2.17) . Then :

1. For any $f \in C(S)\,, \quad \mathcal{K}_n f \to \mathcal{K} f$ as $n \to \infty$;
2. For all $n \geq 1, \quad \|\mathcal{K} - \mathcal{K}_n\| \geq \|\mathcal{K}\|$;
3. $\|(\mathcal{K} - \mathcal{K}_n)\mathcal{K}\| \to 0$ as $n \to \infty$; (2.18)
4. $\|(\mathcal{K} - \mathcal{K}_n)\mathcal{K}_n\| \to 0$ as $n \to \infty$. (2.19)

The following set of assumptions are due to P. Anselone and R. Moore (see Anselone[1, pp. 5–8]) and they will imply (2.18) and (2.19).

A1) X is a Banach space and $\mathcal{K}$, $\mathcal{K}_n$, $n \geq 1$, are linear operators on X to X.

A2) $\mathcal{K}_n x \to \mathcal{K} x, \quad \forall\, x \in X$.

A3) $\{\mathcal{K}_n\}$ is a collectively compact family of operators, i. e., the set

$$S = \{\ \mathcal{K}_n x \mid n \geq 1 \quad \text{and} \quad \|x\| \leq 1\ \}$$

has compact closure in X.

Lemma 2.5 Assume A1–A3. Then

1. $\mathcal{K}$ is compact;

2. The sequence $\{\mathcal{K}_n\}$ is uniformly bounded;

3. $\|(\mathcal{K}-\mathcal{K}_n)\mathcal{M}\| \to 0$ as $n \to \infty$, for any compact operator $\mathcal{M} : X \longrightarrow X$.

4. $\|(\mathcal{K}-\mathcal{K}_n)\mathcal{K}_n\| \to 0$ as $n \to \infty$.

Now we complete the abstract error analysis.

Theorem 2.6 Assume A1–A3. Assume $\lambda \neq 0$ is not an eigenvalue of $\mathcal{K}$. Then for all sufficiently large n, say $n \geq N(\lambda)$, we have

$$\|(\mathcal{K}-\mathcal{K}_n)\mathcal{K}_n\| \leq \frac{|\lambda|}{\|(\lambda-\mathcal{K})^{-1}\|} \quad .$$

Furthermore, $(\lambda-\mathcal{K}_n)^{-1}$ exists for all $n \geq N(\lambda)$, and

$$\|(\lambda-\mathcal{K}_n)^{-1}\| \leq \frac{1+\|(\lambda-\mathcal{K})^{-1}\|\,\|\mathcal{K}_n\|}{|\lambda| - \|(\lambda-\mathcal{K})^{-1}\|\,\|(\mathcal{K}-\mathcal{K}_n)\mathcal{K}_n\|} \quad ;$$

this means that $(\lambda-\mathcal{K}_n)^{-1}$ is uniformly bounded for all $n \geq N(\lambda)$. For the error in $(\lambda-\mathcal{K}_n)f_n = g$ as an approximation to $(\lambda-\mathcal{K})f = g$, we have

$$\|f-f_n\| \ \leq \ \|(\lambda-\mathcal{K}_n)^{-1}\|\|(\mathcal{K}-\mathcal{K}_n)f\| \,. \tag{2.20}$$

This can be applied by using asymptotic error formula for the numerical integration error $(\mathcal{K}-\mathcal{K}_n)f(P)$. Also, the hypotheses A1–A3 allow $\mathcal{K}_n$ to have its definition extended to include product integration, which will include $\mathcal{K}_n = \mathcal{K}\mathcal{P}_n$, $\mathcal{P}_n$ an interpolatory projection.

CHAPTER III

THE COLLOCATION METHOD

3.1 Introduction

In this chapter, we investigate the collocation method for solving an integral equation on a surface in $\mathbf{R}^3$. Then, we try to examine the relation between the collocation method and the Nyström method by using the product integration method.

3.2 The collocation method for a smooth kernel

Consider the integral equation

$$\lambda f(P) - \int_S k(P,Q)\, f(Q)\, dS_Q = g(P), \qquad P \in S \tag{3.1}$$

with $k(P,Q)$ continuous for $P, Q \in S$. We write the equation (3.1) as

$$(\lambda - \mathcal{K})f = g$$

symbolically. We assume λ is nonzero and is not an eigenvalue of the integral operator $\mathcal{K}$ defined implicitly in (3.1). Thus, (3.1) has a unique solution $f \in C(S)$ for each $g \in C(S)$. S is a piecewise smooth surface in $\mathbf{R}^3$, as discussed in Section 2.3.

Following the discussion in Section 2.3, let $\{\triangle_1, \ldots, \triangle_N\}$ be a triangulation of S. Let

$$m_K : \sigma \xrightarrow[\text{onto}]{1\text{–}1} \triangle_K\,, \qquad K = 1, \ldots, N$$

be a parametrization of $\triangle_K$, with σ the unit simplex. To define the collocation method, the solution function $f(m_K(s,t))$, $(s,t) \in \sigma$, is approximated by a quadratic polynomial (as in Section 2.4) in (s,t) :

$$f(m_K(s,t)) \approx f_N(m_K(s,t)) \equiv \sum_{j=1}^{6} f(m_K(\rho_j)) l_j(s,t)\,. \tag{3.2}$$

The functions $l_j(s,t)$ are quadratic Lagrange polynomials satisfying

$$l_i(\rho_j) = \delta_{ij}\,.$$

Denote

$$v_{j,K} = m_K(\rho_j) \quad j = 1, \ldots, 6\,, \quad K = 1, \ldots, N\,.$$

For S a boundary of a bounded simply-connected region in $\mathbf{R}^3$, we have

$$N_v = 2(N+1)$$

node points.

The collocation method for solving (3.1) amounts to:

1. solving the system

$$\lambda f_N(v_i) - \int_S k(v_i, Q)\, f_N(Q)\, dS_Q = g(v_i), \qquad i = 1, \ldots, N_v \tag{3.3}$$

for the nodal values $\{f_N(v_i) \mid i = 1, \ldots, N_v\}$.

2. using the interpolation formula (3.2) to extend the nodal values to $f_N(Q)$, for general $Q \in S$.

Solving (3.3) reduces to solving the linear system

$$\lambda f_N(v_i) - \sum_{K=1}^{N} \sum_{j=1}^{6} f_N(v_{j,K}) \int_\sigma k(v_i, m_K(s,t)) l_j(s,t) \mid D_s m_K(s,t) \times D_t m_K(s,t) \mid ds\, dt$$

$$= g(v_i), \qquad i = 1, \dots, N_v \tag{3.4}$$

For notation,

$$D_s m_K(s,t) = \frac{\partial m_K(s,t)}{\partial s}, \qquad D_t m_K(s,t) = \frac{\partial m_K(s,t)}{\partial t}$$

and

$$\mid D_s m_K(s,t) \times D_t m_K(s,t) \mid$$

is the Jacobian determinant of the mapping $m_K(s,t)$ used in transforming surface integrals over $\triangle_K$ into integrals over σ.

A major problem with (3.4) is that $D_s m_K$ and $D_t m_K$ are not easy to compute for most surfaces S. Therefore, we use an approximate surface $\widetilde{S}_N$ with a parametrization that is easy to differentiate. The approximate surface $\widetilde{S}_N$ is composed of elements $\widetilde{\triangle}_1, \dots, \widetilde{\triangle}_K$, with $\widetilde{\triangle}_K$ an interpolant of $\triangle_K$. Define

$$\begin{aligned} \widetilde{m}_K(s,t) &= \sum_{j=1}^{6} m_K(\rho_j) l_j(s,t) \\ &= \begin{bmatrix} \sum_{j=1}^6 v_{j,K}^1 l_j(s,t) \\ \sum_{j=1}^6 v_{j,K}^2 l_j(s,t) \\ \sum_{j=1}^6 v_{j,K}^3 l_j(s,t) \end{bmatrix} \qquad (s,t) \in \sigma \end{aligned} \tag{3.5}$$

where $v^i_{j,K}$ is the i-th coordinate of $m_K(q_j)$. Thus, $\widetilde{m}_K(s,t)$ interpolates $m_K(s,t)$ at $\{\rho_1,\ldots,\rho_6\}$, and each component is quadratic in (s,t).

Using this surface, we seek $\widetilde{f}_N$:

$$\widetilde{f}_N(m_K(s,t)) = \sum_{j=1}^{6} \widetilde{f}_N(v_{j,K}) l_j(s,t) \qquad (s,t)\in\sigma \qquad K=1,\ldots,N\,. \tag{3.6}$$

It is obtained from the linear system

$$\lambda \widetilde{f}_N(v_i) - \sum_{K=1}^{N}\sum_{j=1}^{6} \widetilde{f}_N(v_{j,K}) \int_\sigma k(v_i, \widetilde{m}_K(s,t)) l_j(s,t) \mid D_s\widetilde{m}_K(s,t)\times D_t\widetilde{m}_K(s,t)\mid ds\,dt$$

$$= g(v_i), \qquad i=1,\ldots,N_v \tag{3.7}$$

The kernel function $k(v_i,Q)$ is being evaluated at points Q not on S, but we assume $k(v_i,Q)$ extends smoothly and easily to such nearby points Q.

3.3 The product integration method

The collocation method can be considered as a product integration method. Define

$$\begin{aligned}
&\mathcal{K}_N f(P) \\
&\quad= \sum_{K=1}^{N}\sum_{j=1}^{6} f_N(v_{j,K}) \int_\sigma k(P, \widetilde{m}_K(s,t)) l_j(s,t) \mid D_s\widetilde{m}_K(s,t)\times D_t\widetilde{m}_K(s,t)\mid ds\,dt \\
&\quad= \sum_{K=1}^{N}\sum_{j=1}^{6} f_N(v_{j,K})\,\omega_{j,K}(P)
\end{aligned}$$

where

$$\omega_{j,K}(P) = \int_\sigma k(P, \widetilde{m}_K(s,t)) l_j(s,t) \mid D_s\widetilde{m}_K(s,t)\times D_t\widetilde{m}_K(s,t)\mid ds\,dt$$

Applying this approximation to the integral equation $(\lambda - \mathcal{K})f = g$, and using the theory of the Nyström method, we obtain the linear system

$$\lambda \widehat{f}_N(v_i) - \sum_{K=1}^{N}\sum_{j=1}^{6} \widehat{f}_N(v_{j,K}) \int_\sigma k(v_i, \widetilde{m}_K(s,t)) l_j(s,t) \mid D_s\widetilde{m}_K(s,t) \times D_t\widetilde{m}_K(s,t) \mid ds\, dt$$

$$= g(v_i), \qquad i = 1, \ldots, N_v$$

This is exactly the same system as in (3.7) for the collocation method. The function $\widehat{f}_N$ is in $C(S)$, and it is given by Nyström interpolation away from the node points. The results of the two methods coincide at the node points, but they differ elsewhere. Write the collocation solutions as

$$\widetilde{f}_N(q) = \sum_{j=1}^{6} \widetilde{f}(v_{j,K}) l_{j,K}(q), \qquad q \in \triangle_K$$

where the $l_{j,K}$'s are defined in Section 2.3.

Then the relationship of the two solutions is

$$\begin{aligned}\widehat{f}_N(q) &= \frac{1}{\lambda}\left\{ g(q) + \sum_{K=1}^{N}\sum_{j=1}^{6} \widehat{f}_N(v_{j,K}) \cdot \right. \\ &\qquad \left. \int_\sigma k(q, \widetilde{m}_K(s,t)) l_j(s,t) \mid D_s\widetilde{m}_K(s,t) \times D_t\widetilde{m}_K(s,t) \mid ds\, dt \right\} \\ &= \frac{1}{\lambda}\left\{ g(q) + \sum_{K=1}^{N}\sum_{j=1}^{6} \widetilde{f}_N(v_{j,K}) \omega_{j,K}(q) \right\}\end{aligned}$$

For the collocation method, $\widetilde{f}_N$ can be shown to satisfy

$$\|f - \widetilde{f}_N\|_\infty = O(\delta_N^3)$$

when S is a smooth surface; this is a result from Nedelec[29]. For piecewise smooth surfaces, it has been shown to be at least $O(\delta_N^2)$ (see Atkinson[6]). But for the Nyström method (see formula (2.20))

$$\|f - \widehat{f}_N\|_\infty \leq C\|(\mathcal{K} - \mathcal{K}_N)f\|_\infty .$$

Thus for the collocation method, we have the new error bound

$$\max_{1\leq i\leq N_v} \mid f(v_i) - \widetilde{f}_N(v_i) \mid \leq C\|(\mathcal{K} - \mathcal{K}_N)f\|_\infty .$$

With this as motivation, we examine the error $\|(\mathcal{K} - \mathcal{K}_N)f\|_\infty$ in the next section.

3.4 Error Analysis

Write

$$\mathcal{K}f(P) = \sum_{K=1}^{N} \int_\sigma k(P, m_K(s,t))f(m_K(s,t)) \mid D_s m_K \times D_t m_K \mid dsdt$$

and

$$\begin{aligned} \mathcal{K}_N f(P) &= \sum_{K=1}^{N} \sum_{j=1}^{6} f(v_{j,K}) \int_\sigma k(P, \widetilde{m}_K(s,t)) l_j(s,t) \mid D_s\widetilde{m}_K \times D_t\widetilde{m}_K \mid dsdt \\ &\equiv \sum_{K=1}^{N} \int_\sigma k(P, \widetilde{m}_K(s,t)) f_N(m_K(s,t)) \mid D_s\widetilde{m}_K \times D_t\widetilde{m}_K \mid dsdt \end{aligned}$$

with f_N denoting the piecewise quadratic interpolant of f, from (3.2).

We break the error analysis into five parts :

$$(\mathcal{K} - \mathcal{K}_N)f(P)$$

$$= \sum_{K=1}^{N} \int_{\sigma} k(P, m_K(s,t)) f(m_K(s,t)) \mid D_s m_K \times D_t m_K \mid dsdt -$$

$$\sum_{K=1}^{N} \int_{\sigma} k(P, \widetilde{m}_K(s,t)) f_N(m_K(s,t)) \mid D_s \widetilde{m}_K \times D_t \widetilde{m}_K \mid dsdt$$

$$= E_1 + E_2 + E_3 + E_4 + E_5$$

$$E_1 = \sum_{K=1}^{N} \int_{\sigma} k(P, m_K(s,t)) f(m_K(s,t)) \mid D_s m_K \times D_t m_K \mid dsdt -$$

$$\sum_{K=1}^{N} \int_{\sigma} k(P, m_K(s,t)) f(m_K(s,t)) \mid D_s \widetilde{m}_K \times D_t \widetilde{m}_K \mid dsdt$$

$$E_2 = \sum_{K=1}^{N} \int_{\sigma} k(P, m_K(s,t)) [f(m_K(s,t)) - f_N(m_K(s,t))] \mid D_s \widetilde{m}_K \times D_t \widetilde{m}_K \mid dsdt$$

$$- \sum_{K=1}^{N} \int_{\sigma} k(P, m_K(s,t)) [f(m_K(s,t)) - f_N(m_K(s,t))] \mid D_s m_K \times D_t m_K \mid dsdt$$

$$E_3 = \sum_{K=1}^{N} \int_{\sigma} k(P, m_K(s,t)) [f(m_K(s,t)) - f_N(m_K(s,t))] \mid D_s m_K \times D_t m_K \mid dsdt$$

$$E_4 = \sum_{K=1}^{N} \int_\sigma [k(P, m_K(s,t)) - k(P, \widetilde{m}_K(s,t))] f_N(m_K(s,t)) \mid D_s\widetilde{m}_K \times D_t\widetilde{m}_K \mid dsdt$$

$$- \sum_{K=1}^{N} \int_\sigma [k(P, m_K(s,t)) - k(P, \widetilde{m}_K(s,t))] f_N(m_K(s,t)) \mid D_s m_K \times D_t m_K \mid dsdt$$

$$E_5 = \sum_{K=1}^{N} \int_\sigma [k(P, m_K(s,t)) - k(P, \widetilde{m}_K(s,t))] f_N(m_K(s,t)) \mid D_s m_K \times D_t m_K \mid dsdt$$

Lemma 3.1 Let $f(s,t) = c_1 s^3 + c_2 s^2 t + c_3 s t^2 + c_4 t^3$ where c_i's are real numbers. Let

$$\mathcal{P}_n(s,t) = \sum_{i=1}^{6} f(q_i) l_i(s,t)$$

be the Lagrange form of the interpolating polynomial. Then

$$\int_\sigma \frac{\partial}{\partial s} [f(s,t) - \mathcal{P}_n(s,t)]\, dsdt = 0$$

$$\int_\sigma \frac{\partial}{\partial t} [f(s,t) - \mathcal{P}_n(s,t)]\, dsdt = 0$$

Proof: Let

$$H(s,t) \equiv f(s,t) - \mathcal{P}_n(s,t)$$

$$= c_1(s^3 - \frac{3}{2}s^2 + \frac{1}{2}s) + c_2(s^2 t - \frac{1}{2}st) + c_3(st^2 - \frac{1}{2}st) + c_4(t^3 - \frac{3}{2}t^2 + \frac{1}{2}t)\,.$$

Then

$$\frac{\partial}{\partial s} H(s,t) = c_1(3s^2 - 3s + \frac{1}{2}) + c_2(2st - \frac{1}{2}) + c_3(t^2 - \frac{1}{2}t)$$

$$\frac{\partial}{\partial t}H(s,t) = c_4(3t^2 - 3t + \frac{1}{2}) + c_2(s^2 - \frac{1}{2}s) + c_3(2st - \frac{1}{2}s)$$

and by direct computation,

$$\int_\sigma \frac{\partial}{\partial s}H(s,t)\, ds\, dt = 0$$

$$\int_\sigma \frac{\partial}{\partial t}H(s,t)\, ds\, dt = 0\ .$$

This property will be used in lemma 3.2. ∎

As in equation (2.8), we let

$$m_K(s,t) = F_j(u\widehat{v}_1 + t\widehat{v}_2 + s\widehat{v}_3) = \begin{bmatrix} x^1(u\widehat{v}_1 + t\widehat{v}_2 + s\widehat{v}_3) \\ x^2(u\widehat{v}_1 + t\widehat{v}_2 + s\widehat{v}_3) \\ x^3(u\widehat{v}_1 + t\widehat{v}_2 + s\widehat{v}_3) \end{bmatrix}$$

for some j and $u = 1 - s - t$, $(s,t) \in \sigma$, $x^i \in C^5(R_j)$, $i = 1,2,3$. Since the x^i are functions of s and t, and also of x and y, we use both $x^i(s,t)$ and $x^i(x,y)$, with the context indicating which is intended.

Lemma 3.2 For each $\triangle_K$,

$$\left| \int_\sigma k(P, m_K(s,t)) f(m_K(s,t)) (\mid D_s m_K \times D_t m_K \mid - \mid D_s \widetilde{m}_K \times D_t \widetilde{m}_K \mid) ds dt \right| \leq C \widehat{\delta}_K^5$$

where $\widehat{\delta}_K$ is the size of $\widehat{\triangle}_K$, and C depends on k, f and $\{ F_j \}$.

Proof: Let

$$\tilde{x}^i(s,t) = \sum_{j=1}^{6} x^i(s_j, t_j) l_j(s,t) \quad \text{where } (s_j, t_j) = \rho_j,\ i = 1,2,3.$$

By using the Taylor error formula, we have

$$x^i(s,t) - \widetilde{x}^i(s,t) = H^i(s,t) + G^i(s,t) + O(\widehat{\delta}_K^5)$$

where

$$H^i(s,t) = \frac{1}{3!}\left[(s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t})^3 x^i(0,0) - \sum_{j=1}^{6}(s_j\frac{\partial}{\partial s} + t_j\frac{\partial}{\partial t})^3 x^i(0,0) l_j(s,t)\right], \quad (3.8)$$

$$G^i(s,t) = \frac{1}{4!}\left[(s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t})^4 x^i(0,0) - \sum_{j=1}^{6}(s_j\frac{\partial}{\partial s} + t_j\frac{\partial}{\partial t})^4 x^i(0,0) l_j(s,t)\right],$$

and $O(\widehat{\delta}_K^5)$ comes from the fifth derivative of $x^i(s,t)$. Note that the derivatives of x^i with respect to (s,t) give rise to formulas involving $\widehat{v}_2 - \widehat{v}_1$ and $\widehat{v}_3 - \widehat{v}_1$. For example

$$\begin{aligned} x_s^i(s,t) &= \frac{\partial}{\partial s} x^i(u\widehat{v}_1 + t\widehat{v}_2 + s\widehat{v}_3) \\ &= \nabla x^i \cdot (\widehat{v}_3 - \widehat{v}_1)\,. \end{aligned}$$

To save space, we omit the subscript K from m_K and $\widetilde{m}_K$.

$$D_s m \times D_t m = \begin{bmatrix} x_s^1(s,t) \\ x_s^2(s,t) \\ x_s^3(s,t) \end{bmatrix} \times \begin{bmatrix} x_t^1(s,t) \\ x_t^2(s,t) \\ x_t^3(s,t) \end{bmatrix} = \begin{bmatrix} x_s^2(s,t)x_t^3(s,t) - x_s^3(s,t)x_t^2(s,t) \\ x_s^3(s,t)x_t^1(s,t) - x_s^1(s,t)x_t^3(s,t) \\ x_s^1(s,t)x_t^2(s,t) - x_s^2(s,t)x_t^1(s,t) \end{bmatrix}$$

$$\mid D_s m(s,t) \times D_t m(s,t) \mid^2 = (x_s^2 x_t^3 - x_s^3 x_t^2)^2 + (x_s^3 x_t^1 - x_s^1 x_t^3)^2 + (x_s^1 x_t^2 - x_s^2 x_t^1)^2$$

$$\mid D_s \widetilde{m}(s,t) \times D_t \widetilde{m}(s,t) \mid^2 = (\widetilde{x}_s^2 \widetilde{x}_t^3 - \widetilde{x}_s^3 \widetilde{x}_t^2)^2 + (\widetilde{x}_s^3 \widetilde{x}_t^1 - \widetilde{x}_s^1 \widetilde{x}_t^3)^2 + (\widetilde{x}_s^1 \widetilde{x}_t^2 - \widetilde{x}_s^2 \widetilde{x}_t^1)^2$$

$$= \mid D_s m(s,t) \times D_t m(s,t) \mid^2$$

$$
\begin{aligned}
&- 2(x_s^2 x_t^3 - x_s^3 x_t^2)\left[x_s^2(H_t^3 + G_t^3) + x_t^3(H_s^2 + G_s^2) - x_s^3(H_t^2 + G_t^2) - x_t^2(H_s^3 + G_s^3)\right] \\
&- 2(x_s^3 x_t^1 - x_s^1 x_t^3)\left[x_s^3(H_t^1 + G_t^1) + x_t^1(H_s^3 + G_s^3) - x_s^1(H_t^3 + G_t^3) - x_t^3(H_s^1 + G_s^1)\right] \\
&- 2(x_s^1 x_t^2 - x_s^2 x_t^1)\left[x_s^1(H_t^2 + G_t^2) + x_t^2(H_s^1 + G_s^1) - x_s^2(H_t^1 + G_t^1) - x_t^1(H_s^2 + G_s^2)\right] \\
&+ O(\widehat{\delta}^8)
\end{aligned}
$$

Since $\mid D_s m(s,t) \times D_t m(s,t) \mid^2$ is of order four and the remaining terms are of at least order six, using

$$
\sqrt{x^4 + x^6} \;=\; x^2\sqrt{1 + x^2} \;=\; x^2[1 - \frac{1}{2}x^2 + O(x^4)]
$$

we get the following.

$$
\begin{aligned}
&\mid D_s\widetilde{m}(s,t) \times D_t\widetilde{m}(s,t) \;= \mid D_s m(s,t) \times D_t m(s,t) \mid \\
&- \Big\{(x_s^2 x_t^3 - x_s^3 x_t^2)\left[x_s^2(H_t^3 + G_t^3) + x_t^3(H_s^2 + G_s^2) - x_s^3(H_t^2 + G_t^2) - x_t^2(H_s^3 + G_s^3)\right] \\
&+ (x_s^3 x_t^1 - x_s^1 x_t^3)\left[x_s^3(H_t^1 + G_t^1) + x_t^1(H_s^3 + G_s^3) - x_s^1(H_t^3 + G_t^3) - x_t^3(H_s^1 + G_s^1)\right] \\
&+ (x_s^1 x_t^2 - x_s^2 x_t^1)\Big[x_s^1(H_t^2 + G_t^2) + x_t^2(H_s^1 + G_s^1) - x_s^2(H_t^1 + G_t^1) \\
&- x_t^1(H_s^2 + G_s^2)\Big]\Big\} \,/\, \mid D_s m(s,t) \times D_t m(s,t) \mid + O(\widehat{\delta}^6)
\end{aligned}
$$

Expanding each x^i, x_s^i, and x_t^i about $(s,t) = (0,0)$.

$$
\begin{aligned}
&\mid D_s m(s,t) \times D_t m(s,t) \mid - \mid D_s\widetilde{m}(s,t) \times D_t\widetilde{m}(s,t) \mid \\
&= \Big\{(x_s^2 x_t^3 - x_s^3 x_t^2)\left[x_s^2(H_t^3 + G_t^3) + x_t^3(H_s^2 + G_s^2) - x_s^3(H_t^2 + G_t^2) - x_t^2(H_s^3 + G_s^3)\right] \\
&+ (x_s^3 x_t^1 - x_s^1 x_t^3)\left[x_s^3(H_t^1 + G_t^1) + x_t^1(H_s^3 + G_s^3) - x_s^1(H_t^3 + G_t^3) - x_t^3(H_s^1 + G_s^1)\right] \\
&+ (x_s^1 x_t^2 - x_s^2 x_t^1)\Big[x_s^1(H_t^2 + G_t^2) + x_t^2(H_s^1 + G_s^1) - x_s^2(H_t^1 + G_t^1)
\end{aligned}
$$

$$-x_t^1(H_s^2+G_s^2)\Big]\Big\} \;/\; \mid D_s m(s,t)\times D_t m(s,t)\mid + O(\widehat{\delta}^6)$$

$$= \Big\{(x_s^2x_t^3 - x_s^3x_t^2)\Big[x_s^2(H_t^3+G_t^3) + x_t^3(H_s^2+G_s^2) - x_s^3(H_t^2+G_t^2) - x_t^2(H_s^3+G_s^3)\Big]$$

$$+ (x_s^3x_t^1 - x_s^1x_t^3)\Big[x_s^3(H_t^1+G_t^1) + x_t^1(H_s^3+G_s^3) - x_s^1(H_t^3+G_t^3) - x_t^3(H_s^1+G_s^1)\Big]$$

$$+ (x_s^1x_t^2 - x_s^2x_t^1)\Big[x_s^1(H_t^2+G_t^2) + x_t^2(H_s^1+G_s^1) - x_s^2(H_t^1+G_t^1) - x_t^1(H_s^2+G_s^2)\Big]$$

$$+ (x_s^2x_t^3 - x_s^3x_t^2)\Big[(sx_{ss}^2+tx_{st}^2)H_t^3 + (sx_{st}^3+tx_{tt}^3)H_s^2$$

$$-(sx_{ss}^3+tx_{st}^3)H_t^2 - (sx_{st}^2+tx_{tt}^2)H_s^3\Big]$$

$$+ (x_s^3x_t^1 - x_s^1x_t^3)\Big[(sx_{ss}^3+tx_{st}^3)H_t^1 + (sx_{st}^1+tx_{tt}^1)H_s^3$$

$$-(sx_{ss}^1+tx_{st}^1)H_t^3 - (sx_{st}^3+tx_{tt}^3)H_s^1\Big]$$

$$+ (x_s^1x_t^2 - x_s^2x_t^1)\Big[(sx_{ss}^1+tx_{st}^1)H_t^2 + (sx_{st}^2+tx_{tt}^2)H_s^1$$

$$-(sx_{ss}^2+tx_{st}^2)H_t^1 - (sx_{st}^1+tx_{tt}^1)H_s^2\Big]$$

$$+ (x_s^2H_t^3 + x_t^3H_s^2 - x_s^3H_t^2 - x_t^2H_s^3)\Big[x_s^2(sx_{st}^3+tx_{tt}^3) + x_t^3(sx_{ss}^2+tx_{st}^2)$$

$$-x_s^3(sx_{st}^2+tx_{tt}^2) - x_t^2(sx_{ss}^3+tx_{st}^3)\Big]$$

$$+ (x_s^3H_t^1 + x_t^1H_s^3 - x_s^1H_t^3 - x_t^3H_s^1)\Big[x_s^3(sx_{st}^1+tx_{tt}^1) + x_t^1(sx_{ss}^3+tx_{st}^3)$$

$$-x_s^1(sx_{st}^3+tx_{tt}^3) - x_t^3(sx_{ss}^1+tx_{st}^1)\Big]$$

$$+ (x_s^1H_t^2 + x_t^2H_s^1 - x_s^2H_t^1 - x_t^1H_s^2)\Big[x_s^1(sx_{st}^2+tx_{tt}^2) + x_t^2(sx_{ss}^1+tx_{st}^1)$$

$$-x_s^2(sx_{st}^1+tx_{tt}^1) - x_t^1(sx_{ss}^2+tx_{st}^2)\Big]\Big\} \;/\; \mid D_s m(s,t)\times D_t m(s,t)\mid + O(\widehat{\delta}^6)$$

$$1/\mid D_s m(s,t)\times D_t m(s,t)\mid = \big(\,1/\mid D_s m(0,0)\times D_t m(0,0)\mid\,\big)\cdot$$

$$\Big\{\;1\;-\;\Big\{(x_s^2x_t^3 - x_s^3x_t^2)\Big[x_s^2(sx_{st}^3+tx_{tt}^3) + x_t^3(sx_{ss}^2+tx_{st}^2)$$

$$-x_s^3(sx_{st}^2+tx_{tt}^2) - x_t^2(sx_{ss}^3+tx_{st}^3)\Big]$$

$$+ (x_s^3 x_t^1 - x_s^1 x_t^3) \Big[x_s^3 (s x_{st}^1 + t x_{tt}^1) + x_t^1 (s x_{ss}^3 + t x_{st}^3)$$

$$- x_s^1 (s x_{st}^3 + t x_{tt}^3) - x_t^3 (s x_{ss}^1 + t x_{st}^1) \Big]$$

$$+ (x_s^1 x_t^2 - x_s^2 x_t^1) \Big[x_s^1 (s x_{st}^2 + t x_{tt}^2) + x_t^2 (s x_{ss}^1 + t x_{st}^1) - x_s^2 (s x_{st}^1 + t x_{tt}^1)$$

$$- x_t^1 (s x_{ss}^2 + t x_{st}^2) \Big] \Big\} / \mid D_s m(0,0) \times D_t m(0,0) \mid^2 + \; O(\widehat{\delta}^2) \Big\} \tag{3.9}$$

$$\mid D_s m(s,t) \times D_t m(s,t) \mid - \mid D_s \widetilde{m}(s,t) \times D_t \widetilde{m}(s,t) \mid$$

$$= \Big\{ (x_s^2 x_t^3 - x_s^3 x_t^2) \Big[x_s^2 (H_t^3 + G_t^3) + x_t^3 (H_s^2 + G_s^2) - x_s^3 (H_t^2 + G_t^2) - x_t^2 (H_s^3 + G_s^3) \Big]$$

$$+ (x_s^3 x_t^1 - x_s^1 x_t^3) \Big[x_s^3 (H_t^1 + G_t^1) + x_t^1 (H_s^3 + G_s^3) - x_s^1 (H_t^3 + G_t^3) - x_t^3 (H_s^1 + G_s^1) \Big]$$

$$+ (x_s^1 x_t^2 - x_s^2 x_t^1) \Big[x_s^1 (H_t^2 + G_t^2) + x_t^2 (H_s^1 + G_s^1) - x_s^2 (H_t^1 + G_t^1) - x_t^1 (H_s^2 + G_s^2) \Big]$$

$$+ (x_s^2 x_t^3 - x_s^3 x_t^2) \Big[(s x_{ss}^2 + t x_{st}^2) H_t^3 + (s x_{st}^3 + t x_{tt}^3) H_s^2$$

$$- (s x_{ss}^3 + t x_{st}^3) H_t^2 - (s x_{st}^2 + t x_{tt}^2) H_s^3 \Big]$$

$$+ (x_s^3 x_t^1 - x_s^1 x_t^3) \Big[(s x_{ss}^3 + t x_{st}^3) H_t^1 + (s x_{st}^1 + t x_{tt}^1) H_s^3$$

$$- (s x_{ss}^1 + t x_{st}^1) H_t^3 - (s x_{st}^3 + t x_{tt}^3) H_s^1 \Big]$$

$$+ (x_s^1 x_t^2 - x_s^2 x_t^1) \Big[(s x_{ss}^1 + t x_{st}^1) H_t^2 + (s x_{st}^2 + t x_{tt}^2) H_s^1$$

$$- (s x_{ss}^2 + t x_{st}^2) H_t^1 - (s x_{st}^1 + t x_{tt}^1) H_s^2 \Big]$$

$$+ (x_s^2 H_t^3 + x_t^3 H_s^2 - x_s^3 H_t^2 - x_t^2 H_s^3) \Big[x_s^2 (s x_{st}^3 + t x_{tt}^3) + x_t^3 (s x_{ss}^2 + t x_{st}^2)$$

$$- x_s^3 (s x_{st}^2 + t x_{tt}^2) - x_t^2 (s x_{ss}^3 + t x_{st}^3) \Big]$$

$$+ (x_s^3 H_t^1 + x_t^1 H_s^3 - x_s^1 H_t^3 - x_t^3 H_s^1) \Big[x_s^3 (s x_{st}^1 + t x_{tt}^1) + x_t^1 (s x_{ss}^3 + t x_{st}^3)$$

$$- x_s^1 (s x_{st}^3 + t x_{tt}^3) - x_t^3 (s x_{ss}^1 + t x_{st}^1) \Big]$$

$$+ (x_s^1 H_t^2 + x_t^2 H_s^1 - x_s^2 H_t^1 - x_t^1 H_s^2) \Big[x_s^1 (s x_{st}^2 + t x_{tt}^2) + x_t^2 (s x_{ss}^1 + t x_{st}^1)$$

$$-x_s^2(sx_{st}^1+tx_{tt}^1)-x_t^1(sx_{ss}^2+tx_{st}^2)\Big]\Big\} \,/\mid D_sm(0,0)\times D_tm(0,0)\mid -$$

$$\Big\{(x_s^2x_t^3-x_s^3x_t^2)\left[x_s^2H_t^3+x_t^3H_s^2-x_s^3H_t^2-x_t^2H_s^3\right]$$

$$+(x_s^3x_t^1-x_s^1x_t^3)\left[x_s^3H_t^1+x_t^1H_s^3-x_s^1H_t^3-x_t^3H_s^1\right]$$

$$+(x_s^1x_t^2-x_s^2x_t^1)\left[x_s^1H_t^2+x_t^2H_s^1-x_s^2H_t^1-x_t^1H_s^2\right]\Big\}\cdot$$

$$\Big\{(x_s^2x_t^3-x_s^3x_t^2)\Big[x_s^2(sx_{st}^3+tx_{tt}^3)+x_t^3(sx_{ss}^2+tx_{st}^2)$$

$$-x_s^3(sx_{st}^2+tx_{tt}^2)-x_t^2(sx_{ss}^3+tx_{st}^3)\Big]$$

$$+(x_s^3x_t^1-x_s^1x_t^3)\Big[x_s^3(sx_{st}^1+tx_{tt}^1)+x_t^1(sx_{ss}^3+tx_{st}^3)$$

$$-x_s^1(sx_{st}^3+tx_{tt}^3)-x_t^3(sx_{ss}^1+tx_{st}^1)\Big]$$

$$+(x_s^1x_t^2-x_s^2x_t^1)\Big[x_s^1(sx_{st}^2+tx_{tt}^2)+x_t^2(sx_{ss}^1+tx_{st}^1)$$

$$-x_s^2(sx_{st}^1+tx_{tt}^1)-x_t^1(sx_{ss}^2+tx_{st}^2)\Big]\Big\} \,/\mid D_sm(0,0)\times D_tm(0,0)\mid^3+\ O(\widehat{\delta}^6)$$

Let

$$E4(s,t)=\Big\{(x_s^2x_t^3-x_s^3x_t^2)\left[x_s^2H_t^3+x_t^3H_s^2-x_s^3H_t^2-x_t^2H_s^3\right]$$

$$+(x_s^3x_t^1-x_s^1x_t^3)\left[x_s^3H_t^1+x_t^1H_s^3-x_s^1H_t^3-x_t^3H_s^1\right]$$

$$+(x_s^1x_t^2-x_s^2x_t^1)\left[x_s^1H_t^2+x_t^2H_s^1-x_s^2H_t^1-x_t^1H_s^2\right]\Big\} \,/\mid D_sm(0,0)\times D_tm(0,0)\mid$$

and

$$E5(s,t)=\Big\{(x_s^2x_t^3-x_s^3x_t^2)\left[x_s^2G_t^3+x_t^3G_s^2-x_s^3G_t^2-x_t^2G_s^3\right]$$

$$+(x_s^3x_t^1-x_s^1x_t^3)\left[x_s^3G_t^1+x_t^1G_s^3-x_s^1G_t^3-x_t^3G_s^1\right]$$

$$+(x_s^1x_t^2-x_s^2x_t^1)\left[x_s^1G_t^2+x_t^2G_s^1-x_s^2G_t^1-x_t^1G_s^2\right]$$

$$+(x_s^2x_t^3-x_s^3x_t^2)\Big[(sx_{ss}^2+tx_{st}^2)H_t^3+(sx_{st}^3+tx_{tt}^3)H_s^2$$

$$-(sx^3_{ss}+tx^3_{st})H^2_t-(sx^2_{st}+tx^2_{tt})H^3_s\Big]$$

$$+\left(x^3_s x^1_t-x^1_s x^3_t\right)\Big[(sx^3_{ss}+tx^3_{st})H^1_t+(sx^1_{st}+tx^1_{tt})H^3_s$$

$$-(sx^1_{ss}+tx^1_{st})H^3_t-(sx^3_{st}+tx^3_{tt})H^1_s\Big]$$

$$+\left(x^1_s x^2_t-x^2_s x^1_t\right)\Big[(sx^1_{ss}+tx^1_{st})H^2_t+(sx^2_{st}+tx^2_{tt})H^1_s$$

$$-(sx^2_{ss}+tx^2_{st})H^1_t-(sx^1_{st}+tx^1_{tt})H^2_s\Big]$$

$$+\left(x^2_s H^3_t+x^3_t H^2_s-x^3_s H^2_t-x^2_t H^3_s\right)\Big[x^2_s(sx^3_{st}+tx^3_{tt})+x^3_t(sx^2_{ss}+tx^2_{st})$$

$$-x^3_s(sx^2_{st}+tx^2_{tt})-x^2_t(sx^3_{ss}+tx^3_{st})\Big]$$

$$+\left(x^3_s H^1_t+x^1_t H^3_s-x^1_s H^3_t-x^3_t H^1_s\right)\Big[x^3_s(sx^1_{st}+tx^1_{tt})+x^1_t(sx^3_{ss}+tx^3_{st})$$

$$-x^1_s(sx^3_{st}+tx^3_{tt})-x^3_t(sx^1_{ss}+tx^1_{st})\Big]$$

$$+\left(x^1_s H^2_t+x^2_t H^1_s-x^2_s H^1_t-x^1_t H^2_s\right)\Big[x^1_s(sx^2_{st}+tx^2_{tt})+x^2_t(sx^1_{ss}+tx^1_{st})$$

$$-x^2_s(sx^1_{st}+tx^1_{tt})-x^1_t(sx^2_{ss}+tx^2_{st})\Big]\Big\} \,/\, \mid D_s m(0,0)\times D_t m(0,0)\mid -$$

$$\Big\{\left(x^2_s x^3_t-x^3_s x^2_t\right)\Big[x^2_s H^3_t+x^3_t H^2_s-x^3_s H^2_t-x^2_t H^3_s\Big]$$

$$+\left(x^3_s x^1_t-x^1_s x^3_t\right)\Big[x^3_s H^1_t+x^1_t H^3_s-x^1_s H^3_t-x^3_t H^1_s\Big]$$

$$+\left(x^1_s x^2_t-x^2_s x^1_t\right)\Big[x^1_s H^2_t+x^2_t H^1_s-x^2_s H^1_t-x^1_t H^2_s\Big]\Big\}\cdot$$

$$\Big\{\left(x^2_s x^3_t-x^3_s x^2_t\right)\Big[x^2_s(sx^3_{st}+tx^3_{tt})+x^3_t(sx^2_{ss}+tx^2_{st})$$

$$-x^3_s(sx^2_{st}+tx^2_{tt})-x^2_t(sx^3_{ss}+tx^3_{st})\Big]$$

$$+\left(x^3_s x^1_t-x^1_s x^3_t\right)\Big[x^3_s(sx^1_{st}+tx^1_{tt})+x^1_t(sx^3_{ss}+tx^3_{st})$$

$$-x^1_s(sx^3_{st}+tx^3_{tt})-x^3_t(sx^1_{ss}+tx^1_{st})\Big]$$

$$+\left(x^1_s x^2_t-x^2_s x^1_t\right)\Big[x^1_s(sx^2_{st}+tx^2_{tt})+x^2_t(sx^1_{ss}+tx^1_{st})$$

$$-x_s^2(sx_{st}^1 + tx_{tt}^1) - x_t^1(sx_{ss}^2 + tx_{st}^2)\Big]\Big\} \,/\, \mid D_s m(0,0) \times D_t m(0,0) \mid^3$$

$E4(s,t)$ and $E5(s,t)$ have no connection to E_4 and E_5 on page 24–25. $E4(s,t)$ and $E5(s,t)$ are the collection of terms which are of order four and order five in $\widehat{\delta}$, respectively, and they will be examined below.

From above computation, we can see that

$$\mid D_s m(s,t) \times D_t m(s,t) \mid - \mid D_s \widetilde{m}(s,t) \times D_t \widetilde{m}(s,t) \mid$$

$$= \; E4(s,t) \; + \; E5(s,t) \; + \; O(\widehat{\delta}^6) \tag{3.10}$$

is at least $O(\widehat{\delta}^4)$ for every $(s,t) \in \sigma$. Expanding $k(P, m_K(s,t))$ and $f(m_K(s,t))$ about $(s,t) = (0,0)$, we have

$$k(P, m_K(s,t)) f(m_K(s,t)) (\mid D_s m(s,t) \times D_t m(s,t) \mid - \mid D_s \widetilde{m}(s,t) \times D_t \widetilde{m}(s,t) \mid)$$

$$= k(P, m_K(0,0)) f(m_K(0,0)) (E4(s,t) + E5(s,t))$$

$$+ k(P, m_K(0,0)) [s f_s(m_K(0,0)) + t f_t(m_K(0,0))] E4(s,t)$$

$$+ [s k_s(P, m_K(0,0)) + t k_t(P, m_K(0,0))] f(m_K(0,0)) E4(s,t) + \; O(\widehat{\delta}^6) \,. \tag{3.11}$$

For notation,

$$\begin{aligned}
f_s(m_K(s,t)) &= \nabla f(m_K(s,t)) \cdot D_s m_K(s,t) \,, \\
f_t(m_K(s,t)) &= \nabla f(m_K(s,t)) \cdot D_t m_K(s,t) \,, \\
k_s(P, \; m_K(s,t)) &= \nabla k(P, \; m_K(s,t)) \cdot D_s m_K(s,t) \,, \\
k_t(P, \; m_K(s,t)) &= \nabla k(P, \; m_K(s,t)) \cdot D_t m_K(s,t) \,.
\end{aligned}$$

By using Lemma 3.1, we know that

$$\int_\sigma E4(s,t)\,ds\,dt \;=\; 0$$

Therefore,

$$\int_\sigma k(P,m_K(s,t))f(m_K(s,t))(\mid D_s m(s,t)\times D_t m(s,t)\mid - \mid D_s\widetilde{m}(s,t)\times D_t\widetilde{m}(s,t)\mid)dsdt$$

$$=\; IE5(s,t) + \;\; O(\widehat{\delta}_K^6)$$

where

$$\begin{aligned} IE5(s,t) \;&=\; \int_\sigma \{k(P,m_K(0,0))f(m_K(0,0))E5(s,t) \\ &\quad + k(P,m_K(0,0))[sf_s(m_K(0,0)) + tf_t(m_K(0,0))]E4(s,t) \\ &\quad + [sk_s(P,m_K(0,0)) + tk_t(P,m_K(0,0))]f(m_K(0,0))E4(s,t)\}\;ds\,dt \end{aligned}$$

Thus, this shows that

$$\int_\sigma k(P,m_K(s,t))f(m_K(s,t))(\mid D_s m(s,t)\times D_t m(s,t)\mid - \mid D_s\widetilde{m}(s,t)\times D_t\widetilde{m}(s,t)\mid)dsdt$$

$$=\; O(\widehat{\delta}^5) \qquad \text{for every } \;\triangle_K\,.$$

∎

Theorem 3.3 $E_1 = O(\widehat{\delta}^4)$.

Proof: For every symmetric pair of triangles (see figure 2.3), let

$$m_1(s,t) = F(u\widehat{v}_1 + t\widehat{v}_2 + s\widehat{v}_3)$$

$$m_2(s,t) = F(u\widehat{v}_1 + t\widehat{v}_4 + s\widehat{v}_5)$$

Let $IE5_1(s,t)$ and $IE5_2(s,t)$ be the $IE5(s,t)$ for $m_1(s,t)$ and $m_2(s,t)$ respectively. The reason of $IE5_1 + IE5_2$ being zero is the occurrence of cancellation. The integrand of $IE5_1$ is a polynomial in s and t, and its coefficients are combinations of $(\widehat{v}_3 - \widehat{v}_1)$ and $(\widehat{v}_2 - \widehat{v}_1)$. For notation,

$$\widehat{v}_i = (v_{i,x},\ v_{i,y}) \qquad \text{for } i = 1,\ 2,\ 3\,,$$

and

$$\begin{aligned} x_s^i(0,0) &= \nabla x^i(\widehat{v}_1) \cdot (\widehat{v}_3 - \widehat{v}_1) \\ &= x_x^i(\widehat{v}_1)(v_{3,x} - v_{1,x}) + x_y^i(\widehat{v}_1)(v_{3,y} - v_{1,y}) \qquad (3.12) \\ x_t^i(0,0) &= \nabla x^i(\widehat{v}_1) \cdot (\widehat{v}_2 - \widehat{v}_1) \\ &= x_x^i(\widehat{v}_1)(v_{2,x} - v_{1,x}) + x_y^i(\widehat{v}_1)(v_{2,y} - v_{1,y}) \qquad (3.13) \end{aligned}$$

We examine the coefficient

$$\frac{s x_s^2 x_t^3 x_{ss}^2 H_t^3}{\mid D_s m(0,0) \times D_t m(0,0) \mid} \tag{3.14}$$

from $E5(s,t)$, and show why the cancellation happens.

Computing from (3.8), the numerator of (3.14) is

$$s x_s^2 x_t^3 x_{ss}^2 H_t^3 = \frac{1}{6} s x_s^2(0,0) x_t^3(0,0) x_{ss}^2(0,0) \cdot$$

$$\left[(3t^2 - 3t + \frac{1}{2}) x_{ttt}^3(0,0) + (6st - \frac{3}{2}s) x_{stt}^3(0,0) + (3s^2 - \frac{2}{3}s) x_{sst}^3(0,0)\right]$$

The term we are examining is

$$x_s^2(0,0) x_t^3(0,0) x_{ss}^2(0,0) x_{ttt}^3(0,0) =$$

$$\Big[x_x^2(\widehat{v}_1)(v_{3,x}-v_{1,x})+x_y^2(\widehat{v}_1)(v_{3,y}-v_{1,y})\Big]\Big[x_x^3(\widehat{v}_1)(v_{2,x}-v_{1,x})+x_y^3(\widehat{v}_1)(v_{2,y}-v_{1,y})\Big]$$
$$\cdot\Big[x_{xx}^2(\widehat{v}_1)(v_{3,x}-v_{1,x})^2+2x_{xy}^2(\widehat{v}_1)(v_{3,x}-v_{1,x})(v_{3,y}-v_{1,y})+x_{yy}^2(\widehat{v}_1)(v_{3,y}-v_{1,y})^2\Big]$$
$$\cdot\Big[x_{xxx}^3(\widehat{v}_1)(v_{2,x}-v_{1,x})^3+3x_{xxy}^3(\widehat{v}_1)(v_{2,x}-v_{1,x})^2(v_{2,y}-v_{1,y})$$
$$+3x_{xyy}^3(\widehat{v}_1)(v_{2,x}-v_{1,x})(v_{2,y}-v_{1,y})^2+x_{yyy}^3(\widehat{v}_1)(v_{2,y}-v_{1,y})^3\Big] \qquad (3.15)$$

The corresponding term in $IE5_2$ is

$$\Big[x_x^2(\widehat{v}_1)(v_{5,x}-v_{1,x})+x_y^2(\widehat{v}_1)(v_{5,y}-v_{1,y})\Big]\Big[x_x^3(\widehat{v}_1)(v_{4,x}-v_{1,x})+x_y^3(\widehat{v}_1)(v_{4,y}-v_{1,y})\Big]$$
$$\cdot\Big[x_{xx}^2(\widehat{v}_1)(v_{5,x}-v_{1,x})^2+2x_{xy}^2(\widehat{v}_1)(v_{5,x}-v_{1,x})(v_{5,y}-v_{1,y})+x_{yy}^2(\widehat{v}_1)(v_{5,y}-v_{1,y})^2\Big]$$
$$\cdot\Big[x_{xxx}^3(\widehat{v}_1)(v_{4,x}-v_{1,x})^3+3x_{xxy}^3(\widehat{v}_1)(v_{4,x}-v_{1,x})^2(v_{4,y}-v_{1,y})$$
$$+3x_{xyy}^3(\widehat{v}_1)(v_{4,x}-v_{1,x})(v_{4,y}-v_{1,y})^2+x_{yyy}^3(\widehat{v}_1)(v_{4,y}-v_{1,y})^3\Big] \qquad (3.16)$$

Since

$$\widehat{v}_1-\widehat{v}_2=-(\widehat{v}_1-\widehat{v}_4)$$
$$\widehat{v}_1-\widehat{v}_3=-(\widehat{v}_1-\widehat{v}_5),$$

(3.15) + (3.16) is zero. Thus,

$$\sum_{j=1}^{2}\int_\sigma k(P,m_K(s,t))f(m_K(s,t))(|\,D_sm(s,t)\times D_tm(s,t)\,|-|\,D_s\widetilde{m}(s,t)\times D_t\widetilde{m}(s,t)\,|)$$
$$=IE5_1(s,t)+IE5_2(s,t)+O(\widehat{\delta}^6)$$
$$=O(\widehat{\delta}^6)$$

Thus, cancellation happens on each symmetric pair of triangles, and the error contributed by each such pair of symmetric $\triangle_K$ is $O(\widehat{\delta}^6)$. If there are n_j^2

triangles for each R_j, we have $(n_j^2 - n_j)/2$ pairs of triangles with error in $O(\widehat{\delta}^6)$, and n_j remaining triangles with error in $O(\widehat{\delta}^5)$. We also can see that

$$\widehat{\delta} \approx 1/n_j \ .$$

Therefore

$$\begin{aligned} E_1 &= (n_j^2 - n_j)O(\widehat{\delta}^6) + n_j O(\widehat{\delta}^5) \\ &= C \cdot O(\widehat{\delta}^4), \end{aligned}$$

i.e., the global error from using the Jacobian determinant of the approximate surface is $O(\widehat{\delta}^4)$. ∎

Theorem 3.4 $E_2 = O(\widehat{\delta}^5)$

Proof: Let

$$f(m_K(s,t)) - f_N(m_K(s,t)) = H_{f,K}(s,t) + O(\widehat{\delta}^4)$$

where

$$H_{f,K}(s,t) = \frac{1}{3!}\left[(s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t})^3 f(m_K(0,0)) - \sum_{j=1}^{6}(s_j\frac{\partial}{\partial s} + t_j\frac{\partial}{\partial t})^3 f(m_K(0,0)) l_j(s,\, t)\right]$$

Since

$$f(m_K(s,t)) - f_N(m_K(s,t)) = O(\widehat{\delta}^3)$$

and

$$\mid D_s m_K \times D_t m_K \mid - \mid D_s \widetilde{m}_K \times D_t \widetilde{m}_K \mid = O(\widehat{\delta}^4)$$

for every $(s,t) \in \sigma$ and for $K = 1, \ldots, N$, we can conclude that

$$\int_\sigma k(P, m_K(s,t))[f(m_K(s,t)) - f_n(m_K(s,t))] \mid D_s m_K \times D_t m_K \mid dsdt$$

$$- \int_\sigma k(P, m_K(s,t))[f(m_K(s,t)) - f_n(m_K(s,t))] \mid D_s \widetilde{m}_K \times D_t \widetilde{m}_K \mid dsdt$$

$$= O(\widehat{\delta}^7)\ .$$

Therefore, $E_2 = O(\widehat{\delta}^5)$. ■

Theorem 3.5 $E_3 = O(\widehat{\delta}^4)$

Proof: Using similar computation as (3.9), we get

$$\begin{aligned}
&\mid D_s m(s,t) \times D_t m(s,t) \mid = \mid D_s m(0,0) \times D_t m(0,0) \mid \cdot \\
&\quad \Big\{\ 1\ +\ \Big\{(x_s^2 x_t^3 - x_s^3 x_t^2)\left[x_s^2(s x_{st}^3 + t x_{tt}^3) + x_t^3(s x_{ss}^2 + t x_{st}^2)\right. \\
&\quad \left. - x_s^3(s x_{st}^2 + t x_{tt}^2) - x_t^2(s x_{ss}^3 + t x_{st}^3)\right] \\
&\quad + (x_s^3 x_t^1 - x_s^1 x_t^3)\left[x_s^3(s x_{st}^1 + t x_{tt}^1) + x_t^1(s x_{ss}^3 + t x_{st}^3)\right. \\
&\quad \left. - x_s^1(s x_{st}^3 + t x_{tt}^3) - x_t^3(s x_{ss}^1 + t x_{st}^1)\right] \\
&\quad + (x_s^1 x_t^2 - x_s^2 x_t^1)\left[x_s^1(s x_{st}^2 + t x_{tt}^2) + x_t^2(s x_{ss}^1 + t x_{st}^1)\right. \\
&\quad \left. - x_s^2(s x_{st}^1 + t x_{tt}^1) - x_t^1(s x_{ss}^2 + t x_{st}^2)\right]\Big\} \,/\, \mid D_s m(0,0) \times D_t m(0,0) \mid^2 +\ O(\widehat{\delta}^2)\Big\}
\end{aligned}$$

For every (s,t) in σ, we can expand about $(0,0)$ to obtain

$$k(P, m_K(s,t))[f(m_K(s,t)) - f_N(m_K(s,t))] \mid D_s m_K \times D_t m_K \mid$$

$$= \left[k(P, m_K(0,0)) + sk_s(P, m_K(0,0)) + tk_t(P, m_K(0,0)) + O(\widehat{\delta}^2)\right] \cdot$$

$$(H_{f,K}(s,t) + O(\widehat{\delta}^4)\ )\mid D_s m(0,0) \times D_t m(0,0) \mid \cdot$$

$$\Big\{\ 1\ +\ \Big\{(x_s^2 x_t^3 - x_s^3 x_t^2)\left[x_s^2(sx_{st}^3 + tx_{tt}^3) + x_t^3(sx_{ss}^2 + tx_{st}^2)\right.$$

$$\left. -x_s^3(sx_{st}^2 + tx_{tt}^2) - x_t^2(sx_{ss}^3 + tx_{st}^3)\right]$$

$$+ (x_s^3 x_t^1 - x_s^1 x_t^3)\left[x_s^3(sx_{st}^1 + tx_{tt}^1) + x_t^1(sx_{ss}^3 + tx_{st}^3)\right.$$

$$\left. -x_s^1(sx_{st}^3 + tx_{tt}^3) - x_t^3(sx_{ss}^1 + tx_{st}^1)\right]$$

$$+ (x_s^1 x_t^2 - x_s^2 x_t^1)\left[x_s^1(sx_{st}^2 + tx_{tt}^2) + x_t^2(sx_{ss}^1 + tx_{st}^1)\right.$$

$$\left. -x_s^2(sx_{st}^1 + tx_{tt}^1) - x_t^1(sx_{ss}^2 + tx_{st}^2)\right]\Big\} / \mid D_s m(0,0) \times D_t m(0,0) \mid^2 +\ O(\widehat{\delta}^2)\Big\}$$

$$= k(P, m_K(0,0)) H_{f,K}(s,t) \mid D_s m(0,0) \times D_t m(0,0) \mid + O(\widehat{\delta}^6)$$

Again, for every symmetric pair of triangles $\triangle_1$ and $\triangle_2$ (see figure 2.3), let

$$m_1(s,t) = F(u\widehat{v}_1 + t\widehat{v}_2 + s\widehat{v}_3)$$

$$m_2(s,t) = F(u\widehat{v}_1 + t\widehat{v}_4 + s\widehat{v}_5).$$

Then by examining the integral of $H_{f,K}(s,t)$ over $\triangle_1$ and $\triangle_2$, we obtain

$$\sum_{K=1}^{2} \int_\sigma k(P, m_K(s,t))[f(m_K(s,t)) - f_N(m_K(s,t))] \mid D_s m_K(s,t) \times D_t m_K(s,t) \mid dsdt$$

$$= \sum_{K=1}^{2} \int_\sigma k(P, m_K(0,0)) H_{f,K}(s,t) \mid D_s m_K(0,0) \times D_t m_K(0,0) \mid dsdt + O(\widehat{\delta}^6)$$

$$= O(\widehat{\delta}^6)$$

Thus, we have proved that E_3 is of order four. ∎

Theorem 3.6 $E_4 = O(\widehat{\delta}^5)$

Proof: Since k is a function of x^1, x^2, and x^3, we first expand it about $\widetilde{m}_K(s,t)$ for each (s,t); and subsequently, we expand the leading terms about $(0,0)$, when we treat it as a function of s and t. Write

$$
\begin{aligned}
& k(P, m_K(s,t)) - k(P, \widetilde{m}_K(s,t)) \\
= \quad & (x^1(s,t) - \widetilde{x}^1(s,t)) k_{x^1}(P, \widetilde{m}_K(s,t)) + (x^2(s,t) - \widetilde{x}^2(s,t)) k_{x^2}(P, \widetilde{m}_K(s,t)) \\
& + (x^3(s,t) - \widetilde{x}^3(s,t)) k_{x^3}(P, \widetilde{m}_K(s,t)) + O(\widehat{\delta}^6) \\
= \quad & (H^1(s,t) + O(\widehat{\delta}^4)) \Big[k_{x^1}(P, \widetilde{m}_K(0,0)) \\
& + s k_{x^1 s}(P, \widetilde{m}_K(0,0)) + t k_{x^1 t}(P, \widetilde{m}_K(0,0)) + O(\widehat{\delta}^2) \Big] \\
& + (H^2(s,t) + O(\widehat{\delta}^4)) \Big[k_{x^2}(P, \widetilde{m}_K(0,0)) \\
& + s k_{x^2 s}(P, \widetilde{m}_K(0,0)) + t k_{x^2 t}(P, \widetilde{m}_K(0,0)) + O(\widehat{\delta}^2) \Big] \\
& + (H^3(s,t) + O(\widehat{\delta}^4)) \Big[k_{x^3}(P, \widetilde{m}_K(0,0)) \\
& + s k_{x^3 s}(P, \widetilde{m}_K(0,0)) + t k_{x^3 t}(P, \widetilde{m}_K(0,0)) + O(\widehat{\delta}^2) \Big] + O(\widehat{\delta}^6) \\
= \quad & H^1(s,t) k_{x^1}(P, \widetilde{m}_K(0,0)) + H^2(s,t) k_{x^2}(P, \widetilde{m}_K(0,0)) \\
& + H^3(s,t) k_{x^3}(P, \widetilde{m}_K(0,0)) + O(\widehat{\delta}^4) \\
= \quad & H^1(s,t) k_{x^1}(P, m_K(0,0)) + H^2(s,t) k_{x^2}(P, m_K(0,0)) \\
& + H^3(s,t) k_{x^3}(P, m_K(0,0)) + O(\widehat{\delta}^4)
\end{aligned}
$$

By using the proof of Theorem 3.2, we know that

$$
\mid D_s m(s,t) \times D_t m(s,t) \mid - \mid D_s \widetilde{m}(s,t) \times D_t \widetilde{m}(s,t) \mid = \; O(\widehat{\delta}^4).
$$

Therefore,

$$\int_\sigma [k(P, m_K(s,t)) - k(P, \widetilde{m}_K(s,t))]\, f_N(m_K(s,t)) \mid D_s\widetilde{m}_K \times D_t\widetilde{m}_K \mid dsdt$$

$$- \int_\sigma [k(P, m_K(s,t)) - k(P, \widetilde{m}_K(s,t))]\, f_N(m_K(s,t)) \mid D_s m_K \times D_t m_K \mid dsdt$$

$$= O(\widehat{\delta}^7)$$

for each $\triangle_K$, and E_4 is of order five. ■

Theorem 3.7 $E_5 = O(\widehat{\delta}^4)$

Proof: Using notation from earlier theorems, we write

$$[k(P, m_K(s,t)) - k(P, \widetilde{m}_K(s,t))]\, f_N(m_K(s,t)) \mid D_s m_K \times D_t m_K \mid =$$

$$\Big[H^1(s,t)k_{x^1}(P, m_K(0,0)) + H^2(s,t)k_{x^2}(P, m_K(0,0)) + H^3(s,t)k_{x^3}(P, m_K(0,0))$$

$$+O(\widehat{\delta}^4)\Big] \Big[f_N(m_K(0,0)) + s f_{N\,s}(m_K(0,0)) + t f_{N\,t}(m_K(0,0)) + O(\widehat{\delta}^2)\Big] \cdot$$

$$\mid D_s m(0,0) \times D_t m(0,0) \mid \Big\{ \ 1 \ + \ \Big\{(x_s^2 x_t^3 - x_s^3 x_t^2)\cdot$$

$$\Big[x_s^2(s x_{st}^3 + t x_{tt}^3) + x_t^3(s x_{ss}^2 + t x_{st}^2) - x_s^3(s x_{st}^2 + t x_{tt}^2) - x_t^2(s x_{ss}^3 + t x_{st}^3)\Big]$$

$$+ (x_s^3 x_t^1 - x_s^1 x_t^3) \Big[x_s^3(s x_{st}^1 + t x_{tt}^1) + x_t^1(s x_{ss}^3 + t x_{st}^3)$$

$$- x_s^1(s x_{st}^3 + t x_{tt}^3) - x_t^3(s x_{ss}^1 + t x_{st}^1)\Big]$$

$$+ (x_s^1 x_t^2 - x_s^2 x_t^1) \Big[x_s^1(s x_{st}^2 + t x_{tt}^2) + x_t^2(s x_{ss}^1 + t x_{st}^1)$$

$$- x_s^2(s x_{st}^1 + t x_{tt}^1) - x_t^1(s x_{ss}^2 + t x_{st}^2)\Big]\Big\} \,/\, \mid D_s m(0,0) \times D_t m(0,0) \mid^2 + \ O(\widehat{\delta}^2)\Big\}$$

$$= \Big[H^1(s,t)k_{x^1}(P, m_K(0,0)) + H^2(s,t)k_{x^2}(P, m_K(0,0))$$

$$+ \ H^3(s,t)k_{x^3}(P, m_K(0,0))\Big] \mid D_s m(0,0) \times D_t m(0,0) \mid + O(\widehat{\delta}^6)$$

for every $(s,t) \in \sigma$ and for every $\triangle_K$.

Now, let us integrate the final expression over σ, the unit simplex, and add the contributions over all $\triangle_K$'s together. We find that cancellation happens again among every symmetric pair of triangles, as in Theorem 3.3. Therefore, E_5 is of order four. ■

After analyzing the five error terms, we have shown that

$$\|(\mathcal{K}-\mathcal{K}_N)f\|_\infty \;\leq\; O(\widehat{\delta}^4)\,.$$

Thus, we have the following new error bound for $\{\widetilde{f}_N(v_i) \mid i=1,\ldots,N_v\}$ of the collocation method:

$$\max_{1\leq i\leq N_v} \mid f(v_i)-\widetilde{f}_N(v_i)\mid \leq C\|(\mathcal{K}-\mathcal{K}_N)f\|_\infty \;=\; O(\widehat{\delta}^4)\,.$$

This is better than the error bound for $\|f-\widetilde{f}_N\|_\infty$ of the collocation method, which only gives us $O(\widehat{\delta}^3)$. The above results also show

$$\|f-\widehat{f}_N\|_\infty \;=\; O(\widehat{\delta}^4)\;,$$

for the Nyström method based on product integration of Section 3.3.

3.5 The Single layer integral

Before we can solve the BIE problems, P1 and P2 in Section 2.2, we have to know the errors from terms in equations (2.2) and (2.4). We will begin with the equation (2.4), because equations (2.2) and (2.4) are different only in their right hand inhomogeneous terms. Consider the right hand side of the equation (2.4), the single layer integral

$$\int_S f(Q) \frac{1}{\mid P - Q \mid} dS_Q, \qquad\qquad P \in S. \tag{3.17}$$

Since the integrand of the integral (3.17) is singular and the singularity only happens at Q equals P, we have to change the way we analyse errors, even if we are going to use the same schema to evaluate the integral. Write

$$\int_S \frac{f(Q)}{\mid P - Q \mid} dS_Q \;\approx\; \sum_{K=1}^{N} \int_\sigma \frac{f(\widetilde{m}_K(s,t))}{\mid P - \widetilde{m}_K(s,t) \mid} \mid D_s \widetilde{m}_K(s,t) \times D_t \widetilde{m}_K(s,t) \mid \, ds\, dt$$

where P is one of node points.

We can see the integrand in (3.17) varies from singular to quite smooth. To handle this varied behavior, we use two ways to study error. The first case is for those $\triangle_K$'s that contain the point P, and the second case is for the remaining $\triangle_K$'s.

Theorem 3.8 Let P be a vertex in $\triangle_K$ for some K. Then

$$\int_\sigma \frac{f(m_K(s,t))}{\mid P - m_K(s,t) \mid} \mid D_s m_K(s,t) \times D_t m_K(s,t) \mid \, ds\, dt -$$

$$\int_\sigma \frac{f_N(\widetilde{m}_K(s,t))}{\mid P - \widetilde{m}_K(s,t) \mid} \mid D_s \widetilde{m}_K(s,t) \times D_t \widetilde{m}_K(s,t) \mid \, ds\, dt \;=\; O(\widehat{\delta}_K^3)$$

where $\widehat{\delta}_K$ is the diameter of $\triangle_K$.

Proof: Without loss of generality, we assume that

$$P \;=\; m_K(0,0) \;=\; \widetilde{m}_K(0,0) \;=\; (p_1, p_2, p_3).$$

Before proving the theorem, we show that

$$\int_\sigma \frac{1}{\mid P - m_K(s,t) \mid}\, ds\, dt \;=\; O(\widehat{\delta}_K^{-1}).$$

$$\int_\sigma \frac{1}{\mid P - m_K(s,t) \mid}\, ds\, dt$$

$$= \int_\sigma \frac{1}{\left[(p_1 - x^1(s,t))^2 + (p_2 - x^2(s,t))^2 + (p_3 - x^3(s,t))^2\right]^{1/2}}\, ds\, dt$$

$$= \int_\sigma \Big[(sx_s^1(0,0) + tx_t^1(0,0) + O(\widehat{\delta}_K^2))^2 + (sx_s^2(0,0) + tx_t^2(0,0) + O(\widehat{\delta}_K^2))^2$$

$$+ (sx_s^3(0,0) + tx_t^3(0,0) + O(\widehat{\delta}_K^2))^2\Big]^{-1/2}\, ds\, dt$$

$$= \int_\sigma \Big[(sx_s^1(0,0) + tx_t^1(0,0))^2 + (sx_s^2(0,0) + tx_t^2(0,0))^2$$

$$+ (sx_s^3(0,0) + tx_t^3(0,0))^2 + O(\widehat{\delta}_K^3)\Big]^{-1/2}\, ds\, dt$$

After integrating the dominant part of the above equation by using polar coordinates in the $st-plane$ about $(0,0)$, we obtain

$$\int_\sigma \frac{1}{\mid P - m_K(s,t) \mid}\, ds\, dt \;=\; O(\widehat{\delta}_K^{-1}).$$

Now, we break the error analysis into three parts.

$$\int_\sigma \frac{f(m_K(s,t))}{\mid P - m_K(s,t) \mid} \mid D_s m_K(s,t) \times D_t m_K(s,t) \mid \; ds\, dt -$$

$$\int_\sigma \frac{f_N(\widetilde{m}_K(s,t))}{\mid P - \widetilde{m}_K(s,t) \mid} \mid D_s \widetilde{m}_K(s,t) \times D_t \widetilde{m}_K(s,t) \mid \; ds\, dt$$

$$= \; E_1 \; + \; E_2 \; + \; E_3$$

with

$$E_1 \;=\; \int_\sigma \frac{f(m_K(s,t)) - f_N(\widetilde{m}_K(s,t))}{\mid P - m_K(s,t) \mid} \mid D_s m_K(s,t) \times D_t m_K(s,t) \mid \; ds\, dt \tag{3.18}$$

$$E_2 \;=\; \int_\sigma \frac{f_N(\widetilde{m}_K(s,t))}{\mid P - m_K(s,t) \mid} (\mid D_s m_K(s,t) \times D_t m_K(s,t) \mid$$

$$- \mid D_s \widetilde{m}_K(s,t) \times D_t \widetilde{m}_K(s,t) \mid) \; ds\, dt \tag{3.19}$$

$$E_3 \;=\; \int_\sigma \left[\frac{1}{\mid P - m_K(s,t) \mid} - \frac{1}{\mid P - \widetilde{m}_K(s,t) \mid} \right] \cdot$$

$$\mid D_s \widetilde{m}_K(s,t) \times D_t \widetilde{m}_K(s,t) \mid f_N(\widetilde{m}_K(s,t))\, ds\, dt \tag{3.20}$$

In equation (3.18),

$$\mid E_1 \mid \;\leq\; \int_\sigma \frac{\mid f(m_K(s,t)) - f_N(\widetilde{m}_K(s,t)) \mid}{\mid P - m_K(s,t) \mid} \mid D_s m_K(s,t) \times D_t m_K(s,t) \mid \; ds\, dt$$

$$\leq\; O(\widehat{\delta}_K^3) \cdot O(\widehat{\delta}_K^2) \cdot \int_\sigma \frac{1}{\mid P - m_K(s,t) \mid} ds\, dt \;\leq\; O(\widehat{\delta}_K^5) \cdot O(\widehat{\delta}_K^{-1}) \;=\; O(\widehat{\delta}_K^4)$$

For the equation (3.19), we can easily see it has order three:

$$| E_2 | \leq \max_{s,t\in\sigma} \Big| \, | D_s m_K(s,t) \times D_t m_K(s,t) | - | D_s \widetilde{m}_K(s,t) \times D_t \widetilde{m}_K(s,t) | \, \Big| \cdot$$

$$\int_\sigma \frac{f_N(\widetilde{m}_K(s,t))}{| P - m_K(s,t) |} \, ds \, dt = O(\widehat{\delta}_K^4) \cdot O(\widehat{\delta}_K^{-1}) = O(\widehat{\delta}_K^3)$$

The equation (3.20) is the most difficult term to analyse. For every $(s,t) \in \sigma$,

$$\frac{1}{| P - m_K(s,t) |} - \frac{1}{| P - \widetilde{m}_K(s,t) |} =$$

$$\frac{(x^1 - p_1)(\widetilde{x}^1 - x^1) + (x^2 - p_2)(\widetilde{x}^2 - x^2) + (x^3 - p_3)(\widetilde{x}^3 - x^3)}{[(x^1 - p_1)^2 + (x^2 - p_2)^2 + (x^3 - p_3)^2]^{3/2}} + O(\widehat{\delta}_K^3) \quad (3.21)$$

By using the Taylor error formula for each x^i about $(0,0)$, equation (3.21) equals

$$\frac{(sx_s^1 + tx_t^1)H^1(s,t) + (sx_s^2 + tx_t^2)H^2(s,t) + (sx_s^3 + tx_t^3)H^3(s,t) + O(\widehat{\delta}_K^5)}{[(sx_s^1 + tx_t^1)^2 + (sx_s^2 + tx_t^2)^2 + (sx_s^3 + tx_t^3)^2 + O(\widehat{\delta}_K^3)]^{3/2}}$$

$$+ \; O(\widehat{\delta}_K^3) \qquad (3.22)$$

Integrating equation (3.22) over σ, then

$$\int_\sigma \left\{ \frac{(sx_s^1 + tx_t^1)H^1(s,t) + (sx_s^2 + tx_t^2)H^2(s,t) + (sx_s^3 + tx_t^3)H^3(s,t) + O(\widehat{\delta}_K^5)}{[(sx_s^1 + tx_t^1)^2 + (sx_s^2 + tx_t^2)^2 + (sx_s^3 + tx_t^3)^2 + O(\widehat{\delta}_K^3)]^{3/2}} + \right.$$

$$\left. O(\widehat{\delta}_K^3) \right\} \left\{ f_N(m_K(0,0)) + O(\widehat{\delta}_K) \right\} \left\{ \, | D_s \widetilde{m}_K(0,0) \times D_t \widetilde{m}_K(0,0) | + O(\widehat{\delta}_K^3) \right\} ds \, dt$$

$$= \int_\sigma \left\{ \frac{(sx_s^1 + tx_t^1)H^1(s,t) + (sx_s^2 + tx_t^2)H^2(s,t) + (sx_s^3 + tx_t^3)H^3(s,t)}{[(sx_s^1 + tx_t^1)^2 + (sx_s^2 + tx_t^2)^2 + (sx_s^3 + tx_t^3)^2]^{3/2}} + O(\widehat{\delta}_K^2) \right\}$$

$$\cdot \left\{ f_N(m_K(0,0)) + O(\widehat{\delta}_K) \right\} \left\{ \mid D_s\widetilde{m}_K(0,0) \times D_t\widetilde{m}_K(0,0) \mid + O(\widehat{\delta}_K^3) \right\} \, ds\, dt$$

$$= \int_\sigma \left\{ \frac{(sx_s^1 + tx_t^1)H^1(s,t) + (sx_s^2 + tx_t^2)H^2(s,t) + (sx_s^3 + tx_t^3)H^3(s,t)}{[(sx_s^1 + tx_t^1)^2 + (sx_s^2 + tx_t^2)^2 + (sx_s^3 + tx_t^3)^2]^{3/2}} \right\} \cdot$$

$$f_N(m_K(0,0)) \mid D_s\widetilde{m}_K(0,0) \times D_t\widetilde{m}_K(0,0) \mid \; ds\, dt \; + \; O(\widehat{\delta}_K^4)$$

$$= \; O(\widehat{\delta}_K) \cdot O(\widehat{\delta}_K^2) + O(\widehat{\delta}_K^4) \; = \; O(\widehat{\delta}_K^3)$$

In the above computation,

$$(sx_s^1 + tx_t^1)H^1(s,t) + (sx_s^2 + tx_t^2)H^2(s,t) + (sx_s^3 + tx_t^3)H^3(s,t)$$

is of order four,

$$(sx_s^1 + tx_t^1)^2 + (sx_s^2 + tx_t^2)^2 + (sx_s^3 + tx_t^3)^2]^{3/2}$$

is of order three, and

$$\mid D_s\widetilde{m}_K(0,0) \times D_t\widetilde{m}_K(0,0) \mid$$

is of order two. Hence, E_3 is of order three. ∎

Corollary 3.9 Let P be a node point in $\triangle_K$ for some K, and let P be a midpoint of a side of $\triangle_K$. Then,

$$\int_\sigma \frac{f(m_K(s,t))}{\mid P - m_K(s,t) \mid} \mid D_s m_K(s,t) \times D_t m_K(s,t) \mid \; ds\, dt -$$

$$\int_\sigma \frac{f_N(\widetilde{m}_K(s,t))}{\mid P - \widetilde{m}_K(s,t) \mid} \mid D_s \widetilde{m}_K(s,t) \times D_t \widetilde{m}_K(s,t) \mid \; ds\, dt \;=\; O(\widehat{\delta}_K^3)$$

where $\widehat{\delta}_K$ is the diameter of $\triangle_K$.

Proof: Without loss of generality, we assume that

$$P \;=\; m_K(0, \frac{1}{2}) \;=\; \widetilde{m}_K(0, \frac{1}{2}) \;=\; (p_1,\, p_2,\, p_3)$$

where $m_K(s,t) \;=\; F_j(uv_1 + tv_2 + sv_3)$ for some j. First, we split σ into two triangles, σ_1 and σ_2, and we put the singular point at a vertex in each of the new triangles; see figure 3.1 for the case with $P \;=\; m_K(\rho_4)$. Let

$$m^1(s',t') \;\equiv\; m_K(s',\, \frac{t'}{2}) \tag{3.23}$$

$$m^2(s',t') \;\equiv\; m_K(s',\, \frac{1}{2}(1 - s' + t')) \tag{3.24}$$

$$\widetilde{m}^1(s',t') \;\equiv\; \widetilde{m}_K(s',\, \frac{t'}{2}) \tag{3.25}$$

$$\widetilde{m}^2(s',t') \;\equiv\; \widetilde{m}_K(s',\, \frac{1}{2}(1 - s' + t')) \tag{3.26}$$

$$\int_\sigma \frac{f(m_K(s,t))}{\mid P - m_K(s,t) \mid} \mid D_s m_K(s,t) \times D_t m_K(s,t) \mid \; ds\, dt$$

$$= \; \sum_{i=1}^{2} \int_{\sigma_i} \frac{f(m_K(s,t))}{\mid P - m_K(s,t) \mid} \mid D_s m_K(s,t) \times D_t m_K(s,t) \mid \; ds\, dt$$

$$= \; \frac{1}{2} \int_\sigma \frac{f(m^1(s',t'))}{\mid P - m^1(s',t') \mid} \mid D_s m^1(s',t') \times D_t m^1(s',t') \mid ds'dt'$$

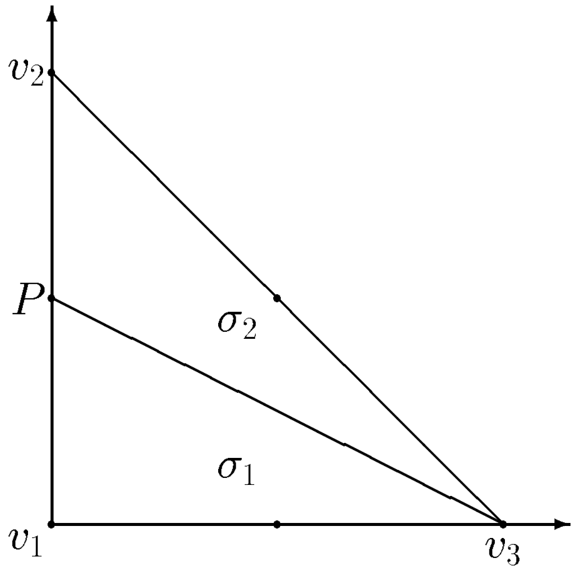

Figure 3.1: Splitting triangles

$$+ \ \frac{1}{2}\int_{\sigma} \frac{f(m^2(s',t'))}{\mid P - m^2(s',t') \mid} \mid D_s m^2(s',t') \times D_t m^2(s',t') \mid \ ds'\,dt'$$

We have

$$\sum_{i=1}^{2}\int_{\sigma_i} \frac{f(m_K(s,t))}{\mid P - m_K(s,t) \mid} \mid D_s m_K(s,t) \times D_t m_K(s,t) \mid \ ds\,dt -$$

$$\sum_{i=1}^{2}\int_{\sigma_i} \frac{f_N(\widetilde{m}_K(s,t))}{\mid P - \widetilde{m}_K(s,t) \mid} \mid D_s \widetilde{m}_K(s,t) \times D_t \widetilde{m}_K(s,t) \mid \ ds\,dt$$

$$= \ E_1 \ + \ E_2$$

with

$$E_i \;=\; \frac{1}{2}\int_\sigma \frac{f(m^i(s',t'))}{\mid P - m^i(s',t') \mid} \mid D_s m^i(s',t') \times D_t m^i(s',t') \mid \; ds'\, dt'$$

$$-\frac{1}{2}\int_\sigma \frac{f_N(\widetilde{m}^i(s',t'))}{\mid P - \widetilde{m}^i(s',t') \mid} \mid D_s \widetilde{m}^i(s',t') \times D_t \widetilde{m}^i(s',t') \mid \; ds'\, dt' \qquad (3.27)$$

for $i = 1,\ 2.$

The m^i,s are degree two polynomials for each component, and $m^i - \widetilde{m}^i$ is of order three. Applying Theorem 3.8, E_1 and E_2 are of order three. Thus, the error contributed by the integral over $\triangle_K$, which contains P, is always of order three, no matter whether P is a vertex or a midpoint of a side of $\triangle_K$. ■

Combining these two results, the local error from the single layer integral is $O(\widehat{\delta}_K^3)$ when $P \in \triangle_K$. In the next theorem, we examine the errors from integrating over those triangles $\triangle_K$ which do not contain P. Then, we can combine these two cases together and give the global error for the single layer integration.

Theorem 3.10 Let P be a node point, and consider all $\triangle_K$ for which $P \notin \triangle_K$. Then

$$\sum_K \int_\sigma \frac{f(m_K(s,t))}{\mid P - m_K(s,t) \mid} \mid D_s m_K(s,t) \times D_t m_K(s,t) \mid \; ds\, dt -$$

$$\sum_K \int_\sigma \frac{f_N(\widetilde{m}_K(s,t))}{\mid P - \widetilde{m}_K(s,t) \mid} \mid D_s \widetilde{m}_K(s,t) \times D_t \widetilde{m}_K(s,t) \mid \; ds\, dt \;=\; O(\widehat{\delta}_K^3)$$

where $\widehat{\delta}_K$ is the diameter of $\triangle_K$.

Proof: Since $P \notin \triangle_K$, we can treat the function $1/\mid P - m_K(s,t)\mid$ as a smooth function. All results from Lemma 3.2 and Theorem 3.3 – 3.7 can be applied with slight changes. By using the same ideas, notation, and results in Section 3.4, $E_1, \ldots, E_5$ will be investigated.

In Lemma 3.2, we had the following formula for every $(s,t) \in \sigma$.

$$
\begin{aligned}
& k(P, m_K(s,t)) f(m_K(s,t))(\mid D_s m(s,t) \times D_t m(s,t) \mid - \mid D_s \widetilde{m}(s,t) \times D_t \widetilde{m}(s,t) \mid) \\
& = k(P, m_K(0,0)) f(m_K(0,0))(E4(s,t) + E5(s,t)) \\
& + k(P, m_K(0,0))[s f_s(m_K(0,0)) + t f_t(m_K(0,0))] E4(s,t) \\
& + [s k_s(P, m_K(0,0)) + t k_t(P, m_K(0,0))] f(m_K(0,0)) E4(s,t) + \ O(\widehat{\delta}^6) \qquad (3.28)
\end{aligned}
$$

In this theorem,

$$
\begin{aligned}
k(P, m_K(s,t)) \ &= \ \frac{1}{\mid P - m_K(s,t) \mid} \\
&= \frac{1}{[(p_1 - x^1(s,t))^2 + (p_2 - x^2(s,t))^2 + (p_3 - x^3(s,t))^2]^{1/2}} \\
&= \frac{1}{[(p_1 - x^1(0,0))^2 + (p_2 - x^2(0,0))^2 + (p_3 - x^3(0,0))^2]^{1/2}} \\
&- \frac{s x_s^1(0,0)(p_1 - x^1(0,0)) + s x_s^2(0,0)(p_2 - x^2(0,0)) + s x_s^3(0,0)(p_3 - x^3(0,0))}{[(p_1 - x^1(0,0))^2 + (p_2 - x^2(0,0))^2 + (p_3 - x^3(0,0))^2]^{3/2}}
\end{aligned}
$$

$$- \frac{tx_t^1(0,0)(p_1 - x^1(0,0)) + tx_t^2(0,0)(p_2 - x^2(0,0)) + tx_t^3(0,0)(p_3 - x^3(0,0))}{[(p_1 - x^1(0,0))^2 + (p_2 - x^2(0,0))^2 + (p_3 - x^3(0,0))^2]^{3/2}}$$

$$+ \ O(\frac{\widehat{\delta}^2}{d_K^3}) \tag{3.29}$$

where d_K is the distance between the point P and the triangle $\triangle_K$. Note that we obtain the error term $O(\widehat{\delta}^2/d_K^3)$ from the Taylor error formula

$$O(\frac{\widehat{\delta}^2}{d_K^3}) \ = \ \left(s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t}\right)^2 \left(\frac{1}{\mid P - m_K(s,t) \mid}\right)\Bigg|_{\substack{s = \zeta \\ t = \eta}}$$

where (ζ, η) is between $(0,0)$ and (s,t). Then, the equation (3.28) becomes

$$k(P, m_K(s,t)) f(m_K(s,t)) (\mid D_s m(s,t) \times D_t m(s,t) \mid - \mid D_s \widetilde{m}(s,t) \times D_t \widetilde{m}(s,t) \mid)$$

$$= k(P, m_K(0,0)) f(m_K(0,0)) (E4(s,t) + E5(s,t))$$

$$+ k(P, m_K(0,0)) [s f_s(m_K(0,0)) + t f_t(m_K(0,0))] E4(s,t)$$

$$+ [s k_s(P, m_K(0,0)) + t k_t(P, m_K(0,0))] f(m_K(0,0)) E4(s,t) \ + \ O(\frac{\widehat{\delta}_K^6}{d_K^3})$$

Let

$$d(P) \ \equiv \ d \ = \ \min \ \{ \ d_K \mid P \notin \triangle_K, \ \ K \ = \ 1, \ldots, N \ \} \ .$$

Therefore, $d_K \ \doteq \ d, 2d, \cdots$, depends on how far the $\triangle_K$ is away from the point P.

Let

$$r \ = \ \frac{\widehat{\delta}}{d} \ .$$

Note the indexing $\triangle_1, \ldots, \triangle_N$ does not indicate distance from P. But, there is an arrangement of $\{\triangle_K\}$ where the number of triangles at a distance R is proportional to R, with $R = d,\ 2d,\ 3d, \cdots$. Then, we claim that E_1 is of order three.

Proof of claim: There are two types of triangles in each R_j of (2.7). Those triangles that are part of symmetric pairs of triangles are of the first type, and the remaining triangles are of the second type.

For the triangles of the first type, the error contributed by each of them is improved from $O(\widehat{\delta}_K^5/d_K^2)$ to $O(\widehat{\delta}_K^6/d_K^3)$, because of the cancellation over symmetric pairs. The number, c_i, of triangles of the first type at a distance $i \cdot d$ is proportional to i for $i = 1, \ldots, t_j$. Note that for some integer t_j, $t_j \cdot d$ is the longest possible distance from P to triangles in R_j. Adding the error contributed by each triangle of the first type in R_j, we have

$$\sum_K O(\frac{\widehat{\delta}_K^6}{d_K^3}) \;=\; \sum_{i=1}^{t_j} c_i \cdot O(\frac{\widehat{\delta}^6}{(i \cdot d)^3}) \;=\; O(\widehat{\delta}^3) \sum_{i=1}^{t_j} r^3 \frac{i}{i^3} \;=\; O(\widehat{\delta}^3).$$

For the triangles of the second type, the error contributed by each of them is $O(\widehat{\delta}_K^5/d_K^2)$. The number of triangles of this type at a distance $i \cdot d$ is a finite number, and it usually is two or three; but the proof is omitted. Therefore,

$$\sum_K O(\frac{\widehat{\delta}_K^5}{d_K^2}) \;=\; \sum_{i=1}^{t_j} c_j' \cdot O(\frac{\widehat{\delta}^5}{(i \cdot d)^2}) \;=\; O(\widehat{\delta}^3) \sum_{i=1}^{t_j} r^2 \frac{1}{i^2} \;=\; O(\widehat{\delta}^3)$$

where c_j' is either two or three. This completes the claim, and this method of computing the error will be used often in the following derivations.

For the second part of the error analysis,

$$\int_\sigma \frac{f(m_K(s,t)) - f_N(m_K(s,t))}{\mid P - m_K(s,t) \mid} \{ \mid D_s m_K \times D_t m_K \mid - \mid D_s \widetilde{m}_K \times D_t \widetilde{m}_K \mid \} \, dsdt$$

$$= O(\widehat{\delta}^3) \cdot O(\widehat{\delta}^4) \cdot \int_\sigma \frac{1}{\mid P - m_K(s,t) \mid} \, ds \, dt \;=\; O(\widehat{\delta}^7) \cdot \frac{1}{d_K}$$

for every K where $P \notin \triangle_K$. Adding errors from each triangle, we have that E_2 is $O(\widehat{\delta}^5)$, as we discussed in computing E_1. Using the proof of Theorem 3.5 carefully can give us that the error from

$$\int_\sigma \frac{f(m_K(s,t)) - f_N(m_K(s,t))}{\mid P - m_K(s,t) \mid} \mid D_s m_K \times D_t m_K \mid \, dsdt$$

is $O(\widehat{\delta}^6 / d_K)$ for every triangle. Thus, following the discussion of the claim, E_3 is $O(\widehat{\delta}^4)$.

Analyzing E_4, from Theorem 3.6, we have

$$k(P, m_K(s,t)) - k(P, \widetilde{m}_K(s,t)) = \frac{1}{\mid P - m_K(s,t) \mid} - \frac{1}{\mid P - \widetilde{m}_K(s,t) \mid}$$

$$= \frac{(H^1 + O(\widehat{\delta}^4))(p_1 - x^1) + (H^2 + O(\widehat{\delta}^4))(p_2 - x^2) + (H^3 + O(\widehat{\delta}^4))(p_3 - x^3)}{[(p_1 - x^1(s,t))^2 + (p_2 - x^2(s,t))^2 + (p_3 - x^3(s,t))^2]^{3/2}}$$

$$+ O(\frac{\widehat{\delta}^6}{d_K^3})$$

$$= \frac{(H^1 + O(\widehat{\delta}^4))(p_1 - x^1) + (H^2 + O(\widehat{\delta}^4))(p_2 - x^2) + (H^3 + O(\widehat{\delta}^4))(p_3 - x^3)}{[(p_1 - x^1(0,0))^2 + (p_2 - x^2(0,0))^2 + (p_3 - x^3(0,0))^2]^{3/2}}$$

$$+ (H^1 + H^2 + H^3 + O(\widehat{\delta}^4)) O(\frac{\widehat{\delta}}{d_K^3}) + O(\frac{\widehat{\delta}^6}{d_K^3})$$

$$= \frac{(H^1 + O(\widehat{\delta}^4))(p_1 - x^1) + (H^2 + O(\widehat{\delta}^4))(p_2 - x^2) + (H^3 + O(\widehat{\delta}^4))(p_3 - x^3)}{[(p_1 - x^1(0,0))^2 + (p_2 - x^2(0,0))^2 + (p_3 - x^3(0,0))^2]^{3/2}}$$

$$+ O(\frac{\widehat{\delta}_K^4}{d_K^3}) \, . \tag{3.30}$$

Therefore, by combining (3.10) and (3.30), we have

$$\int_\sigma \left\{ \frac{1}{| P - m_K(s,t) |} - \frac{1}{| P - \widetilde{m}_K(s,t) |} \right\} f_N(m_K(s,t)) \mid D_s m_K \times D_t m_K \mid \, ds \, dt \, -$$

$$\int_\sigma \left\{ \frac{1}{| P - m_K(s,t) |} - \frac{1}{| P - \widetilde{m}_K(s,t) |} \right\} f_N(m_K(s,t)) \mid D_s \widetilde{m}_K \times D_t \widetilde{m}_K \mid) ds dt$$

$$= O(\frac{\widehat{\delta}_K^7}{d_K^2})$$

for each $\triangle_K$, and after adding up errors, $E_4 = O(\widehat{\delta}^5 \ln \widehat{\delta})$.

For E_5, each triangle give us an error of $O(\widehat{\delta}_K^5 / d_K^2)$. When adding errors together, cancellation happens at every symmetric pair of triangles and errors become $O(\widehat{\delta}_K^6 / d_K^3)$. Thus, as we discussed in computing E_1, E_5 is $O(\widehat{\delta}^3)$. After going through E_1–E_5, the global error for the single layer integral, in which $P - m_K(s,t)$ is nonzero for every K, is $O(\widehat{\delta}^3)$. This result is uniform as P ranges over the node points of the triangulation. ■

Combining the above theorems, we get the following corollary, which gives the total error for evaluating the single layer integral at the node point.

Corollary 3.11 Let S be a piecewise smooth surface, and let P be a node point on S. Then

$$\int_S \frac{f(Q)}{\mid P-Q \mid} dS_Q - \sum_{K=1}^{N} \int_\sigma \frac{f(\widetilde{m}_K(s,t))}{\mid P-\widetilde{m}_K(s,t) \mid} \mid D_s\widetilde{m}_K(s,t) \times D_t\widetilde{m}_K(s,t) \mid \, ds\, dt = O(\widehat{\delta}^3) \, .$$

Proof: Combine Theorem 3.8, Corollary 3.9, and Theorem 3.10. ■

3.6 Computation of the solid angle $\Omega(P)$

Returning to the integral equation (2.2), it can be shown that

$$\int_S \frac{\partial}{\partial \nu_Q} \left[\frac{1}{\mid P-Q \mid} \right] dS_Q + [2\pi - \Omega(P)] \;=\; 2\pi, \qquad P \in S \, . \tag{3.31}$$

In our discretization of (2.4), we force this identity to remain true. This means using the sum of the approximations of the integral operator terms on the left side of (2.4) to approximate $\Omega(P)$ when P is a node point. We denote the approximate value of $\Omega(P)$ by $\Omega_N(P)$.

Rewriting equation (3.31), we have

$$\Omega(P) \;=\; \int_S \frac{\partial}{\partial \nu_Q} \left[\frac{1}{\mid P-Q \mid} \right] dS_Q \qquad P \in S \, , \tag{3.32}$$

where

$$\frac{\partial}{\partial \nu_Q} \left[\frac{1}{\mid P-Q \mid} \right] \;=\; \nu_Q \cdot \nabla_Q \left[\frac{1}{\mid P-Q \mid} \right] .$$

ν_Q denotes the unit normal to S at Q. For $Q \;=\; m_K(s,t)$,

$$\nu(s,t) \;=\; \nu_Q \;=\; \pm \frac{D_s m_K \times D_t m_K}{\mid D_s m_K \times D_t m_K \mid}$$

with the sign chosen so that ν_Q is directed into the bounded domain D.

Approximating $\Omega(P)$ by $\Omega_N(P)$, we write

$$\Omega(P) \;=\; \int_S \frac{\partial}{\partial \nu_Q} \left[\frac{1}{\mid P-Q \mid} \right] dS_Q$$

$$= \sum_{K=1}^{N} \int_{\sigma} \nu_Q \cdot \nabla_Q \left[\frac{1}{\mid P - m_K(s,t) \mid} \right] \mid D_s m_K \times D_t m_K \mid \, ds \, dt$$

$$= \sum_{K=1}^{N} \int_{\sigma} \pm \frac{D_s m_K \times D_t m_K}{\mid D_s m_K \times D_t m_K \mid} \cdot \frac{P - m_K(s,t)}{\mid P - m_K(s,t) \mid^3} \mid D_s m_K \times D_t m_K \mid \, ds \, dt$$

$$= \sum_{K=1}^{N} \int_{\sigma} \pm (D_s m_K \times D_t m_K) \cdot \frac{P - m_K(s,t)}{\mid P - m_K(s,t) \mid^3} \, ds \, dt$$

and

$$\Omega_N(P) = \int_{\widetilde{S}} \frac{\partial}{\partial \widetilde{\nu}_Q} \left[\frac{1}{\mid P - Q \mid} \right] dS_Q$$

$$= \sum_{K=1}^{N} \int_{\sigma} \widetilde{\nu}_Q \cdot \nabla_Q \left[\frac{1}{\mid P - \widetilde{m}_K(s,t) \mid} \right] \mid D_s \widetilde{m}_K \times D_t \widetilde{m}_K \mid \, ds \, dt$$

$$= \sum_{K=1}^{N} \int_{\sigma} \pm \frac{D_s \widetilde{m}_K \times D_t \widetilde{m}_K}{\mid D_s \widetilde{m}_K \times D_t \widetilde{m}_K \mid} \cdot \frac{P - \widetilde{m}_K(s,t)}{\mid P - \widetilde{m}_K(s,t) \mid^3} \mid D_s \widetilde{m}_K \times D_t \widetilde{m}_K \mid \, ds \, dt$$

$$= \sum_{K=1}^{N} \int_{\sigma} \pm (D_s \widetilde{m}_K \times D_t \widetilde{m}_K) \cdot \frac{P - \widetilde{m}_K(s,t)}{\mid P - \widetilde{m}_K(s,t) \mid^3} \, ds \, dt$$

Without loss of generality, we assume the signs of ν_Q and $\widetilde{\nu}_Q$ are always positive, and compute

$$\Omega(P) - \Omega_N(P) = \sum_{K=1}^{N} \int_{\sigma} (D_s m_K \times D_t m_K) \cdot \frac{P - m_K(s,t)}{\mid P - m_K(s,t) \mid^3} \, ds \, dt$$

$$- \sum_{K=1}^{N} \int_{\sigma} (D_s \widetilde{m}_K \times D_t \widetilde{m}_K) \cdot \frac{P - \widetilde{m}_K(s,t)}{\mid P - \widetilde{m}_K(s,t) \mid^3} \, ds \, dt$$

where P is a node point.

The integrand has one singularity when $Q = P$, and it is quiet smooth over those $\triangle_K$ with $P \notin \triangle_K$, although it is increasingly peaked as P and $\triangle_K$ become closer together. This behavior has been seen in the single layer integral, and we can use the same approach as we used for single layer integral. Computing the error

$$\int_\sigma \left[(D_s m_K \times D_t m_K) \cdot \frac{P - m_K(s,t)}{\mid P - m_K(s,t) \mid^3} - (D_s \widetilde{m}_K \times D_t \widetilde{m}_K) \cdot \frac{P - \widetilde{m}_K(s,t)}{\mid P - \widetilde{m}_K(s,t) \mid^3} \right] ds\, dt$$

with two steps, as in the case of single layer integral, the first step is for $\triangle_K$'s which contain P, and the second part is for triangles which do not contain the node point P.

Theorem 3.12 Let P be a vertex of $\triangle_K$ for some K. Then

$$\int_\sigma \left[(D_s m_K \times D_t m_K) \cdot \frac{P - m_K(s,t)}{\mid P - m_K(s,t) \mid^3} - \right.$$

$$\left. (D_s \widetilde{m}_K \times D_t \widetilde{m}_K) \cdot \frac{P - \widetilde{m}_K(s,t)}{\mid P - \widetilde{m}_K(s,t) \mid^3} \right] ds\, dt \;=\; O(\widehat{\delta}_K^3) \quad (3.33)$$

where $\widehat{\delta}_K$ is the diameter of $\triangle_K$.

Proof: Without loss of generality, assume $P = m_K(0,0)$. Let

$$P \;=\; (p_1, p_2, p_3) \;=\; m_K(0,0) \;=\; \widetilde{m}_K(0,0)\,.$$

We break this proof into two parts:

$$E_1 = \int_\sigma \Bigg[(D_s m_K \times D_t m_K) \cdot \frac{P - m_K(s,t)}{\mid P - m_K(s,t) \mid^3} -$$

$$(D_s \widetilde{m}_K \times D_t \widetilde{m}_K) \cdot \frac{P - \widetilde{m}_K(s,t)}{\mid P - m_K(s,t) \mid^3} \Bigg] \, ds \, dt \qquad (3.34)$$

and

$$E_2 = \int_\sigma \Bigg[(D_s \widetilde{m}_K \times D_t \widetilde{m}_K) \cdot \frac{P - \widetilde{m}_K(s,t)}{\mid P - m_K(s,t) \mid^3} -$$

$$(D_s \widetilde{m}_K \times D_t \widetilde{m}_K) \cdot \frac{P - \widetilde{m}_K(s,t)}{\mid P - \widetilde{m}_K(s,t) \mid^3} \Bigg] \, ds \, dt \ . \qquad (3.35)$$

We now check the first part of the integrand of (3.34).

$$(D_s m_K(s,t) \times D_t m_K(s,t)) \cdot \frac{P - m_K(s,t)}{\mid P - m_K(s,t) \mid^3}$$

$$= \frac{(x_s^2 x_t^3 - x_s^3 x_t^2,\ x_s^3 x_t^1 - x_s^1 x_t^3,\ x_s^1 x_t^2 - x_s^2 x_t^1) \cdot (p_1 - x^1,\ p_2 - x^2,\ p_3 - x^3)}{[(p_1 - x^1)^2 + (p_2 - x^2)^2 + (p_3 - x^3)^2]^{3/2}} \qquad (3.36)$$

Using the Taylor error formula for the x^i about $(s,t) = (0,0)$, the numerator of equation (3.36) becomes

$$\begin{aligned}
& (x_s^2 x_t^3 - x_s^3 x_t^2,\ x_s^3 x_t^1 - x_s^1 x_t^3,\ x_s^1 x_t^2 - x_s^2 x_t^1) \cdot (p_1 - x^1,\ p_2 - x^2,\ p_3 - x^3) \\
= \ & (x_s^2 x_t^3 - x_s^3 x_t^2)(p_1 - x^1) + (x_s^3 x_t^1 - x_s^1 x_t^3)(p_2 - x^2) + (x_s^1 x_t^2 - x_s^2 x_t^1)(p_3 - x^3) \\
= \ & [(x_s^2 + s x_{ss}^2 + t x_{st}^2 + O(\widehat{\delta}_K^3))(x_t^3 + s x_{st}^3 + t x_{tt}^3 + O(\widehat{\delta}_K^3)) \\
& - (x_s^3 + s x_{ss}^3 + t x_{st}^3 + O(\widehat{\delta}_K^3))(x_t^2 + s x_{st}^2 + t x_{tt}^2 + O(\widehat{\delta}_K^3))](s x_s^1 + t x_t^1 + O(\widehat{\delta}_K^2)) \\
& + [(x_s^3 + s x_{ss}^3 + t x_{st}^3 + O(\widehat{\delta}_K^3))((x_t^1 + s x_{st}^1 + t x_{tt}^1 + O(\widehat{\delta}_K^3))
\end{aligned}$$

$$
\begin{aligned}
& -(x_s^1 + sx_{ss}^1 + tx_{st}^1 + O(\widehat{\delta}_K^3))(x_t^3 + sx_{st}^3 + tx_{tt}^3 + O(\widehat{\delta}_K^3))](sx_s^2 + tx_t^2 + O(\widehat{\delta}_K^2)) \\
& + [(x_s^1 + sx_{ss}^1 + tx_{st}^1 + O(\widehat{\delta}_K^3))(x_t^2 + sx_{st}^2 + tx_{tt}^2 + O(\widehat{\delta}_K^3)) \\
& -(x_s^2 + sx_{ss}^2 + tx_{st}^2 + O(\widehat{\delta}_K^3))(x_t^1 + sx_{st}^1 + tx_{tt}^1 + O(\widehat{\delta}_K^3))](sx_s^3 + tx_t^3 + O(\widehat{\delta}_K^2)) \\
= \; & (sx_s^1 + tx_t^1 + O(\widehat{\delta}_K^2))[x_s^2 x_t^3 - x_s^3 x_t^2 + x_s^2(sx_{st}^3 + tx_{tt}^3 + O(\widehat{\delta}_K^3)) \\
& + x_t^3(sx_{ss}^2 + tx_{st}^2 + O(\widehat{\delta}_K^3)) + (sx_{ss}^2 + tx_{st}^2 + O(\widehat{\delta}_K^3))(sx_{st}^3 + tx_{tt}^3 + O(\widehat{\delta}_K^3)) \\
& - x_s^3(sx_{st}^2 + tx_{tt}^2 + O(\widehat{\delta}_K^3)) - x_t^2(sx_{ss}^3 + tx_{st}^3 + O(\widehat{\delta}_K^3)) \\
& -(sx_{ss}^3 + tx_{st}^3 + O(\widehat{\delta}_K^3))(sx_{st}^2 + tx_{tt}^2 + O(\widehat{\delta}_K^3))] \; + \\
& (sx_s^2 + tx_t^2 + O(\widehat{\delta}_K^2))[x_s^3 x_t^1 - x_s^1 x_t^3 + x_s^3(sx_{st}^1 + tx_{tt}^1 + O(\widehat{\delta}_K^3)) \\
& + x_t^1(sx_{ss}^3 + tx_{st}^3 + O(\widehat{\delta}_K^3)) + (sx_{ss}^3 + tx_{st}^3 + O(\widehat{\delta}_K^3))(sx_{st}^1 + tx_{tt}^1 + O(\widehat{\delta}_K^3)) \\
& - x_s^1(sx_{st}^3 + tx_{tt}^3 + O(\widehat{\delta}_K^3)) - x_t^3(sx_{ss}^1 + tx_{st}^1 + O(\widehat{\delta}_K^3)) \\
& -(sx_{ss}^1 + tx_{st}^1 + O(\widehat{\delta}_K^3))(sx_{st}^3 + tx_{tt}^3 + O(\widehat{\delta}_K^3))] \; + \\
& (sx_s^3 + tx_t^3 + O(\widehat{\delta}_K^2))[x_s^1 x_t^2 - x_s^2 x_t^1 + x_s^1(sx_{st}^2 + tx_{tt}^2 + O(\widehat{\delta}_K^3)) \\
& + x_t^2(sx_{ss}^1 + tx_{st}^1 + O(\widehat{\delta}_K^3)) + (sx_{ss}^1 + tx_{st}^1 + O(\widehat{\delta}_K^3))(sx_{st}^2 + tx_{tt}^2 + O(\widehat{\delta}_K^3)) \\
& - x_s^2(sx_{st}^1 + tx_{tt}^1 + O(\widehat{\delta}_K^3)) - x_t^1(sx_{ss}^2 + tx_{st}^2 + O(\widehat{\delta}_K^3)) \\
& -(sx_{ss}^2 + tx_{st}^2 + O(\widehat{\delta}_K^3))(sx_{st}^1 + tx_{tt}^1 + O(\widehat{\delta}_K^3))] \\
= \; & (x_t^1 x_s^2 - x_t^2 x_s^1)(s^2 x_{ss}^3 + 2st x_{st}^3 + t^2 x_{tt}^3) + (x_t^2 x_s^3 - x_t^3 x_s^2)(s^2 x_{ss}^1 + 2st x_{st}^1 + t^2 x_{tt}^1) \\
& + (x_t^3 x_s^1 - x_t^1 x_s^3)(s^2 x_{ss}^2 + 2st x_{st}^2 + t^2 x_{tt}^2) + O(\widehat{\delta}_K^5)\,. \qquad (3.37)
\end{aligned}
$$

Computing the corresponding part of the second term of (3.34) with the same formula as we had above,

$$
\begin{aligned}
&(D_s\widetilde{m}_K \times D_t\widetilde{m}_K)\cdot(P-\widetilde{m}_K(s,t))\\
&= (\widetilde{x}_t^1\widetilde{x}_s^2 - \widetilde{x}_t^2\widetilde{x}_s^1)(s^2\widetilde{x}_{ss}^3 + 2st\widetilde{x}_{st}^3 + t^2\widetilde{x}_{tt}^3) + (\widetilde{x}_t^2\widetilde{x}_s^3 - \widetilde{x}_t^3\widetilde{x}_s^2)(s^2\widetilde{x}_{ss}^1 + 2st\widetilde{x}_{st}^1 + t^2\widetilde{x}_{tt}^1)\\
&\quad + (\widetilde{x}_t^3\widetilde{x}_s^1 - \widetilde{x}_t^1\widetilde{x}_s^3)(s^2\widetilde{x}_{ss}^2 + 2st\widetilde{x}_{st}^2 + t^2\widetilde{x}_{tt}^2) + O(\widehat{\delta}_K^5)\\
&= \Big[(x_t^1 + O(\widehat{\delta}_K^3))(x_s^2 + O(\widehat{\delta}_K^3)) - (x_t^2 + O(\widehat{\delta}_K^3))(x_s^1 + O(\widehat{\delta}_K^3))\Big]\cdot\\
&\quad \Big[s^2(x_{ss}^3 + O(\widehat{\delta}_K^3)) + 2st(x_{st}^3 + O(\widehat{\delta}_K^3)) + t^2(x_{tt}^3 + O(\widehat{\delta}_K^3))\Big]\\
&\quad + \Big[(x_t^2 + O(\widehat{\delta}_K^3))(x_s^3 + O(\widehat{\delta}_K^3)) - (x_t^3 + O(\widehat{\delta}_K^3))(x_s^2 + O(\widehat{\delta}_K^3))\Big]\cdot\\
&\quad \Big[s^2(x_{ss}^1 + O(\widehat{\delta}_K^3)) + 2st(x_{st}^1 + O(\widehat{\delta}_K^3)) + t^2(x_{tt}^1 + O(\widehat{\delta}_K^3))\Big]\\
&\quad + \Big[(x_t^3 + O(\widehat{\delta}_K^3))(x_s^1 + O(\widehat{\delta}_K^3)) - (x_t^1 + O(\widehat{\delta}_K^3))(x_s^3 + O(\widehat{\delta}_K^3))\Big]\cdot\\
&\quad \Big[s^2(x_{ss}^2 + O(\widehat{\delta}_K^3)) + 2st(x_{st}^2 + O(\widehat{\delta}_K^3)) + t^2(x_{tt}^2 + O(\widehat{\delta}_K^3))\Big] + O(\widehat{\delta}_K^5)\\
&= (x_t^1x_s^2 - x_t^2x_s^1)(s^2x_{ss}^3 + 2stx_{st}^3 + t^2x_{tt}^3) + (x_t^2x_s^3 - x_t^3x_s^2)(s^2x_{ss}^1 + 2stx_{st}^1 + t^2x_{tt}^1)\\
&\quad + (x_t^3x_s^1 - x_t^1x_s^3)(s^2x_{ss}^2 + 2stx_{st}^2 + t^2x_{tt}^2) + O(\widehat{\delta}_K^5)
\end{aligned}
\tag{3.38}
$$

Thus,

$$
(D_sm_K \times D_tm_K)\cdot(P-m_K(s,t)) - (D_s\widetilde{m}_K \times D_t\widetilde{m}_K)\cdot(P-\widetilde{m}_K(s,t)) = O(\widehat{\delta}_K^5). \tag{3.39}
$$

Expanding each x^i about $(0,0)$, the denominator of (3.36) is

$$
\begin{aligned}
&[(p_1-x^1)^2 + (p_2-x^2)^2 + (p_3-x^3)^2]^{-3/2}\\
&= [(sx_s^1 + tx_t^1 + O(\widehat{\delta}_K^2))^2 + (sx_s^2 + tx_t^2 + O(\widehat{\delta}_K^2))^2 + (sx_s^3 + tx_t^3 + O(\widehat{\delta}_K^2))^2]^{-3/2}\\
&= [(sx_s^1 + tx_t^1)^2 + (sx_s^2 + tx_t^2)^2 + (sx_s^3 + tx_t^3)^2 + O(\widehat{\delta}_K^3)]^{-3/2} \;=\; O(|\,\widehat{\delta}_K\,|^{-3})
\end{aligned}
$$

Then,

$$\int_\sigma \frac{(D_s m_K \times D_t m_K)\cdot(P - m_K(s,t)) - (D_s\widetilde{m}_K \times D_t\widetilde{m}_K)\cdot(P - \widetilde{m}_K(s,t))}{\mid P - m_K(s,t)\mid^3}\, ds\, dt$$

$$= \quad O(\widehat{\delta}_K^2) \tag{3.40}$$

Furthermore, examining (3.39) carefully, we can see that cancellation happens again on every symmetric pair of triangles. Thus, the error from (3.39) is $O(\widehat{\delta}_K^6)$; and therefore, the error in (3.40) can be improved to be $O(\widehat{\delta}_K^3)$. Since there are at most six triangles containing the node point P, and the error from (3.40) should be $O(\widehat{\delta}_K^3)$.

To finish this theorem, we need to know the error from the following:

$$\mid \frac{1}{\mid P - m_K(s,t)\mid^3} - \frac{1}{\mid P - \widetilde{m}_K(s,t)\mid^3} \mid$$

$$\leq \quad \mid \frac{1}{\mid P - m_K(s,t)\mid} - \frac{1}{\mid P - \widetilde{m}_K(s,t)\mid} \mid \cdot \mid \frac{1}{\mid P - m_K(s,t)\mid^2}$$

$$+ \frac{1}{(\mid P - m_K(s,t)\mid)(\mid P - \widetilde{m}_K(s,t)\mid)} + \frac{1}{\mid P - \widetilde{m}_K(s,t)\mid^2} \mid \tag{3.41}$$

Using the result from (3.21), (3.41) has an error of $O(\widehat{\delta}_K^{-1})$, i.e.,

$$\mid \frac{1}{\mid P - m_K(s,t)\mid^3} - \frac{1}{\mid P - \widetilde{m}_K(s,t)\mid^3} \mid$$

$$\leq O(\widehat{\delta}_K) \cdot O(\widehat{\delta}_K^{-2}) \;=\; O(\widehat{\delta}_K^{-1})$$

Then, from the above result and (3.38), we have the following error analysis:

$$| \int_\sigma (D_s\widetilde{m}_K \times D_t\widetilde{m}_K) \cdot (P - \widetilde{m}_K(s,t)) \left[\frac{1}{| P - m_K(s,t) |^3} - \frac{1}{| P - \widetilde{m}_K(s,t) |^3} \right] ds\, dt \, |$$

$$\leq O(\widehat{\delta}_K^4) \cdot O(\widehat{\delta}_K^{-1}) \; = \; O(\widehat{\delta}_K^3) \tag{3.42}$$

Combining (3.40) and (3.42), we complete the proof of this theorem. ■

Theorem 3.13 Let P be a node point, and consider all $\triangle_K$ for which $P \notin \triangle_K$. Then

$$\sum_K \int_\sigma \left[(D_s m_K \times D_t m_K) \cdot \frac{P - m_K(s,t)}{| P - m_K(s,t) |^3} - \right.$$

$$\left. (D_s\widetilde{m}_K \times D_t\widetilde{m}_K) \cdot \frac{P - \widetilde{m}_K(s,t)}{| P - \widetilde{m}_K(s,t) |^3} \right] ds\, dt \; = \; O(\widehat{\delta}_K^2) \tag{3.43}$$

Proof: Since $P \notin \triangle_K$, again, we can treat the function $1/ \mid P - m_K(s,t) \mid^3$ as a smooth function. This proof will have two parts, as with Theorem 3.10, and we use results from the latter. Let d_K, d, and r be the same as in Theorem 3.9.

As in Theorem 3.10, we write equation (3.43) as $E_1 + E_2$ where

$$E_1 \; = \; \sum_K \int_\sigma \left[\frac{(D_s m_K \times D_t m_K) \cdot (P - m_K(s,t))}{| P - m_K(s,t) |^3} \right.$$

$$\left. - \frac{(D_s\widetilde{m}_K \times D_t\widetilde{m}_K) \cdot (P - \widetilde{m}_K(s,t))}{| P - m_K(s,t) |^3} \right] ds\, dt \tag{3.44}$$

$$E_2 = \sum_K \int_\sigma (D_s\widetilde{m}_K \times D_t\widetilde{m}_K) \cdot (P - \widetilde{m}_K(s,t)) \cdot$$

$$\left[\frac{1}{\mid P - m_K(s,t) \mid^3} - \frac{1}{\mid P - \widetilde{m}_K(s,t) \mid^3}\right] ds\, dt \tag{3.45}$$

In the previous theorem, we assumed that $P = m_K(0,0)$; and we assume

$$P \neq m_K(s,t), \qquad \forall\ (s,t) \in \sigma\ .$$

in this theorem. Therefore, we compute (3.36) again; but with a different approach.

$$
\begin{aligned}
&(D_s\widetilde{m}_K \times D_t\widetilde{m}_K) \cdot (P - \widetilde{m}_K(s,t)) \\
&= (\tilde{x}_s^2\tilde{x}_t^3 - \tilde{x}_s^3\tilde{x}_t^2,\ \tilde{x}_s^3\tilde{x}_t^1 - \tilde{x}_s^1\tilde{x}_t^3,\ \tilde{x}_s^1\tilde{x}_t^2 - \tilde{x}_s^2\tilde{x}_t^1) \cdot (p_1 - \tilde{x}^1,\ p_2 - \tilde{x}^2,\ p_3 - \tilde{x}^3) \\
&= \Big[(x_s^2 - H_s^2 - G_s^2 + O(\widehat{\delta}_K^5))(x_t^3 - H_t^3 - G_t^3 + O(\widehat{\delta}_K^5)) \\
&\quad - (x_s^3 - H_s^3 - G_s^3 + O(\widehat{\delta}_K^5))(x_t^2 - H_t^2 - G_t^2 + O(\widehat{\delta}_K^5))\Big] \cdot \\
&\quad (p_1 - x^1 + H^1 + G^1 + O(\widehat{\delta}_K^5)) \\
&\quad + \Big[(x_s^3 - H_s^3 - G_s^3 + O(\widehat{\delta}_K^5))(x_t^1 - H_t^1 - G_t^1 + O(\widehat{\delta}_K^5)) \\
&\quad - (x_s^1 - H_s^1 - G_s^1 + O(\widehat{\delta}_K^5))(x_t^3 - H_t^3 - G_t^3 + O(\widehat{\delta}_K^5))\Big] \cdot \\
&\quad (p_2 - x^2 + H^2 + G^2 + O(\widehat{\delta}_K^5)) \\
&\quad + \Big[(x_s^1 - H_s^2 - G_s^2 + O(\widehat{\delta}_K^5))(x_t^2 - H_t^3 - G_t^3 + O(\widehat{\delta}_K^5)) \\
&\quad - (x_s^2 - H_s^2 - G_s^2 + O(\widehat{\delta}_K^5))(x_t^1) - H_t^1 - G_t^1 + O(\widehat{\delta}_K^5))\Big] \cdot \\
&\quad (p_3 - x^3 + H^2 + G^2 + O(\widehat{\delta}_K^5)) \\
&= (x_s^2x_t^3 - x_s^3x_t^2,\ x_s^3x_t^1 - x_s^1x_t^3,\ x_s^1x_t^2 - x_s^2x_t^1) \cdot (p_1 - x^1,\ p_2 - x^2,\ p_3 - x^3) \\
&\ + \Big[x_s^3(H_t^2 + G_t^2) + x_t^2(H_s^3 + G_s^3) - x_s^2(H_t^3 + G_t^3) - x_t^3(H_s^2 + G_s^2) + O(\widehat{\delta}_K^6)\Big]\,(p_1 - x^1)
\end{aligned}
$$

$$+\left[x_s^1(H_t^3+G_t^3)+x_t^3(H_s^1+G_s^1)-x_s^3(H_t^1+G_t^1)-x_t^1(H_s^3+G_s^3)+O(\widehat{\delta}_K^6)\right](p_2-x^2)$$
$$+\left[x_s^2(H_t^1+G_t^1)+x_t^1(H_s^2+G_s^2)-x_s^1(H_t^2+G_t^2)\right.$$
$$\left.-\ x_t^2(H_s^1+G_s^1)+O(\widehat{\delta}_K^6)\right](p_3-x^3)+O(\widehat{\delta}_K^7) \tag{3.46}$$

Expand each x^i about $(s,t)=(0,0)$ and compute

$$(D_s m_K\times D_t m_K)\cdot(P-m_K(s,t))-(D_s\widetilde{m}_K\times D_t\widetilde{m}_K)\cdot(P-\widetilde{m}_K(s,t))$$
$$=\left[x_s^2(H_t^3+G_t^3)+x_t^3(H_s^2+G_s^2)-x_s^3(H_t^2+G_t^2)-x_t^2(H_s^3+G_s^3)+O(\widehat{\delta}_K^6)\right](p_1-x^1)$$
$$+\left[x_s^3(H_t^1+G_t^1)+x_t^1(H_s^3+G_s^3)-x_s^1(H_t^3+G_t^3)-x_t^3(H_s^1+G_s^1)+O(\widehat{\delta}_K^6)\right](p_2-x^2)$$
$$+\left[x_s^1(H_t^2+G_t^2)+x_t^2(H_s^1+G_s^1)\right.$$
$$\left.-\ x_s^2(H_t^1+G_t^1)-x_t^1(H_s^2+G_s^2)+O(\widehat{\delta}_K^6)\right](p_3-x^3)+O(\widehat{\delta}_K^7)$$
$$=\ E4(s,t)+E5(s,t)+O(\widehat{\delta}_K^6)$$

with

$$\begin{aligned}E4(s,t) &= \left[x_s^2H_t^3+x_t^3H_s^2-x_s^3H_t^2-x_t^2H_s^3\right](p_1-x^1)\\ &\quad+\left[x_s^3H_t^1+x_t^1H_s^3-x_s^1H_t^3-x_t^3H_s^1\right](p_2-x^2)\\ &\quad+\left[x_s^1H_t^2+x_t^2H_s^1-x_s^2H_t^1-x_t^1H_s^2\right](p_3-x^3)\end{aligned} \tag{3.47}$$

and

$$\begin{aligned}E5(s,t) &= \left[x_s^2G_t^3+x_t^3G_s^2-x_s^3G_t^2-x_t^2G_s^3+(sx_{ss}^2+tx_{st}^2)H_t^3+(sx_{st}^3+tx_{tt}^3)H_s^2\right.\\ &\quad\left.-\ (sx_{ss}^3+tx_{st}^3)H_t^2-(sx_{st}^2+tx_{tt}^2)H_s^3\right](p_1-x^1)\\ &\quad+\left[x_s^3G_t^1+x_t^1G_s^3-x_s^1G_t^3-x_t^3G_s^1+(sx_{ss}^3+tx_{st}^3)H_t^1+(sx_{st}^1+tx_{tt}^1)H_s^3\right.\end{aligned}$$

$$
\begin{aligned}
&- (sx_{ss}^1 + tx_{st}^1)H_t^3 - (sx_{st}^3 + tx_{tt}^3)H_s^1\Big] (p_2 - x^2) \\
&+ \Big[x_s^1 G_t^2 + x_t^2 G_s^1 - x_s^2 G_t^1 - x_t^1 G_s^2 + (sx_{ss}^1 + tx_{st}^1)H_t^2 + (sx_{st}^2 + tx_{tt}^2)H_s^1 \\
&- (sx_{ss}^2 + tx_{st}^2)H_t^1 - (sx_{st}^1 + tx_{tt}^1)H_s^2\Big] (p_3 - x^3) \\
&- \Big[x_s^2 H_t^3 + x_t^3 H_s^2 - x_s^3 H_t^2 - x_t^2 H_s^3\Big] (sx_s^1 + tx_t^1) \\
&- \Big[x_s^3 H_t^1 + x_t^1 H_s^3 - x_s^1 H_t^3 - x_t^3 H_s^1\Big] (sx_s^2 + tx_t^2) \\
&- \Big[x_s^1 H_t^2 + x_t^2 H_s^1 - x_s^2 H_t^1 - x_t^1 H_s^2\Big] (sx_s^3 + tx_t^3)\,. \qquad (3.48)
\end{aligned}
$$

$E4$ and $E5$ are not the same as the $E4$ and $E5$ in the previous derivation, but they have very similar properties. $E4$ is a collection of terms that are $O(\widehat{\delta}_K^4 \cdot d_K)$; and integrating $E4$ over σ yields zero. $E5$ contains terms which are of order five, and with it, cancellation happens.

Before examining E_1, we need to expand the function $\mid P - m_K(s,t) \mid^{-3}$ about $(s,t) = (0,0)$.

$$
\begin{aligned}
&\frac{1}{\mid P - m_K(s,t) \mid^3} \\
&= \frac{1}{[(p_1 - x^1(s,t))^2 + (p_2 - x^2(s,t))^2 + (p_3 - x^3(s,t))^2]^{3/2}} \\
&= \frac{1}{[(p_1 - x^1(0,0))^2 + (p_2 - x^2(0,0))^2 + (p_3 - x^3(0,0))^2]^{3/2}} \\
&- \frac{sx_s^1(0,0)(p_1 - x^1(0,0)) + sx_s^2(0,0)(p_2 - x^2(0,0)) + sx_s^3(0,0)(p_3 - x^3(0,0))}{[(p_1 - x^1(0,0))^2 + (p_2 - x^2(0,0))^2 + (p_3 - x^3(0,0))^2]^{5/2}}
\end{aligned}
$$

$$- \frac{tx_t^1(0,0)(p_1 - x^1(0,0)) + tx_t^2(0,0)(p_2 - x^2(0,0)) + tx_t^3(0,0)(p_3 - x^3(0,0))}{[(p_1 - x^1(0,0))^2 + (p_2 - x^2(0,0))^2 + (p_3 - x^3(0,0))^2]^{5/2}}$$

$$+ \; O(\frac{\widehat{\delta}_K^2}{d_K^5}) \, . \tag{3.49}$$

Combining (3.47), (3.48), and (3.49), we obtain

$$\frac{(D_s m_K \times D_t m_K) \cdot (P - m_K(s,t)) - (D_s \widetilde{m}_K \times D_t \widetilde{m}_K) \cdot (P - \widetilde{m}_K(s,t))}{\mid P - m_K(s,t) \mid^3}$$

$$= \quad \frac{E4 + E5 + O(\widehat{\delta}_K^6)}{[(p_1 - x^1(0,0))^2 + (p_2 - x^2(0,0))^2 + (p_3 - x^3(0,0))^2]^{3/2}}$$

$$- \, (E4 + E5 + O(\widehat{\delta}_K^6))e + O(\frac{\widehat{\delta}_K^6}{d_K^4}) \tag{3.50}$$

where

$$e \quad = \quad \frac{sx_s^1(0,0)(p_1 - x^1(0,0)) + sx_s^2(0,0)(p_2 - x^2(0,0)) + sx_s^3(0,0)(p_3 - x^3(0,0))}{[(p_1 - x^1(0,0))^2 + (p_2 - x^2(0,0))^2 + (p_3 - x^3(0,0))^2]^{5/2}}$$

$$+ \, \frac{tx_t^1(0,0)(p_1 - x^1(0,0)) + tx_t^2(0,0)(p_2 - x^2(0,0)) + tx_t^3(0,0)(p_3 - x^3(0,0))}{[(p_1 - x^1(0,0))^2 + (p_2 - x^2(0,0))^2 + (p_3 - x^3(0,0))^2]^{5/2}} \, .$$

Note that $E4$ is $O(\widehat{\delta}_K^4 \cdot d_K)$, E_5 is $O(\widehat{\delta}_K^5)$, and e is $O(\widehat{\delta}_K / d_K^4)$. Integrating (3.50) over σ, the error contributed by each $\triangle_K$ is $O(\widehat{\delta}_K^5 / d_K^3) + O(\widehat{\delta}_K^6 / d_K^4)$. Thus, the global error E_1 is $O(\widehat{\delta}^2)$.

For computing E_2, we first calculate

$$\mid \frac{1}{\mid P - m_K(s,t) \mid^3} - \frac{1}{\mid P - \widetilde{m}_K(s,t) \mid^3} \mid$$

$$\leq \ \left| \frac{1}{| P - m_K(s,t) |} - \frac{1}{| P - \widetilde{m}_K(s,t) |} \right| \cdot \left| \frac{1}{| P - m_K(s,t) |^2} \right.$$

$$\left. + \frac{1}{(| P - m_K(s,t) |)(| P - \widetilde{m}_K(s,t) |)} + \frac{1}{| P - \widetilde{m}_K(s,t) |^2} \right| \qquad (3.51)$$

Using (3.30), (3.51) is $O(\widehat{\delta}_K^3/d_K^2) \cdot O(1/d_K^2) = O(\widehat{\delta}_K^3/d_K^4)$. Therefore, the integrand of (3.45) is

$$\left| (D_s\widetilde{m}_K \times D_t\widetilde{m}_K) \cdot (P - \widetilde{m}_K(s,t)) \left[\frac{1}{| P - m_K(s,t) |^3} - \frac{1}{| P - \widetilde{m}_K(s,t) |^3} \right] \right|$$

$$\leq \ O(\widehat{\delta}_K^2) \cdot O(d_K) \cdot O(\frac{\widehat{\delta}_K^3}{d_K^4}) \ = \ O(\frac{\widehat{\delta}_K^5}{d_K^3})$$

Thus, the error contributed by each $\triangle_K$ is $O(\widehat{\delta}_K^5/d_K^3)$, and E_2 is of order two. This proves the theorem. ∎

The above theorems show the difference between the value of solid angle and the approximate value of solid angle, and we have the following corollary.

Corollary 3.14 Let S be a piecewise smooth surface, and let P be a node point on S. Then

$$\Omega(P) - \Omega_N(P) = O(\widehat{\delta}^2) \, .$$

Proof: Combine Theorem 3.12 and 3.13. ∎

This result is not as good as desired (see Atkinson[2, pp. 22–24]), but that the empirical results for piecewise smooth surfaces do not clearly indicate a convergence rate. For smooth surfaces, the empirical rate seems to be $O(\widehat{\delta}^3)$, see Atkinson[2, p. 23].

3.7 Generalizations

We have only presented results for using polynomial of degree two to approximate the unknown function f and the surface S. There are other degrees of interpolation that can be used, and the assumption on the smoothness of S and the definition of the nodes will change appropriately. In order to obtain the results we have in this thesis, we found that the following two properties have to hold:

1. No matter what kind of node points and basis functions have been chosen, a generalized Lemma 3.1 has to be satisfied.

2. Cancellation happens over symmetric pairs of triangles.

We first state the generalized Lemma 3.1, and then we give the general theorem for any degree of interpolation. Suppose we use interpolation of degree d to approximate both the unknown function and the piecewise smooth surface S.

Lemma 3.15 Let $f(s,t) = c_1 s^{d+1} + c_2 s^d t + c_3 s^{d-1} t^2 + \cdots + c_{d+2} t^{d+1}$, where c_i's are real numbers. Let $\{q_1, \ldots, q_v\}$ be node points in the unit simplex and $\{l_1, \ldots, l_v\}$ be basis functions in the Lagrange form, where v depends on d. Let

$$\mathcal{P}(s,t) = \sum_{i=1}^{v} f(q_i) l_i(s,t)$$

Then

$$\int_\sigma \frac{\partial}{\partial s}\left[f(s,t) - \mathcal{P}(s,t)\right] dsdt = 0$$

$$\int_\sigma \frac{\partial}{\partial t}\left[f(s,t) - \mathcal{P}(s,t)\right] dsdt = 0$$

■

Theorem 3.16 Suppose the interpolation satisfies the previous lemma then

$$\max_{1\le i\le N_v} \mid f(v_i) - \tilde{f}_N(v_i) \mid \le C\|(\mathcal{K} - \mathcal{K}_N)f\|_\infty = O(\hat{\delta}^e)\ ,$$

where $e = d + 1$ when d is an odd number and $e = d + 2$ when d is an even number.

Proof: When d is an odd number, the cancellation does not occur; and cancellation does occur over symmetric pairs of triangles when d is an even number. The remaining proofs are completely analogous to those earlier for the quadratic case. ■

CHAPTER IV

THE DISCRETE COLLOCATION METHOD

4.1 Introduction

In chapter III, we discussed the collocation method for solving integral equations. In practice, we have to evaluate a lot of integrals when we try to solve integral equations by using the collocation method, and usually these must be done by time-consuming numerical integrations. Therefore, we introduce a discrete collocation method in this chapter to study the effects of the numerical integration errors.

4.2 The discrete collocation method for a smooth kernel

As discussed in Section 3.1, we consider the integral equation

$$\lambda f(P) - \int_S k(P,Q)\, f(Q)\, dS_Q \;=\; g(P), \qquad P \in S \tag{4.1}$$

The assumptions for the surface S and the kernel function k are the same as in Section 3.2. Following the discussion in Section 2.3 and 3.2, let $\{\triangle_1, \ldots, \triangle_N\}$ be a triangulation of S. Let m_K be a parametrization of $\triangle_K$, and let $\widetilde{m}_K$ be the approximation of m_K as in equation (3.5),

$$\widetilde{m}_K(s,t) \;=\; \sum_{j=1}^{6} m_K(\rho_j) l_j(s,t)\,.$$

We try to obtain $\widetilde{f}_N$ by solving the linear system (3.7)

$$\lambda \tilde{f}_N(v_i) - \sum_{K=1}^{N} \sum_{j=1}^{6} \tilde{f}_N(v_{j,K}) \int_\sigma k(v_i, \widetilde{m}_K(s,t)) l_j(s,t) \mid D_s\widetilde{m}_K(s,t) \times D_t\widetilde{m}_K(s,t) \mid ds\, dt$$

$$= g(v_i), \qquad i = 1, \ldots, N_v \tag{4.2}$$

As noted earlier, the integrals in (4.2),

$$\int_\sigma k(v_i, \widetilde{m}_K(s,t)) l_j(s,t) \mid D_s\widetilde{m}_K(s,t) \times D_t\widetilde{m}_K(s,t) \mid ds\, dt \tag{4.3}$$

must still be evaluated, and numerical integration is the only practical course. The principal method we have used is the 3-point rule

$$\int_\sigma h(s,t) ds dt \;\approx\; \frac{1}{6} \sum_{j=4}^{6} h(\rho_j)\ . \tag{4.4}$$

This method has degree of precision two, integrating exactly all quadratic polynomials.

The method (4.4) is used to evaluate the integrals in (4.3). The resulting linear system is

$$\lambda \check{f}_N(v_i) - \frac{1}{6} \sum_{K=1}^{N} \sum_{j=4}^{6} \check{f}_N(v_{j,K}) k(v_i, v_{j,K}) \mid D_s\widetilde{m}_K(\rho_j) \times D_t\widetilde{m}_K(\rho_j) \mid$$

$$= g(v_i), \qquad i = 1, \ldots, N_v \tag{4.5}$$

The values $\{\check{f}_N(v_i) \mid i = 1, \ldots, N_v\}$ can be used to construct a quadratic interpolant $\check{f}_N$. We call $\check{f}_N$ the *discrete collocation solution*, and it is more explicitly computable than $\tilde{f}_N$ or f_N. For smooth surfaces S, it has been shown that

$$\|f - \check{f}_N\|_\infty \;=\; O(\widehat{\delta}^3)\ ,$$

but we have only $O(\widehat{\delta}^2)$ convergence for piecewise smooth surfaces; see Atkinson[6].

4.3 The Nyström method

The system (4.5) can also be interpreted as the linear system for a Nyström method for solving (4.1). Introduce the integration scheme

$$\int_\sigma h(m_K(s,t)) \mid D_s\widetilde{m}_K(s,t) \times D_t\widetilde{m}_K(s,t) \mid dsdt$$

$$\approx \sum_{j=4}^{6} \omega_{j,K} h(m_K(\rho_j)), \qquad K = 1, \ldots, N_v$$

$$\omega_{j,K} = \frac{1}{6} \mid D_s\widetilde{m}_K(\rho_j) \times D_t\widetilde{m}_K(\rho_j) \mid \; . \tag{4.6}$$

Then define a numerical integral for all of S:

$$\int_S F(Q)dS_Q = \sum_{K=1}^{N} \int_\sigma F(m_K(s,t)) \mid D_s m_K(s,t) \times D_t m_K(s,t) \mid dsdt$$

$$\approx \sum_{K=1}^{N} \sum_{j=4}^{6} \omega_{j,K} F(m_K(\rho_j)) \tag{4.7}$$

Use this integration method to approximate the integral in (4.1). Define

$$\mathcal{K}_N f(P) = \sum_{K=1}^{N} \sum_{j=4}^{6} f_N(v_{j,K}) \omega_{j,K} k(P, v_{j,K}) \tag{4.8}$$

This leads to an approximating numerical integral equation,

$$(\lambda - \mathcal{K}_N) h_N = g \; . \tag{4.9}$$

The function $h_N \in C(S)$, and it is given by Nyström interpolation away from the nodes. $\check{f}_N$ is also a function in $C(S)$, and it is given by the formula for quadratic isoparametric interpolation given in (3.6). The functions $\check{f}_N$ and h_N coincide at node points, but they differ elsewhere. Following the discussion in Section 3.3, we use the error bound for Nyström method,

$$\|f - h_N\|_\infty \leq C\|(\mathcal{K} - \mathcal{K}_N)f\|_\infty$$

in order to examine the error bound for the discrete collocation method at the node points $\{v_i\}$:

$$\max_{1 \leq i \leq N_v} \mid f(v_i) - \check{f}_N(v_i) \mid \leq C\|(\mathcal{K} - \mathcal{K}_N)f\|_\infty .$$

4.4 Error analysis

Write

$$\mathcal{K}f(P) = \sum_{K=1}^{N} \int_\sigma k(P, m_K(s,t))f(m_K(s,t)) \mid D_s m_K \times D_t m_K \mid dsdt$$

and

$$\mathcal{K}_N f(P) \equiv \sum_{K=1}^{N} \sum_{j=4}^{6} f(v_{j,K}) k(P, \widetilde{m}_K(\rho_j))\omega_{j,K}$$

$$= \sum_{K=1}^{N} \sum_{j=4}^{6} \frac{1}{6} f(v_{j,K}) k(P, \widetilde{m}_K(\rho_j)) \mid D_s\widetilde{m}_K(\rho_j) \times D_t\widetilde{m}_K(\rho_j) \mid$$

We study the error $(\mathcal{K} - \mathcal{K}_N)f(P)$ using the following decomposition:

$$
\begin{aligned}
&(\mathcal{K}-\mathcal{K}_N)f(P) \\
&= \sum_{K=1}^{N}\int_\sigma k(P,m_K(s,t))f(m_K(s,t)) \mid D_s m_K \times D_t m_K \mid dsdt - \\
&\quad \sum_{K=1}^{N}\sum_{j=4}^{6}\frac{1}{6}f(v_{j,K})k(P,\widetilde{m}_K(\rho_j)) \mid D_s\widetilde{m}_K(\rho_j)\times D_t\widetilde{m}_K(\rho_j) \mid \\
&= E_1 + E_2
\end{aligned}
$$

$$
\begin{aligned}
E_1 &= \sum_{K=1}^{N}\int_\sigma k(P,m_K(s,t))f(m_K(s,t)) \mid D_s m_K \times D_t m_K \mid dsdt \; - \\
&\quad \frac{1}{6}\sum_{K=1}^{N}\sum_{j=4}^{6} f(v_{j,K})k(P,m_K(\rho_j)) \mid D_s m_K(\rho_j)\times D_t m_K(\rho_j) \mid \\
E_2 &= \frac{1}{6}\sum_{K=1}^{N}\sum_{j=4}^{6} f(v_{j,K})k(P,m_K(\rho_j)) \mid D_s m_K(\rho_j)\times D_t m_K(\rho_j) \mid \; - \\
&\quad \frac{1}{6}\sum_{K=1}^{N}\sum_{j=4}^{6} f(v_{j,K})k(P,\widetilde{m}_K(\rho_j)) \mid D_s\widetilde{m}_K(\rho_j)\times D_t\widetilde{m}_K(\rho_j) \mid
\end{aligned}
$$

E_1 is the error from the numerical integration, and E_2 is the error from the approximate surface $\widetilde{m}$.

Lemma 4.1 Let h be defined on S and $h \in C^4(S)$. Let $m_K(s,t)$ be the parametrization of $\triangle_K$. Then, for each $\triangle_K$,

$$
\int_\sigma h(m_K(s,t))dsdt \; - \; \frac{1}{6}\sum_{j=4}^{6} h(m_K(\rho_j)) \; = \; O(\widehat{\delta}_K^3) \tag{4.10}
$$

where $\widehat{\delta}_K$ is the size of $\widehat{\triangle}_K$.

Proof: Define

$$\widetilde{h}(m_K(s,t)) \equiv \sum_{j=1}^{6} h(m_K(\rho_j))l_j(s,t) .$$

Since

$$\int_\sigma \widetilde{h}(m_K(s,t))dsdt = \frac{1}{6}\sum_{j=4}^{6} h(m_K(\rho_j)) ,$$

we rewrite equation (4.10) as

$$\int_\sigma h(m_K(s,t))dsdt - \frac{1}{6}\sum_{j=4}^{6} h(m_K(\rho_j))$$

$$= \int_\sigma h(m_K(s,t))dsdt - \int_\sigma \widetilde{h}(m_K(s,t))dsdt$$

$$= \int_\sigma \left[h(m_K(s,t)) - \widetilde{h}(m_K(s,t))\right] dsdt$$

By using the Taylor error formula, we get

$$h(m_K(s,t)) - \widetilde{h}(m_K(s,t)) = H_{h,K}(s,t) + O(\widehat{\delta}_K^4) \tag{4.11}$$

where

$$H_{h,K}(s,t) = \frac{1}{3!}\left[(s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t})^3 h(m_K(0,0)) - \sum_{j=1}^{6}(s_j\frac{\partial}{\partial s} + t_j\frac{\partial}{\partial t})^3 h(m_K(0,0))l_j(s,t)\right] .$$

Note that $O(\widehat{\delta}_K^4)$ in (4.11) is from the fourth derivative of $h(m_K(s,t))$, i.e.,

$$O(\widehat{\delta}_K^4) = \frac{1}{4!}(s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t})^4 h(m_K(s,t))\bigg|_{\substack{s=\zeta \\ t=\eta}} \tag{4.12}$$

where (ζ, η) is in the line segment from $(0,0)$ to (s,t). $H_{h,K}$ is a degree three polynomial and its coefficients are $O(\widehat{\delta}_K^3)$. Hence

$$\int_\sigma h(m_K(s,t))dsdt \;-\; \frac{1}{6}\sum_{j=4}^{6} h(m_K(\rho_j))$$

$$= \int_\sigma \left[h(m_K(s,t)) \;-\; \widetilde{h}(m_K(s,t))\right] dsdt$$

$$= \int_\sigma \left[H_{h,K}(s,t) + O(\widehat{\delta}_K^4)\right] dsdt \;=\; O(\widehat{\delta}_K^3)\;. \tag{4.13}$$

■

Theorem 4.2 $E_1 \;=\; O(\widehat{\delta}^4)\;.$

Proof: Let

$$m_K(s,t) = F_j(u\widehat{v}_1 + t\widehat{v}_2 + s\widehat{v}_3) = \begin{bmatrix} x^1(u\widehat{v}_1 + t\widehat{v}_2 + s\widehat{v}_3) \\ x^2(u\widehat{v}_1 + t\widehat{v}_2 + s\widehat{v}_3) \\ x^3(u\widehat{v}_1 + t\widehat{v}_2 + s\widehat{v}_3) \end{bmatrix}$$

for some j and $u = 1 - s - t,\ (s,t) \in \sigma,\ x^i \in C^5(R_j),\ i = 1,2,3.$ We can write $\widehat{v}_i \;=\; (v_{i,x},\ v_{i,y})$ because $\widehat{v}_i$'s are points in the $xy-plane$. Hence,

$$x_s^i \;=\; \frac{\partial x^i}{\partial s} = \frac{\partial x^i}{\partial x}(v_{3,x} - v_{1,x}) + \frac{\partial x^i}{\partial y}(v_{3,y} - v_{1,y})$$

and

$$x_t^i \;=\; \frac{\partial x^i}{\partial t} = \frac{\partial x^i}{\partial x}(v_{2,x} - v_{1,x}) + \frac{\partial x^i}{\partial y}(v_{2,y} - v_{1,y})\;.$$

Write

$$D_s m_K \times D_t m_K = \begin{bmatrix} x_s^1(s,t) \\ x_s^2(s,t) \\ x_s^3(s,t) \end{bmatrix} \times \begin{bmatrix} x_t^1(s,t) \\ x_t^2(s,t) \\ x_t^3(s,t) \end{bmatrix} = \begin{bmatrix} x_s^2(s,t)x_t^3(s,t) - x_s^3(s,t)x_t^2(s,t) \\ x_s^3(s,t)x_t^1(s,t) - x_s^1(s,t)x_t^3(s,t) \\ x_s^1(s,t)x_t^2(s,t) - x_s^2(s,t)x_t^1(s,t) \end{bmatrix}$$

$$= \|(\widehat{v}_3 - \widehat{v}_1) \times (\widehat{v}_2 - \widehat{v}_1)\| \begin{bmatrix} x_x^2 x_y^3 - x_x^3 x_y^2 \\ x_x^3 x_y^1 - x_x^1 x_y^3 \\ x_x^1 x_y^2 - x_x^2 x_y^1 \end{bmatrix}$$

$$= \|(\widehat{v}_3 - \widehat{v}_1) \times (\widehat{v}_2 - \widehat{v}_1)\| \cdot J(m_K(s,t))$$

where $\|(\widehat{v}_3 - \widehat{v}_1) \times (\widehat{v}_2 - \widehat{v}_1)\|$ is the area of $\widehat{\triangle}_K$. After the above computation, the integrals in E_1 can be expressed as

$$\int_\sigma k(P, m_K(s,t)) f(m_K(s,t)) \mid D_s m_K \times D_t m_K \mid dsdt$$

$$= \|(\widehat{v}_3 - \widehat{v}_1) \times (\widehat{v}_2 - \widehat{v}_1)\| \cdot \int_\sigma k(P, m_K(s,t)) f(m_K(s,t)) \mid J(m_K(s,t)) \mid dsdt$$

Let

$$h(m_K(s,t)) = k(P, m_K(s,t)) f(m_K(s,t)) \mid J(m_K(s,t)) \mid . \tag{4.14}$$

Then, h satisfies the assumptions in Lemma 4.1, and

$$\int_\sigma k(P, m_K(s,t)) f(m_K(s,t)) \mid D_s m_K \times D_t m_K \mid dsdt -$$

$$\frac{1}{6} \sum_{j=4}^{6} f(v_{j,K}) k(P, m_K(\rho_j)) \mid D_s m_K(\rho_j) \times D_t m_K(\rho_j) \mid$$

$$= \|(\widehat{v}_3 - \widehat{v}_1) \times (\widehat{v}_2 - \widehat{v}_1)\| \cdot \int_\sigma k(P, m_K(s,t)) f(m_K(s,t)) \mid J(m_K(s,t)) \mid dsdt$$

$$- \frac{1}{6} \|(\widehat{v}_3 - \widehat{v}_1) \times (\widehat{v}_2 - \widehat{v}_1)\| \sum_{j=4}^{6} f(v_{j,K}) k(P, m_K(\rho_j)) \mid J(m_K(\rho_j)) \mid$$

$$= \|(\widehat{v}_3 - \widehat{v}_1) \times (\widehat{v}_2 - \widehat{v}_1)\| \cdot \left[\int_\sigma h(m_K(s,t)) dsdt - \frac{1}{6} \sum_{j=1}^{4} h(m_K(\rho_j)) \right] \qquad (4.15)$$

Since all the $\triangle_K$'s in the same R_j are congruent, they all have the same area, i.e., this quntity $\|(\widehat{v}_3 - \widehat{v}_1) \times (\widehat{v}_2 - \widehat{v}_1)\|$ is the same for every triangle in R_j. From Lemma 4.1, the quantity in the brackets of the equation (4.15) is of order three. We examine (4.13) again and we can find the following. At first, $H_{h,K}(s,t)$ is a polynomial with degree three. Secondly, the coefficients of it are combinations of $(\widehat{v}_2 - \widehat{v}_1)$ and $(\widehat{v}_3 - \widehat{v}_1)$. For instance, the coefficient of s^3 of $H_{h,K}$ is

$$\frac{\partial^3}{\partial s^3} h(m_K(0,0)) = \frac{\partial^3}{\partial x^3} h(m_K(\rho_1))(v_{3,x} - v_{1,x})^3$$

$$+3 \frac{\partial^3}{\partial x^2 \partial y} h(m_K(\rho_1))(v_{3,x} - v_{1,x})^2 (v_{3,y} - v_{1,y})$$

$$+\, 3\, \frac{\partial^3}{\partial x \partial^2 y} h(m_K(\rho_1))(v_{3,x} - v_{1,x})(v_{3,y} - v_{1,y})^2 + \frac{\partial^3}{\partial y^3} h(m_K(\rho_1))(v_{3,y} - v_{1,y})^3. \quad (4.16)$$

For every symmetric pair of triangles (see figure 2.3), let

$$\begin{aligned} m_1(s,t) &= F_j(u\widehat{v}_1 + t\widehat{v}_2 + s\widehat{v}_3) \\ m_2(s,t) &= F_j(u\widehat{v}_1 + t\widehat{v}_4 + s\widehat{v}_5) \end{aligned}$$

and

$$\begin{aligned} \widehat{v}_1 - \widehat{v}_2 &= -\left(\widehat{v}_1 - \widehat{v}_4\right) \\ \widehat{v}_1 - \widehat{v}_3 &= -\left(\widehat{v}_1 - \widehat{v}_5\right). \end{aligned}$$

We now have for $H_{h,1}$ and $H_{h,2}$, that the coefficient of s^3 in $H_{h,1}$ is

$$\frac{\partial^3}{\partial x^3} h(m_1(\rho_1))(v_{3,x} - v_{1,x})^3 + 3\, \frac{\partial^3}{\partial x^2 \partial y} h(m_1(\rho_1))(v_{3,x} - v_{1,x})^2(v_{3,y} - v_{1,y})$$

$$+\, 3\, \frac{\partial^3}{\partial x \partial^2 y} h(m_1(\rho_1))(v_{3,x} - v_{1,x})(v_{3,y} - v_{1,y})^2 + \frac{\partial^3}{\partial y^3} h(m_1(\rho_1))(v_{3,y} - v_{1,y})^3, \quad (4.17)$$

and the coefficient of s^3 in $H_{h,2}$ is

$$\frac{\partial^3}{\partial x^3} h(m_2(\rho_1))(v_{3,x} - v_{1,x})^3 + 3\, \frac{\partial^3}{\partial x^2 \partial y} h(m_2(\rho_1))(v_{3,x} - v_{1,x})^2(v_{3,y} - v_{1,y})$$

$$+\, 3\, \frac{\partial^3}{\partial x \partial^2 y} h(m_2(\rho_1))(v_{3,x} - v_{1,x})(v_{3,y} - v_{1,y})^2 + \frac{\partial^3}{\partial y^3} h(m_2(\rho_1))(v_{3,y} - v_{1,y})^3. \quad (4.18)$$

Adding (4.17) and (4.18) gives us zero, and an analogous argument holds for the remaining coefficients. This means that cancellation happens on any symmetric pair of triangles, and the order of error can be improved from $\widehat{\delta}^5$ to $\widehat{\delta}^6$.

As we mentioned in Theorem 3.3, if there are n_j^2 triangles in R_j, then, there are $(n_j^2 - n_j)/2$ symmetric pairs and n_j remaining triangles that are not part of a symmetric pair. Hence, $n_j^2 - n_j$ triangles lead to integration errors of order $\widehat{\delta}^6$, and n_j triangles lead to integration errors of oder $\widehat{\delta}^5$, from each R_j. Adding them up, we get a global error of order $\widehat{\delta}^4$. This proves that E_1 is $O(\widehat{\delta}^4)$. ■

Theorem 4.3 $E_2 = O(\widehat{\delta}^4)$

Proof: For each K,

$$m_K(\rho_j) = \widetilde{m}_K(\rho_j) \qquad j = 1, \ldots, 6 .$$

Then,

$$\frac{1}{6}\sum_{j=4}^{6} f(v_{j,K})k(P, m_K(\rho_j)) \mid D_s m_K(\rho_j) \times D_t m_K(\rho_j) \mid -$$

$$\frac{1}{6}\sum_{j=4}^{6} f(v_{j,K})k(P, \widetilde{m}_K(\rho_j)) \mid D_s \widetilde{m}_K(\rho_j) \times D_t \widetilde{m}_K(\rho_j) \mid$$

$$= \frac{1}{6}\sum_{j=4}^{6} f(v_{j,K})k(P, m_K(\rho_j)) \Big[\mid D_s m_K(\rho_j) \times D_t m_K(\rho_j) \mid$$

$$- \mid D_s \widetilde{m}_K(\rho_j) \times D_t \widetilde{m}_K(\rho_j) \mid \Big] . \qquad (4.19)$$

Using equation (3.11),

$$k(P, m_K(s,t))f(m_K(s,t))(\mid D_s m(s,t) \times D_t m(s,t) \mid - \mid D_s \widetilde{m}(s,t) \times D_t \widetilde{m}(s,t) \mid)$$

$$= k(P, m_K(0,0))f(m_K(0,0))(E4(s,t) + E5(s,t))$$

$$+ k(P, m_K(0,0))[sf_s(m_K(0,0)) + tf_t(m_K(0,0))]E4(s,t)$$

$$+ [sk_s(P, m_K(0,0)) + tk_t(P, m_K(0,0))]f(m_K(0,0))E4(s,t) + \quad O(\widehat{\delta}^6) \qquad (4.20)$$

In this formula, replace (s,t) by ρ_4, ρ_5, and ρ_6. We compute $E4(s,t)$ first. In Section 3.4,

$$\begin{aligned} E4(s,t) \; = \; & \Big\{ (x_s^2 x_t^3 - x_s^3 x_t^2) \left[x_s^2 H_t^3 + x_t^3 H_s^2 - x_s^3 H_t^2 - x_t^2 H_s^3 \right] \\ & + (x_s^3 x_t^1 - x_s^1 x_t^3) \left[x_s^3 H_t^1 + x_t^1 H_s^3 - x_s^1 H_t^3 - x_t^3 H_s^1 \right] \\ & + (x_s^1 x_t^2 - x_s^2 x_t^1) \left[x_s^1 H_t^2 + x_t^2 H_s^1 - x_s^2 H_t^1 - x_t^1 H_s^2 \right] \Big\} / \mid D_s m(0,0) \times D_t m(0,0) \mid \end{aligned}$$

and

$$H^i(s,t) = \frac{1}{3!} \left[(s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t})^3 x^i(0,0) - \sum_{j=1}^{6} (s_j \frac{\partial}{\partial s} + t_j \frac{\partial}{\partial t})^3 x^i(0,0) l_j(s,t) \right] .$$

We have

$$\begin{aligned} H_s^i(s,t) \; &= \; \frac{1}{3!} \left[\frac{\partial^3 x^i(0,0)}{\partial s^3}(3s^2 - 3s + \frac{1}{2}) \; + \; \frac{\partial^3 x^i(0,0)}{\partial s^2 \partial t}(6st - \frac{3t}{2}) \right. \\ & \left. \qquad + \; \frac{\partial^3 x^i(0,0)}{\partial s \partial t^2}(3t^2 - \frac{3t}{2}) \right] \end{aligned}$$

$$\begin{aligned} H_t^i(s,t) \; &= \; \frac{1}{3!} \left[\frac{\partial^3 x^i(0,0)}{\partial t^3}(3t^2 - 3t + \frac{1}{2}) \; + \; \frac{\partial^3 x^i(0,0)}{\partial s \partial t^2}(6st - \frac{3s}{2}) \right. \\ & \left. \qquad + \; \frac{\partial^3 x^i(0,0)}{\partial s^2 \partial t}(3s^2 - \frac{3s}{2}) \right] . \end{aligned}$$

$$H_s^i(0,\frac{1}{2}) \;=\; \frac{1}{3!}\left[\,\frac{1}{2}\,\frac{\partial^3 x^i(0,0)}{\partial s^3} \;-\; \frac{3}{4}\,\frac{\partial^3 x^i(0,0)}{\partial s^2 \partial t}\right]$$

$$H_s^i(\frac{1}{2},\frac{1}{2}) \;=\; \frac{1}{3!}\left[\,\frac{-1}{4}\,\frac{\partial^3 x^i(0,0)}{\partial s^3} \;+\; \frac{3}{4}\,\frac{\partial^3 x^i(0,0)}{\partial s^2 \partial t}\right]$$

$$H_s^i(\frac{1}{2},0) \;=\; \frac{1}{3!}\left(\,\frac{-1}{4}\,\frac{\partial^3 x^i(0,0)}{\partial s^3}\,\right)$$

$$H_t^i(0,\frac{1}{2}) \;=\; \frac{1}{3!}\left(\,\frac{-1}{4}\,\frac{\partial^3 x^i(0,0)}{\partial t^3}\,\right)$$

$$H_t^i(\frac{1}{2},\frac{1}{2}) \;=\; \frac{1}{3!}\left[\,\frac{-1}{4}\,\frac{\partial^3 x^i(0,0)}{\partial t^3} \;+\; \frac{3}{4}\,\frac{\partial^3 x^i(0,0)}{\partial s \partial t^2}\right]$$

$$H_t^i(\frac{1}{2},0) \;=\; \frac{1}{3!}\left[\frac{1}{2}\,\frac{\partial^3 x^i(0,0)}{\partial t^3} \;-\; \frac{3}{4}\,\frac{\partial^3 x^i(0,0)}{\partial s \partial t^2}\right]$$

$$H_s^i(0,\frac{1}{2}) \;+\; H_s^i(\frac{1}{2},\frac{1}{2}) \;+\; H_s^i(\frac{1}{2},0) \;=\; 0 \tag{4.21}$$

$$H_t^i(0,\frac{1}{2}) \;+\; H_t^i(\frac{1}{2},\frac{1}{2}) \;+\; H_t^i(\frac{1}{2},0) \;=\; 0 \tag{4.22}$$

Hence,

$$\sum_{j=4}^{6} E4(\rho_j) \;=\; 0\,.$$

Thus, equation (4.19) is at least of order five for each K. But examining carefully the terms which are of order five in (4.19), cancellation happens between every symmetric pair of triangles, just as we explained in Theorem 4.2. Therefore, E_2 is of order four globally, completing the proof. ■

With Theorem 4.2 and 4.3, we complete the error analysis for smooth kernel functions. The new error bound for the discrete collocation method at node points is

$$\max_{1\leq i\leq N_v} \mid f(v_i) - \check{f}_N(v_i) \mid \leq C\|(\mathcal{K}-\mathcal{K}_N)f\|_\infty = O(\widehat{\delta}^4) .$$

This also gives us

$$\|f - h_N\|_\infty = O(\widehat{\delta}^4) ,$$

for the Nyström method of Section 4.3.

4.5 The numerical integration for the single layer integral

The single layer function on the right hand side of equation (2.4) needs to be evaluated numerically. Consider the single layer integral

$$\int_S f(Q)\frac{1}{\mid P-Q \mid}dS_Q \tag{4.23}$$

$$= \sum_{K=1}^{N}\int_\sigma \frac{f(m_K(s,t))}{\mid P - m_K(s,t) \mid} \mid D_s m_K(s,t)\times D_t m_K(s,t) \mid dsdt$$

for $P \in S$. The single layer kernel $\mid P - m_K(s,t) \mid^{-1}$ in (4.23) varies from singular to quite smooth. To handle this varied behavior, we use two forms of numerical integration over σ. The first case is when $P \in \triangle_K$, and therefore the integrand in (4.23) is singular at $Q = P$. Note that for our purposes, P is one of the node points. For this case, we need to use a very accurate numerical method (see

Atkinson[2, pp. 17–19]). The second case is for those $\triangle_K$'s for which $P \notin \triangle_K$. For these integrals, the numerical integration method we have used is (4.4),

$$\int_\sigma h(s,t)dsdt \;\approx\; \frac{1}{6}\sum_{j=4}^{6} h(\rho_j)\;.$$

Thus, we have for those $\triangle_K$ with $P \notin \triangle_K$,

$$\sum_K \int_\sigma \frac{f(m_K(s,t))}{\mid P - m_K(s,t) \mid} \mid D_s m_K(s,t) \times D_t m_K(s,t) \mid dsdt$$

$$\approx \frac{1}{6}\sum_K \sum_{j=4}^{6} \frac{f(\widetilde{m}_K(\rho_j))}{\mid P - \widetilde{m}_K(\rho_j) \mid} \mid D_s\widetilde{m}_K(\rho_j) \times D_t\widetilde{m}_K(\rho_j) \mid$$

$$= \frac{1}{6}\sum_K \sum_{j=4}^{6} \frac{f(m_K(\rho_j))}{\mid P - m_K(\rho_j) \mid} \mid D_s\widetilde{m}_K(\rho_j) \times D_t\widetilde{m}_K(\rho_j) \mid$$

Using this numerical integration method, we will prove the following theorem.

Theorem 4.4 Let P be a node point, and consider all $\triangle_K$ for which $P \notin \triangle_K$. Then

$$\sum_K \int_\sigma \frac{f(m_K(s,t))}{\mid P - m_K(s,t) \mid} \mid D_s m_K(s,t) \times D_t m_K(s,t) \mid dsdt$$

$$- \frac{1}{6}\sum_K \sum_{j=4}^{6} \frac{f(\widetilde{m}_K(\rho_j))}{\mid P - \widetilde{m}_K(\rho_j) \mid} \mid D_s\widetilde{m}_K(\rho_j) \times D_t\widetilde{m}_K(\rho_j) \mid = O(\widehat{\delta}) \qquad (4.24)$$

Proof: Recall definitions in Section 3.5, and they will be used below.

1. d_K is the distance from P to $\triangle_K$.

2. d is the minimum value of d_K for $K = 1, \ldots, N$.

3. r is $\widehat{\delta}/d$.

Dividing this error analysis into two steps, as we did for Theorem 4.2 and Theorem 4.3,

$$\sum_K \int_\sigma \frac{f(m_K(s,t))}{\mid P - m_K(s,t) \mid} \mid D_s m_K(s,t) \times D_t m_K(s,t) \mid dsdt$$

$$- \ \frac{1}{6} \sum_K \sum_{j=4}^{6} \frac{f(\widetilde{m}_K(\rho_j))}{\mid P - \widetilde{m}_K(\rho_j) \mid} \mid D_s \widetilde{m}_K(\rho_j) \times D_t \widetilde{m}_K(\rho_j) \mid = \ E_1 \ + \ E_2$$

with

$$\begin{aligned} E_1 \ &= \ \sum_K \int_\sigma \frac{f(m_K(s,t))}{\mid P - m_K(s,t) \mid} \mid D_s m_K(s,t) \times D_t m_K(s,t) \mid dsdt \\ &\quad - \ \frac{1}{6} \sum_K \sum_{j=4}^{6} \frac{f(m_K(\rho_j))}{\mid P - m_K(\rho_j) \mid} \mid D_s m_K(\rho_j) \times D_t m_K(\rho_j) \mid \end{aligned} \tag{4.25}$$

$$\begin{aligned} E_2 \ &= \ \frac{1}{6} \sum_K \sum_{j=4}^{6} \frac{f(m_K(\rho_j))}{\mid P - m_K(\rho_j) \mid} \mid D_s m_K(\rho_j) \times D_t m_K(\rho_j) \mid \\ &\quad - \frac{1}{6} \sum_K \sum_{j=4}^{6} \frac{f(\widetilde{m}_K(\rho_j))}{\mid P - \widetilde{m}_K(\rho_j) \mid} \mid D_s \widetilde{m}_K(\rho_j) \times D_t \widetilde{m}_K(\rho_j) \mid \ . \end{aligned} \tag{4.26}$$

As in Theorem 4.2, let

$$\begin{aligned} h(m_K(s,t)) \ &= \ k(P, m_K(s,t)) f(m_K(s,t)) \mid J(m_K(s,t)) \mid \\ &= \ \frac{f(m_K(s,t))}{\mid P - m_K(s,t) \mid} \mid J(m_K(s,t)) \mid \end{aligned} \tag{4.27}$$

where

$$J(m_K(s,t)) = \begin{bmatrix} x_x^2(s,t)x_y^3(s,t) - x_x^3(s,t)x_y^2(s,t) \\ x_x^3(s,t)x_y^1(s,t) - x_x^1(s,t)x_y^3(s,t) \\ x_x^1(s,t)x_y^2(s,t) - x_x^2(s,t)x_y^1(s,t) \end{bmatrix}. \tag{4.28}$$

Since $P \notin \triangle_K$, h satisfies the assumptions in Lemma 4.1. The last term in the error of (4.11) is $O(\widehat{\delta}_K^4)$, which is the fourth derivative of $h(m_K(s,t))$ as we explained in (4.12). $h(m_K(s,t))$ is (4.27) in this theorem, and hence the last term in the error of it is $O(\widehat{\delta}_K^4/d_K^5)$. Recall the leading term in the error,

$$H_{h,K}(s,t) = \frac{1}{3!}\left[(s\frac{\partial}{\partial s}+t\frac{\partial}{\partial t})^3 h(m_K(0,0)) - \sum_{j=1}^{6}(s_j\frac{\partial}{\partial s}+t_j\frac{\partial}{\partial t})^3 h(m_K(0,0))l_j(s,t)\right]$$

We discuss part of the coefficient of s^3 in this equation, and this will show the general behavior of $H_{h,K}$. The term we would like to examine is

$$\frac{\partial^3}{\partial s^3}h(m_K(0,0)) = \frac{\partial^3}{\partial s^3}\left(\frac{f(m_K(s,t))}{\mid P - m_K(s,t)\mid}\mid J(m_K(s,t))\mid\right)\Bigg|_{\substack{s=0\\t=0}}$$

$$= f(m_K(0,0)) \mid J(m_K(0,0)) \mid \frac{\partial^3}{\partial s^3}\left(\frac{1}{\mid P - m_K(s,t)\mid}\right)\Bigg|_{\substack{s=0\\t=0}} \tag{4.29}$$

$$+3\left\{\frac{\partial}{\partial s}\left[f(m_K(s,t)) \mid J(m_K(s,t)) \mid\right]\frac{\partial^2}{\partial s^2}\left(\frac{1}{\mid P - m_K(s,t)\mid}\right)\right\}\Bigg|_{\substack{s=0\\t=0}} \tag{4.30}$$

$$+3\left\{\frac{\partial^2}{\partial s^2}\left[f(m_K(s,t)) \mid J(m_K(s,t)) \mid\right]\frac{\partial}{\partial s}\left(\frac{1}{\mid P - m_K(s,t)\mid}\right)\right\}\Bigg|_{\substack{s=0\\t=0}} \tag{4.31}$$

$$+\frac{1}{\mid P - m_K(0,0)\mid}\frac{\partial^3}{\partial s^3}\left(f(m_K(s,t)) \mid J(m_K(s,t)) \mid\right)\Bigg|_{\substack{s=0\\t=0}} \tag{4.32}$$

(4.30), (4.31), and (4.32) are $O(\widehat{\delta}_K^3/d_K^3)$, $O(\widehat{\delta}_K^3/d_K^2)$, and $O(\widehat{\delta}_K^3/d_K)$, respectively, and (4.29) *is* $O(\widehat{\delta}_K^3/d_K^4)$, the dominant term in (4.29)–(4.32). Hence,

$$h(m_K(s,t)) - \widetilde{h}(m_K(s,t)) = O(\frac{\widehat{\delta}_K^3}{d_K^4}) + O(\frac{\widehat{\delta}_K^4}{d_K^5}) \ ,$$

and

$$\int_\sigma \frac{f(m_K(s,t))}{\mid P - m_K(s,t) \mid} \mid D_s m_K(s,t) \times D_t m_K(s,t) \mid dsdt$$

$$- \ \frac{1}{6} \sum_{j=4}^{6} \frac{f(m_K(\rho_j))}{\mid P - m_K(\rho_j) \mid} \mid D_s m_K(\rho_j) \times D_t m_K(\rho_j) \mid$$

$$= \ \ O(\frac{\widehat{\delta}_K^5}{d_K^4}) + O(\frac{\widehat{\delta}_K^6}{d_K^5})$$

Thus, errors contributed by each $\triangle_K$ are $O(\widehat{\delta}_K^5/d_K^4)$. For each symmetric pair of triangles, the error $O(\widehat{\delta}_K^5/d_K^4)$ can be improved to $O(\widehat{\delta}_K^6/d_K^5)$ because of cancellation of the type discussed in Theorem 4.2. Following the methods used in Theorem 3.10, we obtain that E_1 is of order one.

We now examine each term of E_2.

$$\frac{1}{6} \sum_{j=4}^{6} \frac{f(m_K(\rho_j))}{\mid P - m_K(\rho_j) \mid} \mid D_s m_K(\rho_j) \times D_t m_K(\rho_j) \mid$$

$$- \frac{1}{6} \sum_{j=4}^{6} \frac{f(\widetilde{m}_K(\rho_j))}{\mid P - \widetilde{m}_K(\rho_j) \mid} \mid D_s \widetilde{m}_K(\rho_j) \times D_t \widetilde{m}_K(\rho_j) \mid$$

$$= \frac{1}{6} \sum_{j=4}^{6} \frac{f(m_K(\rho_j)) \Big[\mid D_s m_K(\rho_j) \times D_t m_K(\rho_j) \mid - \mid D_s \widetilde{m}_K(\rho_j) \times D_t \widetilde{m}_K(\rho_j) \mid \Big]}{\mid P - m_K(\rho_j) \mid}.$$

Use (3.22) with the single layer kernel, and obtain

$$
\begin{aligned}
e_{2,K}(s,t) \;\equiv\; & \frac{f(m_K(s,t))}{\mid P - m_K(s,t) \mid} \Big[\mid D_s m_K(s,t) \times D_t m_K(s,t) \mid \\
& \qquad\qquad - \mid D_s \widetilde{m}_K(s,t) \times D_t \widetilde{m}_K(s,t) \mid \Big] \\
= \; & \frac{f(m_K(0,0))}{\mid P - m_K(0,0) \mid} (E4(s,t) \;+\; E5(s,t)) \\
& + \frac{s f_s(m_K(0,0)) \;+\; t f_t(m_K(0,0))}{\mid P - m_K(0,0) \mid} E4(s,t) \\
& + \; (s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t}) \left(\frac{1}{|P - m_K(s,t)|} \right) \Bigg|_{\substack{s\,=\,0 \\ t\,=\,0}} \cdot f(m_K(0,0)) E4(s,t) \\
& + \; O(\frac{\widehat{\delta}_K^6}{d_K^3})
\end{aligned}
$$

where $E4$ and $E5$ are defined in Lemma 3.2. Hence,

$$
\sum_{j=4}^{6} e_{2,K}(\rho_j) \;=\; O(\frac{\widehat{\delta}_K^5}{d_K^2}) \;+\; O(\frac{\widehat{\delta}_K^6}{d_K^3}) \,. \tag{4.33}
$$

With the usual cancellation over symmetric pairs and the discussion of the claim in Theorem 3.10, E_2 is $O(\widehat{\delta}^3)$. With E_1 and E_2 proven, we have $O(\widehat{\delta})$ for the error analysis of the numerical integration method computing the single layer integral. ■

After going through the last theorem carefully, we find those triangles which are quite close to the point P contribute most of the error. To improve on this, We can choose a fixed number ϵ and then use another more accurate numerical integration method for those triangles within ϵ of P (see Atkinson[2, pp. 17–19]). Applying this numerical integration method on each triangle for which the distance between P and the triangle is less than ϵ, we have the following corollary.

Corollary 4.5 Consider those triangles $\triangle_K$ for which the distance from P to $\triangle_K$ is greater than a fixed number ϵ. Then,

$$\sum_K \int_\sigma \frac{f(m_K(s,t))}{\mid P - m_K(s,t) \mid} \mid D_s m_K(s,t) \times D_t m_K(s,t) \mid dsdt$$

$$- \frac{1}{6} \sum_K \sum_{j=4}^{6} \frac{f(\widetilde{m}_K(\rho_j))}{\mid P - \widetilde{m}_K(\rho_j) \mid} \mid D_s \widetilde{m}_K(\rho_j) \times D_t \widetilde{m}_K(\rho_j) \mid = O(\widehat{\delta}^4)$$

Proof: After choosing a fixed number ϵ, this case amounts to integrating a smooth function over the region

$$S \setminus \{ Q \mid |P - Q| \leq \epsilon \} .$$

We apply Theorem 4.3 and Theorem 4.4 to this case, and this completes the corollary. ■

4.6 The numerical integration method for computing the solid angle $\Omega(P)$

Consider the equation (3.26),

$$\Omega(P) = \int_S \frac{\partial}{\partial \nu_Q} \left[\frac{1}{\mid P - Q \mid} \right] dS_Q \qquad P \in S$$

$$= \sum_{K=1}^{N} \int_\sigma \pm \frac{D_s m_K \times D_t m_K}{\mid D_s m_K \times D_t m_K \mid} \cdot \frac{P - m_K(s,t)}{\mid P - m_K(s,t) \mid^3} \mid D_s m_K \times D_t m_K \mid dsdt$$

$$= \sum_{K=1}^{N} \int_{\sigma} \pm (D_s m_K \times D_t m_K) \cdot \frac{P - m_K(s,t)}{\mid P - m_K(s,t) \mid^3} ds dt \ . \tag{4.34}$$

Because S is piecewise smooth, we can assume the direction of the normal vector does not change and that the sign in the above integral is positive. Therefore, we write (4.34) as

$$\Omega(P) = \sum_{K=1}^{N} \int_{\sigma} (D_s m_K \times D_t m_K) \cdot \frac{P - m_K(s,t)}{\mid P - m_K(s,t) \mid^3} ds dt \ . \tag{4.35}$$

As we discussed with the single layer integral, the integrand of (4.35) varies from singular to quite smooth. We handle this varied behavior by using two forms of numerical integration method over σ. The first case is for $P \in \triangle_K$ and thus the integrand in (4.35) is singular at $Q = P$. We will consider only the case that P is a node point. For this case, we use a very accurate numerical integration method to get a very good approximation which is of order four (see Atkinson[2, pp. 17–19]). The second case is for those $\triangle_K$'s which do not contain the point P. For the second case, the numerical integration scheme we have used is (4.4), the 3-point rule

$$\int_{\sigma} h(s,t) ds dt \ \approx \ \frac{1}{6} \sum_{j=4}^{6} h(\rho_j) \ .$$

Thus, we have

$$\sum_{K=1}^{N} \int_{\sigma} (D_s m_K \times D_t m_K) \cdot \frac{P - m_K(s,t)}{\mid P - m_K(s,t) \mid^3} ds dt \qquad P \notin \triangle_K$$

$$\approx \ \frac{1}{6} \sum_{K=1}^{N} \sum_{j=4}^{6} (D_s \widetilde{m}_K(\rho_j) \times D_t \widetilde{m}_K(\rho_j)) \cdot \frac{P - \widetilde{m}_K(\rho_j)}{\mid P - \widetilde{m}_K(\rho_j) \mid^3}$$

$$= \frac{1}{6} \sum_{K=1}^{N} \sum_{j=4}^{6} (D_s \widetilde{m}_K(\rho_j) \times D_t \widetilde{m}_K(\rho_j)) \cdot \frac{P - m_K(\rho_j)}{\mid P - m_K(\rho_j) \mid^3} \tag{4.36}$$

Theorem 4.6 Let P be a node point, and consider all $\triangle_K$ for which $P \notin \triangle_K$. Then,

$$\sum_K \int_\sigma (D_s m_K(s,t) \times D_t m_K(s,t)) \cdot \frac{P - m_K(s,t)}{\mid P - m_K(s,t) \mid^3} dsdt$$

$$- \frac{1}{6} \sum_K \sum_{j=4}^{6} (D_s \widetilde{m}_K(\rho_j) \times D_t \widetilde{m}_K(\rho_j)) \cdot \frac{P - \widetilde{m}_K(\rho_j)}{\mid P - \widetilde{m}_K(\rho_j) \mid^3} = O(1) \tag{4.37}$$

Proof: Two steps will be used to prove this theorem.

$$\sum_K \int_\sigma (D_s m_K \times D_t m_K) \cdot \frac{P - m_K(s,t)}{\mid P - m_K(s,t) \mid^3} dsdt$$

$$- \frac{1}{6} \sum_K \sum_{j=4}^{6} (D_s \widetilde{m}_K(\rho_j) \times D_t \widetilde{m}_K(\rho_j)) \cdot \frac{P - \widetilde{m}_K(\rho_j)}{\mid P - \widetilde{m}_K(\rho_j) \mid^3}$$

$$= E_1 + E_2$$

with

$$E_1 = \sum_K \int_\sigma (D_s m_K \times D_t m_K) \cdot \frac{P - m_K(s,t)}{\mid P - m_K(s,t) \mid^3} dsdt$$

$$- \frac{1}{6} \sum_K \sum_{j=4}^{6} (D_s m_K(\rho_j) \times D_t m_K(\rho_j)) \cdot \frac{P - m_K(\rho_j)}{\mid P - m_K(\rho_j) \mid^3}$$

$$E_2 \;=\; \frac{1}{6}\sum_K\sum_{j=4}^{6}(D_s m_K(\rho_j)\times D_t m_K(\rho_j))\cdot\frac{P-m_K(\rho_j)}{\mid P-m_K(\rho_j)\mid^3}$$

$$-\;\frac{1}{6}\sum_K\sum_{j=4}^{6}(D_s\widetilde{m}_K(\rho_j)\times D_t\widetilde{m}_K(\rho_j))\cdot\frac{P-\widetilde{m}_K(\rho_j)}{\mid P-\widetilde{m}_K(\rho_j)\mid^3}\,.$$

To prove E_1, let

$$h(m_K(s,t)) \;=\; J(m_K(s,t))\cdot\frac{P-m_K(s,t)}{\mid P-m_K(s,t)\mid^3}$$

where $J(m_K(s,t))$ is defined as (4.28). Because $P\notin\triangle_K$, $h(m_K(s,t))$ satisfies the assumptions in Lemma 4.1.

$$H_{h,K}(s,t) \;=\; \frac{1}{3!}\left[(s\frac{\partial}{\partial s}+t\frac{\partial}{\partial t})^3 h(m_K(0,0))-\sum_{j=1}^{6}(s_j\frac{\partial}{\partial s}+t_j\frac{\partial}{\partial t})^3 h(m_K(0,0))l_j(s,t)\right]$$

We would like to examine one term of the coefficient s^3 in $H_{h,K}$, and it will show the general behavior of $H_{h,K}$.

$$\frac{\partial^3}{\partial s^3}h(m_K(0,0)) \;=\; \frac{\partial^3}{\partial s^3}\left(J(m_K(s,t))\cdot\frac{P-m_K(s,t)}{\mid P-m_K(s,t)\mid^3}\right)\Bigg|_{\substack{s\,=\,0\\ t\,=\,0}}$$

$$= J(m_K(0,0))\cdot(P-m_K(0,0))\frac{\partial^3}{\partial s^3}\left(\frac{1}{\mid P-m_K(s,t)\mid^3}\right)\Bigg|_{\substack{s\,=\,0\\ t\,=\,0}} \tag{4.38}$$

$$+\,3\left\{\frac{\partial}{\partial s}\left[(P-m_K(s,t))\cdot J(m_K(s,t))\right]\frac{\partial^2}{\partial s^2}\left(\frac{1}{\mid P-m_K(s,t)\mid^3}\right)\right\}\Bigg|_{\substack{s\,=\,0\\ t\,=\,0}} \tag{4.39}$$

$$+\,3\left\{\frac{\partial^2}{\partial s^2}\left[(P-m_K(s,t))\cdot J(m_K(s,t))\right]\frac{\partial}{\partial s}\left(\frac{1}{\mid P-m_K(s,t)\mid^3}\right)\right\}\Bigg|_{\substack{s\,=\,0\\ t\,=\,0}} \tag{4.40}$$

$$+\;\frac{1}{\mid P-m_K(0,0)\mid^3}\frac{\partial^3}{\partial s^3}\left[(P-m_K(s,t))\cdot J(m_K(s,t))\right]\Bigg|_{\substack{s\,=\,0\\ t\,=\,0}} \tag{4.41}$$

The errors from (4.38), ..., (4.41) vary from $O(\widehat{\delta}_K^3/d_K^5)$ to $O(\widehat{\delta}_K^3/d_K^2)$. Hence,

$$\int_\sigma (D_s m_K \times D_t m_K) \cdot \frac{P - m_K(s,t)}{\mid P - m_K(s,t) \mid^3} ds dt$$

$$- \frac{1}{6} \sum_{j=4}^{6} (D_s m_K(\rho_j) \times D_t m_K(\rho_j)) \cdot \frac{P - m_K(\rho_j)}{\mid P - m_K(\rho_j) \mid^3}$$

$$= \ O(\frac{\widehat{\delta}_K^5}{d_K^5}) \ + \ O(\frac{\widehat{\delta}_K^6}{d_K^6})$$

where $O(\widehat{\delta}_K^5/d_K^5)$ is the dominant term in (4.38)–(4.41), and $O(\widehat{\delta}_K^6/d_K^6)$ comes from the fourth derivative of $h(m_K(s,t))$. d_K is the distance from P to $\triangle_K$, and $d_K = i \cdot d$ where i depends on how far $\triangle_K$ is away from P, and d is defined in Theorem 3.10. Note $r = \widehat{\delta}/d$, therefore

$$O(\frac{\widehat{\delta}_K^5}{d_K^5}) \ = \ O(\frac{r^5}{i^5}) \ = \ r^5 O(\frac{1}{i^5}) \ .$$

The usual cancellation over symmetric pairs does happen here, but it does not improve the rate of convergence of this numerical method because of the term $O(\widehat{\delta}_K^6/d_K^6)$. As discussed in theorem 3.10, there are c_i triangles at a distance $i \cdot d$, and c_i is proportional to i for $i = 1, \ldots, t_j$. Note that $t_j \cdot d$ is the longest possible distance from P to triangles in R_j. Suppose each R_j has n_j^2 triangles, then t_j is proportional to n_j,. We now compute E_1 contributed by R_j as

$$E_1 \ = \ \sum_K O(\frac{\widehat{\delta}_K^5}{d_K^5}) \ = \ \sum_{i=1}^{t_j} c_i \cdot r^5 O(\frac{1}{i^5}) \ = \ r^5 \ = \ \text{constant} \tag{4.42}$$

Since $m_K(\rho_j) = \widetilde{m}_K(\rho_j)$ for $j = 1, \ldots, 6$, E_2 can be rewritten as

$$E_2 = \frac{1}{6} \sum_{K=1}^{N} \sum_{j=4}^{6} (D_s m_K(\rho_j) \times D_t m_K(\rho_j)) \cdot \frac{P - m_K(\rho_j)}{\mid P - m_K(\rho_j) \mid^3}$$

$$- \frac{1}{6} \sum_{K=1}^{N} \sum_{j=4}^{6} (D_s \widetilde{m}_K(\rho_j) \times D_t \widetilde{m}_K(\rho_j)) \cdot \frac{P - m_K(\rho_j)}{\mid P - m_K(\rho_j) \mid^3} \quad (4.43)$$

We compute $D_s m_K(\rho_j) \times D_t m_K(\rho_j)$ first.

$$D_s m_K(s,t) \times D_t m_K(s,t) - D_s \widetilde{m}_K(s,t) \times D_t \widetilde{m}_K(s,t)$$

$$= \begin{bmatrix} x_s^2(s,t)x_t^3(s,t) - x_s^3(s,t)x_t^2(s,t) \\ x_s^3(s,t)x_t^1(s,t) - x_s^1(s,t)x_t^3(s,t) \\ x_s^1(s,t)x_t^2(s,t) - x_s^2(s,t)x_t^1(s,t) \end{bmatrix} - \begin{bmatrix} \widetilde{x}_s^2(s,t)\widetilde{x}_t^3(s,t) - \widetilde{x}_s^3(s,t)\widetilde{x}_t^2(s,t) \\ \widetilde{x}_s^3(s,t)\widetilde{x}_t^1(s,t) - \widetilde{x}_s^1(s,t)\widetilde{x}_t^3(s,t) \\ \widetilde{x}_s^1(s,t)\widetilde{x}_t^2(s,t) - \widetilde{x}_s^2(s,t)\widetilde{x}_t^1(s,t) \end{bmatrix}$$

$$= \begin{bmatrix} x_s^2(H_t^3 + G_t^3) + x_t^3(H_s^2 + G_s^2) - x_s^3(H_t^2 + G_t^2) + x_t^2(H_s^3 + G_s^3) + O(\widehat{\delta}_K^6) \\ x_s^3(H_t^1 + G_t^1) + x_t^1(H_s^3 + G_s^3) - x_s^1(H_t^3 + G_t^3) + x_t^3(H_s^1 + G_s^1) + O(\widehat{\delta}_K^6) \\ x_s^1(H_t^2 + G_t^2) + x_t^2(H_s^1 + G_s^1) - x_s^2(H_t^1 + G_t^1) + x_t^1(H_s^2 + G_s^2) + O(\widehat{\delta}_K^6) \end{bmatrix}$$

Expanding each x_s^i and x_t^i about $(0,0)$, the above equation equals

$V_1(s,t) + V_2(s,t) - V_3(s,t) + O(\widehat{\delta}_K^6)$ where

$$V_1(s,t) = \begin{bmatrix} x_s^2 H_t^3 + x_t^3 H_s^2 - x_s^3 H_t^2 + x_t^2 H_s^3 \\ x_s^3 H_t^1 + x_t^1 H_s^3 - x_s^1 H_t^3 + x_t^3 H_s^1 \\ x_s^1 H_t^2 + x_t^2 H_s^1 - x_s^2 H_t^1 + x_t^1 H_s^2 \end{bmatrix}$$

$$V_2(s,t) \;=\; \begin{bmatrix} x_s^2 G_t^3 + x_t^3 G_s^2 + (s x_{ss}^2 + t x_{st}^2) H_t^3 + (s x_{st}^3 + t x_{tt}^3) H_s^2 \\ x_s^3 G_t^1 + x_t^1 G_s^3 + (s x_{ss}^3 + t x_{st}^3) H_t^1 + (s x_{st}^1 + t x_{tt}^1) H_s^3 \\ x_s^1 G_t^2 + x_t^2 G_s^1 + (s x_{ss}^1 + t x_{st}^1) H_t^2 + (s x_{st}^2 + t x_{tt}^2) H_s^1 \end{bmatrix}$$

$$V_3(s,t) \;=\; \begin{bmatrix} x_s^3 G_t^2 + x_t^2 G_s^3 + (s x_{ss}^3 + t x_{st}^3) H_t^2 + (s x_{st}^2 + t x_{tt}^2) H_s^3 \\ x_s^1 G_t^3 + x_t^3 G_s^1 + (s x_{ss}^1 + t x_{st}^1) H_t^3 + (s x_{st}^3 + t x_{tt}^3) H_s^1 \\ x_s^2 G_t^1 + x_t^1 G_s^2 + (s x_{ss}^2 + t x_{st}^2) H_t^1 + (s x_{st}^1 + t x_{tt}^1) H_s^2 \end{bmatrix}$$

V_1 is of order four, and V_2 and V_3 are of order five. Computing (4.43), and let

$$\begin{aligned} e_{2,K}(s,t) \;&\equiv\; \frac{(P - m_K(s,t)) \cdot [D_s m_K \times D_t m_K - D_s \widetilde{m}_K \times D_t \widetilde{m}_K]}{\mid P - m_K(s,t) \mid^3} \\ &=\; \frac{(P - m_K(0,0)) \cdot [V_1(s,t) + V_2(s,t) - V_3(s,t)]}{\mid P - m_K(0,0) \mid^3} \\ &\quad -\; \frac{(s D_s m_K(0,0) + t D_t m_K(0,0)) \cdot V_1}{\mid P - m_K(s,t) \mid^3} \\ &\quad +\; (P - m_K(0,0)) \cdot V_1(s,t)) (s \frac{\partial}{\partial s} + t \frac{\partial}{\partial t}) \left(\frac{1}{\mid P - m_K(0,0) \mid^3} \right) \Bigg|_{\substack{s\,=\,0 \\ t\,=\,0}} \\ &\quad +\; O(\frac{\widehat{\delta}_K^6}{d_K^4}) \end{aligned}$$

Hence,

$$\sum_{j=4}^{6} e_{2,K}(\rho_j) \;=\; O(\frac{\widehat{\delta}_K^5}{d_K^3}) \;+\; O(\frac{\widehat{\delta}_K^6}{d_K^4}) \,.$$

Because of cancellation of the type discussed in theorem 4.2 and calculation of the type discussed in Theorem 3.10, E_2 is of order two. This prove the theorem ■

The result from above theorem is not very useful, because it is not convergent. But, we also notice that the triangles which contribute the most error are triangles which are very close to the point P. Therefore, we can choose a fixed number $\epsilon > 0$, and another more accurate numerical integration method to apply on those triangles for which their distance to P is less than ϵ (see Atkinson[2, pp. 17–19]). Then, we have the following corollary for triangles which are far away from P.

Corollary 4.7 Consider those triangles $\triangle_K$ for which the distance from P to $\triangle_K$ is greater than ϵ. Then,

$$\sum_K \int_\sigma (D_s m_K \times D_t m_K) \cdot \frac{P - m_K(s,t)}{\mid P - m_K(s,t) \mid^3} ds dt$$

$$- \frac{1}{6} \sum_K \sum_{j=4}^{6} (D_s \widetilde{m}_K(\rho_j) \times D_t \widetilde{m}_K(\rho_j)) \cdot \frac{P - \widetilde{m}_K(\rho_j)}{\mid P - \widetilde{m}_K(\rho_j) \mid^3} = O(\widehat{\delta}^4)$$

Proof: As discussed in Corollary 4.5, this case amounts to integrating a smooth function over the region

$$S \setminus \{ Q \mid |P - Q| \leq \epsilon \} .$$

Thus, $O(\widehat{\delta}^4)$ is expected and is followed by Theorem 4.3 and Theorem 4.4. This completes the corollary. ■

Section 4.5 and 4.6 show that it is necessary to carefully choose a numerical integration method for those triangles within a short distance of P.

4.7 Generalizations

In analogy with Section 3.7, we also give a generalization for the discrete collocation method to other degrees of interpolation. In order to get the earlier results of this chapter, we found that the following two properties have to hold:

1. No matter what kind of node points and basis functions have been chosen, generalized forms of (4.21) and (4.22) have to be satisfied.

2. Cancellation happens over symmetric pairs of triangles.

We first state the generalization of (4.21)–(4.22), and then we give the general theorem for any degree of interpolation. Suppose we use interpolation of degree d to approximate both the unknown function and the piecewise smooth surface S.

Lemma 4.8 Let $f(s,t) = c_1 s^{d+1} + c_2 s^d t + c_3 s^{d-1} t^2 + \cdots + c_{d+2} t^{d+1}$, where c_i's are real numbers. Let $\{q_1, \ldots, q_v\}$ be node points in the unit simplex and $\{l_1, \ldots, l_v\}$ be basis functions in the Lagrange form, where v depends on d. Define $w_1, \ldots, w_v$ as

$$w_i = \int_\sigma l_i(s,t) ds dt \qquad i = 1, \ldots, v\ .$$

Let

$$\mathcal{P}(s,t) = \sum_{i=1}^{v} f(q_i) l_i(s,t)$$

Then

$$\sum_{i=1}^{v} w_i \frac{\partial}{\partial s}(f - \mathcal{P})(q_i) = 0$$

$$\sum_{i=1}^{v} w_i \frac{\partial}{\partial t}(f - \mathcal{P})(q_i) = 0$$

■

The numerical integration method we use for this case is based on the interpolation, i.e.,

$$\int_{\sigma} h(s,t)dsdt \approx \sum_{i=1}^{v} h(q_i)w_i \ . \tag{4.44}$$

Theorem 4.9 Suppose the interpolation satisfies Lemma 4.8, and use the numerical integration method (4.44). Then

$$\max_{1\le i\le N_v} \mid f(v_i) - \check{f}_N(v_i) \mid \le C\|(\mathcal{K} - \mathcal{K}_N)f\|_\infty = O(\widehat{\delta}^e) \ ,$$

where $e = d+1$ when d is an odd number and $e = d+2$ when d is an even number.

Proof: When d is an odd number, the cancellation does not occur; and cancellation does occur over symmetric pairs of triangles when d is an even number. the remaining proofs are completely analogous to those earlier for the quadratic case. ■

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