GALERKIN METHODS FOR SOLVING SINGLE LAYER INTEGRAL EQUATIONS IN THREE DIMENSIONS

by

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An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics in the
Graduate College of The
University of Iowa

December1994

Thesis supervisor: Professor Kendall E. Atkinson

ABSTRACT

We consider numerically solving three classical equations: the Laplace equation, the Helmholtz equation, and the biharmonic equation, with various boundary conditions in three dimensions. The problems are reformulated as Fredholm integral equations of the first kind instead of the second kind, the classical way of solving these three equations. They are solved using the Galerkin method with spherical polynomials as the approximating functions. This method was used by K. Atkinson in the paper of The numerical solution of Laplace's equation in three dimensions, SIAM. J. on Numer. Anal. 19, in 1982, for solving the Laplace equation in three dimensions, and the method is restricted to simply-connected regions with smooth boundaries.

In Chapter 2, we formulate the interior Dirichlet problem of the Laplace equation as a boundary integral equation, using a single-layer potential to represent the solution; and we give a complete convergence and error analysis. In Chapter 3, we extend the results in Chapter 2 to solve the exterior Dirichlet problem, and we give some numerical examples. A similar method is applied to solve the interior and the exterior Neumann problems. In addition, we show that for the Laplace equation with the Dirichlet boundary data, we can solve directly for the Neumann boundary data. Numerical examples are presented.

In Chapter 4, we study one approach to solving the exterior Dirichlet problem for the Helmholtz equation, again involving an integral equation of the first kind. A complete error analysis is given, based on results obtained in Chapter 2. In Chapter 5, we solve the Dirichlet problem for the biharmonic equation. We obtain a new formulation of the problem, converting it to a system of two integral equations. A complete error analysis is given and numerical examples are presented.

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To my daughter Christina

ACKNOWLEDGEMENTS

First and foremost, I would like to express my sincere gratitude to my advisor, Professor Kendall E. Atkinson, for introducing me to this topic of investigation, for his years of encouragement, his infinite patience, and his adept guidance, without whose generous assistance this thesis could not have been written.

I am also indebted to Professors W. Han who gave his valuable help.

I owe many thanks to my parents and my other family members for their many years of support in all possible means.

Finally, my deepest gratitude goes to my husband, Chaoxin, whose love and support have motivated me to persevere with my study.

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CHAPTER I

INTRODUCTION

1.1 Boundary integral equations

Many applications are concerned with numerically solving boundary integral equations. In this thesis, we mainly consider three classical equations in three dimensions: the Laplace equation, the Helmholtz equation, and the biharmonic equation, together with various boundary conditions. The problems are all converted to equivalent boundary integral equations, using single-layer or double-layer potential representations. We use the Galerkin method to numerically solve these equations. More detailed descriptions are presented in the following.

1.1.1. Laplace's Equation

Many physical applications, especially in fluid dynamics and elasticity theory, are concerned with finding the solution of various boundary value problems for Laplace's equation,

$$\triangle u \equiv 0, \quad \text{in } \Omega$$

in which Ω is a simply connected bounded domain in $\mathbf{R}^3$, with a smooth boundary Γ. We consider the following two cases of solving this equation, according to two different boundary conditions:

1. Dirichlet problem: $u = f$ on Γ is given.

2. Neumann problem: $\frac{\partial u}{\partial n}$ is given on Γ, where n is the inward unit normal vector

along Γ.

For an example of using integral equations to develop the existence and uniqueness theory for Laplace's equation, see [20, Chapter 6], [16].

Let

$$g(p,q) = \frac{1}{|p-q|}$$

A classical way to solve the Dirichlet problem is to represent the solution as a double-layer potential:

$$u(p) = \int_\Gamma \frac{\partial g(p,q)}{\partial n_q}\sigma(q)ds_q$$

with density function σ. The single-layer potential

$$u(p) = \int_\Gamma g(p,q)\sigma(q)ds_q$$

with density function σ, is used for solving the Neumann problem. Both cases lead to solving a well-known Fredholm second kind integral equation; see [3], [9], [20]. Recently, people have also begun to use the single-layer potential to solve the Dirichlet problem and the double-layer potential to solve the Neumann problem, and these lead to first kind integral equations; see [25], [14], [10], [26], [27]. For numerically solving these first kind integral equations, people have used a variety of methods. For example, for finite element methods, see [25], [10].

In 1982, Atkinson [3] first applied Galerkin's method to second kind boundary integral equation reformulations of Laplace's equation, with spherical polynomials as the approximating functions. That method has recently become more popular. For example, see Ganesh and Graham [13]; and for related work, see Sloan [30].

In this thesis, in Chapter 2, we use this Galerkin's method to solve the interior Dirichlet problem. Representing the solution as a single-layer potential, we convert the problem to that of solving a Fredholm first kind integral equation on a smooth boundary Γ. When compared to boundary element methods, this approach leads to a small linear system; and it has both good theoretical and numerical results. In Chapter 3, we extend these results to the solution of other boundary value problems. We begin by discussing the exterior Dirichlet problem, including some examples. A similar method is applied to solve the interior and exterior Neumann problems. In addition, in Chapter 3, we describe a method to calculate the normal derivative of solutions of both the interior and exterior Dirichlet problems, again using the same idea of solving a first kind integral equation with a single-layer potential. Numerical examples are presented.

1.1.2. The exterior Dirichlet problem for the Helmholtz Equation

Consider the numerical solution of boundary integral equations for the Helmholtz equation. There is an enormous literature for this problem; for example, see David Colton and Rainer Kress [8]. Much of the work has been for the two dimensional problem; for example, see Rainer Kress and Ian Sloan [21], in which the authors applied a discrete Galerkin's method to a planar problem. In [22, Chapter V], the author T.C. Lin applied Galerkin's method to the exterior Dirichlet problem for the Helmholtz equation in the two dimensional case. These authors use first kind and second kind integral equations, respectively, and they all obtain both good theoretical and numerical results.

In Chapter 4 of this thesis, we consider the Helmholtz equation

$$\triangle u + k^2 u = 0 \text{ in } \Omega_e, \quad k \text{ a non zero real number}$$

in three dimension with exterior Dirichlet boundary conditions, with the limited goal of studying one approach to solving the three dimensional problem. In the above equation, Ω_e is an unbounded domain with a smooth connected boundary and a bounded complement. Many methods have been used to study the Helmholtz equation in the three dimensional case; for example, see [22, Chapter III], in which the author used a Fredholm second kind integral equation. We convert the problem to that of solving a Fredholm first kind integral equation, and we use Galerkin's method with spherical polynomials as the approximating functions. A complete error analysis is given, based on using results obtained in Chapter 2.

1.1.3. The Biharmonic Equation

Consider the biharmonic equation

$$\triangle^2 u = 0 \text{ in } \Omega$$

for a smooth region Ω. In general, there are three standard boundary value problems for the biharmonic equation; and if u, $\frac{\partial u}{\partial n}$ are given on the boundary of Ω, then the problem is called the Dirichlet problem(see [17]). For a planar region Ω, the problem is well-developed. For example, in [17] the author describes a new formulation of the equation for Dirichlet boundary conditions; also see [28], [19]. There are some papers concerning the biharmonic equation for the three dimensional case. S. Kesavan in [19] uses a finite element framework; and B. Fuglede in [29] presented a so-called "direct-method". Both papers deal with the Dirichlet problem for the biharmonic equation. In Chapter 5 of this thesis, we consider the Dirichlet problem for the biharmonic equation on certain domains with a smooth boundary in $\mathbf{R}^3$. We obtain a new formulation of the problem, converting it to a system of two integral equations, which has two density functions that are determined by imposing the Dirichlet boundary conditions. We give a complete error analysis and we also give some numerical examples. For the special case of Ω, the unit sphere, we give the exact solution of our integral equations for the Dirichlet problem for the biharmonic equation.

1.2 The numerical method-Preliminaries

1.2.1 Sobolev spaces

Let $G \subseteq \mathbf{R}^m, m \in \mathbf{N}$, with G having a non-empty interior, and let $t \in \mathbf{R}$. We define the Sobolev space H^t as follows: For an integer $t \geq 0$, let

$$H^t(G) = \{f \mid \|f\|_t^2 = \sum_{|j| \leq t} \|\partial^j f\|_{L^2(G)}^2 < \infty\}$$

The derivatives are distributional derivatives. For $t > 0$ and not an integer, let $t = p + b$, p an integer, and $0 < b < 1$. Define

$$H^t(G) = \{f \mid \|f\|_t^2 = \|f\|_p^2 + \sum_{|j|=p} \int_G \int_G \frac{|\partial^j f(x) - \partial^j f(y)|^2}{|x-y|^{m+2b}} d\sigma(x) d\sigma(y) < \infty\}$$

for $j = (j_1, \ldots, j_m)$, $|j| = j_1 + \cdots + j_m$, $\partial^j f(x) = \frac{\partial^{|j|} f(x_1, \ldots, x_m)}{\partial x_1^{j_1} \cdots \partial x_m^{j_m}}$. For $t < 0$, let

$$H^t = (H^{-t})^*$$

the dual space of $H^{-t}(G)$; see [15, p. 17].

We suppose $\Gamma = \partial G$ is regular (see [25, p. 367]). This means that for $m = 3$,

$$\Gamma = \bigcup_{i=1}^{k} \Gamma_i \quad \text{and} \quad \Gamma_i = \{(x, y, F_i(x, y)) \mid (x, y) \in \Lambda_i\} \tag{1.1}$$

where Λ_i is an open bounded simply-connected region in R^2, and F_i is a smooth function on Λ_i, $i = 1, \cdots, k$. For any function g on Γ, define

$$\tilde{g}_i(x, y) = g_i(x, y, F_i(x, y)), \qquad g_i = g|\Gamma_i, \quad i = 1, \cdots, k$$

The Sobolev space $H^t(\Gamma)$ for any real number t is defined as follows:

$$H^t(\Gamma) \equiv \{g \mid \|g\|_{H^t(\Gamma)} = \sqrt{\sum_{i=1}^{k} \|\tilde{g}_i\|_{H^t(\Lambda_i)}^2} < \infty\}$$

The spaces $H^t(G)$ have the following important properties:

1) If $p < q$, then $H^q \subset H^p$ and the inclusion is compact.

2) For $p > \frac{m}{2}$, $H^p \subset C(\overline{G})$, where $C(\overline{G})$ is the space of continuous functions on $\overline{G}$. This result is called the Sobolev embedding theorem.

We will use the Sobolev spaces defined above throughout this thesis.

1.2.2 Spherical harmonics

Let U denote the unit sphere in $\mathbf{R}^3$,

$$U = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$$

¿From [23, Chap. 5,7], there is a standard orthogonal basis of spherical harmonics for $L^2(U)$. Let $P_n(u)$ denote the Legendre polynomials, defined by

$$P_n(u) = \frac{1}{2^n n!} \frac{d^n}{du^n}(u^2 - 1)^n, \quad \text{for all } u \in [-1, 1]$$

and let $P_n^m(u)$ denote the associated Legendre functions on $[-1, 1]$, defined by

$$P_n^m(u) = (1 - u^2)^{\frac{m}{2}} \frac{d^m}{du^m} P_n(u)$$

for any $n \geq 0, 1 \leq m \leq n$. If $(x, y, z) \in U$, let

$$(x, y, z) = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$$

for $0 \leq \phi \leq 2\pi,\ 0 \leq \theta \leq \pi$. The standard basis functions for the spherical harmonics of degree n are

$$P_n(\cos\theta), P_n^m(\cos\theta)\cos m\phi, P_n^m(\cos\theta)\sin m\phi, \quad 1 \leq m \leq n$$

The total number of basis functions of degree $\leq N$ is $d_N = (N + 1)^2$. If μ is defined on U, we usually use $\mu(\phi, \theta)$ instead of $\mu(x, y, z)$.

If μ is a spherical polynomial of degree N, then

$$\mu(\phi, \theta) = \sum_{n=0}^{N} \{A_n P_n(\cos\theta) + \sum_{m=1}^{n} [A_n^m \cos(m\phi) + B_n^m \sin(m\phi)] P_n^m(\cos\theta)\}$$

where

$$A_n = \frac{2n + 1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \mu(\phi, \theta) P_n(\cos\theta) \sin\theta \ d\theta \ d\phi$$

and

$$\begin{bmatrix} A_n^m \\ B_n^m \end{bmatrix} = \frac{2n + 1}{2\pi} \frac{(n - m)!}{(n + m)!} \int_0^{2\pi} \int_0^{\pi} \mu(\phi, \theta) \begin{bmatrix} \cos m\phi \\ \sin m\phi \end{bmatrix} P_n^m(\cos\theta) \sin\theta \ d\theta \ d\phi$$

Let n denote the inward normal derivative of a point on U. ¿From [32, pp. 6-7], we have the following:

$$\begin{aligned}
\int_U P_n^m(Y)\frac{1}{|X-Y|}ds_Y &= \frac{4\pi}{2n+1}P_n^m(X) \\
\int_U P_n^m(Y)\frac{\partial}{\partial n_Y}\left(\frac{1}{|X-Y|}\right)ds_Y &= \frac{2\pi}{2n+1}P_n^m(X)
\end{aligned}$$

for any $n \geq 0$, $0 \leq m \leq n$, where $P_n^0 = P_n, n \geq 0$. The basis functions P_n^m are eigenfunctions of the single and double layer potentials on the unit sphere.

1.2.3 The Galerkin Method

We begin with the general definition of the projection method.

Definition 1.2.1 Let E and F be Banach spaces and let $A : E \to F$ be an injective bounded linear operator. Let $E_n \subset E$ and $F_n \subset F$ be two sequences of subspaces with $dim E_n = dim F_n = n$, and let $\mathcal{P}_n : F \to F_n$ be projection operators. The projection method, generated by E_n and $\mathcal{P}_n$, approximates the equation

$$A\varphi = f$$

by the projection equation

$$\mathcal{P}_n A\varphi_n = \mathcal{P}_n f, \quad \varphi_n \in E_n$$

This projection method is called convergent for the operator A if there exists an index N such that for each $f \in A(E)$ the approximating equation $\mathcal{P}_n A\varphi_n = \mathcal{P}_n f$ has a unique solution $\varphi_n \in E_n$ for all $n \geq N$ and if these solutions converge to the unique solution φ of $A\varphi = f$, $\varphi_n \to \varphi$ as $n \to \infty$. See [20, p. 185].

The Galerkin method is defined by the following Galerkin's equation (1.2).

Definition 1.2.2 Let E and F be Hilbert spaces and let $A : E \to F$ be an injective bounded linear operator. Let $E_n \subset E$ and $F_n \subset F$ be two subspaces with $dim E_n = dim F_n = n$, and let $\mathcal{P}_n : F \to F_n$ be the orthogonal projection operators. Then $\varphi_n \in E_n$ is a solution to the projection method, generated by E_n and $\mathcal{P}_n$, for the equation $A\varphi = f$ if and only if

$$(A\varphi_n, g) = (f, g) \tag{1.2}$$

for all $g \in F_n$. This is also called a variational equation. It follows that (1.2) is equivalent to $\mathcal{P}_n(A\varphi_n - f) = 0$. The equation (1.2) is called the Galerkin equation; see [20, p. 199].

In this thesis, for any numerical solution of integral equations, we use Galerkin's method with E, F as Sobolev spaces. The finite dimensional spaces are $E_n = F_n = V_n$, with V_n the spherical polynomials of degree $\leq n$ (defined in Chapter 2); and $\mathcal{P}_n$ denotes the orthogonal projection operator from the Sobolev space $L^2(\Gamma)$ to V_n.

1.3 Mathematical background results

Now we list some basic and important results, which we will use frequently in the succeeding chapters in this thesis.

Theorem 1.3.1 The Lax-Milgram Theorem

Let V be a real Hilbert space, let $a(\cdot,\cdot) : V \times V \to \mathbf{R}$ be a bilinear form, continuous and V-elliptic, i.e.

(1) There exists a constant $\alpha > 0$, such that

$$\alpha\|v\|^2 \leq a(v,v), \ \textit{for any } v \in V,$$

and let $f : V \to \mathbf{R}$ be a continuous linear form. Then there exists a unique element $u \in V$ such that:

(2) $a(u,v) = f(v)$, for any $v \in V$.

Proof See [9, p. 171].

Theorem 1.3.2 The Geometric Series Theorem

Let $A : X \to X$ be a bounded linear operator mapping a Banach space X into itself with $\|A\| < 1$, and let $I : X \to X$ denote the identity operator. Then $I - A$ has a bounded inverse operator on X which is given by the Neumann series

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$$

and which satisfies

$$\|(I - A)^{-1}\| \le \frac{1}{1 - \|A\|}$$

The iterated operators A^n are defined by $A^0 = I$ and $A^n = AA^{n-1}$ for $n = 1, 2, \cdots$.

Proof See [20, p. 16].

Theorem 1.3.3 *Let E be a normed space, and let $A : E \to E$ be a compact operator. If $I - A$ is injective, then the inverse operator $(I - A)^{-1} : E \to E$ exists and is bounded.*

Proof See [20, p. 43].

Theorem 1.3.4 The Trace Theorem

Suppose the surface Γ is either compact or a portion of a hyperplane. We introduce the restriction operator

$$\gamma : C^{\infty}(\bar{\Omega}) \to C^{\infty}(\Gamma)$$

by

$$\gamma(u) = u|\Gamma$$

Then for $s > \frac{1}{2}$, the operator γ can be extended to a continuous operator

$$\gamma : H^s(\Omega) \to H^{s-\frac{1}{2}}(\Gamma)$$

If the surface Γ is smooth (in particular, if Γ is part of a hyperplane), this extension is an epimorphism.

Proof See [11, pp. 119-120] or [33, pp. 120-122].

Before we state Cea's Lemma, let $\mathcal{X}$ be a complex Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$. Assume $\mathcal{A}:\mathcal{X}\times\mathcal{X}\to C$ is a function satisfying:

A1 For all $v_1, v_2, w\in\mathcal{X}$ and all complex numbers c_1, c_2,

$$\mathcal{A}(c_1v_1+c_2v_2,w)=c_1\mathcal{A}(v_1,w)+c_2\mathcal{A}(v_2,w)$$

Also,

$$\mathcal{A}(v,w)=\overline{\mathcal{A}(w,v)},\quad v,w\in\mathcal{X}$$

A2 $\mathcal{A}$ is bounded,

$$|\mathcal{A}(v,w)|\le c_{\mathcal{A}}\|v\|\|w\|,\quad v,w\in\mathcal{X}$$

for some finite constant $c_{\mathcal{A}}$.

A3 $\mathcal{A}$ is strongly elliptic:

$$|\mathcal{A}(v,v)|\ge c_e\|v\|^2,\quad v\in\mathcal{X}$$

with some constant $c_e>0$.

A4 Further, let l be a bounded linear functional on $\mathcal{X}$.

We are interested in solving the variational problem of finding $w\in\mathcal{X}$ such that

$$\mathcal{A}(v,w)=l[v],\quad\text{for all } v\in\mathcal{X}\tag{1.3}$$

Consider the finite element method for solving (1.3). Let $\mathcal{X}_n$ be a finite dimensional subspace of the Hilbert space $\mathcal{X}$. Then we are going to find $w\in\mathcal{X}_n$ for which

$$\mathcal{A}(v,w)=l[v],\quad v\in\mathcal{X}_n\tag{1.4}$$

Theorem 1.3.5 Cea's Lemma

Assume A1-A4 for the original space $\mathcal{X}$ and the original variational problem (1.3). Let $\mathcal{X}_n$ have a basis $\{\varphi_{n,1},\cdots,\varphi_{n,d_n}\}$. Then the finite dimensional variational problem (1.4) has a unique solution $u_n\in\mathcal{X}_n$, and it satisfies

$$\|u_n\|\le\frac{1}{c_e}\|l\|$$

The matrix A_n, with $(A_n)_{i,j} = \mathcal{A}(\varphi_{n,i}, \varphi_{n,j})$, is Hermitian and positive definite, and

$$\eta^* A_n \eta \geq c_e \|v\|^2, \quad v = \sum_{j=1}^{d_n} \bar{\eta}_j \varphi_{n,j}$$

For the error in u_n,

$$\|u - u_n\| \leq \frac{c_{\mathcal{A}}}{c_e} \inf_{v \in \mathcal{X}_n} \|u - v\|$$

Proof See [7, p. 104].

□

CHAPTER II

SOLVING THE INTERIOR DIRICHLET PROBLEM FOR THE LAPLACE EQUATION ON SURFACES IN $\mathbf{R^3}$

2.1 Introduction

In this chapter, we consider solving the interior Dirichlet problem for Laplace's equation on a closed surface in R^3. Using the single-layer potential representation, the problem is converted to that of solving a first kind integral equation. The numerical method is based on Galerkin's method with spherical polynomials as the approximating functions. We study the error of the approximating solution in suitable Sobolev spaces.

Consider the solution of the following Dirichlet problem:

$$\begin{cases} \triangle u = 0 & \text{in } \Omega \\ u = u_0 & \text{on } \Gamma \end{cases} \tag{2.1}$$

with Ω an open bounded simply-connected region in R^3. Assume that $\Gamma = \partial\Omega$, the boundary of Ω, is sufficiently smooth. From [25], the solution can be represented by the expression

$$u(p) = \frac{1}{4\pi}\int_\Gamma \frac{\rho(q)}{\mid p-q \mid} dS_q \quad , \quad p \in \overline{\Omega} \tag{2.2}$$

in which $\rho(q)$ is called a single-layer density function and is determined by solving:

$$\frac{1}{4\pi}\int_\Gamma \frac{\rho(q)}{\mid p-q \mid} dS_q = u_0 \quad , \quad p \in \Gamma \tag{2.3}$$

The Galerkin method with approximations based on spherical polynomials on the unit sphere will be applied to solve equation (2.3). We study the error of the solution in the Sobolev spaces defined in Chapter 1, and we give both a practical implementation of the numerical method and numerical examples. The details are presented below. The numerical method is presented in Section 2.2, and the practical implementation of the numerical method is covered in Section 2.3. In Section 2.4, we give numerical examples.

2.2 The numerical method

Let

$$\mathcal{K}\rho(p) = \frac{1}{4\pi}\int_\Gamma \frac{\rho(q)}{\mid p-q \mid} dS_q$$

Then (2.3) can be written as

$$\mathcal{K}\rho(p) = u_0(p), \qquad p \in \Gamma \tag{2.4}$$

¿From [25], there is a unique bounded extension of $\mathcal{K}$ to each negative ordered Sobolev spaces, $\mathcal{K}$ is an isomorphism of the Sobolev space $H^{-\frac{1}{2}}(\Gamma)$ onto the Sobolev space $H^{\frac{1}{2}}(\Gamma)$, and the associated bilinear form

$$a(\rho,\rho') = \frac{1}{4\pi}\int_\Gamma\int_\Gamma \frac{\rho(p)\rho'(q)}{\mid p-q \mid} dS_p dS_q$$

satisfies a strong ellipticity condition:

$$a(\rho,\rho') \geq C\|\rho\|^2_{H^{-\frac{1}{2}}(\Gamma)}, \qquad \rho \in H^{-\frac{1}{2}}(\Gamma), \quad C > 0. \tag{2.5}$$

In addition, it can be shown that solving (2.4) converts to the problem of finding $\rho \in H^{-\frac{1}{2}}(\Gamma)$ for which

$$a(\rho,\rho') = (u_0,\rho'), \quad \text{all} \quad \rho' \in H^{-\frac{1}{2}}(\Gamma) \tag{2.6}$$

2.2.1 Construction of a finite dimensional subspace of $H^{-\frac{1}{2}}(\Gamma)$

Let U denote the unit sphere in R^3,

$$U = \{(x,y,z) \mid x^2+y^2+z^2 = 1\}$$

Recall that from [23], there is a standard orthogonal basis of spherical harmonics for $L^2(U)$. Let $P_n(u)$ and $P_n^m(u)$ be the Legendre polynomials and associated Legendre functions on $[-1,1]$, $n \geq 0$, $1 \leq m \leq n$. For $(x,y,z) \in U$, let

$$(x,y,z) = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$$

for $0 \leq \phi \leq 2\pi$, $0 \leq \theta \leq \pi$. Then the basis functions for the spherical harmonics of degree n are

$$P_n(\cos\theta), P_n^m(\cos\theta)\cos m\phi, P_n^m(\cos\theta)\sin m\phi, \quad 1 \leq m \leq n$$

Now let $H_1^0 \cdots H_{d_N}^0$ be normalized basis functions in $L^2(U)$ and let $\mathcal{X}_N$ be the space of spherical polynomials of degree $\leq N$:

$$\mathcal{X}_N = \text{Span}\{H_1^0 \cdots H_{d_N}^0\} = \{\sum_{i=1}^{d_N} a_i H_i^0 \mid a_i \in R\}$$

Then easily $\mathcal{X}_N \subseteq H^{-\frac{1}{2}}(U)$.

Assume that there is a C^∞ mapping $M : U \xrightarrow[\text{onto}]{1-1} \Gamma$. For any function g defined on Γ, we define $\hat{g}$ on U, by

$$\hat{g}(\hat{Q}) = g(M(\hat{Q}))$$

for any $\hat{Q} \in U$. This defines a map $\mathcal{M} : L^2(\Gamma) \to L^2(U)$, with $\mathcal{M}g = \hat{g}$. Let V_N be the finite dimensional space:

$$V_N = \text{Span}\{\eta_1 \cdots \eta_{d_N}\}$$

where

$$\eta_i(Q) = H_i^0(M^{-1}(Q))$$

for $Q \in \Gamma$, $i = 1, \cdots, d_N$. The result that V_N is a subspace of $H^{-\frac{1}{2}}(\Gamma)$ or that V_N is a subspace of all Sobolev spaces $H^t(\Gamma)$ $(t \in R)$, is obtained immediately from the following theorem.

Theorem 2.2.2 *Assume $M : U \xrightarrow[\text{onto}]{1-1} \Gamma$ is a C^∞ mapping. Then for any $t \geq 0$, the associated linear mapping $\mathcal{M} : H^t(\Gamma) \xrightarrow[\text{onto}]{1-1} H^t(U)$; and $\|\mathcal{M} \cdot\|_{H^t(U)}$ is equivalent to*

$\| \cdot \|_{H^t(\Gamma)}$. *By duality, the map $\mathcal{M}$ can be extended uniquely to a bounded map of* $H^{-t}(\Gamma) \xrightarrow[\text{onto}]{1-1} H^{-t}(U)$.

In order to prove the theorem, we need some lemmas.

Lemma 2.2.3 *Let $r \in \mathbf{R}$. Then $g \in H^r(U)$ if and only if its Laplace expansion satisfies*

$$\sum_{n=0}^{\infty}(2n+1)^{2r}\{| \, a_n \, |^2 + \sum_{m=1}^{n}[| \, a_n^m \, |^2 + | \, b_n^m \, |^2]\} < \infty$$

where

$$g = \sum_{n=0}^{\infty}\{a_n H_n + \sum_{m=1}^{n}[a_n^m H_{n1}^m + b_n^m H_{n2}^m]\}$$

In this H_n, H_{n1}^m, H_{n2}^m, $1 \le m \le n$, are orthonormal spherical harmonic basis functions, discussed earlier.

Proof See [24, Theorem 6.5 , p. 264]. □

Lemma 2.2.4 *Let $g \in H^r(U)$, $r \in \mathbf{R}$, and define*

$$\|g\|_{*H^r(U)} = (\sum_{n=0}^{\infty}(2n+1)^{2r}\{|a_n|^2 + \sum_{m=1}^{n}[|a_n^m|^2 + |b_n^m|^2]\})^{\frac{1}{2}}$$

where

$$g = \sum_{n=0}^{\infty}\{a_n H_n + \sum_{m=1}^{n}[a_n^m H_{n1}^m + b_n^m H_{n2}^m]\}$$

Then $\| \cdot \|_{*H^r}$ is equivalent to the usual H^r norm $\| \cdot \|_{H^r}$.

Proof See [24, p. 215 and p. 259] □

Lemma 2.2.5 *Define*

$$\langle g_1, g_2 \rangle_{*L^2(\Gamma)} = \int_{\Gamma} W(Q) g_1(Q) g_2(Q) ds$$

where

$$W(M(\hat{Q})) = \frac{1}{J_M(\hat{Q})}$$

for $\hat{Q} \in U$ with J_M the Jacobian of the mapping M. Then $\| \cdot \|_{*L^2(\Gamma)}$ is equivalent to

$\|\cdot\|_{L^2(\Gamma)}$ *and* $\|g\|_{*L^2(\Gamma)} = \|\hat{g}\|_{L^2(U)}$ *with*

$$\hat{g}(\hat{Q}) = g(M(\hat{Q})), \quad \hat{Q} \in U.$$

Proof Since M is a one-to-one C^∞ mapping of U onto Γ, we can assume

$$0 < D_1 \leq J_M(\hat{Q}) \leq D_2 < \infty$$

for some constants D_1, D_2. Hence

$$0 < C_1 \leq W(Q) \leq C_2 < \infty$$

for C_1, C_2 be constants and $Q \in \Gamma$. Then

$$\begin{aligned} C_1\|g\|_{L^2(\Gamma)} &= C_1 \int_\Gamma |g|^2 ds \\ &\leq \|g\|_{*L^2(\Gamma)} = \int_\Gamma |W(Q)||g|^2 ds \\ &\leq C_2 \int_\Gamma |g|^2 = C_2\|g\|_{L^2(\Gamma)} \end{aligned}$$

Thus $\|\cdot\|_{*L^2(\Gamma)}$ is equivalent to $\|\cdot\|_{L^2(\Gamma)}$, and

$$\begin{aligned} \langle g_1, g_2\rangle_{*L^2(\Gamma)} &= \int_\Gamma W(Q)g_1(Q)g_2(Q)ds \\ &= \int_U W(M(\hat{Q}_1))g_1(M(\hat{Q}_1))g_2(M(\hat{Q}_1))J_M(\hat{Q})ds \\ &= \int_U \hat{g}_1(\hat{Q})\hat{g}_2(\hat{Q})ds \\ &= \langle \hat{g}_1, \hat{g}_2\rangle_{L^2(U)} \end{aligned}$$

Hence

$$\|g\|_{*L^2(\Gamma)} = \|\hat{g}\|_{L^2(U)}$$

□

Proof of Theorem 2.2.2 We have assumed Γ to be regular as mentioned in Chapter 1, section 1.2, for any $\Gamma \subseteq R^3$. For this reason, we can prove this theorem for the case of only one of the sub-surfaces Γ_i of (1.1),

$$\Gamma_i = \{(x, y, F_i(x, y)) \mid (x, y) \in \Lambda_i\}$$

with F_i infinitely differentiable and Λ_i an open region in R^2. For any

$g \in H^t(\Gamma)$, define

$$\tilde{g}(x,y) = g(x,y,F_i(x,y))$$

and

$$\|g\|_{H^t(\Gamma_i)} = \sqrt{\|\tilde{g}\|^2_{H^t(\Lambda_i)}}$$

Recall that $\hat{g}(Q) = g(M(Q))$ where

$$M : U \to \Gamma$$

is a 1-1 and onto smooth map. We can assume that

$$U_i = M^{-1}(\Gamma_i)$$

and

$$U_i = \{(\xi,\eta,G_i(\xi,\eta)) | (\xi,\eta) \in \Delta_i\}$$

where Δ_i is an open region in R^2, G_i is infinitely differentiable and

$$\hat{\hat{g}}(\xi,\eta) = \hat{g}(\xi,\eta,G_i(\xi,\eta))$$

Hence

$$\|\hat{g}\|_{H^t(U_i)} = \sqrt{\|\hat{\hat{g}}\|^2_{H^t(\Delta_i)}} \quad \text{if } \hat{g} \in H^t(U_i).$$

Define a projection map P on Λ_i by

$$P : \Gamma_i \to \Lambda_i, \quad \text{by } P(x,y,F_i(x,y)) = (x,y);$$

a map $L : \Delta_i \to \Lambda_i$ by

$$L(\xi,\eta) = PM(\xi,\eta,G_i(\xi,\eta))$$

and $T : \Lambda_i \to R^2$ by

$$T(x,y) = (\frac{\partial L^{-1}(x,y)}{\partial x}, \frac{\partial L^{-1}(x,y)}{\partial y})$$

Since G_i, P, M are all smooth maps, so are L and T.

Case 1. $t = 1$,

$$\begin{aligned}
\|g\|^2_{H^1(\Gamma_i)} &= \int_{\Lambda_i} |\,\tilde{g}|^2 ds_{\Lambda_i} + \int_{\Lambda_i} |grad(\tilde{g})|^2\, ds_{\Lambda_i} \\
&= \int_{\Delta_i} |\tilde{g}(L(\xi,\eta))|^2\, |J_L| ds_{\Delta_i} + \int_{\Delta_i} \left| grad(\tilde{g}(L(\xi,\eta))) \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix} \right|^2 |J_L| ds_{\Delta_i}
\end{aligned}$$

where J_L is the Jacobian matrix for the mapping L. By the smoothness of L and L^{-1}, we can suppose that

$$|J_L| \leq C_2 < \infty$$

$$\left| \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix} \right| \leq C_3 < \infty$$

for C_2, C_3 are constants, so

$$\begin{aligned} \|g\|^2_{H^1(\Gamma_i)} &\leq \int_{\Delta_i} C_2 |\hat{g}|^2 ds_{\Delta_i} + \int_{\Delta_i} |grad(\hat{g})|^2 C_3^2 C_2 ds_{\Delta_i} \\ &\leq C\left(\int_{\Delta_i} |\hat{g}|^2 ds_{\Delta_i} + \int_{\Delta_i} |grad(\hat{g})|^2 ds_{\Delta_i}\right) \\ &= C\|\hat{g}\|^2_{H^1(U_i)} \end{aligned}$$

for some constant $C < \infty$.

Similarly, we have

$$\|\hat{g}\|^2_{H^1(U_i)} \leq C\|g\|^2_{H^1(\Gamma_i)}$$

for C some constant.

Case 2. $t = 1 + b, \quad 0 < b < 1$. Recall that T, L, L^{-1} are all smooth, for any $(\xi_1, \eta_1), (\xi_2, \eta_2) \in \Lambda_i$, we have the following:

$$|T(L(\xi_1, \eta_1)) - T(L(\xi_2, \eta_2))| \leq C|(\xi_1, \eta_1) - (\xi_2, \eta_2)|$$

$$|L(\xi_1, \eta_1) - L(\xi_2, \eta_2)| \geq C|(\xi_1, \eta_1) - (\xi_2, \eta_2)|$$

and

$$\max_{(\xi,\eta)\in\Lambda_i} |T(L(\xi, \eta))| \leq C$$

with C a generic constant.

$$\begin{aligned} \|g\|^2_{H^{1+b}(\Gamma_i)} &= \|\tilde{g}\|^2_{H^1(\Gamma_i)} + \int_{\Lambda_i}\int_{\Lambda_i} \frac{|\partial\tilde{g}(x_1, y_1) - \partial\tilde{g}(x_2, y_2)|^2}{|(x_1, y_1) - (x_2, y_2)|^{2+2b}} dS \\ &\leq C\|\hat{\tilde{g}}\|^2_{H^1(U_i)} + \int_{\Delta_i}\int_{\Delta_i} \frac{\partial\tilde{g}(L(\xi_1, \eta_1)) - \partial\tilde{g}(L(\xi_2, \eta_2))|^2}{|L(\xi_1, \eta_1)) - L(\xi_2, \eta_2)|^{2+2b}} |J_L| dS \end{aligned}$$

$$\leq \quad C\|\hat{\hat{g}}\|^2_{H^1(U_i)}$$

$$+ \quad C\int_{\Delta_i}\int_{\Delta_i} \frac{|\partial\hat{\hat{g}}(\xi_1,\eta_1)-\partial\hat{\hat{g}}(\xi_2,\eta_2)|^2}{|(\xi_1,\eta_1)-(\xi_2,\eta_2)|^{2+2b}} \frac{|(\xi_1,\eta_1)-(\xi_2,\eta_2)|^{2+2b}}{|L(\xi_1,\eta_1)-L(\xi_2,\eta_2)|^{2+2b}}$$

$$\cdot |T(L(\xi_1,\eta_1))|\cdot|J_L|dS$$

$$+ \quad \int_{\Delta_i}\int_{\Delta_i} |\partial\hat{\hat{g}}(\xi_2,\eta_2)|^2 \frac{|T(L(\xi_1,\eta_1))-T(L(\xi_2,\eta_2))|^2}{|L(\xi_1,\eta_1)-L(\xi_2,\eta_2)|^{2+2b}}|J_L|dS$$

$$\leq \quad C\|\hat{\hat{g}}\|^2_{H^1(U_i)} + C\int_{\Delta_i}\int_{\Delta_i} \frac{|\partial\hat{\hat{g}}(\xi_1,\eta_1)-\partial\hat{\hat{g}}(\xi_2,\eta_2)|^2}{|(\xi_1,\eta_1)-(\xi_2,\eta_2)|^{2+2b}}dS$$

$$+ \quad C\int_{\Delta_i}\int_{\Delta_i} |\partial\hat{\hat{g}}(\xi_2,\eta_2)|^2 \frac{1}{|(\xi_1,\eta_1)-(\xi_2,\eta_2)|^{2b}}dS$$

$$\leq \quad C\|\hat{\hat{g}}\|^2_{H^{1+b}(U_i)} + C\int_{\Delta_i}|\partial\hat{\hat{g}}|^2 dS \int_{\Delta_i} \frac{1}{|(\xi_1,\eta_1)-(\xi_2,\eta_2)|^{2b}}d\xi_1 d\eta_1$$

$$\leq \quad C\|\hat{\hat{g}}\|^2_{H^{1+b}(U_i)} + C\|\hat{\hat{g}}\|^2_{H^1(U_i)} \int_{\Delta_i} \frac{1}{|(\xi_1,\eta_1)-(\xi_2,\eta_2)|^{2b}}d\xi_1 d\eta_1$$

$$\leq \quad C\|\hat{g}\|^2_{H^{1+b}(U_i)}$$

since

$$\int_{\Delta_i} \frac{1}{|(\xi_1,\eta_1)-(\xi_2,\eta_2)|^{2b}}d\xi_1 d\eta_1 \leq C < \infty$$

for $0 < b < 1$.

Similarly,

$$\|\hat{g}\|^2_{H^{1+b}(U_i)} \leq C\|g\|^2_{H^{1+b}(\Gamma_i)}$$

Case 3. $t > 0$, an integer. This is similar to **Case 1**, and therefore we omit it.

Case 4. $t > 0$, non-integer. Let $t = m + b$ with m an integer and $0 < b < 1$. This is similar to **Case 2**.

Case 5. $t < 0, t \in \mathbf{R}$. Then we treat $H^t(\Gamma)$ as the dual space of $H^{-t}(\Gamma)$. Let $t = -\tau$ with $\tau \geq 0$, then

$$H^{-\tau}(\Gamma) = (H^{\tau}(\Gamma))'$$
$$H^{-\tau}(U) = (H^{\tau}(U))'$$

For any $f \in H^{-\tau}(\Gamma)$,

$$\|f\| = \sup_{v \in H^{\tau}(\Gamma)} \frac{f(v)}{\|v\|_{H^{\tau}(\Gamma)}}$$

is equivalent to

$$\|\hat{f}\| = \sup_{u \in H^{\tau}(U)} \frac{\hat{f}(u)}{\|u\|_{H^{\tau}(U)}}$$

with $\hat{f} \in H^{-\tau}(U)$ defined by

$$\hat{f}(u) = f(v)$$

where $\hat{v} = u$ for any $u \in H^{\tau}(U)$.

Now we define :

$$\mathcal{M} : H^t(\Gamma) \to H^t(U)$$

by

$$\mathcal{M}f = \hat{f}, \quad \text{for any} \quad f \in H^t(\Gamma)$$

Then $\mathcal{M}$ is a 1-1 and onto linear map, and $\|\hat{f}\|_{H^t(U)}$ is equivalent to $\|f\|_{H^t(\Gamma)}$.

$\square$

2.2.2 The Galerkin method and error analysis

We approximate the solution ρ of (2.6) on the finite dimensional subspace

$$V_N = \text{Span}\{\eta_1, \cdots \eta_{d_N}\}$$

where

$$\eta_i = H_i^0 M^{-1} = \mathcal{M}^{-1} H_i^0$$

from Section 2.2. Let $\rho_N = \sum_{j=1}^{d_N} \alpha_j \eta_j \in V_N$ be the solution of the Galerkin equation

$$a(\rho_N, \rho') = (u_0, \rho'), \quad \text{all} \quad \rho' \in V_N$$

In particular,

$$a(\rho_N, \eta_i) = (u_0, \eta_i), \quad i = 1, 2, \cdots d_N.$$

Hence our Galerkin's method for solving (2.6) is given by

$$\begin{cases} \rho_N = \sum_{j=1}^{d_N} \alpha_j \eta_j \\ \\ \sum_{j=1}^{d_N} \alpha_j a_j(\eta_j, \eta_i) = (u_0, \eta_i), \quad i = 1, 2, \cdots, d_N \end{cases} \tag{2.7}$$

Theorem 2.2.6 *Let ρ, ρ_N be defined on Γ as above, and let $\rho \in L^2(\Gamma)$. Then ρ_N exists, is unique, and*

$$\|\rho - \rho_N\|_{H^{-\frac{1}{2}}(\Gamma)} \to 0, \quad (N \to \infty)$$

This says the method (2.7) is convergent.

Proof ρ_N exists by using the strong ellipticity condition (2.5),

$$a(\rho, \rho) \geq C\|\rho\|^2_{H^{-\frac{1}{2}}(\Gamma)}, \quad \rho \in H^{-\frac{1}{2}}(\Gamma)$$

and Cea's Lemma; see Chapter 1, Theorem 1.3.5. From this reference,

$$\|\rho - \rho_N\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C \inf_{\sigma \in V_N} \|\rho - \sigma\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C \inf_{\sigma \in V_N} \|\rho - \sigma\|_{*L^2(\Gamma)} \tag{2.8}$$

with C a constant independent of N. Note that

$$\|\rho - \sigma\|_{*L^2(\Gamma)} = \|\hat{\rho} - \hat{\sigma}\|_{L^2(U)}$$

with

$$\hat{\sigma} = \sum_1^{d_N} \alpha_i H_i^0$$

Thus

$$\inf_{\sigma \in V_N} \|\rho - \sigma\|_{*L^2(\Gamma)} = \inf_{\sigma \in \mathcal{X}_N} \|\hat{\rho} - \hat{\sigma}\|_{L^2(U)} = \|\hat{\rho} - \mathcal{P}_N \hat{\rho}\|_{L^2(U)}$$

In this chapter, we let $\mathcal{P}_N$ denote the orthogonal projection from $L^2(U)$ to $\mathcal{X}_N$. Returning to (2.8)

$$\|\rho - \rho_N\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C\|\hat{\rho} - \mathcal{P}_N \hat{\rho}\|_{L^2(U)} \to 0.$$

□

In order to get a more useful error bound, we need the following two properties.

Lemma 2.2.7 *(Approximation property) Let $s, t \in \mathbf{R}$ with $t \leq s$, Then for any $u \in H^s(U)$, there is an element $u_N \in \mathcal{X}_N$ with*

$$\|u - u_N\|_{H^t(U)} \leq C N^{t-s} \|u\|_{H^s(U)}$$

namely $u_N = \mathcal{P}_N u$.

Proof By Lemma 2.2.3 and Lemma 2.2.4, for $r \in \mathbf{R}$,

$$H^r(U) = \{g \mid \|g\|_{*H^r(U)} < \infty\}$$

where

$$g = \sum_{n=0}^{\infty} \{a_n H_n + \sum_{m=1}^{n} [a_n^m H_{n1}^m + b_n^m H_{n2}^m]\}$$

and

$$\|g\|_{*H^r(U)} = \left[\sum_{n=0}^{\infty} (2n+1)^{2r} \{| a_n |^2 + \sum_{m=1}^{n} [| a_n^m |^2 + | b_n^m |^2]\}\right]^{\frac{1}{2}}$$

is equivalent to $\|g\|_{H^r(U)}$. Now for $u \in H^s(U)$, suppose

$$u = \sum_{n=0}^{\infty} \{a_n H_n + \sum_{m=1}^{n} [a_n^m H_{n1}^m + b_n^m H_{n2}^m]\}$$

Let

$$u_N = \sum_{n=0}^{N} \{a_n H_n + \sum_{m=1}^{n} [a_n^m H_{n1}^m + b_n^m H_{n2}^m]\}$$

Hence $u_N \in \mathcal{X}_N$, and

$$\begin{aligned}
& \|u - u_N\|_{H^t(U)}^2 \leq C \|u - u_N\|_{*H^t(U)}^2 \\
= \; & C \sum_{n=N}^{\infty} (2n+1)^{2t} \{| a_n |^2 + \sum_{m=1}^{n} [| a_n^m |^2 + | b_n^m |^2]\} \\
= \; & C \sum_{n=N}^{\infty} (2n+1)^{2t-2s} (2n+1)^{2s} \{| a_n |^2 + \sum_{m=1}^{n} [| a_n^m |^2 + | b_n^m |^2]\} \\
\leq \; & C (2N)^{2(t-s)} \sum_{n=0}^{\infty} (2n+1)^{2s} \{| a_n |^2 + \sum_{m=1}^{n} [| a_n^m |^2 + | b_n^m |^2]\} \\
\leq \; & C N^{2(t-s)} \|u\|_{*H^s(U)}^2
\end{aligned}$$

$$\leq \quad CN^{2(t-s)}\|u\|^2_{H^s(U)} \quad \text{for some constant C}$$

So

$$\|u-u_N\|_{H^t(U)} \leq CN^{t-s}\|u\|_{H^s(U)}$$

□

Lemma 2.2.8 *(Inverse property) Let $t, s \in \mathbf{R}$ with $t \leq s$. Then for any $v \in \mathcal{X_N}$,*

$$\|v\|_{H^s(U)} \leq CN^{s-t}\|v\|_{H^t(U)}$$

with C independent of v and N.

Proof: For any $v \in \mathcal{X_N}$,

$$v = \sum_{n=0}^{N}\{a_n H_n + \sum_{m=1}^{n}[a_n^m H_{n1}^m + b_n^m H_{n2}^m]\}$$

and

$$\begin{aligned}
\|v\|^2_{H^s(U)} &\leq C\|v\|^2_{*H^s(U)} \\
&= C\sum_{n=0}^{N}(2n+1)^{2s}\{|\ a_n\ |^2 + \sum_{m=1}^{n}[|\ a_n^m\ |^2 + |\ b_n^m\ |^2]\} \\
&\leq C\sum_{n=0}^{N}(2n+1)^{2s-2t}(2n+1)^{2t}\{|\ a_n\ |^2 + \sum_{m=1}^{n}[|\ a_n^m\ |^2 + |\ b_n^m\ |^2]\} \\
&\leq C(2N+1)^{2(s-t)}\sum_{n=0}^{N}(2n+1)^{2t}\{|\ a_n\ |^2 + \sum_{m=1}^{n}[|\ a_n^m\ |^2 + |\ b_n^m\ |^2]\} \\
&\leq CN^{2(s-t)}\|v\|^2_{*H^t(U)} \leq CN^{2(s-t)}\|v\|^2_{H^t(U)}
\end{aligned}$$

So

$$\|v\|_{H^s(U)} \leq CN^{s-t}\|v\|_{H^t(U)}$$

□

Theorem 2.2.9 *Let ρ and ρ_N be defined as above in (2.6) and (2.7). If $\rho \in H^s(\Gamma)$, any $s \in \mathbf{R}, s \geq -\frac{1}{2}$, then for any $-\frac{1}{2} \leq r \leq s$,*

$$\|\rho - \rho_N\|_{H^r(\Gamma)} = O(\frac{1}{N^{s-r}})$$

Proof Define a projection

$$L_N : H^{-\frac{1}{2}}(\Gamma) \to V_N \subset H^{-\frac{1}{2}}(\Gamma)$$

such that

$$L_N\rho = \rho_N \quad \text{for any } \rho \in H^{-\frac{1}{2}}(\Gamma)$$

Then L_N is a bounded projection. In fact, by the strong ellipticity condition (2.5),

$$a(\rho,\rho) \geq \alpha\|\rho\|^2_{H^{-\frac{1}{2}}(\Gamma)}, \quad \alpha > 0$$

This means that for any $\rho \in H^{-\frac{1}{2}}(\Gamma)$ and

$$\mathcal{K}\rho(p) = \frac{1}{4\pi}\int_\Gamma \frac{\rho(q)}{|p-q|}$$

we get

$$(\mathcal{K}\rho,\rho) \geq \alpha\|\rho\|^2_{H^{-\frac{1}{2}}(\Gamma)}$$

¿From [7, p. 9],

$$\|L_N\rho\|_{H^{-\frac{1}{2}}(\Gamma)} = \|\rho_N\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \frac{\|\mathcal{K}\|}{\alpha}\|\rho\|_{H^{-\frac{1}{2}}(\Gamma)}$$

That is

$$\|L_N\| \leq \frac{\|\mathcal{K}\|}{\alpha} < \infty$$

and L_N is bounded.

If $\rho \in V_N$ then the solution of

$$a(\rho,\rho') = (u_0,\rho') \quad , \quad \text{all } \rho' \in V_N$$

is ρ itself. Hence

$$\rho_N = \rho \quad \text{or} \quad L_N\rho = \rho, \quad \text{any} \quad \rho \in V_N.$$

Thus L_N is a projection.

Now for $\rho \in H^r(\Gamma)$, $r \in \mathbf{R}$, by Theorem 2.2.2 we have a unique $\hat{\rho} \in H^r(U)$ with $\mathcal{M}\rho = \hat{\rho}$. By the Approximation Property, for any $s \geq r$, we have $\hat{\rho}_{1,N} \in \mathcal{X}_\mathcal{N}$, ($\rho_{1,N} \in V_N$) for the orthogonal projection that

$$\|\hat{\rho} - \hat{\rho}_{1,N}\|_{H^r(U)} \leq CN^{r-s}\|\hat{\rho}\|_{H^s(U)}$$

Also

$$\begin{aligned}
\|\rho - \rho_N\|_{H^r(\Gamma)} &\leq C\|\hat{\rho} - \hat{\rho}_N\|_{H^r(U)} \\
&\leq C\left[\|\hat{\rho} - \hat{\rho}_{1,N}\|_{H^r(U)} + \|\hat{\rho}_{1,N} - \hat{\rho}_N\|_{H^r(U)}\right]
\end{aligned}$$

$$\leq C\left[\|\rho-\rho_{1,N}\|_{H^r(\Gamma)}+\|\rho_{1,N}-\rho_N\|_{H^r(\Gamma)}\right]$$
$$\leq C\left[\|\rho-\rho_{1,N}\|_{H^r(\Gamma)}+\|L_N(\rho_{1,N}-\rho)\|_{H^r(\Gamma)}\right]$$

¿From the Inverse Property,

$$\|\hat{\rho}_{1,N}-\hat{\rho}_N\|_{H^r(U)}\leq CN^{r+\frac{1}{2}}\|\hat{\rho}_{1,N}-\hat{\rho}_N\|_{H^{-\frac{1}{2}}(U)}$$

Then we have

$$\|L_N(\rho_{1,N}-\rho)\|_{H^r(\Gamma)}\leq CN^{r+\frac{1}{2}}\|L_N(\rho_{1,N}-\rho)\|_{H^{-\frac{1}{2}}(\Gamma)}$$

Hence

$$\begin{aligned}
&\|\rho-\rho_{1,N}\|_{H^r(\Gamma)}+\|L_N(\rho_{1,N}-\rho)\|_{H^r(\Gamma)}\\
\leq\;& C\|\rho-\rho_{1,N}\|_{H^r(\Gamma)}+CN^{r+\frac{1}{2}}\|L_N(\rho_{1,N}-\rho)\|_{H^{-\frac{1}{2}}(\Gamma)}\\
\leq\;& C\|\rho-\rho_{1,N}\|_{H^r(\Gamma)}+CN^{r+\frac{1}{2}}\|\rho_{1,N}-\rho\|_{H^{-\frac{1}{2}}(\Gamma)}\\
\leq\;& C\|\rho-\rho_{1,N}\|_{H^r(\Gamma)}+CN^{r+\frac{1}{2}}N^{-\frac{1}{2}-r}\|\rho_{1,N}-\rho\|_{H^r(\Gamma)}\\
\leq\;& C\|\rho_{1,N}-\rho\|_{H^r(\Gamma)}\\
\leq\;& C\|\hat{\rho}_{1,N}-\hat{\rho}\|_{H^r(U)}\\
\leq\;& CN^{r-s}\|\hat{\rho}\|_{H^s(U)}\\
\leq\;& CN^{r-s}\|\rho\|_{H^s(\Gamma)}
\end{aligned}$$

So

$$\|\rho-\rho_N\|_{H^r(\Gamma)}=O(\frac{1}{N^{s-r}})$$

for any $s\geq r\geq -\frac{1}{2}$, and $\rho\in H^s(\Gamma)$.

□

Corollary 2.2.10 *For any $t\geq -\frac{1}{2}$,*

$$L_N: H^t\to V_N$$

is a bounded projection. That is

$$\|L_N\|_{H^t(\Gamma)\to V_N}\leq C$$

with C a constant independent of N.

$\square$

2.2.3 Condition number of the linear system

We approximate the solution ρ of (2.6) by Galerkin's method on V_N. The method leads to solving the usually small linear system

$$B\alpha = b$$

where

$$B = (a_{i,j})_{d_N \times d_N}, \qquad a_{i,j} = a(\eta_i, \eta_j), \quad i,j = 1,2,\cdots,d_N$$

$$\alpha^T = (\alpha_1, \cdots, \alpha_{d_N}), \qquad b^T = ((u_0,\eta_1), \cdots, (u_0, \eta_{d_N}))$$

The following theorem gives an upper bound for the condition number of the matrix B.

Theorem 2.2.11 *The condition number of the matrix B is $O(N)$:*

$$Cond(B) = \|B\|\|B^{-1}\| \leq CN$$

for C a constant.

Proof We define the matrix norm by

$$\|B\| = \max_{\eta \in R^M, \eta \neq 0} \frac{|B\eta|}{|\eta|}$$

It is known that the smallest and largest eigenvalues of the symmetric matrix B satisfy

$$0 < \lambda_{min} = \min_{\eta \in R^M, \eta \neq 0} \frac{\eta B \eta}{|\eta|^2}, \qquad \lambda_{max} = \max_{\eta \in R^M, \eta \neq 0} \frac{\eta B \eta}{|\eta|^2}$$

where $|\cdot|$ denotes the usual Euclidean norm,

$$|\eta| = (\sum_{i=1}^{M} \eta_i^2)^{\frac{1}{2}}$$

Thus the condition number of B is given by

$$Cond(B) = \|B\|\|B^{-1}\| = \frac{\lambda_{max}}{\lambda_{min}}$$

See [18, pp. 126-128].

Recall that

$$(\eta_i, \eta_j)_{*L^2(\Gamma)} = (\hat{\eta}_i, \hat{\eta}_j)_{L^2(U)} = (H_i^0, H_j^0)_{L^2(U)} = \delta_{i,j} \quad , \qquad i, j = 1, 2, \cdots, d_N.$$

So for any $q_N \in V_N$ with $q_N = \sum_{i=1}^{d_N} \alpha_i \eta_i$, there is a unique

$$\alpha = (\alpha_1, \cdots \alpha_{d_N}) \in R^{d_N}$$

and

$$|\alpha| = \sqrt{\sum_{i=1}^{d_N} |\alpha_i|^2} = \sqrt{\alpha^T \alpha}$$

In addition,

$$\|\rho_N\|^2_{*L^2(\Gamma)} = (\rho_N, \rho_N)_{*L^2(\Gamma)} = (\sum \alpha_i \eta_i, \sum \alpha_i \eta_i)_{*L^2(\Gamma)} = \sum_{i=1}^{d_N} \sum_{j=1}^{d_N} \alpha_i \alpha_j (\eta_i, \eta_j)_{*L^2(\Gamma)} = \alpha^T \alpha$$

From Lemma 2.2.5, there are constants C_1, C_2 (independent of N) for which

$$C_1 \|\rho_N\|_{*L^2(\Gamma)} \leq \|\rho_N\|_{L^2(\Gamma)} \leq C_2 \|\rho_N\|_{*L^2(\Gamma)}, \quad N \geq 1$$

So

$$(1) \qquad C_1 \alpha^T \alpha \leq \|\rho_N\|^2_{L^2(\Gamma)} \leq C_2 \alpha^T \alpha$$

From previous results (in the proof of Theorem 2.2.9), we have

$$a(\rho_N, \rho_N) \geq C_3 \|\rho_N\|^2_{H^{-\frac{1}{2}}(\Gamma)}$$

and

$$\|\rho_N\|_{L^2(\Gamma)} = \|L_N \rho\|_{L^2(\Gamma)} \leq C_4 N^{\frac{1}{2}} \|L_N \rho\|_{H^{-\frac{1}{2}}(\Gamma)} = C_4 N^{\frac{1}{2}} \|\rho_N\|_{H^{-\frac{1}{2}}(\Gamma)}$$

with C_3, C_4 are constant. So

$$a(\rho_N, \rho_N) \geq C_3 \|\rho_N\|^2_{H^{-\frac{1}{2}(\Gamma)}} \geq C_5 N^{-1} \|\rho_N\|^2_{L^2(\Gamma)}$$

with C_5 constant, and

$$\begin{aligned} a(\rho_N, \rho_N) &= (\mathcal{K}\rho_N, \rho_N) \\ &\leq \|\mathcal{K}\rho_N\|_{H^{\frac{1}{2}}(\Gamma)} \|\rho_N\|_{H^{-\frac{1}{2}}(\Gamma)} \\ &\leq \|\mathcal{K}\| \|\rho_N\|^2_{H^{-\frac{1}{2}}(\Gamma)} \leq C_6 \|\rho_N\|^2_{L^2(\Gamma)} \end{aligned}$$

where C_6 is a constant. Thus

$$(2) \qquad C_5 N^{-1} \|\rho_N\|^2_{L^2(\Gamma)} \leq a(\rho_N, \rho_N) \leq C_5 \|\rho_N\|^2_{L^2(\Gamma)}$$

Hence we have the following from (1) and (2):

$$\frac{\alpha^T B\alpha}{\alpha^T\alpha} = \frac{a(\rho_N,\rho_N)}{\alpha^T\alpha} \le \frac{C_6\|\rho_N\|^2_{L^2(\Gamma)}}{\alpha^T\alpha} \le C$$

$$\frac{\alpha^T B\alpha}{\alpha^T\alpha} = \frac{a(\rho_N,\rho_N)}{\alpha^T\alpha} \ge \frac{C_5 N^{-1}\|\rho_N\|^2_{L^2(\Gamma)}}{\alpha^T\alpha} \ge C\frac{1}{N}$$

Then

$$\lambda_{max} = \sup_{\alpha\neq 0}\frac{\alpha^T B\alpha}{\alpha^T\alpha} \le C$$

$$\lambda_{min} = \inf_{\alpha\neq 0}\frac{\alpha^T B\alpha}{\alpha^T\alpha} \ge C\frac{1}{N}$$

Hence

$$Cond(B) = \frac{\lambda_{max}}{\lambda_{min}} \le CN$$

for some constant C.

$\square$

2.2.4 Error bound for u_N

The error bound for solving (2.4) is obtained as follows. Let

$$u_N(p) = \frac{1}{4\pi}\int_\Gamma \frac{\rho_N(q)}{|p-q|}dy, \qquad p\in\Omega \tag{2.9}$$

Using the maximum principle for harmonic functions,

$$\max_{p\in\Omega}|u(p)-u_N(p)| = \max_{Z\in\Gamma}|u(Z)-u_N(Z)|$$

with respect to the true solution u of (2.4). Let $p\to Z\in\Gamma$ from (2.9), we have

$$u_N(Z) = \mathcal{K}\rho_N(Z)$$

So

$$|u(Z)-u_N(Z)| = |\mathcal{K}(\rho-\rho_N)(Z)|$$

Recall that for any $0<\epsilon\le 1$, $H^{1+\epsilon}(\Gamma)\subset C(\Gamma)$ and

$$\|g\|_\infty \le \|g\|_{H^{1+\epsilon}}$$

Also

$$\mathcal{K}: H^\epsilon \to H^{1+\epsilon}$$

is a bounded operator; see [9, p.124]. Thus for $Z \in \Gamma$,

$$\begin{aligned} |\mathcal{K}(\rho - \rho_N)(Z)| &\leq C\|\mathcal{K}(\rho - \rho_N)\|_{H^{1+\epsilon}} \\ &\leq C\|\rho - \rho_N\|_{H^\epsilon} \end{aligned}$$

By Theorem 2.2.9

$$\|\rho - \rho_N\|_{H^\epsilon} \leq CN^{\epsilon - r}\|\rho\|_{H^r}$$

for any $\rho \in H^r, \quad r \geq \epsilon$.

Hence

$$\max_{p \in \Omega} |u(p) - u_N(p)| \leq CN^{\epsilon - r}\|\rho\|_{H^r(\Gamma)}$$

and this converges to zero as $N \to \infty$, provided $\rho \in H^r(\Gamma)$ with $r > \epsilon$.

2.3 Implementation of Galerkin's method

The Galerkin method we described above converts our problem (2.4) to the usually small linear system (2.7). The most difficult part of the implementation is the calculation of the Galerkin coefficients $(\eta_i, \mathcal{K}\eta_j)$, for $1 \leq i, j \leq d_N$. Each of these is a double surface integral over Γ, with $\mathcal{K}\eta_i$ involving a singular integrand; and both integrals must be calculated numerically. With a proper transformation (see [3, p.4]), we can assume that all integrals are over U, the unit sphere. We use a 'product Gaussian quadrature method', which is discussed in paper [2].

Let

$$I(f) = \int_U f(Q)ds_Q = \int_0^{2\pi} \int_0^\pi f(\phi, \theta)\sin(\theta)d\theta d\phi$$

with $f(\phi, \theta)$ denoting $f(Q)$ with $Q = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$. Approximate this by the product Gaussian formula

$$I_M(f) = \delta \sum_{i=1}^{2M} \sum_{j=1}^{M} \omega_j f(\phi_i, \theta_j) \tag{2.10}$$

Here $M \geq 1, \quad \delta = \frac{\pi}{M}, \ \phi_i = i\delta$ for $i = 1, 2, \cdots 2M$; and $\{\omega_j\}, \ \{\cos(\theta_j)\}$ are the Gauss-Legendre weights and nodes of order M on [-1,1]. The degree of precision of the formula is $2M-1$ (see Stroud [31, p. 40], or see [2]).

Theorem 2.3.1 *For $f \in H^t(U)$ with $t>1,\ t \in \mathbf{R}$,*

$$I_M(f) \to I(f)$$

as $M \to \infty$. Moreover,

$$|I(f) - I_M(f)| \leq \frac{C}{(4M+1)^{t-1}} \|f\|_{H^t(U)}.$$

Proof For $t>1,\ \ H^t(U) \subset C(U)$ ¿From [2, Theorem 3],

$$I_M(f) \to I(f) \ \text{ as } \ M \to \infty$$

for $f \in C(U)$. Also, we can write $f \in H^t(U)$ as

$$f = \sum_{n=0}^{\infty} \left[a_n H_n + \sum_{m=1}^{n} (a_n^m H_{n1}^m + b_n^m H_{n2}^m) \right] \tag{2.11}$$

where $H_n, H_{n1}^m, H_{n2}^m,\ \ m = 1, \cdots n, n = 0, 1, \cdots \infty$ are an orthonormal basis from the normalization of the orthogonal basis functions

$$P_n(\cos\theta), P_n^m(\cos\theta)\cos(m\phi), P_n^m(\cos\theta)\sin(m\phi)$$

for $1 \leq m \leq n, n = 0, 1, \cdots$.

Using these basis functions, we can write

$$\begin{aligned} f(\phi,\theta) &= \sum_{n=0}^{\infty} \left[A_n P_n(\cos\theta) + \sum_{m=1}^{n} \{A_n^m \cos(m\phi) + B_n^m \sin(m\phi)\} P_n^m(\cos\theta) \right] \\ &= g(\phi,\theta) + \sum_{n=2M}^{\infty} \left[A_n P_n(\cos\theta) + \sum_{m=1}^{n} \{A_n^m \cos(m\phi) + B_n^m \sin(m\phi)\} P_n^m(\cos\theta) \right] \end{aligned}$$

where $\{A_n^m, B_n^m\}$ are given below in (2.12) and (2.13), and each such coefficient is an appropriate multiple of the corresponding coefficient in (2.11). Also, $g(\phi,\theta)$ is a polynomial degree $\leq 2M-1$,

$$g(\phi,\theta) = \sum_{n=0}^{2M-1} \left[A_n P_n(\cos\theta) + \sum_{m=1}^{n} \{A_n^m \cos(m\phi) + B_n^m \sin(m\phi)\} P_n^m(\cos\theta) \right]$$

Since

$$\int_U P_n^m(cos\theta) sin m\pi dS = 0$$
$$\int_U P_n^m(cos\theta) cos m\pi dS = 0$$

$$\int_U P_n(cos\theta)dS = 0$$

for $n \geq 1$. Hence

$$\begin{aligned}
I(f) &= I(g) + \sum_{n=2M}^{\infty} [\int_0^{2\pi} \int_0^{\pi} A_n P_n(\cos\theta) \sin\theta d\theta d\phi \\
&+ \sum_{m=1}^{n} \{A_n^m \int_0^{2\pi} \int_0^{\pi} \cos(m\phi) P_n^m(\cos\theta) \sin\theta d\theta d\phi \\
&+ B_n^m \int_0^{2\pi} \int_0^{\pi} \sin(m\phi) P_n^m(\cos\theta) \sin(\theta) d\theta d\phi\}] \\
&= I(g)
\end{aligned}$$

For the numerical integral $I_M(f)$,

$$\begin{aligned}
I_M(f) &= I_M(g) + \sum_{n=2M}^{\infty} [A_n I_M(P_n(\cos\theta)) \\
&+ \sum_{m=1}^{n} \{A_n^m I_M(\cos m\phi P_n^m(\cos\theta)) + B_n^m I_M(\sin m\phi P_n^m(\cos\theta))\}]
\end{aligned}$$

So we have

$$\begin{aligned}
I(f) - I_M(f) &= -\sum_{n=2M}^{\infty} [A_n I_M(P_n(\cos\theta)) \\
&+ \sum_{m=1}^{n} \{A_n^m I_M(\cos m\phi P_n^m(\cos\theta)) + B_n^m I_M(\sin m\phi P_n^m(\cos\theta))\}]
\end{aligned}$$

By [23, pp. 88,118],

$$\int_{-1}^{1} P_n^2(x) dx = \frac{2}{2n+1}$$

$$\int_{-1}^{1} |P_n^m(x)|^2 dx = \frac{(n+m)!}{(n-m)!} \frac{2}{2n+1}$$

Let

$$A_n^0 = A_n, \quad a_n^0 = a_n$$

Then

$$A_n^m = \sqrt{\frac{2n+1}{2} \frac{(n-m)!}{(n+m)!}}\, a_n^m \quad \text{for} \quad m = 0, \cdots n \tag{2.12}$$

$$B_n^m = \sqrt{\frac{2n+1}{2} \frac{(n-m)!}{(n+m)!}}\, b_n^m \quad \text{for} \quad m = 1, \cdots n \tag{2.13}$$

The trapezoidal rule is given by

$$\hat{T_m}(f) = h\sum_{j=0}^{m-1} f(t_j)$$

with $h = \frac{2\pi}{m}$, $t_j = jh$, for $0 \le j < \infty$. We have

$$\hat{T_m}(e^{ikt}) = \begin{cases} 2\pi \quad , & k = 0 \pmod{m} \\ 0 \quad , & k \ne 0 \pmod{m} \end{cases}$$

see [6, p. 17]. So $\hat{T_m}(\sin kt) = 0$, for all k, that is

$$h\sum_{j=0}^{m-1} \sin(kt_j) = 0, \quad k = 1, 2, \cdots, m-1.$$

Hence we have the following:

$$\begin{aligned} I_M(\sin(m\phi)P_n^m(\cos(\theta))) &= (\frac{h}{2}\sum_{j=0}^{2M-1} \sin(mj\frac{h}{2}))(\sum_{i=1}^{M}\omega_i P_n^m(\cos\theta_i)) \\ &= 0 \end{aligned}$$

for integer $m = 1, 2, \cdots, m-1$.

Since $\|I_M\| = 4\pi$ as a linear function on $C(U)$, we have

$$\delta\sum_{i=1}^{2M}\sum_{j=1}^{M}\omega_j = 4\pi$$

Then

$$\begin{aligned} |I(f) - I_M(f)| &= \left|\delta\sum_{i=1}^{2M}\sum_{j=1}^{M}\omega_j \sum_{n=2M}^{\infty}\sum_{m=0}^{n} A_n^m P_n^m(\cos\theta_j)\cos m\phi_i\right| \\ &\le \left|\delta\sum_{i=1}^{2M}\sum_{j=1}^{M}\omega_j\right| \left|\sum_{n=2M}^{\infty}\sum_{m=0}^{n}\sqrt{\frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}}a_n^m \cos m\phi_i P_n^m(\cos\theta_j)\right| \\ &\le \delta\sum_{i=1}^{2M}\sum_{j=1}^{M}\omega_j \sum_{n=2M}^{\infty}\sum_{m=0}^{n}\sqrt{\frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}}|P_n^m(\cos(\theta_j))|\ |a_n^m| \\ &\le 4\pi \max_{j=1,\cdots,m}\left(\sum_{n=2M}^{\infty}[\sum_{m=0}^{n}\frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}\ |P_n^m(\cos\theta_j)|^2]^{\frac{1}{2}}(\sum_{m=0}^{n}|a_n^m|^2)^{\frac{1}{2}}\right) \\ &= 4\pi\sum_{n=2M}^{\infty}\sqrt{\frac{2n+1}{2}}\ (\sum_{m=0}^{n}|a_n^m|^2)^{\frac{1}{2}} \end{aligned}$$

This last step uses a result from [32, p. 7]. Then

$$\begin{aligned} |I(f) - I_M(f)| &\le 4\pi\sum_{n=2M}^{\infty}[\frac{(2n+1)^{-2t+1}}{2}\sum_{m=0}^{n}(2n+1)^{2t}|a_n^m|^2]^{\frac{1}{2}} \\ &\le 4\pi(\sum_{n=2M}^{\infty}\frac{(2n+1)^{-2t+1}}{2})^{\frac{1}{2}}(\sum_{n=2M}^{\infty}\sum_{m=0}^{n}(2n+1)^{2t}|a_n^m|^2)^{\frac{1}{2}} \end{aligned}$$

$$\leq \quad C(\int_{2M}^{\infty}(2x+1)^{-2t+1}dx)^{\frac{1}{2}}\|f\|_{H^t(U)}$$
$$\leq \quad \frac{C}{(4M+1)^{t-1}}\|f\|_{H^t(U)}$$

where C is a constant depending on t. □

Note: The result obtained here is slightly stronger than the convergence result from [2, p. 334].

To evaluate $\mathcal{K}\eta_j$, we use a method described in [4]. Write

$$\mathcal{K}\eta_j = \hat{\mathcal{K}}\hat{\eta}_j = \hat{\mathcal{K}}H_j^0 = \int_U H_j^0(Q_u)\hat{K}(P_u, Q_u)ds_{Q_u} \tag{2.14}$$

with $P_u \in U$ and an appropriate $\hat{\mathcal{K}}$ defined on U, $\hat{K}$ is the kernel function for mapping $\hat{\mathcal{K}}$. Note this integrand has an unbounded discontinuity in $\hat{K}(P_u, Q_u)$ at $Q_u = P_u$. If we apply method (2.10) to (2.14) directly, the convergence will be slow. So in order to avoid this, we use a new spherical coordinates representation for the integral, see [4, pp. 89-90], with

$$Q_u = Q_u(\phi'', \theta'') = (\cos\phi''\sin\theta'', \sin\phi''\sin\theta'', \cos\theta'')$$

with $Q_u = P_u$ corresponding to $\theta'' = 0$ or $\theta'' = \pi$. Then

$$\begin{aligned}\int_U H_j^0(Q_u)\hat{K}(P_u, Q_u)ds_{Q_u} &= \int_0^{2\pi}\int_0^{\pi} H_j^0(Q_u(\phi'', \theta''))\hat{K}(P_u, Q_u(\phi'', \theta''))\sin\theta'' d\theta'' d\phi'' \\ &= \int_0^{2\pi}\int_{-1}^{1} H_j^0(Q_u(\phi'', \cos^{-1} z))\hat{K}(P_u, Q_u(\phi'', \cos^{-1} z))dz d\phi''\end{aligned}$$

The singularity in the integrand now occurs along the line $z = -1$ or $z = 1$.

The convergence for applying the method (2.10) to the above integral is quite good in our examples. For the entire coefficient, we use

$$(\eta_i, \mathcal{K}\eta_j) = (H_i^0, \hat{\mathcal{K}}H_j^0) \approx I_M(H_i^0, \hat{\mathcal{K}}H_j^0)$$

and we use the standard rule I_M for the inner product integral. Since $\hat{\mathcal{K}}H_j^0$ is quite smooth, the method is quite accurate with a smaller $M = M_o$(outer integral parameter) than is used for the singular integration (inner integral parameter) $M = M_i$. In general, we choose $M > N$ in our examples. For more details, see [2, pp. 89-90].

The integral for $u_N(p),\ p \in \Omega$ is evaluated using I_M in (2.9). Note that when $p \notin \Gamma$, the integration kernel is smooth; and when p approaches to the boundary, the integrand is increasingly peaked, with consequences as shown in later examples.

2.4 Numerical examples

We use the algorithm described in [4] and we choose three kinds of surfaces in R^3. The true solutions are from the following:

$$\begin{aligned} u^1 &= 1 \\ u^2 &= x \\ u^3 &= x^2 + y^2 + z^2 \\ u^4 &= e^x \cos y + e^z \sin x \\ u^5 &= ((x-5)^2 + (y-4)^2 + (z-3)^2)^{-\frac{1}{2}} \end{aligned}$$

Example 2.1 Let $\Gamma = U$, the unit sphere, $x^2 + y^2 + z^2 = 1$. We calculate the error $u - u_4$, in which the degree of the approximating polynomial is 4. The integration parameters for calculating the Galerkin coefficient are: inner integration parameter $M_i = 32$ and outer integration parameter $M_o = 16$. In Table 1, the first error column is based on the boundary function u^1, and the second on u^5. Both have an accurate solution, and we do not need a higher degree of approximation. Note that for $u = u^1$, the errors in the table are entirely due to the numerical integration of the various inner products.

Example 2.2 Let Γ be the ellipsoid

$$x^2 + \left(\frac{y}{1.5}\right)^2 + \left(\frac{z}{2}\right)^2 = 1 \quad \text{(E1)}$$

Table 1: $\Gamma =$ Unit sphere

x	y	z	$\mathrm{u}^1 - \mathrm{u}^1_4$	$\mathrm{u}^5 - \mathrm{u}^5_4$
0.0000	0.0000	0.0000	$-1.408D-06$	$-1.992D-07$
0.1000	0.1000	0.1000	$-1.408D-06$	$-2.131D-07$
0.2500	0.2500	0.2500	$-1.408D-06$	$-1.465D-07$
0.5000	0.5000	0.5000	$-2.507D-06$	$2.602D-06$

Table 2: Maximum Laplace coefficients($u = u^5$,$N = 8$)

degree I	the maximum coefficient for the degree I
0	$1.1082094548D+00$
1	$6.6129416332D-02$
2	$1.3768464426D-01$
3	$-1.1407979356D-02$
4	$8.6900303697D-03$
5	$4.5775822458D-04$
6	$1.3675609828D-03$
7	$8.3698632603D-05$
8	$2.6810746805D-04$

We still use $M_i = 32$, $M_o = 16$ for calculating the Galerkin coefficients. First, we calculate the Laplace expansion for u^5 , to see how the Laplace coefficients decrease as N increases. See Table 2.

Second, we want to calculate $u_N(p) = \int_\Gamma \frac{\rho_N(q)}{|p-q|}$, $p \in \Omega$. When p approaches the boundary Γ, the integrand becomes increasingly ill-behaved; and we need to increase the degree N of the approximation and the integration parameter (say M) for getting $u_N(p)$, in order to have sufficient accuracy near the boundary. For this example, the worst point at which we evaluated $u_N(p)$ was around $(0.7, 0.7, 0.7)$. In Table 3, the first error column is based on $N = 4$ with an integration parameter $M = 32$, and the second error column is based on $N = 6, M = 64$.

Table 3 shows an increase in accuracy when the approximation degree is increased. For different functions u, we need different degrees in order to have comparable sized error. This can be seen in comparing Table 3 and Table 4. Finally, we show in Figure 1 how the error behaviour varies with the approximation degree. In the graph, the solid line '-' is the maximum of the absolute value of the errors among the 4 points $(0, 0, 0)$,$(0.1, 0.1, 0.1)$,$(0.25, 0.25, 0.25)$ and $(0.5, 0.5, 0.5)$ for $u = u^5$; the star line '*' is that for $u = u^1$; and the circle line 'o', is that for $u = u^3$.

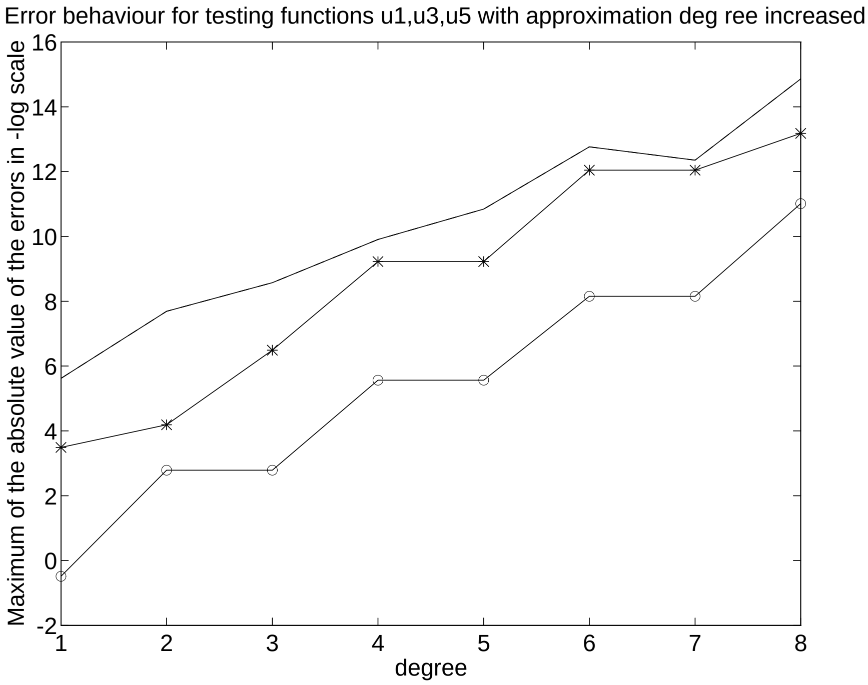

Figure 1: The error behaviour on ellipsoid (E1)

Table 3: The error for $u = u^1$ on the ellipsoid (E1)

	x	y	z	$\mathrm{u}^1 - \mathrm{u}^1_4$	$\mathrm{u}^1 - \mathrm{u}^1_6$
	0.00	0.00	0.00	$-9.876D-05$	$5.878D-06$
	0.10	0.10	0.10	$-9.844D-05$	$5.809D-06$
	0.25	0.25	0.25	$-8.664D-05$	$3.337D-06$
	0.50	0.50	0.50	$8.048D-05$	$-2.641D-05$
	0.70	0.70	0.70	$7.435D-04$	$-7.526D-05$

Table 4: The error on the ellipsoid ($E1$)

	x	y	z	$\mathrm{u}^3 - \mathrm{u}^3_9$	$\mathrm{u}^5 - \mathrm{u}^5_7$
	0.00	0.00	0.00	$1.652D-05$	$7.107D-07$
	0.10	0.10	0.10	$1.606D-05$	$7.263D-07$
	0.25	0.25	0.25	$-9.898D-07$	$2.301D-07$
	0.50	0.50	0.50	$-2.080D-04$	$-4.309D-06$

If we define Γ to be the ellipsoid

$$x^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{5}\right)^2 = 1 \quad \text{(E2)}$$

it is more ill-behaved than E1 in the sense of being more thin and narrow in shape. Consequently the accuracy is worse for comparable values of N. This is shown in Table 5 which uses $N = 7$, $M = 32$.

Table 5: Errors for ellipsoid ($E2$)

	x	y	z	$u^3 - u^3_7$	$u^5 - u^5_7$
	0.00	0.00	0.00	$-2.067D-01$	$1.686D-04$
	0.10	0.10	0.10	$-2.066D-01$	$1.788D-04$
	0.25	0.25	0.25	$-2.056D-01$	$1.876D-04$
	0.50	0.50	0.50	$-2.073D+01$	$1.834D-04$

Table 6: Improved errors for ellipsoid ($E2$)

	x	y	z	$u^3 - u^3_9$	$u^5 - u^5_9$
	0.00	0.00	0.00	$6.650D-02$	$-5.576D-05$
	0.10	0.10	0.10	$6.648D-02$	$-5.988D-05$
	0.25	0.25	0.25	$6.559D-02$	$-6.356D-05$
	0.50	0.50	0.50	$5.272D-02$	$-5.259D-05$

Now if we increase the degree of the approximation ($N = 9$) and the integration parameter($M = 64$) for calculating $u_N(A),\ A \in \Omega$, we obtain better results, as shown in Table 6.

Example 2.3 We use a peanut-shaped region, based on the ovals of Cassini. Γ is defined by

$$R = \sqrt{\cos(2\theta) + \sqrt{\alpha + 1 - \sin(2\theta)^2}}$$

$$(x, y, z) = R(\sin\theta\cos\phi, 2\sin\theta\cos\phi, \cos\theta)$$

The larger α is, the better is the behaviour of the solution procedure, based on

Table 7: $\alpha = 0.8$ for "peanut region"

	x	y	z	$\mathrm{u}^4 - \mathrm{u}_8^4$	$\mathrm{u}^5 - \mathrm{u}_8^5$
	0.00	0.00	0.00	$-8.958D-06$	$3.566D-07$
	0.10	0.10	0.10	$-2.554D-05$	$2.882D-07$
	0.25	0.25	0.25	$-1.613D-04$	$-1.130D-06$
	0.00	0.00	0.30	$-1.634D-05$	$7.020D-07$

Table 8: $\alpha = 0.1$ for "peanut" region

	x	y	z	$u^4 - u_9^4$	$u^5 - u_9^5$
	0.00	0.00	0.00	$-7.256D-05$	$-3.609D-06$
	0.00	0.00	0.40	$-2.041D-04$	$-1.313D-05$
	0.80	0.80	0.80	$-4.001D-04$	$7.169D-05$
	0.03	0.06	0.04	$-8.324D-05$	$-3.498D-06$
	0.07	0.14	0.10	$-1.494D-04$	$-4.669D-06$

comparing the results in Table 7 and Table 8. Figures 2 and 3 are the cross-sections in the xz-plane with $\alpha = 0.8$ and $\alpha = 0.1$ respectively. Also, for both the $\alpha = 0.8$ and $\alpha = 0.1$ cases, function u^5 gives smaller errors than function u^4 does, since u^4 varies greatly and more rapidly over some sections of this region.

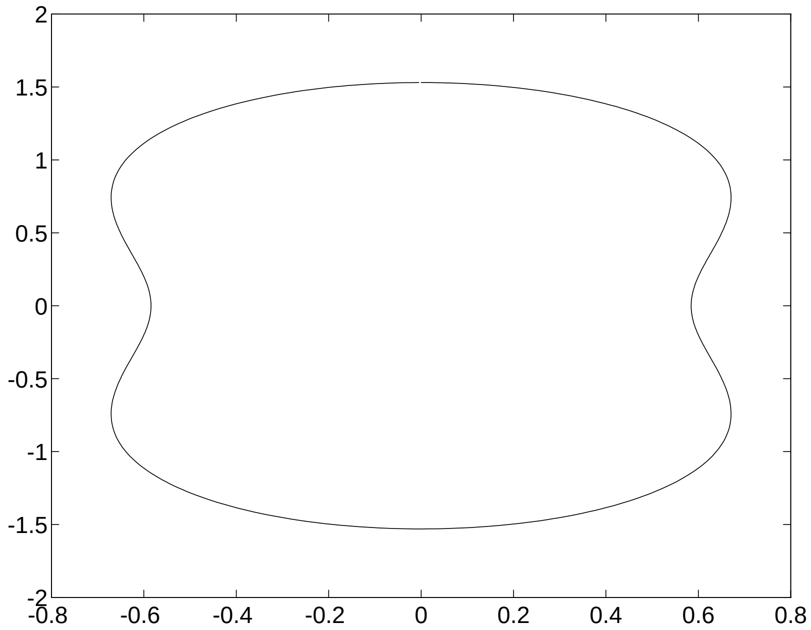

Figure 2: The peanut region for $\alpha = 0.8$

Example 2.4 Finally, we define a heart-shaped region:

$$(x, y, z) = R(2 \sin\theta \cos\phi, \sin\theta\cos\phi, \cos\theta)$$

$$R(x, y, z) = 2 - (1 + 100(z+1)^2)^{-1}, \quad (x, y, z) \in U$$

In Table 9, the first error column is based on using the boundary function u^1; and the second and third error columns are based on u^5 with $N = 4$ and $N = 8$ respectively. Once we choose sufficiently large inner and outer integration parameters M_i and M_o for calculating the Galerkin coefficients, (for example: $M_i = 32$ and $M_o = 16$), the accuracy of the error depends on the approximation degree N.

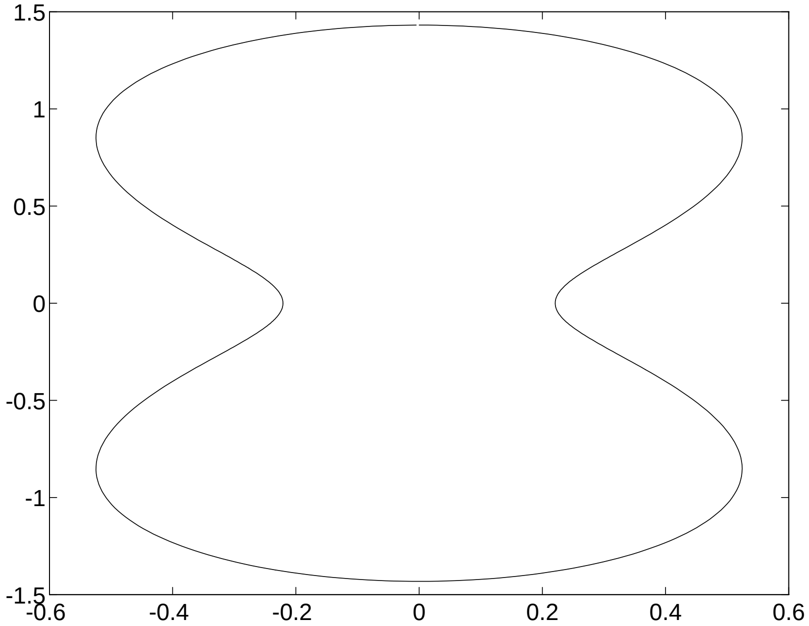

Figure 3: The peanut region for $\alpha = 0.1$

Table 9: Using different boundary functions for testing the error

	x	y	z	$u^1 - u^1_8$	$u^5 - u^5_4$	$u^5 - u^5_8$
	0.10	0.10	0.10	$-8.724D-05$	$-1.159D-04$	$-6.187D-06$
	2.00	0.00	0.00	$-2.370D-05$	$2.891D-05$	$-2.728D-06$
	0.00	1.00	0.00	$-9.478D-05$	$-2.213D-05$	$-6.197D-06$
	2.00	0.00	1.00	$-2.903D-05$	$-1.516D-04$	$-4.301D-06$

CHAPTER III

SOLVING OTHER FIRST KIND INTEGRAL EQUATIONS ON SURFACES IN $\mathbf{R}^3$

3.1 Introduction

In Chapter 2, we used a Fredholm first kind integral equation to solve the interior Dirichlet problem for Laplace's equation using Galerkin's method. In this chapter, we extend the results of chapter 2 to solve the exterior Dirichlet and both the interior and exterior Neumann problems for Laplace's equation. The same method can be used to calculate the inward normal derivative of the solutions of both the interior and the exterior Dirichlet problem. Numerical examples are given.

3.2 The exterior Dirichlet problem for Laplace's equation

3.2.1 Transformation of the problem

Consider solving the equation

$$\begin{cases} \triangle u = 0 & \text{in } \Omega_e \\ u = f & \text{on } \Gamma \\ u(p) \to 0 & \text{as } |p| \to \infty \end{cases} \tag{3.1}$$

It can be shown that the problem (3.1) has a unique solution by converting it to an equivalent interior Dirichlet problem, using the "Kelvin transformation"; see [1, p.

10]. The solution of (3.1) can be represented by

$$u(p) = \frac{1}{4\pi}\int_\Gamma \frac{\rho(q)}{|p-q|}dS_q, \quad p \in \Omega_e$$

where ρ is called the density function and is determined by solving

$$\frac{1}{4\pi}\int_\Gamma \frac{\rho(q)}{|p-q|}ds_q = f(p), \quad p \in \Gamma \qquad (3.1a)$$

3.2.2 Numerical Examples

The following tables show results for numerically solving the equation (3.1) by using Galerkin's method to solve (3.1a). We use the algorithm described in Chapter 2, and we obtain test functions by applying the Kelvin transform to the boundary functions

$$\begin{aligned} u^1 &= 1 \\ u^2 &= x \\ u^3 &= e^x \cos y \\ u^4 &= ((x-5)^2 + (y-4)^2 + (z-3)^2)^{-\frac{1}{2}} \end{aligned}$$

inverting them with respect to the unit sphere. In the following tables, we continue to call the boundary functions u^1,u^2 u^3,u^4; but they refer to the functions obtained using the Kelvin transform.

Let

$$(x, y, z) = (A\cos\phi\sin\theta, B\sin\phi\sin\theta, C\cos\theta)$$

with $(A, B, C) = (1, 1, 1)$ for surface 1 (the unit sphere), $(A, B, C) = (1, 1.5, 2)$ and $(1, 2, 5)$ for surfaces 2 and 3 respectively, which are both ellipsoids. The Galerkin coefficients were calculated with inner integration parameter $M_i = 32$ and outer integration parameter $M_o = 16$. Table 10 contains results for Surface 1, with $N = 5$ in the first error column and $N = 8$ in the second error column. Note that the error in $u^1 - u^1_5$ in Table 10 is just that due to numerical integration, since $\mathcal{P}_N\rho = \rho$ because

ρ is a constant. These indicate the effect of the integration errors in calculating the various Galerkin coefficients.

Table 10: Exterior Dirichlet Problem on Surface 1

x	y	z	$\mathrm{u}^1 - \mathrm{u}^1_5$	$\mathrm{u}^3 - \mathrm{u}^3_8$
1.0000	1.5000	2.0000	$-5.230D-07$	$-6.463D-07$
1.5000	0.0000	0.0000	$-9.389D-07$	$-4.209D-06$
0.0000	2.0000	0.0000	$-7.042D-07$	$-7.042D-07$
3.0000	0.0000	0.0000	$-4.694D-07$	$-1.090D-06$

Table 11: Exterior Dirichlet Problem on Surface 2

x	y	z	$\mathrm{u}^3 - \mathrm{u}^3_5$	$\mathrm{u}^3 - \mathrm{u}^3_8$
1.5000	0.0000	0.0000	$4.293D-02$	$6.952D-03$
2.0000	0.0000	0.0000	$7.853D-03$	$6.897D-04$
3.0000	3.0000	3.0000	$5.893D-04$	$1.800D-05$
1.0000	1.5000	2.0000	$5.997D-04$	$-1.904D-05$

Table 11 contains results for the exterior Dirichlet problem on Surface 2, The true solution is u^3, and the approximation degree is $N = 5$ and $N = 8$. The other parameters are same as in Table 10. The high the degree, the better the error result.

The last table, Table 12, contains results on a more elongated ellipsoid. The parameters are integration the same as in Table 11. The higher the approximation degree, the more stable and accurate is the result.

Table 12: Exterior Dirichlet Problem on Surface 3

x	y	z	$u^4 - u^4_5$	$u^4 - u^4_8$
0.0000	0.0000	6.0000	$-3.639D-04$	$-1.587D-05$
4.0000	0.0000	0.0000	$1.815D-04$	$1.009D-05$
1.0000	2.0000	5.0000	$1.217D-04$	$-4.741D-06$
0.0000	3.0000	0.0000	$4.545D-04$	$4.105D-05$
4.0000	8.0000	20.000	$-5.033D-09$	$-6.660D-09$

3.3 The normal derivative of the solution of the Dirichlet problem

From Chapter 2, we know that the solution of the interior Dirichlet problem for Laplace's equation

$$\begin{cases} \triangle u = 0 & \text{in } \Omega \\ u = f & \text{on } \Gamma \end{cases}$$

can be represented by

$$u(p) = \frac{1}{4\pi} \int_\Gamma \frac{\rho(q)}{|p-q|} dS_q, \quad p \in \Omega$$

where Ω is a simple connected domain in $\mathbf{R^3}$ with smooth boundary Γ, and ρ is called the density function and is determined by solving

$$\frac{1}{4\pi} \int_\Gamma \frac{\rho(q)}{|p-q|} ds_q = f(p), \quad p \in \Gamma$$

It is hard to compute the normal derivative of u directly from the above equation. But with the Green's formula, we have

$$-\int_\Gamma \frac{\partial u}{\partial n_q} \frac{1}{|p-q|} dS_q + \int_\Gamma u(q) \frac{\partial}{\partial n_q} (\frac{1}{|p-q|}) dS_q = 2\pi u(p), \quad p \in \Gamma$$

So

$$\int_\Gamma \frac{\partial u}{\partial n_q} \frac{1}{|p-q|} dS_q = -2\pi f(p) + \int_\Gamma f(q) \frac{\partial}{\partial n_q} (\frac{1}{|p-q|}) dS_q$$

Let

$$g(p) = -2\pi f(p) + \int_\Gamma f(q) \frac{\partial}{\partial n_q}(\frac{1}{|p-q|}) dS_q$$

Then solving the equation

$$\int_\Gamma \frac{\partial u}{\partial n_q} \frac{1}{|p-q|} dS_q = g(p), \quad p \in \Gamma \tag{3.2}$$

will yield the desired normal derivative of u on Γ. We must modify the algorithm of Chapter 2, before solving the equation (3.2). We need two kinds of Galerkin coefficients: $(S\eta_i, \eta_j)$ and $(D\eta_i, \eta_j)$, where S is the single layer integral operator and D is the double layer integral operator. Once the Galerkin coefficients are calculated, the computing time for calculating the solution of the equation (3.2) is small. We give some numerical examples below.

Example 3.3.1 Consider the case $\Gamma = U$, the unit sphere. From Chapter 1, Section 1.2, we know that spherical polynomials are eigenfunctions of both the single layer and double layer potential. Actually, $D = \frac{1}{2}S$ in this case. Then we can calculate the exact solution of the above equation (3.2). Let S_n be any spherical harmonic of degree n, for $n \geq 0$. If the given boundary function f has the Laplace series expansion

$$f = \sum_{n=0}^{\infty} a_n S_n$$

Then the solution of the equation (3.2), $\frac{\partial u}{\partial n}$, will have the expansion

$$\frac{\partial u}{\partial n} = \sum_{n=0}^{\infty} b_n S_n$$

where

$$b_n = -n a_n$$

The following Table 13 gives the Laplace expansion for the given boundary function $f = x$, and Table 14 is the Laplace expansion for the solution. The coefficient of

Table 13: The Laplace expansion for the function $f = x$

	The coefficient	The Laplace expansion
	of $P_1^1(\cos\theta)\cos\phi$ term	$-2.05D+00$
	of the others, the maximum one is	$-2.25D-15$

Table 14: The Laplace expansion for the solution function

	The coefficient	The Laplace expansion
	of $P_1^1(\cos\theta)\cos\phi$ term	$2.05D+00$
	of the others, the maximum one is	$6.27D-15$

$P_1^1(\cos\theta)\cos\phi$ term is different only in sign ($b_1 = -a_1$). The degree of the expansion is 8, whereas using degree=1 would be sufficient in this case.

Recall that in Chapter 2, we used a "product Gaussian quadrature method" to approximate the integration. In Table 15, we evaluate the error of the true solution (i.e. the true normal derivative) and the numerical solution at every Gaussian node point over the sphere, and then we take the maximum of the absolute values of the error among these points (Total $2 * M_o^2$ many points) as our maximum error. The approximation degree here is 10.

Example 3.3.2. Let Γ be the ellipsoid

$$x^2 + \left(\frac{y}{1.5}\right)^2 + \left(\frac{z}{2}\right)^2 = 1 \ (E1)$$

Since we solve the equation (3.2) on the boundary Γ, the integral operator of $\frac{1}{|p-q|}$ is

Table 15: The maximum error for solving equation (3.2)

The given boundary function	The maximum error
$f = 1$	$6.80E - 15$
$f = x$	$4.20E - 06$
$f = x^2 + y^2 - 2z^2$	$2.79E - 05$

Table 16: The maximum error with $f = x$ on (E1)

The approximation degree	The maximum error
$N = 7$	$4.78E - 03$
$N = 9$	$1.42E - 03$
$N = 11$	$4.23E - 04$
$N = 13$	$1.29E - 04$
$N = 15$	$3.43E - 05$

singular and the normal derivative of u varies rapidly over the Γ, then if we choose the boundary function $f = x$ or $f = x^2 + y^2 + z^2$, in order to have an accurate answer, we should have sufficiently high approximation degree and integration parameters (inner and outer parameters) to calculate the Galerkin coefficients $(S\eta_i, \eta_j)$ and $(D\eta_i, \eta_j)$. For Table 16 and Table 17, we choose the inner integration parameter $M_i = 32$ and the outer integration parameter $M_o = 16$ (so that we have the maximum errors resulted from computation with 512 points),; and the approximation degrees are $N = 10$ and $N = 15$, respectively. The higher the degree, the better the error results.

Table 17: The maximum error on $(E1)$ $(f = x^2 + y^2 - 2z^2)$

The approximation degree N	The maximum error
$N = 10$	$9.08E - 03$
$N = 15$	$8.71E - 04$

Table 18: The maximum error for $f = x$

The inner integral parameter	$M_i = 32$
The outer integral parameter	$M_o = 16$
The approximation degree	$N = 10$
The given boundary function	$f = x$
The maximum error	$6.84E - 02$

Example 3.3.3. Let the boundary Γ be the ellipsoid

$$x^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{5}\right)^2 = 1$$

which is more ill-behaved than that in Example 3.3.2 in the sense of being more thin and narrow in shape. If the boundary function is $f = x$ or $f = x^2 + y^2 - 2z^2$, we need a higher approximation degree than for the previous ellipsoidal surface. See the numerical results in Table 18 and Table 19. Also note that the maximum of $\frac{\partial u}{\partial n}$ over Γ is 1.

By an analogous analysis, we can calculate the normal derivative of the solution of the exterior Dirichlet problem. Hence, when solving the Laplace equation with a

Table 19: The maximum error for $f = x$

	The inner integral parameter	$M_i = 32$
	The outer integral parameter	$M_o = 16$
	The approximation degree	$N = 15$
	The given boundary function	$f = x$
	The maximum error	$1.14E - 02$

Dirichlet boundary condition, we can solve for the Neumann boundary data at the same time as we solve for the density in the double layer representation of the solution u.

3.4 The Neumann problem for Laplace's equation

3.4.1 Introduction

We used the Galerkin method to solve the interior and exterior Dirichlet problems for Laplace's equation, by converting the problem to that of solving first kind integral equations. In contrast to the classical procedure, the Dirichlet problem has been solved with the help of a single layer potential instead of a double layer potential. Now a similar idea applies to the Neumann problem for Laplace's equation. We represent the unknown solution as a double layer potential, and it will lead to an integral equation of the first kind. The variational formulation of the problem comes from J. Giroire and J. C. Nedelec [14], and Tuong Ha Duong [10].

3.4.2 Variational formulation of the Neumann problem

As in [10], let Ω be a bounded open set in $\mathbf{R}^3$, the boundary Γ of which is sufficiently smooth, and let Ω_e be the set complementary to $\bar{\Omega}$. The normal to Γ is denoted by n, r is the distance from the origin to a point in $\mathbf{R}^3$, and $[v] = v|_{\Gamma}^{int} - v|_{\Gamma}^{ext}$ denotes the jump through Γ of a function v defined in $\mathbf{R}^3$. We consider the exterior Neumann problem

$$\begin{cases} \triangle u = 0 & \text{in } \Omega_e \\ \dfrac{\partial u}{\partial n} = g \in H^{-\frac{1}{2}}(\Gamma) & \end{cases} \tag{3.3}$$

in the following Sobolev space

$$W_0^1(\Omega_e) = \{u \in \mathcal{D}'(\Omega_e) \mid (1+r^2)^{-\frac{1}{2}} u \in L^2(\Omega_e) \text{ and } Du \in L^2(\Omega_e)\}$$

We consider the interior Neumann problem

$$\begin{cases} \triangle u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = g \in H^{-\frac{1}{2}(\Gamma)} & \end{cases} \tag{3.4}$$

in the space $H^1(\Omega)/\mathbf{R}$, with the compatibility condition

$$< g, 1 >= 0$$

Here $<,>$ denotes the duality of the spaces inner product $(H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma))$. From [10], the two problems are well-posed; and the solution u can be represented by a double layer potential:

$$u(p) = -\frac{1}{4\pi}\int_{\Gamma} \rho(q) \frac{\partial}{\partial n_q}(\frac{1}{|p-q|}) dq, \quad \text{for any } p \in \Omega \cup \Omega_e$$

Here the function ρ can be obtained by solving the following:

$$\begin{cases} \text{Find } \rho \in H^{\frac{1}{2}}(\Gamma)/\mathbf{R} \text{ such that} \\ a(\rho, \rho') =< g, \rho' >_{H_0^{-\frac{1}{2}}(\Gamma)\times H^{\frac{1}{2}}(\Gamma)/\mathbf{R}} \quad \text{for any } \rho' \in H^{\frac{1}{2}}(\Gamma)/\mathbf{R} \end{cases} \tag{3.5}$$

where the bilinear form a has the following expression

$$a(\rho, \rho') = \frac{1}{8\pi}\int_{\Gamma}\int_{\Gamma} (\rho(q) - \rho(p))(\rho'(q) - \rho'(p)) \frac{\partial^2}{\partial n_p \partial n_q}(\frac{1}{|p-q|}) ds_q ds_p$$

See [14], [10]. By the Lax-Milgram theorem, the equation (3.5) is uniquely solvable since the bilinear form a satisfies a strong ellipticity condition

$$a(\rho, \rho) \geq C\|\rho\|^2_{H^{\frac{1}{2}}(\Gamma)/\mathbf{R}}$$

for a constant C; see [14, p. 975].

3.4.3 Galerkin's method and error analysis

We use the same approximation space as in Chapter 2, namely $V_N = Span\{\eta_1, \cdots, \eta_{d_N}\}$. Then the Galerkin method for solving equation (3.3) is

$$\begin{cases} \text{Find } \rho_N \in V_N/R \ \text{ such that} \\ \quad a(\rho_N, \rho_N^{'}) =< g, \rho_N^{'} > \qquad \text{for any } \rho_N^{'} \in V_N/R \end{cases} \tag{3.6}$$

In a practical computation of (3.6), we expand ρ_N and $\rho_N^{'}$ in V_N, and we use every term of Laplace's expansion of ρ_N and $\rho_N^{'}$, except the constant term.

The method is convergent since we have the following theorem

Theorem 3.4.1 *Let ρ, ρ_N be defined on Γ as above, then ρ_N exists and is unique; and if $\rho \in H^{s+1}(\Gamma)/\mathbf{R}$, for any $s \in \mathbf{R}$, $s > -\frac{1}{2}$, then*

$$\|\rho - \rho_N\|_{H^{\frac{1}{2}}(\Gamma)/\mathbf{R}} = O(\frac{1}{N^{s+\frac{1}{2}}})$$

Proof From [14, p.980], we have

$$\|\rho - \rho_N\|_{H^{\frac{1}{2}}(\Gamma)/\mathbf{R}} \leq C \inf_{\sigma \in V_N/\mathbf{R}} \|\rho - \sigma\|_{H^{\frac{1}{2}}(\Gamma)/\mathbf{R}}$$

with C a constant independent of N. Using similar ideas to those used in proving Theorem 2.2.6 in Chapter 2, we have

$$\|\rho - \rho_N\|_{H^{\frac{1}{2}}(\Gamma)/\mathbf{R}} \to 0, \text{ as } N \to \infty$$

Furthermore, note the fact that if $\mathcal{L}_N$ is a projection map on V_N, defined as in Chapter 2 with $\rho \to \rho_N$, then

$$\sigma + R - \mathcal{L}_N(\sigma + R) = \sigma - \mathcal{L}_N(\sigma)$$

for any $\sigma \in H^{\frac{1}{2}}(\Gamma)$ and $R \in \mathbf{R}$. By a proof similar to that of Theorem 2.2.9 in Chapter 2, we have

$$\|\rho - \rho_N\|_{H^{\frac{1}{2}}(\Gamma)/\mathbf{R}} = O(\frac{1}{N^{s+\frac{1}{2}}})$$

□

CHAPTER IV

THE EXTERIOR DIRICHLET PROBLEM FOR THE HELMHOLTZ EQUATION

4.1 Introduction

The purpose of this chapter is to study the exterior Dirichlet problem for the Helmholtz equation on an unbounded domain, with a smooth connected boundary and a bounded complement. We use Galerkin's method to solve the problem, and the problem is formulated as a first kind of integral equation, based on representing the solution as a single layer potential. A complete numerical analysis is given.

4.2 Integral equation formulation

The Helmholtz equation is

$$\triangle u + k^2 u = 0, \text{ on } \Omega_e, \quad k \text{ real} \neq 0 \tag{4.1}$$

in which Ω_e is an unbounded domain in $\mathbf{R^3}$ with a smooth connected boundary Γ and a bounded complement. We associate with it boundary conditions on the surface Γ which are of the Dirichlet type; that is, given g on Γ and a positive number k, find a solution u of (4.1) satisfying the boundary condition

$$u = g \text{ on } \Gamma$$

and the Sommerfeld radiation condition

$$\frac{\partial u(X)}{\partial r} - iku(X) = o(\frac{1}{\sqrt{r}}), \quad r = |X| \to \infty$$

¿From [9, p. 144, Theorem 2], the solution u can be expressed by
$$u(p) = \frac{1}{4\pi}\int_\Gamma \frac{e^{-ik|p-q|}}{|p-q|}\sigma(q)ds_q, \quad p \in \Omega_e. \tag{4.2}$$
The density function σ is determined by the equation
$$\frac{1}{4\pi}\int_\Gamma \frac{e^{-ik|p-q|}}{|p-q|}\sigma(q)ds_q = g(p), \quad p \in \Gamma, \tag{4.3}$$
provided that k^2 is not an eigenvalue of the interior Dirichlet problem for the operator $\triangle$. We use Galerkin's method with approximations based on spherical harmonics on the unit sphere to solve (4.3) in the Sobolev space $H^t(\Gamma)$, $t \in \mathbf{R}$.

Denote
$$\begin{aligned} L\sigma(p) &= \frac{1}{4\pi}\int_\Gamma \frac{\sigma(q)}{|p-q|}ds_q, \qquad p \in \Gamma \\ E_k\sigma(p) &= \frac{1}{4\pi}\int_\Gamma \frac{e^{-ik|p-q|}-1}{|p-q|}\sigma(q)ds_q, \qquad p \in \Gamma \end{aligned}$$
where k^2 is not an eigenvalue of the interior Dirichlet problem for the operator $\triangle$. Then (4.3) is equivalent to
$$(L + E_k)\sigma(p) = g(p), \quad p \in \Gamma \tag{4.4}$$
Recall that L is a one-to-one map from $H^t(\Gamma)$ to $H^{t+1}(\Gamma)$ (see [9, p. 124]) and E_k is a continuous map from $H^t(\Gamma)$ to $H^{t+2}(\Gamma)$ (see [9, p.145]). So by using the Fredholm Alternative Theorem, the following can be proven.

Theorem 4.2.1 *The integral operator $L + E_k$ is one-to-one and onto from $H^t(\Gamma)$ to $H^{t+1}(\Gamma)$, for $t \in \mathbf{R}$. Equivalently, (4.3) has a unique solution $\sigma \in H^t$ for given $g \in H^{t+1}(\Gamma)$.*

Proof See [9, p. 144, Theorem 2]. □

Then the existence and uniqueness of the solution in (4.4) is guaranteed.

4.3 The numerical method and its error analysis

Recall the earlier notation of Chapter 2. In particular, $H^t(\Gamma)$ is a Sobolev space on Γ, $V_{d_N} = Span\{\eta_{d_1}, \cdots, \eta_{d_N}\}$ is the space of spherical polynomials of degree $\leq N$, and $\mathcal{P}_N$ is the orthogonal projection of $L^2(\Gamma)$ onto V_{d_N}. The Galerkin method for solving (4.4) is

$$\mathcal{P}_N(L + E_k)\sigma_N = \mathcal{P}_N g$$

for $\sigma_N \in V_N$. Since it is difficult to get an explicit expression for L^{-1}, we need the following theorem in order to produce an equivalent Galerkin's equation.

Theorem 4.3.1

$$\mathcal{P}_N L : \ V_N \rightarrow \ \mathcal{P}_N(H^t), \quad t \geq 0$$

is an isomorphism, and

$$\|(\mathcal{P}_N L)^{-1}\|_{H^{t+1} \rightarrow H^t} \leq C < \infty$$

with C independent of N, and for any $t \geq -\frac{1}{2}$.

Proof For any $g \in H^t$ and $\sigma_N \in V_N$,

$$\mathcal{P}_N L \sigma_N = \mathcal{P}_N g$$

is uniquely solvable if and only if

$$(L\sigma_N, \eta_i) = (g, \eta_i), \ i = 1, \cdots, d_N$$

is uniquely solvable, and this follows from using the results in Chapter 2. So

$$\mathcal{P}_N L : V_N \xrightarrow[\text{onto}]{1-1} \mathcal{P}_N(H^t)$$

Let

$$L\sigma = g$$

¿From Chapter 2, Corollary 2.2.10

$$\mathcal{L}_N \sigma = \sigma_N$$

with $\|\mathcal{L}_N\|_{H^t(\Gamma)} \leq C$ and C independent of N. Hence

$$\begin{aligned}
\|(\mathcal{P}_N L)^{-1}\mathcal{P}_N g\|_{H^t} &= \|\sigma_n\|_{H^t} \\
&\leq \|\mathcal{L}_N\|\|\sigma\|_{H^t} \\
&\leq C\|L^{-1}\|\|g\|_{H^{t+1}} \\
&\leq C\|g\|_{H^{t+1}}
\end{aligned}$$

for some constant C independent of N. Letting $g \in V_N$, we have

$$\|(\mathcal{P}_N L)^{-1}\|_{H^{t+1}\to H^t} \leq C < \infty$$

with C independent of N.

□

From the above theorem, the Galerkin method for solving (4.4) is equivalent to

$$(I + (\mathcal{P}_N L)^{-1}\mathcal{P}_N E_k)\sigma_N = (\mathcal{P}_N L)^{-1}\mathcal{P}_N g$$

and then

$$(I + \mathcal{P}_N(\mathcal{P}_N L)^{-1}\mathcal{P}_N E_k)\sigma_N = \mathcal{P}_N(\mathcal{P}_N L)^{-1}\mathcal{P}_N g \tag{4.5}$$

Before we give an error analysis of the Galerkin method, we need the following results.

Theorem 4.3.2 *For any $\sigma \in H^t(\Gamma)$, with $r \geq t \geq 0$,*

$$\|\mathcal{P}_N\sigma - \sigma\|_{H^t(\Gamma)} \leq CN^{t-r}\|\sigma\|_{H^r(\Gamma)}$$

Proof See Chapter 2, Lemma 2.2.7, we know that the theorem is true when $\Gamma = U$. For a general case, use Chapter 2, Theorem 2.2.2, similar to the proof in Theorem 2.2.9, we will get the result. □

Lemma 4.3.3 *For any $\sigma \in H^t$, with $r \geq t \geq 0$,*

$$\|\mathcal{P}_N L^{-1}E_k\sigma - L^{-1}E_k\sigma\|_{H^{t+1}(\Gamma)} \leq CN^{t-r}\|\sigma\|_{H^r(\Gamma)}$$

Proof By Theorem 4.3.2

$$\|\mathcal{P}_N L^{-1} E_k \sigma - L^{-1} E_k \sigma\|_{H^{t+1}(\Gamma)} \le C N^{t-r} \|L^{-1} E_k \sigma\|_{H^{r+1}(\Gamma)}$$

with C independent of N. From earlier, L is bounded one to one and onto from $H^r(\Gamma)$ to $H^{r+1}(\Gamma)$; and E_k is bounded from $H^t(\Gamma)$ to $H^{t+2}(\Gamma)$, see [9, p.145]. So $L^{-1}E_k$ is bounded from $H^r(\Gamma)$ to $H^t(\Gamma)$. Hence

$$\|L^{-1} E_k \sigma\|_{H^{t+1}(\Gamma)} \le C \|\sigma\|_{H^t(\Gamma)}$$

and we have the result. □

Lemma 4.3.4 *For sufficiently large N, the map*

$$I + \mathcal{P}_N (\mathcal{P}_N L)^{-1} \mathcal{P}_N E_k$$

is one-to-one and onto on $H^t(\Gamma)$, and

$$\|\left(I + \mathcal{P}_N (\mathcal{P}_N L)^{-1} \mathcal{P}_N E_k\right)^{-1}\| \le C$$

with C a constant independent of N. Hence the Galerkin's method (4.5) has a unique solution.

Proof

$$\begin{aligned} I + \mathcal{P}_N (\mathcal{P}_N L)^{-1} \mathcal{P}_N E_k &= I + \mathcal{P}_N (\mathcal{P}_N L)^{-1} \mathcal{P}_N L L^{-1} E_k \\ &= I + \mathcal{P}_N L^{-1} E_k \\ &= I + L^{-1} E_k + \left(\mathcal{P}_N L^{-1} E_k - L^{-1} E_k\right) \end{aligned}$$

By Lemma 4.3.3 (with t replaced by $t-1$, and r replaced by t),

$$\|\mathcal{P}_N L^{-1} E_k - L^{-1} E_k\|_{H^t \to H^t} \to 0$$

¿From Theorem 4.2.1, $I + L^{-1}E_k = L^{-1}(L + E_k)$ is a one to one and onto map from $H^t(\Gamma)$ to $H^t(\Gamma)$. So by the Geometric Series Theorem 1.3.2, we have

$$\|\left(I + \mathcal{P}_N (\mathcal{P}_N L)^{-1} \mathcal{P}_N E_k\right)^{-1}\| \le C$$

with C a constant independent of N.

□

We finish this section by showing the solvability of our approximating equation and giving an error analysis result.

Theorem 4.3.5 *If the numerical problem (4.4) be as described as above, and let* $g \in H^t(\Gamma), t \geq -\frac{1}{2}$*. Then* σ_N *exists and is unique; and for any* $r \geq t \geq -\frac{1}{2}$*,*

$$\|\sigma - \sigma_N\|_{H^t(\Gamma)} \leq CN^{t-r}\|\sigma\|_{H^r(\Gamma)}$$

with C *a generic constant.*

Proof By the above Lemma 4.3.4, we have,

$$\begin{aligned}
\sigma - \sigma_N &= \left[I + \mathcal{P}_N(\mathcal{P}_N L)^{-1}(\mathcal{P}_N E_k)\right]^{-1}\left[\sigma + \mathcal{P}_N(\mathcal{P}_N L)^{-1}(\mathcal{P}_N E_k)\sigma - \mathcal{P}_N(\mathcal{P}_N L)^{-1}\mathcal{P}_N g\right] \\
&= \left[I + \mathcal{P}_N(\mathcal{P}_N L)^{-1}(\mathcal{P}_N E_k)\right]^{-1}\left[\sigma + \mathcal{P}_N(\mathcal{P}_N L)^{-1}\mathcal{P}_N(E_k\sigma - g)\right] \\
&= \left[I + \mathcal{P}_N(\mathcal{P}_N L)^{-1}(\mathcal{P}_N E_k)\right]^{-1}\left[\sigma - \mathcal{P}_N(\mathcal{P}_N L)^{-1}\mathcal{P}_N L\sigma\right] \\
&= \left[I + \mathcal{P}_N(\mathcal{P}_N L)^{-1}(\mathcal{P}_N E_k)\right]^{-1}\left[\sigma - \mathcal{P}_N\sigma\right]
\end{aligned}$$

and from Theorem 4.3.2, for any $r \geq t \geq 0$, we have

$$\|\sigma - \sigma_N\|_{H^t(\Gamma)} \leq CN^{t-r}\|\sigma\|_{H^r(\Gamma)}$$

with C a generic constant. □

4.4 The approximation function

The approximate solution of (4.2) is given by

$$\begin{aligned}
u_N(p) &= \frac{1}{4\pi}\int_\Gamma \frac{e^{-ik|p-q|}}{|p-q|}\sigma_N(q)dS_q, \quad p \in \Omega_e \\
&= (L + E_k)\sigma_N(p)
\end{aligned}$$

Since

$$K(p,q) = \frac{1}{4\pi}\frac{e^{-ik|p-q|} - 1}{|p-q|}$$

is bounded, for $p, q \in \overline{\Omega_e}$,

$$\max_{p\in\overline{\Omega_e}} |u(p) - u_N(p)| \leq \frac{1}{4\pi}\max_{p\in\overline{\Omega_e}}\left|\int_\Gamma \frac{\sigma(q) - \sigma_N(q)}{|p-q|}dS\right|$$

$$+\frac{1}{4\pi}\max_{p\in\overline{\Omega_e}}\left|\int_\Gamma(\sigma-\sigma_N)K(p,q)dS\right|$$

we only need to estimate the error on the boundary, by using the maximum principle for the first term above. Since

$$\|L+E_k\|_{H^t(\Gamma)\to H^{t+1}(\Gamma)}\le C$$

with C a generic constant, using the result in Chapter 2 on Section 2.2.4, if $\sigma\in H^r(\Gamma)$, we have

$$\max_{p\in\overline{\Omega_e}}|u(p)-u_N(p)|\le CN^{\epsilon-r}\|\sigma\|_{H^r(\Gamma)}$$

with $r>\epsilon$.

As a special case, if we consider $\Gamma=U$, the unit sphere, then we may obtain an exact solution of the Dirichlet problem for the Helmholtz equation. By using the results from [22, pp. 122-123], we have that if μ is a spherical polynomial of degree n, then

$$(L+E_k)\mu(A)=-ikj_n(-k)h_n^{(1)}(-k|A|)\mu(\frac{A}{|A|}),\qquad |A|\ge 1$$

where the definition of functions j_n and $h_n^{(1)}$ can be found in [22, p.122]. So the Laplace expansion coefficients $\{A_n,A_n^m,B_n^m\}$ of σ are given by

$$(A_n,A_n^m,B_n^m)=\frac{1}{-ikj_n(-k)h_n^{(1)}(-k)}(\alpha_n,\alpha_n^m,\beta_n^m)$$

where $\{\alpha_n,\alpha_n^m,\beta_n^m\}$ are the Laplace expansion coefficients of the boundary function g. If we write

$$p=(r\cos\phi\sin\theta,r\sin\phi\sin\theta,r\cos\theta)$$

with $|r|>1$, we have

$$\begin{aligned}u(p) &= \sum_{n=0}^{\infty}\left(-ikj_n(-k)h_n^{(1)}(-kr)\right)r^n\left\{A_nP_n(\cos\theta)\right.\\ &\quad\left.+\sum_{m=1}^{n}[A_n^m\cos(m\phi)+B_n^m\sin(m\phi)]P_n^m(\cos\theta)\right\}\end{aligned}$$

CHAPTER V

A NUMERICAL METHOD FOR SOLVING THE BIHARMONIC EQUATION ON SMOOTH SURFACES IN $\mathbf{R^3}$

5.1 Introduction

Using Galerkin's method with spherical polynomials as approximations, we consider the Dirichlet problem for the biharmonic equation on domains in $\mathbf{R^3}$ with a smooth boundary. The problem can be converted to the solution of a system of two integral equations with two unknown density functions. Consider the solution of the following Dirichlet problem:

$$\begin{cases} \triangle^2 \mathcal{X} = 0 & \text{in } \Omega \\ \mathcal{X} = f_1 \ , \ \dfrac{\partial \mathcal{X}}{\partial n} = f_2 & \text{on } \Gamma \end{cases} \tag{5.1}$$

where Ω is a bounded set in $\mathbf{R^3}$ with a smooth boundary Γ. As before, we assume that there is a diffeomorphism map between Γ and the unit sphere. The inward unit normal vector on Γ is denoted by n; and f_1 and f_2 are given functions on Γ. We will solve the integral equations using Galerkin's method with globally defined basis functions. Although the numerical method uses globally defined approximations, instead of less expensive boundary element functions, we obtain a small approximating linear system and very adequate accuracy. Once the Galerkin coefficients for the region have been calculated, the computation time is small for any particular boundary function.

Denote

$$G(p, q) = \frac{1}{8\pi}|p - q|$$

$$g(p,q)=\frac{1}{4\pi}\frac{1}{|p-q|}$$

¿From [16, p. 108], or by direct computation, $\triangle G=g,$ and we can uniquely express the solution $\mathcal{X}$ of (5.1) as follows:

$$\mathcal{X}(p)=\int_\Gamma \sigma(q)G(p,q)ds_q+\int_\Gamma \mu(q)g(p,q)ds_q, \quad p\in\Omega \tag{5.2}$$

The two density functions σ,μ are to be determined by imposing the boundary conditions of (5.1). In fact, we have the following two theorems:

Theorem 5.1.1 *Assume $\mathcal{X}\in C^4(\Omega)\bigcap C^3(\bar{\Omega})$, with Ω a simply connected bounded region with a smooth boundary Γ. If $\mathcal{X}$ is a biharmonic function with $\mathcal{X}=\frac{\partial\mathcal{X}}{\partial n}=0$ on Γ, then $\mathcal{X}=0$ in Ω.*

Proof See [17, p.71].

□

Theorem 5.1.2 *Under the same assumptions on Ω as in Theorem 5.1.1, every biharmonic function $\mathcal{X}$ has a unique representation $\mathcal{X}=\phi+\psi$, where*

$$\phi(p)=\int_\Gamma \sigma(q)G(p,q)ds_q,$$
$$\psi(p)=\int_\Gamma \mu(q)g(p,q)ds_q$$

Proof Following an argument similar to the proof from [17, p.72], let $v=\triangle\mathcal{X}$. Then v is harmonic in Ω. By the theory of harmonic functions, there exists a unique single layer density function σ such that

$$v(p)=\int_\Gamma g(p,q)\sigma(q)ds_q$$

Introduce

$$\phi(p)=\int_\Gamma \sigma(q)G(p,q)ds_q$$

Then $\triangle(\mathcal{X}-\phi)=v-v=0$. So

$$\mathcal{X}=\phi+\psi$$

with ϕ defined as above, and with $\triangle\psi = 0$. By the theory of harmonic functions, there is a unique single-layer density function μ such that

$$\psi(p) = \int_\Gamma \mu(q) g(p,q) ds_q$$

By the above theorem, if there is a biharmonic function satisfying the given Dirichlet boundary data, then the solution is unique. This theorem shows that the integral equation formula (5.2) is uniquely determined by the given Dirichlet boundary data, and hence the representation (5.2) is uniquely determined by the given Dirichlet boundary data.

□

5.2 The biharmonic single layer integral operator

Recall the definition of the Sobolev space $H^t(\Gamma)$ in Chapter 1; and let S_n denote any spherical harmonic of degree n, for $n \geq 0$.

Theorem 5.2.1 *Let B be the unit ball in $\mathbf{R^3}$, and define*

$$F(p) = \int_{\partial B} |p-q| S_n(q) ds_q, \quad \textit{for } p \in \partial B$$

Then

$$F(p) = -\frac{16\pi}{(2n+1)(2n-1)(2n+3)} S_n(p)$$

for any $p \in \partial B$.

Proof From the Rayleigh-Green identity (see [16, pp. 271-272]), any biharmonic function ϕ on B satisfies

$$-2\int_{\partial B} \left\{ \phi(\frac{1}{R})' - \frac{1}{R}\phi' \right\} dS_q + \int_{\partial B} \left\{ R(\triangle\phi)' - (\triangle\phi)R' \right\} dS_q = -4\pi\phi(p), \quad \text{for } p \in \partial B$$

In this, $R = |p-q|$, and all derivatives are inner normal derivatives. Let

$$\phi(p) = r_p^{n+2} S_n(p), \quad r_p = |p|$$

Since $r_p^n S_n(p)$ is harmonic, from [16, pp. 105-108], we have the following:

1. ϕ is biharmonic.

2. $\phi'(p) = -(n+2)r_p^{n+1}S_n(p)$, with $\phi' = \frac{\partial\phi}{\partial n}$.

3. $\triangle\phi(p) = \triangle(r_p^2 r_p^n S_n(p)) = (6+4n)r_p^n S_n(p)$.

4. $(\triangle\phi)'(p) = -n(6+4n)r_p^{n-1}S_n(p)$.

For any $p \in B + \partial B$. In addition, we have:

$$\begin{aligned} R' &= -\frac{1}{2}R \\ \left(\frac{1}{R}\right)' &= \frac{1}{2R} \end{aligned}$$

Hence

$$-2\int_{\partial B}\frac{1}{R}\left(\frac{\phi}{2}-\phi\right)dS_q + \int_{\partial B} R\left[(\triangle\phi)' + \frac{1}{2}(\triangle\phi)\right]dS_q = -4\pi\phi$$

$$-2\int_{\partial B}\frac{1}{R}\left(\frac{1}{2}S_n + (n+2)S_n\right)dS_q + \int_{\partial B} R\left[-n(6+4n) + \frac{1}{2}(6+4n)\right]S_n dS_q = -4\pi S_n$$

and

$$(-2n-5)\frac{4\pi}{2n+1}S_n + (6+4n)(-n+\frac{1}{2})\int_{\partial B} RS_n dS_q = -4\pi S_n$$

That is

$$\begin{aligned} F(p) &= \int_{\partial B} RS_n dS_q \\ &= \frac{-4\pi + \frac{2n+5}{2n+1}4\pi}{(2n+3)(-2n+1)}S_n \\ &= -\frac{16\pi}{(2n+1)(2n-1)(2n+3)}S_n(p) \end{aligned}$$

for any $p \in \partial B$.

□

In general, if we suppose that there is a C^∞ mapping between Γ and unit sphere, we have the following assumption.

ASSUMPTION 1: Let S_b be defined by

$$S_b f(p) = \int_\Gamma |p-q| f(q) ds_q, \qquad f \in H^t(\Gamma)$$

and assume that

1. $S_b : H^t(\Gamma) \xrightarrow[\text{onto}]{1-1} H^{t+3}(\Gamma)$ for $t \geq -\frac{3}{2}$

2. $|(S_b f, f)| \geq C\|f\|^2_{-\frac{3}{2}}, \ f \in H^{-\frac{3}{2}}$

Note that

(1) By the Lax-Milgram Theorem, the second assumption implies the part of the first assumption when $t = -\frac{3}{2}$.

(2) The assumption is true if $\Gamma = U$, the unit sphere (Theorem 5.2.1). So by a continuity argument in the space of bounded operators, if Γ is a boundary that is close enough to any unit sphere, then the assumption is satisfied. We hypothesize it is true for any smooth surface Γ that is the boundary of a bounded simply-connected region in $\mathbf{R^3}$.

5.3 A boundary integral equation reformulation

Applying the boundary conditions in (5.1) to the representation (5.2), we obtain a system of two integral equations:

$$\left\{ \begin{array}{ll} \displaystyle\int_\Gamma G(p,q)\sigma(q)ds_q + \int_\Gamma g(p,q)\mu(q)ds_q = f_1(p), & p \in \Gamma \\ \\ \displaystyle\int_\Gamma \frac{\partial G(p,q)}{\partial n_p}\sigma(q)ds_q + \int_\Gamma \frac{\partial g(p,q)}{\partial n_p}\mu(q)ds_q - \frac{1}{2}\mu(p) = f_2(p) & p \in \Gamma \end{array} \right. \tag{5.3}$$

Let

$$\mathcal{A} = \begin{pmatrix} \frac{1}{8\pi}S_b & \frac{1}{4\pi}S \\ \\ 0 & -\frac{1}{2}I \end{pmatrix}$$

where

$$S\mu(p) = \int_\Gamma \frac{1}{|p-q|}\mu(q)ds_q, \quad p \in \Gamma$$

Also introduce

$$\begin{aligned} K_1\sigma(p) &= \int_\Gamma \frac{\partial}{\partial n_p}\left(|p-q|\right)\sigma(q)ds_q \\ K_2\mu(p) &= \int_\Gamma \frac{\partial}{\partial n_p}\left(\frac{1}{|p-q|}\right)\mu(q)ds_q \\ \mathcal{K} &= \begin{pmatrix} 0 & 0 \\ & \\ \frac{1}{8\pi}K_1 & \frac{1}{4\pi}K_2 \end{pmatrix} \end{aligned}$$

Then (5.3) becomes

$$(\mathcal{A}+\mathcal{K})(\sigma,\mu)^T = (f_1, f_2)^T \tag{5.4}$$

Introduce

$$\|(\sigma,\mu)^T\|_{H^m(\Gamma)\times H^n(\Gamma)} = \max(\|\sigma\|_{H^m(\Gamma)}, \|\mu\|_{H^n(\Gamma)})$$

for any $m, n \in \mathbf{R}$. Then $\|(\cdot,\cdot)^T\|_{H^m(\Gamma)\times H^n(\Gamma)}$ is a norm on $H^m(\Gamma)\times H^n(\Gamma)$, and this latter space is complete.

For any $t \geq 0$, from [9, p. 124]

$$S: \ H^t(\Gamma) \xrightarrow[\text{onto}]{1-1} H^{t+1}(\Gamma).$$

¿From [9, p. 364], K_2 is a continous map from $H^t(\Gamma)$ to $H^{t+1}(\Gamma)$; and by the earlier ASSUMPTION 1,

$$S_b: \ H^t(\Gamma) \xrightarrow[\text{onto}]{1-1} H^{t+3}(\Gamma),$$

Let

$$W(p) = \int_\Gamma G(p,q)\sigma(q)ds_q, \quad p \in \Omega$$

By the Trace Theorem 1.3.4 in Chapter 1,

$$W: H^t(\Omega) \to H^{t+3+\frac{1}{2}}(\Omega)$$

so the integral operator

$$\int_\Gamma \frac{\partial}{\partial n_p}(|p-q|)\sigma(q)ds_q, \quad p \in \Omega$$

will map from $H^t(\Omega)$ to $H^{t+2+\frac{1}{2}}(\Omega)$. Hence we have

$$K_1: \ H^t(\Gamma) \to H^{t+2}(\Gamma)$$

by the Trace Theorem (see Chapter 1). However, this can be improved.

If $\Gamma = U$, the unit sphere, we have that $K_1 = -\frac{1}{2}S_b$, so K_1 will map from $H^t(U)$ to $H^{t+3}(U)$. By the continuity argument in the space of bounded operators, if Γ is a boundary that is close enough to any sphere, then K_1 will map from $H^t(\Gamma)$ to $H^{t+3}(\Gamma)$. Hence it is reasonable that we make the following assumption for any smooth surface Γ that is the boundary of a bounded simply-connected region in $\mathbf{R}^3$. Actually, from

$$(p-q, n_p) = O(|p-q|^2)$$

we have:

$$\begin{aligned} \left|\frac{\partial}{\partial n_p}(|p-q|)\right| &= \left|\frac{(p-q, n_p)}{|p-q|}\right| \\ &\leq C|p-q| \end{aligned}$$

with C a constant. So K_1 and S_b have the same mapping property in the space $L^2(\Gamma)$ or $C(\Gamma)$. In general,

ASSUMPTION 2:

K_1: $H^t(\Gamma) \to H^{t+3}(\Gamma)$, for $t \geq 0$.

Theorem 5.3.1 *$\mathcal{A} + \mathcal{K}$ is a bijective map from $H^t(\Gamma) \times H^{t+2}(\Gamma)$ onto $H^{t+3}(\Gamma) \times H^{t+2}(\Gamma)$, for any $t \geq 0$.*

Proof By the ASSUMPTION 1, we have

$$\mathcal{A}^{-1} = \begin{pmatrix} 8\pi S_b^{-1} & 4(S_b)^{-1}S \\ & \\ 0 & -2I \end{pmatrix}$$

Recall that $H^m \subset H^n$, if $m > n$, $m, n \in \mathbf{R}$, and the inclusion map is compact. Also, by ASSUMPTION 2, $\mathcal{K}$ maps from $H^t(\Gamma) \times H^{t+2}(\Gamma)$ to $0 \times H^{t+3}(\Gamma)$, so $\mathcal{K}$ is a compact operator from $H^t(\Gamma) \times H^{t+2}(\Gamma)$ to $H^{t+3}(\Gamma) \times H^{t+2}(\Gamma)$.

Note that $\mathcal{A}+\mathcal{K}=\mathcal{A}(I+\mathcal{A}^{-1}\mathcal{K})$ with $\mathcal{A}^{-1}\mathcal{K}$ compact on $H^t(\Gamma)\times H^{t+2}(\Gamma)$ to itself. By the Fredholm Alternative Theorem, we only need to prove the injectiveness of $\mathcal{A}+\mathcal{K}$, and that follows from Theorem 5.1.1 and Theorem 5.1.2.

□

For the case $\Gamma=\partial B$, the unit sphere, we can carry out the exact solution of σ,μ from system (5.3), and hence obtain the exact solution of our biharmonic equation described in (5.1). For the unit sphere,

$$\frac{\partial}{\partial n_p}(|p-q|) = -\frac{1}{2}|p-q|$$

$$\frac{\partial}{\partial n_p}\left(\frac{1}{|p-q|}\right) = \frac{1}{2}\frac{1}{|p-q|}$$

This simplifies the integral operators in (5.3). To solve (5.1) on ∂B, write

$$\begin{aligned}\sigma &= \sum_{n=0}^{\infty} a_n S_n\\ \mu &= \sum_{n=0}^{\infty} b_n S_n\\ f_1 &= \sum_{n=0}^{\infty} c_n S_n\\ f_2 &= \sum_{n=0}^{\infty} d_n S_n\end{aligned}$$

Recall from Theorem 5.2.1 that

$$\int_{\partial B}|p-q|S_n(q)dS_q = -\frac{16\pi}{(2n+1)(2n-1)(2n+3)}S_n(p)$$

and recall

$$\int_{\partial B}\frac{1}{|p-q|}S_n(q)dS_q = \frac{4\pi}{2n+1}S_n(p)$$

(see Chapter 1, Section 1.2, pp. 6-7). Then we have we have

$$\begin{aligned}a_n &= -(2n+1)(2n+3)(d_n+nc_n);\\ b_n &= -\frac{2n+1}{2n-1}(c_n+2d_n)\end{aligned}$$

Hence

$$\mathcal{X}(rp) = \sum_{n=0}^{\infty} r^n[-\frac{2}{(2n-1)(2n+1)(2n+3)}a_n + \frac{1}{2n+1}b_n]S_n(p)$$

for any $p \in \partial B$ and $|r| \leq 1$.

5.4 Galerkin's method and error analysis

Consider now the case that Γ is the smooth boundary of a bounded simply-connected domain Ω in $\mathbf{R^3}$, and that there is a diffeomorphism between Γ and ∂B. Let

$$V_N = Span\{\eta_1, \cdots, \eta_{d_N}\}$$

as defined in Chapter 2, and denote $\mathcal{P}_N$ as the orthogonal projection from $L^2(\Gamma)$ to $V_N \subset L^2(\Gamma)$.

Theorem 5.4.1 $\mathcal{P}_N S_b : V_N \rightarrow \mathcal{P}_N(H^{t+3}(\Gamma))$, *for any* $t \geq 0$, *is an isomorphism, and*

$$\|(\mathcal{P}_N S_b)^{-1}\|_{H^{t+3}(\Gamma)\rightarrow H^t(\Gamma)} \leq C < \infty$$

with C *independent of* N.

Proof By using the strong ellipticity condition for S_b in ASSUMPTION 1, by using the same techniques as in proving Theorem 2.2.9 in Chpater 2, and by using a very similar proof of that for Theorem 4.3.1.

□

If we define

$$\mathcal{P}_N^1 = \mathcal{P}_N \times \mathcal{P}_N$$

then we have the following theorems

Theorem 5.4.2 *For any* $\Phi = (\sigma, \mu) \in H^t(\Gamma) \times H^{t+2}(\Gamma)$, *with* $r \geq t \geq 0$,

$$\|\mathcal{P}_N^1\Phi - \Phi\|_{H^t \times H^{t+2}} \leq CN^{t-r}\|\Phi\|_{H^r \times H^{r+2}}$$

Proof We have

$$\|\mathcal{P}_N^1\Phi - \Phi\|_{H^t \times H^{t+2}} = \max\{\|\mathcal{P}_N\sigma - \sigma\|_{H^t}, \|\mathcal{P}_N\mu - \mu\|_{H^{t+2}}\}$$

By using Theorem 4.3.2, we have the result. □.

Theorem 5.4.3

$$\mathcal{P}_N^1\mathcal{A}: \; V_N \times V_N \to \mathcal{P}_N^1(H^{t+3} \times H^{t+2}), \quad t \geq 0$$

is an isomorphism.

Proof For any $(f_1, f_2)^T \in H^{t+3} \times H^{t+2}$, and $\sigma_N, \mu_N \in V_N$,

$$\mathcal{P}_N^1\mathcal{A}(\sigma_N, \mu_N)^T = \mathcal{P}_N^1(f_1, f_2)^T$$

is uniquely solvable if and only if

$$\begin{cases} \dfrac{1}{8\pi}\mathcal{P}_N S_b \sigma_N + \dfrac{1}{4\pi}\mathcal{P}_N S \mu_N = \mathcal{P}_N f_1 & \text{(i)} \\ \\ -\frac{1}{2}\mu_N = \mathcal{P}_N f_2 & \text{(ii)} \end{cases}$$

¿From (ii), we have the unique solution

$$\mu_N = -2\mathcal{P}_N f_2$$

Substituting it into (i), we have

$$\mathcal{P}_N S_b \sigma_N = \mathcal{P}_N g_N$$

for $g_N = 4S\mathcal{P}_N f_2 + 8\pi f_1$ a given function in H^{t+3}. Using Theorem 5.4.1, we have a unique σ_N from the above. □

The Galerkin method for solving (5.4) is

$$\mathcal{P}_N^1(\mathcal{A} + \mathcal{K})\Phi_N = \mathcal{P}_N^1 f$$

where

$$\Phi_N = (\sigma_N, \mu_N)^T \in V_N \times V_N$$

and

$$f = (f_1, f_2)^T \in H^{t+3} \times H^{t+2}$$

is given. From Theorem 5.4.3, it is equivalent to

$$(I + (\mathcal{P}_N^1\mathcal{A})^{-1}\mathcal{P}_N^1\mathcal{K})\Phi_N = (\mathcal{P}_N^1\mathcal{A})^{-1}\mathcal{P}_N^1 f$$

and

$$(I + \mathcal{P}_N^1(\mathcal{P}_N^1\mathcal{A})^{-1}\mathcal{P}_N^1\mathcal{K})\Phi_N = \mathcal{P}_N^1(\mathcal{P}_N^1\mathcal{A})^{-1}\mathcal{P}_N^1 f$$

Theorem 5.4.4 *For sufficiently large N, say $N \geq N_0$,*

$$(I + \mathcal{P}_N(\mathcal{P}_N\mathcal{A})^{-1}\mathcal{P}_N\mathcal{K})^{-1}$$

exists and is uniformly bounded on space $H^t(\Gamma) \times H^{t+2}(\Gamma)$ to itself.

Proof

$$\begin{aligned} I + \mathcal{P}_N^1(\mathcal{P}_N^1\mathcal{A})^{-1}\mathcal{P}_N^1\mathcal{K} &= I + \mathcal{P}_N^1(\mathcal{P}_N^1\mathcal{A})^{-1}(\mathcal{P}_N^1\mathcal{A})\mathcal{A}^{-1}\mathcal{K} \\ &= I + \mathcal{P}_N^1\mathcal{A}^{-1}\mathcal{K} \\ &= I + \mathcal{A}^{-1}\mathcal{K} + \left(\mathcal{P}_N^1\mathcal{A}^{-1}\mathcal{K} - \mathcal{A}^{-1}\mathcal{K}\right) \end{aligned}$$

Since $I + \mathcal{A}^{-1}\mathcal{K}$ is invertible from $H^t \times H^{t+2}$ to $H^t \times H^{t+2}$ from Theorem 5.3.1, and

$$\|\mathcal{P}_N^1\mathcal{A}^{-1}\mathcal{K} - \mathcal{A}^{-1}\mathcal{K}\|_{H^t\times H^{t+2}\to H^{t+3}\times H^{t+2}} \to 0$$

from using the fact that

$$\mathcal{A}^{-1}\mathcal{K} : H^t \times H^{t+2} \to H^{t+1} \times H^{t+3}$$

is bounded, and using Theorem 5.4.2. So by the Geometric Series Theorem 1.3.2 in Chapter 1, we have that for sufficiently large N, in the space $H^t(\Gamma) \times H^{t+2}(\Gamma)$,

$$\left(I + \mathcal{P}_N^1(\mathcal{P}_N^1\mathcal{A})^{-1}\mathcal{P}_N^1\mathcal{K}\right)^{-1}$$

is uniformly bounded. □

Hence, we have the framework for proving the solvability of the approximating system and convergence of the method.

Theorem 5.4.5 *If $\Phi = (\sigma, \mu)^T$ is a real solution and $\Phi_N = (\sigma_N, \mu_N)^T$ is the numerical solution of (5.4), described as above, with $f = (f_1, f_2)^T \in H^{t+3}(\Gamma) \times H^{t+2}(\Gamma)$, then for any $r \geq t \geq 0$, $r, t \in \mathbf{R}$*

$$\|\Phi - \Phi_N\|_{H^t(\Gamma)\times H^{t+2}(\Gamma)} \leq CN^{t-r}\|\Phi\|_{H^r(\Gamma)\times H^{r+2}(\Gamma)}$$

with C a generic constant.

Proof

$$\begin{aligned}
\Phi - \Phi_N &= \left[I + \mathcal{P}_N^1(\mathcal{P}_N^1\mathcal{A})^{-1}(\mathcal{P}_N^1\mathcal{K})\right]^{-1}\left[\Phi + \mathcal{P}_N^1(\mathcal{P}_N^1\mathcal{A})^{-1}(\mathcal{P}_N^1\mathcal{K})\Phi - \mathcal{P}_N^1(\mathcal{P}_N^1\mathcal{A})^{-1}\mathcal{P}_N^1 f\right] \\
&= \left[I + \mathcal{P}_N^1(\mathcal{P}_N^1\mathcal{A})^{-1}(\mathcal{P}_N^1\mathcal{K})\right]^{-1}\left[\Phi + \mathcal{P}_N^1(\mathcal{P}_N^1\mathcal{A})^{-1}\mathcal{P}_N^1(\mathcal{K}\Phi - f)\right] \\
&= \left[I + \mathcal{P}_N^1(\mathcal{P}_N^1\mathcal{A})^{-1}(\mathcal{P}_N^1\mathcal{K})\right]^{-1}\left[\Phi + \mathcal{P}_N^1(\mathcal{P}_N^1\mathcal{A})^{-1}\mathcal{P}_N^1\mathcal{A}\Phi\right] \\
&= \left[I + \mathcal{P}_N^1(\mathcal{P}_N^1\mathcal{A})^{-1}(\mathcal{P}_N^1\mathcal{K})\right]^{-1}\left[\Phi - \mathcal{P}_N^1\Phi\right]
\end{aligned}$$

¿From Theorem 5.4.4 , for any $r \geq t \geq 0$, we have:

$$\begin{aligned}
\|\Phi - \mathcal{P}_N^1\Phi\|_{H^t(\Gamma)\times H^{t+2}(\Gamma)} &\leq C\max(\|\sigma - \mathcal{P}_N\sigma\|_{H^t(\Gamma)}, \|\mu - \mathcal{P}_N\mu\|_{H^{t+2}(\Gamma)}) \\
&\leq C\max(N^{t-r}\|\sigma\|_{H^r(\Gamma)}, N^{t-r}\|\mu\|_{H^{r+2}(\Gamma)}) \\
&\leq CN^{t-r}\max(\|\sigma\|_{H^r(\Gamma)}, \|\mu\|_{H^{r+2}(\Gamma)}) \\
&= CN^{t-r}\|\Phi\|_{H^r(\Gamma)\times H^{r+2}(\Gamma)}
\end{aligned}$$

Hence

$$\|\Phi - \Phi_N\|_{H^t(\Gamma)\times H^{t+1}(\Gamma)} \leq CN^{t-r}\|\Phi\|_{H^r(\Gamma)\times H^{r+1}(\Gamma)}$$

with C a generic constant.

□

5.5 Numerical examples

We modify the algorithm described in [4], and we choose three kinds of surfaces Γ in R^3: an ellipsoid(including the unit sphere), a peanut-shaped region, and a heart-shaped region, which are the same surfaces as we chose for the numerical calculations in Chapter 2.

Example 5.1. Let $\Gamma = U$, the unit sphere,

$$x^2 + y^2 + z^2 = 1$$

In Table 20, we choose the exact solution $f_1 = 1$, and in Table 21, we choose $f_1 = \frac{1}{2} + \frac{1}{6}(x^2 + y^2 + z^2)$. By directly calculation from the integral equation system (5.3), the two density functions σ and μ should be $\sigma = 0, \mu = 1$ for Table 20, and $\sigma = 1, \mu = 0$ for Table 21. So the first term of the Laplace's expansion,based on orthonormal basis, for $\mu = 1$ in Table 20 and $\sigma = 1$ in Table 21, should be $\sqrt{4\pi}$, we show the numerical results in Table 22. In both Table 20 and Table 21, the major errors in getting the potential solution come from calculating the Galerkin coefficients $(S\eta_i, \eta_j), (S_b\eta_i, \eta_j), (K_1\eta_i, \eta_j)$, and $(K_2\eta_i, \eta_j)$, Here we use: inner integration parameter $M_i = 32$, and outer integration parameter $M_o = 16$ for calculating the above four Galerkin coefficients. When calculating the right hand side of the system, note that the inner products (f_1, η_j) and (f_2, η_j) involve only a smooth integrand. For that reason, we approximate the integral of the inner products with the Gaussian scheme discussed in Chapter 2. The degree of the approximating polynomials we use here is 8, although degree 0 should be sufficient.

Table 20: Example for Γ a Unit sphere and $\sigma = 0$, $\mu = 1$

	x	y	z	potential solution $\mathcal{X}$	error
	0.0000	0.0000	0.0000	$1.0000014019D+00$	$-1.402D-06$
	0.2000	0.2000	0.2000	$1.0000014023D+00$	$-1.402D-06$
	0.2500	0.2500	0.2500	$1.0000014024D+00$	$-1.402D-06$
	0.5000	0.5000	0.5000	$1.0000025035D+00$	$-2.503D-06$
	0.1000	0.1000	0.1000	$1.0000014020D+00$	$-1.402D-06$

Table 21: Example for Γ a Unit sphere and $\sigma = 1$, $\mu = 0$

	x	y	z	potential solution $\mathcal{X}$	error
	0.0000	0.0000	0.0000	$5.0000000308D-01$	$-3.083D-09$
	0.2000	0.2000	0.2000	$5.2000000321D-01$	$-3.206D-09$
	0.2500	0.2500	0.2500	$5.3125000328D-01$	$-3.275D-09$
	0.5000	0.5000	0.5000	$6.2500000272D-01$	$-2.723D-09$
	0.1000	0.1000	0.1000	$5.0500000311D-01$	$-3.113D-09$

Table 22: The first terms of the Laplace expansion

	Thedensity function	for Table 1	for Table 2
	σ	$1.3605627109D-07$	$3.5449077237D+00$
	μ	$3.5449126034D+00$	$-2.010653358D-14$

Example 5.2. Let Γ be the ellipsoid

$$x^2 + \left(\frac{y}{1.5}\right)^2 + \left(\frac{z}{2}\right)^2 = 1 \quad \text{(E1)}$$

Table 23: The true solution $\mathcal{X} = 1$ on (E1)

	x	y	z	$\mathcal{X} - \mathcal{X}^4$	$\mathcal{X} - \mathcal{X}^8$
	0.00	0.00	0.00	$1.466D-05$	$-6.126D-07$
	0.10	0.10	0.10	$1.129D-05$	$-6.746D-07$
	0.30	0.30	0.30	$3.235D-06$	$-9.593D-07$
	0.50	0.00	0.50	$8.895D-05$	$-2.194D-09$

Table 24: The true solution $\mathcal{X} = x^2 + y^2 + z^2$ on $(E1)$

	x	y	z	$\mathcal{X} - \mathcal{X}^4$	$\mathcal{X} - \mathcal{X}^8$
	0.00	0.00	0.00	$-6.065D-06$	$7.327D-06$
	0.10	0.10	0.10	$5.617D-06$	$7.555D-06$
	0.30	0.30	0.30	$1.342D-05$	$8.386D-06$
	0.50	0.00	0.50	$-4.033D-04$	$4.669D-06$

In Table 23, we let $f_1 = 1$ and $f_2 = 0$. We see in Table 23 that the higher the approximation degree N, the higher the accuracy of the solution. $\mathcal{X}^4$ denotes the approximant of degree 4, and analogously for $\mathcal{X}^8$.

Table 24 shows the error results for choosing the true solution function $\mathcal{X} = x^2 + y^2 + z^2$, and again the accuracy improves when the degree increases.

If we set Γ to be

$$x^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{5}\right)^2 = 1 \quad \text{(E2)}$$

Table 25: Errors for the biharmonic equation on (E2)

	x	y	z	$\mathcal{X} - \mathcal{X}^6$	$\mathcal{X} - \mathcal{X}^8$
	0.00	0.00	0.00	$3.219D-03$	$-1.719D-03$
	0.10	0.10	0.10	$3.030D-03$	$-1.629D-03$
	0.30	0.30	0.30	$1.737D-03$	$-9.763D-04$
	0.50	0.50	0.50	$-1.352D-03$	$-1.385D-04$

it is even more ill-behaved in the sense of being thin and narrow in shape (see [3]). We obtain the convergence results shown in the Table 25. The true solution $\mathcal{X}$ is the same as in Table 24, namely $\mathcal{X} = x^2 + y^2 + z^2$.

Example 5.3. Recall that a peanut-shaped region, based on the ovals of Cassini, is defined by

$$R = \sqrt{\cos(2\theta) + \sqrt{\alpha + 1 - \sin^2(2\theta)}}$$

$$(x, y, z) = R(\sin\theta\cos\phi, 2\sin\theta\cos\phi, \cos\theta)$$

α is a parameter determining the narrowness of the "neck" in the shape (the case of bigger α is more well-behaved than the smaller α case). We require $\alpha > 1$. In the following Table 26, we choose the exact solutions $\mathcal{X}_1 = \frac{1}{2} + \frac{1}{6}(x^2 + y^2 + z^2)$ and $\mathcal{X}_2 = x^2 + y^2 + z^2$ for the first error column and the second error column, respectively.

Example 5.4. Finally, we show error results on a heart-shaped region:

$$(x, y, z) = R(2\sin\theta\cos\phi, \sin\theta\cos\phi, \cos\theta)$$

$$R(x, y, z) = 2 - (1 + 100(z+1)^2)^{-1}, \quad (x, y, z) \in U$$

Table 26: The errors for $\alpha = 0.1$ on "peanut" region

	x	y	z	$\mathcal{X} - \mathcal{X}_1^8$	$\mathcal{X} - \mathcal{X}_2^8$
	0.00	0.00	0.00	$-1.973D-05$	$-5.080D-05$
	0.00	0.00	4.00	$1.312D-04$	$5.715D-04$
	0.14	0.28	0.20	$-2.304D-05$	$1.118D-04$
	0.03	0.06	0.04	$-1.935D-05$	$-8.908D-05$
	0.07	0.14	0.10	$-2.682D-05$	$-2.555D-04$

Table 27: The errors on a heart-shape region

	x	y	z	$\mathcal{X} - \mathcal{X}_1^8$	$\mathcal{X} - \mathcal{X}_2^8$	$\mathcal{X} - \mathcal{X}_3^8$
	0.00	0.00	0.00	$8.644D-04$	$4.301D-10$	$-2.535D-02$
	2.00	0.00	0.00	$2.490D-05$	$2.132D-04$	$-4.402D-04$
	0.00	1.00	0.00	$3.411D-04$	$-2.085D-09$	$-9.900D-03$
	0.00	0.00	1.00	$2.397D-04$	$-8.853D-10$	$-7.209D-03$
	2.00	0.00	1.00	$1.282D-05$	$-4.177D-05$	$-7.244D-04$

In Table 27, the first error column is based on the boundary function $\mathcal{X}_1 = 1$, the second, $\mathcal{X}_2 = x^2$, and the third, $\mathcal{X}_3 = x^2 + y^2 + z^2$.

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