DYNAMIC FRICTIONLESS CONTACT PROBLEMS WITH LINEARLY ELASTIC BODIES

by

Jeongho Ahn

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
The University of Iowa

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Thesis Supervisor: Associate Professor David E. Stewart

ABSTRACT

This thesis consists of two parts. For the first part, we formulate dynamic frictionless contact problem with an elastic body, based on Signorini's contact condition, and consider how to solve this formulation. First, we set up a time-discretization of this problem, which, for each time-step, is a variational inequality. We also derive the minimization problem equivalent to the variational inequality for each-step. After the energy functional for an elastic body is defined, it is shown that the energy functional is increased or decreased, depending on our numerical scheme. Especially, employing the implicit Euler method, the convergence for the time-discretization is investigated. For that numerical method, we obtain an estimate of the magnitude of the normal contact force in the Sobolev space $H^{-1/2}(\partial\Omega)$, depending on the time step size h. Indeed, we need more investigation to determine the boundedness of the contact force and finer regularity properties and conservation of energy, and then implement our numerical scheme. These will be future works.

For the second part, we set up the dynamic frictionless Euler–Bernoulli equation with Signorini contact conditions along the length of a thin beam and consider how to solve this equation. The existence of solutions is shown, based on a penalty method. While existence of solutions is shown, there are no results on whether energy is conserved in the limit. We formulate a time-discretization, using the implicit Euler method for contact conditions and the midpoint rule for the elastic part of the equations. The energy functional is defined, and convergence for the time-discretization is investigated. Our time-discretization leads to energy dissipation. Using this time discretization and the finite element method with B-spline basis functions, we compute numerical solutions. In order to solve the linear complementarity problem that arises in the numerical method, we use a smoothed guarded Newton method. The numerical results are guaranteed to be dissipative. We also investigate numerically the question of whether the numerical solutions converge strongly to their limit, and if energy is conserved for the limit. Our numerical results give some evidence that this is so.

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PH.D. THESIS

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CHAPTER 1
INTRODUCTION

The contact problem has been an important issue in solid mechanics. After the foundation of continuum mechanics, whose concepts provide a framework for studying the behavior of solids was established early in the nineteenth century, contact problems in solid mechanics began appearing in the literature. See [44]. In fact, friction problem had been studied before the continuum concept was developed. However, the use of Coulomb's law applied point-wise in contact problems with the theory of elasticity has caused mathematical difficulties. See [46, 12, 18]. Furthermore, when the frictional contact problem is considered together with dynamic effects, it leads to extremely complicated physical mechanism and mathematical models.

According to these issues involved with friction, in this thesis we first consider simpler problem, a dynamic contact problem without friction. However, the current existence theory is not very satisfactory as it either deals with simplified problems with special geometry [36, 51, 50] or gives no indication as to whether energy is conserved, dissipated, or even produced in impact [33]. There has been considerable focus on problems with Coulomb friction and viscoelastic bodies [9, 13, 12, 14, 30, 31, 32, 38], although these too have often involved unrealistic assumptions in order to obtain existence of solutions (e.g., penalty approximation for contact, non-local friction laws, "viscous" contact laws). There has also been considerable work investigating static and quasistatic approximations of contact problems [10, 11, 14, 27, 29, 28, 32].

The study of contact problems in elasticity had started in 1881 by Hertz [26].

At that time he analyzed a static frictionless contact problem of two elastic bodies. After around half a century, Signorini [52] in 1933 formulated unilaterally the equilibrium of an elastic body in contact with frictionless rigid foundation (obstacle), which has been called Sognorini's problem.

Signorini's problem was first solved rigorously by Fichera. He considered a question of existence and uniqueness of a variational inequality characterizing the minimization of the total potential energy on convex subsets of Banach spaces. Since his paper [22] was published, many important contact problems in mechanic and physics have been formulated in terms of variational inequality. For more on variational inequalities and their applications, see, for example, [17, 5]. In fact, variational inequalities have been recognized to play an important role to develop powerful numerical scheme. It turns out that a number of advanced studies of contact problem in solid mechanic resulted from development of theory of variational inequalities.

A number of papers have discussed the problem of various kinds of dynamic elastic bodies making contact with rigid foundations. For the case of the wave equation with frictionless contact at a boundary, there are the results of Kim [33]. The group of Cocu, Raous and Pratt and their students have carried out the analysis of many dynamic and quasi-static problems involving viscoelastic bodies in contact with a frictional rigid foundation (using a non-local frictional law) [10, 13, 14]. Others working on quasi-static and dynamic contact problems include Han and Sofonea *et al.* [25, 7], Jarušek and Eck [30, 31], for contact on a boundary although the boundary conditions are not always Signorini boundary conditions. Dynamic contact problems with compliant foundations, where the normal contact force is modeled as being due to a linear or nonlinear spring at each contact points, have been used since the work of Oden and Martins [37, 38] and others. However, these models do not address the question of how to handle truly rigid foundations, or the behavior of the solutions as the stiffness of the foundation goes to infinity. The question of energy conservation, or even of a complete accounting for energy, is not addressed in the above work.

In the first part of this thesis, the dynamic contact formulations that we consider are based on Signorini's formulation. The contact in our mathematical model is unilaterally occurring between an elastic body and a frictionless rigid foundation. Also, an elastic body does not penetrate a rigid foundation. Furthermore, if the body and the foundation do not touch at a point, then there is no contact force at that point. The actual contact surface on which the body touches the foundation is unknown in advance.

In the second part of this thesis, we consider a one-dimensional system where contact (modeled using Signorini conditions) can occur anywhere within the spatial extent of the system, and not only at its boundaries. The closest system to this that has been studied in the literature was analyzed by Schatzman [51] which considered a string which moved according to the one-dimensional wave equation and could make contact with a rigid *concave* obstacle. In that case, energy was shown to be conserved; the analysis was based on the use of characteristics. Recent work by Shillor *et al.* has addressed questions relating to contact problems with Euler–Bernoulli beams. Andrews, Shillor and Wright [2] treats frictional contact with both compliant and Signorini contact *at an end point* of the beam. In that paper they show that for a Euler–Bernoulli beam with a model of Kelvin–Voigt viscoelasticity and compliant contact, then existence and uniqueness hold (even if there is viscosity). In the Signorini contact case, existence is shown, but uniqueness is not. In our thesis, we consider contact that is *distributed* along the beam. García, Han, Shillor and Sofonea [23] consider a quasi-static frictional contact problem with an Euler–Bernoulli beam, but include the effect of wear due to the contact.

Finally, we mention that the emphasis on this thesis is on dynamic aspects of contact without friction.

1.1 Notation and some basic concepts

Mathematically, physical quantities such as displacement, velocity, strain, and stress which are used in continuum mechanics are represented by tensors. Then scalar can be considered as zero-order tensor and vector can be considered as first-order tensor.

Throughout this thesis, we employ some notations and conventions which are standard in modern mathematics. The Signorini's problem and dynamic contact problems that we deal with are presented in symbolic (vector) notation. However, sometimes we will express our system in terms of index (indical) notation, instead of vector notation, since the index notation may give us more clear, concise and useful expression. Vectors and tensors are generally designated by bold face characters such as $\mathbf{a}$, $\mathbf{b}$, etc. In the index notation vector $\mathbf{a}$ is represented by single subscript, i.e.,

$[a_i]$.

In three-dimensional Euclidean space such as ordinary physical space, vector $[a_i]$ which is denoted by the index notation can be displayed by the form

$$[a_i] = (a_1, a_2, a_3) \quad \text{or} \quad [a_i] = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}. \tag{1.1}$$

Then the first form in (1.1) is the transpose of $\mathbf{a}$ denoted by $\mathbf{a}^T$. Similarly, a second-order tensor $[A_{ij}]$ which is denoted by the index notation represents nine components and is presented as

$$[A_{ij}] = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}.$$

The good example of a second-order tensor in continuum mechanics is strain and stress tensor.

Under the rules of index notation, the summation convention is stated as follows: when we have a repeated index, called dummy index, in given term, we sum over all values of the index. Also the unrepeated index, called free index, must appear in every term in an equation correctly.

We introduce an (inner) product of tensors and the differential vector operator ∇ with tensors. When we define product of two vectors, we use the single dot and write

$$\mathbf{a} \cdot \mathbf{b} = [a_i] \cdot [b_i] = \sum_{i=1}^{d} a_i \cdot b_i.$$

Note that throughout this thesis, we use the Einstein summation convention, which omits the summation signs (we sum over repeated indices):

$$\mathbf{a} \cdot \mathbf{b} = a_i \cdot b_i.$$

In this thesis, there is no distinction between $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a}^T \cdot \mathbf{b}$. Similarly, when we define product of two second-order tensors, we use the double dot and write

$$\mathbf{A} : \mathbf{B} = [A_{ij}] : [B_{ij}] = \sum_{i,j=1}^{d} A_{ij} B_{ij}.$$

For a vector $\mathbf{a}$ and tensor $\mathbf{A}$, divergence of those can be written as

$$\nabla \cdot \mathbf{a} = a_{i,i} = \frac{\partial a_i}{\partial x_i} \text{ and } \nabla \cdot \mathbf{A} = A_{ij,j} = \frac{\partial A_{ij}}{\partial x_j},$$

where $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \cdots, \partial/\partial x_d)$. Note that for scalar function ϕ, we define a gradient of ϕ as

$$\nabla \phi = \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \cdots, \frac{\partial \phi}{\partial x_d} \right).$$

In order to simplify the algebraic expression in the analysis, we present some special notations:

1. For defined functions f, g, we write $f = O(g)$ as $t \to t_0$, if there exists a constant C such that $|f(t)| \leq C\,|g(t)|$.

2. We write $f(t) \sim g(t)$ if

$$\lim_{t \to \infty} \frac{f(t)}{g(t)} = 1.$$

In general, we will consider certain spaces of functions defined on a bounded open domain Ω in $\mathbf{R}^d (1 \leq d \leq 3$ in application). We will mention function space in Section 2.2. In application, we consider Ω as an open domain representing an interior of a deformable elastic material in $\mathbf{R}^d$ and a connected domain with points on only one side of a boundary $\partial\Omega$.

Suppose that a Cartesian coordinate system is established in $\mathbf{R}^d$ and any vector $\mathbf{x}$ in $\mathbf{R}^d$ is specified by $\mathbf{x} = (x_1, x_2, \cdots, x_d)$. We introduce a definition of Lipschitz domain from which most of basic results will be obtained.

1. Let a boundary $\partial\Omega$ cover with a collection $\{U_{1,}U_2, \cdots U_M\}$ of open subsets of $\mathbf{R}^d$ and $\partial\Omega \subset \bigcup_{r=1}^{M} U_r$ such that

$$\partial\Omega_r = U_r \cap \partial\Omega \neq \emptyset, \; r = 1, 2, \cdots, M.$$

2. After an affine change of local coordinates such as translation and rotation, assume that there are an $\alpha > 0$ and $\beta > 0$ such that, locally, the smoothness of the boundary $\partial\Omega$ can be described in terms of hypersurfaces defined by functions f_r on sets S_r, where

$$\begin{aligned}
S_r &= \{\mathbf{y}_r = (y_{r1}, y_{r2}, \cdots, y_{rd-1}) \mid |y_{ri}| < \alpha, \; i = 1, 2, \cdots, d-1\}, \\
\partial\Omega_r &= U_r \cap \partial\Omega = \{(\mathbf{y}_r, f_r(\mathbf{y}_r)) \mid \mathbf{y}_r \in S_r\}, \\
U_r^+ &= U_r \cap \Omega = \{y_r \mid \mathbf{y}_r \in S_r, \; f_r(\mathbf{y}_r) < y_{rd} < f_r(\mathbf{y}_r) + \beta\}, \\
U_r^+ &= U_r - \overline{\Omega} = \{y_r \mid \mathbf{y}_r \in S_r, \; f_r(\mathbf{y}_r) - \beta < y_{rd} < f_r(\mathbf{y}_r)\}.
\end{aligned}$$

3. Function f_r is Lipschitz continuous, which satisfies the estimate

$$|f_r(\mathbf{x}_r) - f_r(\mathbf{y}_r)| \leq C\|\mathbf{x}_r - \mathbf{y}_r\| \quad \text{for } \mathbf{x}_r, \mathbf{y}_r \in S_r,$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbf{R}^d$.

Then if the above conditions 1, 2, 3 are satisfied, Ω is said to be a Lipschitzian domain. Under Lipschitzian domain, outward normal vector $\mathbf{n}(\mathbf{x})$ on $\partial\Omega$ exists almost everywhere on $\partial\Omega$. Note that each occurrence, C denotes a constant (a quantity that depends only on the data of the problem), which may differ at each occurrence of this thesis.

We denote by $\mathbf{u}$ a displacement vector field on an open domain Ω which describes the deformation of the elastic body Ω ($\mathbf{u} : \Omega \to \mathbf{R}^d$) and denote by $\mathbf{f}$ ($\mathbf{f} : \Omega \to \mathbf{R}^d$) a body force which acts on Ω. Also we denote the magnitude of contact force by N and so $N\mathbf{n}$ is a contact force, which acts only on the boundary $\partial\Omega$. The *gap function* g is used to describe a measure of the "gap" between the elastic body and rigid foundation. Since we assume that deformation is assumed small, we use the linearized strain tensor given by

$$\boldsymbol{\varepsilon}[\mathbf{u}] = \varepsilon_{ij}[\mathbf{u}] = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{1.2}$$

where $u_{i,j} = \partial u_i / \partial x_j$. Note that $\phi_{,j} = \partial \phi / \partial x_j$.

A elastic solid materials are characterized by constitutive equation which relates the strain tensor and stress tensor on the specific form

$$\sigma_{ij}[\mathbf{u}] = E_{ijkl}\varepsilon_{kl}[\mathbf{u}] = E_{ijkl}u_{k,l} \quad \text{for } 1 \leq i, j, k, l \leq d,$$

where the above expression is known as the generalized Hooke's law. Now we assume that the fourth-order Hooke's tensor E_{ijkl} satisfies the following conditions:

1. $E_{ijkl} \in L^\infty(\Omega)$, i.e., there is a constant number M such that

$$\max_{1\le i,j,k,l\le d} \|E_{ijkl}\| \le M,$$

2. E_{ijkl} has the symmetry properties:

$$E_{ijkl} = E_{jikl} = E_{ijlk} = E_{klij} \quad \text{almost everywhere in } \Omega,$$

3. There is a constant number $m > 0$ such that almost everywhere in Ω,

$$E_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \ge m\varepsilon_{ij}\varepsilon_{ij}.$$

Note that $\varepsilon_{kl} = \varepsilon_{lk}$. Since E_{ijkl} and ε_{kl} have these symmetry properties, the stress tensor σ_{ij} is symmetric.

1.2 The equations of elasticity

The equations of elasticity are given by

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla \cdot \boldsymbol{\sigma}[\mathbf{u}] + \mathbf{f} \qquad \text{in } \Omega,$$

where ρ is the density of the elastic body; $\mathbf{u}$ is a displacement; $\boldsymbol{\sigma}$ is a stress tensor; $\mathbf{f}$ is the body force applied to the elastic body. The boundary conditions that are used depend on the physical situation. If the boundary is fixed or clamped on $\Gamma_D \subset \partial\Omega$, then

$$\mathbf{u}(\mathbf{x}) = \mathbf{k}(\mathbf{x}), \qquad \mathbf{x} \in \Gamma_D$$

for some given function $\mathbf{k}$ on Γ_D. If the boundary is not fixed in position, but has a boundary traction $\mathbf{t}(\mathbf{x})$ applied to it over $\Gamma_F \subset \partial\Omega$, then

$$\mathbf{n} \cdot \boldsymbol{\sigma}[\mathbf{u}](\mathbf{x}) = \mathbf{t}(\mathbf{x}), \qquad \mathbf{x} \in \Gamma_F.$$

Frictionless Signorini contact conditions for a piece of the boundary $\Gamma_C \subset \partial\Omega$ can be written as

$$0 \le \mathbf{u}(\mathbf{x}) \cdot \mathbf{n} + g(\mathbf{x}) \quad \perp \quad N(\mathbf{x}) \ge 0 \qquad \text{for almost all } \mathbf{x} \in \Gamma_C,$$

where

$$\mathbf{n}(\mathbf{x}) \cdot \boldsymbol{\sigma}[\mathbf{u}](\mathbf{x}) = N(\mathbf{x})\, \mathbf{n}(\mathbf{x}) \qquad \text{for almost all } \mathbf{x} \in \Gamma_C.$$

Note that the condition "$0 \le a(\mathbf{x}) \perp b(\mathbf{x}) \ge 0$ for almost all $\mathbf{x} \in \Gamma$" means that $a(\mathbf{x}),\ b(\mathbf{x}) \ge 0$ for almost all $\mathbf{x} \in \Gamma_C$ and that $\int_{\Gamma_C} a(\mathbf{x})\, b(\mathbf{x})\, d\mathbf{x} = 0$. Note that $\Gamma_C \cup \Gamma_F \cup \Gamma_D = \partial\Omega$.

1.3 The Euler–Bernoulli beam in contact

The Euler–Bernoulli beam equation is an approximate equation for long, slender rods and beams under small deformation in a vertical plane. What is immediately evident when you try to bend rod or beam is that they are generally much stronger and stiffer along the rod or beam, while they bend much more easily in the transverse direction. The Euler–Bernoulli beam equation ignores any deformation in the axial direction (along the beam), and considers only transverse deformation, since this is usually much larger. If we consider the forces acting on a cross-section of the beam, then there are vertical transverse and axial forces. The axial forces arise to counter bending of the beam; since the beam is assumed slender, we can take a simple linear approximation for the axial forces on the cross-section. Since the deformation in axial direction is assumed negligible, the integral of the axial forces over the cross section must be zero. Thus the axial forces have the functional form "function(x) $\cdot$ $(y - \overline{y})$" where $\overline{y}$ is the transverse component of the centroid of the cross-section. Since the axial deformation is negligible, the deformation generating the axial forces is bending. For small deformations (of any kind), the axial forces have the "function(x)" above would be proportional to the radius of curvature, or $\partial^2 u/\partial x^2$ plus higher order terms, where $u(x, t)$ is the vertical displacement of the beam's centroid. More refined calculation give the elastic energy in such a beam to be well approximated by

$$\frac{1}{2}\int_0^L EI\left(\frac{\partial^2 u}{\partial x^2}\right)^2\,dx,$$

where E is the Young's modulus of the material and

$$I = \int_{\mathcal{A}} (y - \overline{y})^2\,dxdy$$

is the *second moment of area* for the cross-section $\mathcal{A}$.

Applying standard variational techniques, we can obtain the equations for the vertical displacement u of an Euler–Bernoulli beam with constant cross section:

$$\rho A\frac{\partial^2 u}{\partial t^2} = -EI\frac{\partial^4 u}{\partial x^4} + f(x, t), \qquad 0 < x < L,$$

where A is the area of the cross section; f is the body force applied to the beam. Note that this is a fourth order rather than a second order equation, as is usual in elasticity. Initial and boundary conditions need to be included. There must be an

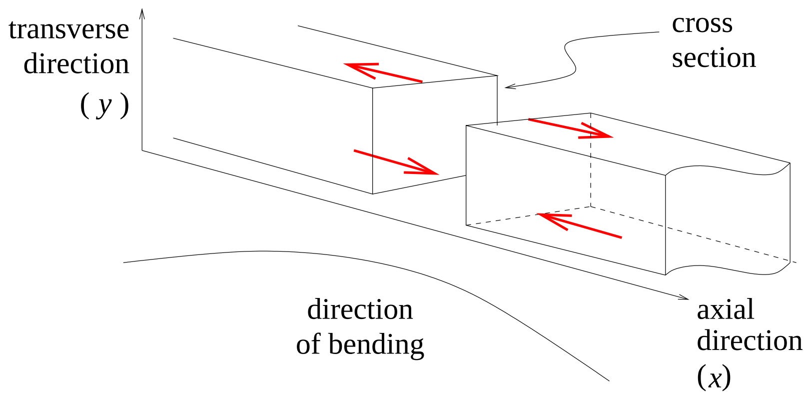

Figure 1.1: Cross-section of a slender beam.

initial configuration, $u(x, 0) = u_0(x)$, and an initial velocity $\partial u/\partial t(x, 0) = v_0(x)$. The boundary conditions that we use are clamped boundary conditions at $x = 0$: $u(0, t) = 0$, $\partial u/\partial x(0, t) = 0$, and free boundary conditions at $x = L$: $\partial^2 u/\partial x^2(L, t) = 0$ and $\partial^3 u/\partial x^3(L, t) = 0$. Note that the free boundary conditions are obtained from the variational conditions.

If we consider contact that can occur along the beam (rather than just at the end-points), we have the following version of Signorini's contact conditions:

$$0 \leq u(x, t) + g(x) \quad \perp \quad N(x, t) \geq 0,$$

where $N(x, t)$ is the (vertical) contact force, and the equations of motion need to be modified to include it:

$$\rho A \frac{\partial^2 u}{\partial t^2} = -EI \frac{\partial^4 u}{\partial x^4} + f(x, t) + N(x, t), \qquad 0 < x < L.$$

This problem is easier to study both theoretically and numerically as it is a spatially one-dimensional problem. If we considered only axial deformations and used a linear elastic model, we would arrive at the wave equation:

$$\rho A\frac{\partial^2 u}{\partial t^2} = -EA\frac{\partial^2 u}{\partial x^2} + f(x,t), \qquad 0 < x < L.$$

The corresponding contact problem:

$$\begin{aligned} \rho A\frac{\partial^2 u}{\partial t^2} &= -EA\frac{\partial^2 u}{\partial x^2} + f(x,t) + N(x,t), \qquad 0 < x < L \\ 0 \le u(x,t) + g(x) &\perp N(x,t) \ge 0 \end{aligned}$$

has been studied by Schatzman [51], and complete results were obtained (including conservation of energy). However, the methods used in [51] relied on studying characteristics for the wave equation. The Euler–Bernoulli equation on the other hand does not have characteristics as it is second order in time and fourth order in space; waves can travel arbitrarily rapidly. Also, it should be noted that Schatzman [51] did not consider time-discretization or efficient numerical methods for the solution of the wave equation with unilateral contact. Andrews, Shillor and Wright [2] only occurred at one end, not along the length of the beam. The best known of these are Timoshenko's beam equations [57, 58]. The Timoshenko equations are a system of two second order equations which have characteristics, just as the wave equation has characteristics. The main difference is that there are two characteristic speeds, which would complicate the analysis in [51]. These equations are a topic for later investigation.

1.4 Outline

The structure of this thesis is as follows. In Chapter 2, we provide some preliminaries which are applied throughout this thesis. In Section 2.1, we introduce basic concepts of functional analysis and convex analysis. In Section 2.2, we present the useful notions of space which are relevant to partial differential equations. In Section 2.3, we review a few theorems. In Section 2.4, we discuss penalty method and linear complementarity problem.

In Chapter 3, we establish a continuous formulation of dynamic frictionless contact condition with an elastic body, based on Signorini's contact condition. From Section 3.1 to Section 3.2 we see how to derive the dynamic frictionless contact problem. Employing time discretization, we set up three numerical formulations for the equations of motion and contact conditions in Section 3.3 and Section 3.4. In Section 3.5, we obtain a variational inequality equivalent to our contact problem and also derive minimization problem equivalent to a variational inequality. This plays an important role in finding numerical solutions at each time step. Total energy functional for elastic bodies is defined in Section 3.7.

In Chapter 4, we discuss the convergence of our time discretization. In Section 4.1, it is shown that the continuous linear interpolants of velocity and displacement are bounded in appropriate spaces. In Section 4.2, we derive an estimate of magnitude of contact force at one time step, depending our numerical schemes.

In Chapter 5, we begin considering the Euler–Bernoulli beam equation with Signorini contact conditions. In Section 5.1, imposing Signorini contact condition along the Euler–Bernoulli beam, we formulate Euler–Bernoulli beam equation with Signorini contact conditions. In Section 5.2, the existence of a solution is shown, based on penalty method.

In Chapter 6, we set up a time discretization, using the midpoint rule for the elastic part and the implicit Euler method for contact conditions and investigate the convergence of time discretization. In Section 6.2, it is shown that our time discretization leads to energy dissipation. This gives us a crucial step for analyzing the convergence theory. In Section 6.3, we show that the continuous linear interpolants of u converge to a solution.

In Chapter 7, we consider how to implement our numerical scheme. In Section 7.1, we discuss the Finite Element Method for the Euler–Bernoulli beam equation with B-spline basis functions. In Section 7.2, we show that energy is dissipated in fully discrete case. In Section 7.3, we solve the linear complementarity problem that arises in the numerical method, using smoothed guarded Newton method. Also the relevant theory of semi-smooth functions is discussed. In Section 7.4, numerical evidence for strong convergence is presented. In Section 7.6, while numerical results (simulation) are presented, we discuss our numerical experience and the numerical results.

Finally we list conclusions and future works in Chapter 8. We discuss the issues related to elastic bodies in and the Euler–Bernoulli beam in dynamic frictionless contact. This thesis is concluded with discussion of future works in Section 8.3.

CHAPTER 2
PRELIMINARIES

2.1 Linear operators and weak concepts

Let X and Y be real Banach spaces.

Definition 2.1. We say that X is compactly imbedded in Y if

1. X is continuously imbedded in Y, i.e., $X \subset Y$ and there is a constant C with $\|x\|_Y \leq C\|x\|_X$ for every $x \in X$,

2. Any bounded sequence in X is precompact, i.e., every bounded sequence in X has a subsequence that converges in Y.

Also we define a bounded linear operator.

Definition 2.2. A linear operator $A : X \to Y$ is bounded if there exists a constant C such that

$$\|Ax\|_Y \leq C\|x\|_X \quad \text{for every } x \in X.$$

If no such C exists, the operator is unbounded. Then we call the smallest such C the norm of A.

Definition 2.3. We denote the set of all bounded linear operators from X to Y by $\mathcal{L}(X, Y)$. We also use the notation $\mathcal{L}(X)$ for $\mathcal{L}(X, X)$.

Thus if $A \in \mathcal{L}(X, Y)$, we define the norm of a linear bounded operator A as

$$\|A\| := \sup_{\|x\|_x \neq 0} \frac{\|Ax\|_Y}{\|x\|_X} = \sup_{\|x\| \leq 1} \|Ax\|_Y.$$

Definition 2.4. A bounded linear operator $f \in \mathcal{L}(X, \mathbf{R})$ is called a bounded linear functional on X. The space of all bounded linear functionals on X is called the dual space and denoted by X^*.

For space X and the dual space X^* we introduce the notation: If $x \in X$ and $f \in X^*$, we write $\langle f, x \rangle$ to denote $f(x)$. So the the symbol $\langle \cdot, \cdot \rangle$ denote the duality paring on X^* and X.

Let H be a Hilbert space with inner product $(\cdot, \cdot)$.

Definition 2.5. If $A \in \mathcal{L}(H)$ satisfies $(Au, v) = (u, A^*v)$ for all $u, v \in H$, A^* is called its adjoint. Furthermore A is said to be self-adjoint if $A^* = A$.

Indeed for a bounded linear operator A from one Hilbert space H_1 to another H_2, its adjoint and self-adjoint can be defined. See [41] for the detailed discussion.

As usual, when we solve partial differential equation we involves a sequences of function which approach to solution. But it not easy to show that they converges in Banach space. For such difficulty, weak convergence is extremely useful. We introduce the following definition.

Definition 2.6. A sequence x_n in X is said to converge weakly to x if $f(x_n)$ converges to $f(x)$ for every $f \in X^*$. A sequence f_n in X^* is said to converges weakly* to f if $f_n(x)$ converges to $f(x)$ for every $x \in X$.

In order to distinguish notations, we write $x_n \to x$ for strong convergence in norm, $x_n \rightharpoonup x$ for weak convergence, and $f_n \rightharpoonup^* f$ for weak* convergence.

Let K be a subset of X. Then K is said to be convex if $(1-\lambda)x + \lambda y \in K$ for $x, y \in K$ and $0 < \lambda < 1$. Let F be a functional from K to $(-\infty, \infty]$. Then F is said to be convex if

$$F(\lambda x + (1-\lambda)y) \leq \lambda F(x) + (1-\lambda)F(y), \quad \text{for } x, y \in K \text{ and } 0 < \lambda < 1.$$

Now, we introduce weak semicontinuity and semicontinuity. Let K be a nonempty closed convex subset of X.

Definition 2.7. A functional F is said to be weakly lower semicontinuous on K if for any sequence x_n in K with the property that $x_n \rightharpoonup x$ in K, we have

$$\liminf_{n\to\infty} F(x_n) \geq F(x). \tag{2.1}$$

F is weakly upper semicontinuous on X if $\limsup_{n\to\infty} F(x_n) \leq F(x)$. Also F is said to be lower semicontinuous if for every sequence $x_n \to x$ in X and (2.1) holds. Upper semicontinuous is defined in an analogous way. The detailed arguments can found in [48, 32].

Finally, we mention the Mazur's Lemma which is applicable to solving partial differential equations. Mazur's Lemma asserts that a closed convex subset K is weakly closed and if F is convex lower semicontinuous, it is weakly sequentially lower semicontinuous. See [3, 20]. More generalized Mazur's Lemma can be found in [35].

2.2 Function spaces

In this section, we introduce some definitions and results related to function space for later references.

2.2.1 Multi-index notation

Generally, the notation of multi-index is very convenient to denote partial derivatives of function $u(\mathbf{x})$ defined on $\mathbf{x} \in \Omega \subset \mathbf{R}^d$. A multi-index is a vector of the form

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_d),$$

where each component α_i is a nonnegative integer. For any multi-index $\boldsymbol{\alpha}$, we define the multi-index order as

$$|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_d.$$

For any vector $\mathbf{x} = (x_1, x_2, \cdots, x_d)$, w set

$$\mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}.$$

From now on, we denote the multi-index α instead of $\boldsymbol{\alpha}$. Given a multi-index α, we define the α partial derivative as

$$D^\alpha u(\mathbf{x}) = \frac{\partial^{|\alpha|} u(\mathbf{x})}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_d^{\alpha_d}}.$$

If k is a nonnegative integer, we define

$$D^k u(\mathbf{x}) = \{D^\alpha u(\mathbf{x}) \mid |\alpha| = k\},$$

the set of all partial derivatives of order k.

For vector-valued function $\mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), u_2(\mathbf{x}), \cdots, u_m(\mathbf{x}))$, we define

$$D^\alpha \mathbf{u}(\mathbf{x}) = (D^\alpha u_1(\mathbf{x}), D^\alpha u_2(\mathbf{x}), \cdots, D^\alpha u_m(\mathbf{x})).$$

Then we have

$$D^k \mathbf{u}(\mathbf{x}) = \{D^\alpha \mathbf{u}(\mathbf{x}) \mid |\alpha| = k\}.$$

Throughout this thesis, we will use del operator ∇ for the special case $k = 1$.

2.2.2 Well-known function spaces

For a normed vector space X, we denote its norm by $\|\cdot\|_X$. If X is a Hilbert space, its inner product and associated norm are denoted by $(\cdot,\cdot)_X$ and $\|\cdot\|_X$, respectively. First given $u \in C(\Omega)$, we define its support as a closed set

$$\mathrm{supp}(u) = \overline{\{\mathbf{x} \in \Omega \mid u(\mathbf{x}) \neq 0\}}$$

Let U be an open subset of $\mathbf{R}^d$. Then we write $U \subset\subset \Omega$ if $U \subset \overline{U} \subset \Omega$ and $\overline{U}$ is compact set.

We make an introduction of the well-known (scalar) function spaces used very often in partial differential equation.

1. $C^k(\Omega)$ is the space of functions u with continuous derivatives $D^\alpha u$ on Ω for all multi-index α, $|\alpha| \leq k$.

2. $C^\infty(\Omega) = \cap_{k=0}^\infty C^k(\Omega)$.

3. $C_0^k(\Omega)$ is the space of functions in $C^k(\Omega)$ with compact support.

4. $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ is the space of test functions defined on Ω.

5. $\mathcal{D}'(\Omega)$ is the space of distributions, i.e., the topological dual space of $\mathcal{D}(\Omega)$.

6. $L^p(\Omega)$ is the space of Lebesgue measurable functions u for which the norm

$$\|u\|_{L^p(\Omega)} := \left[\int_\Omega |u(\mathbf{x})|^p \, dx\right]^{1/p} < \infty,$$

where $1 \leq p < \infty$ and $dx = dx_1 \, dx_2 \, \cdots dx_d$.

7. $L^\infty(\Omega)$ is the space of Lebesgue measurable function u for which norm

$$\|u\|_{L^\infty(\Omega)} := \text{ess}\,\text{sup}_{\mathbf{x}\in\Omega} |u(\mathbf{x})| < \infty.$$

We extend those space to some Banach spaces consisting of mappings

$$u : [0, T] \to X,$$

where X is a real Banach space, with the norm $\|\cdot\|_X$. See [20]. Now we list some of those space.

1. $L^p(0, T; X)$ is the space of all measurable functions $u : [0, T] \to X$ with the norm

$$\|u\|_{L^P(0,T;X)} := \left(\int_0^T \|u(t)\|_X^p \, dt\right)^{1/p} < \infty \quad \text{for } 1 \leq p < \infty$$

and $L^\infty(0, T; X)$ is the space of measurable $u : [0, T] \to X$ with the norm

$$\|u\|_{L^\infty(0,T;X)} := \text{ess sup}_{0\leq t\leq T} \|u(t)\|_X.$$

2. $C(0, T; X)$ is the continuous functions $u : [0, T] \to X$ with the norm

$$\|u\|_{C([0,T];X)} := \max_{0\leq t\leq T} \|u(t)\|_X < \infty.$$

Similarly we will present the Sobolev spaces $W^{1,p}(0,T;X)$ in the next Subsection.

A function f defined on Ω is called a test function if $f \in C^\infty(\Omega)$ and there exists a compact set $K \subset \Omega$ such that the support of f lies in K. A distribution is a linear mapping $\phi \mapsto (f,\phi)$ from $\mathcal{D}(\Omega) \to \mathbf{R}$ such that if $\phi_n \to \phi$ in $\mathcal{D}(\Omega)$, then $(f,\phi_n) \to (f,\phi)$ as $n \to \infty$.

2.2.3 Hölder spaces

It turns out to be useful to consider Hölder continuous functions. We present the definition of Hölder continuous function. Let U be a subset of $\mathbf{R}^d$. Let X be a Banach space.

Definition 2.8. Let $0 < p \leq 1$. $u : U \to \mathbf{R}$ is said to be Hölder continuous function with exponent p if there is a constant C such that

$$|u(x) - u(y)| \leq C|x-y|^p \quad \text{for } x, y \in U.$$

We also define Hölder space $C^p(0,T;X)$ which consists of all functions with the norm

$$\|u\|_{C^p(0,T;X)} = \|u\|_{C(0,T;X)} + \sup_{t_1 \neq t_2} \frac{\|u(t_2) - u(t_1)\|_X}{|t_2 - t_1|^p}.$$

In fact, we can use other norms instead of $\|\cdot\|_{C(0,T;X)}$. Note that the Hölder space $C^p(0,T;X)$ is a Banach space.

2.2.4 Sobolev spaces

Sobolev spaces provide an elegant and systematic mathematical framework such as regularity. Partial differential equations are analyzed naturally not only in terms of properties of the function spaces, but also of their derivatives. These derivatives are defined in the weak sense (in the sense of distributions). Thus after we define the weak derivative, we will define the Sobolev spaces $W^{k,p}(\Omega)$. In this section, Ω will be a open and locally measurable set on $\mathbf{R}^d$.

Definition 2.9. Let $1 \leq p < \infty$. We say $u \in L^p_{\mathbf{loc}}(\Omega)$, i.e., u is locally p-integrable, if for $\mathbf{x} \in \Omega$ there is an open neighborhood U of $\mathbf{x}$ such that $U \subset\subset \Omega$ with $u \in L^p(U)$.

Under the assumption, we can define the weak derivative.

Definition 2.10. Suppose that $u, w \in L^1_{\mathbf{loc}}(\Omega)$ locally and α is a multi-index. Then we say that w is the αth-weak partial derivative of u, denoted by $D^\alpha u = w$, provided

$$\int_\Omega u(\mathbf{x}) D^\alpha \phi(\mathbf{x})\, dx = (-1)^{|\alpha|} \int_\Omega w(\mathbf{x})\phi(\mathbf{x})\, dx \text{ for all } \phi \in \mathcal{D}(\Omega).$$

Let $1 \leq p \leq \infty$ and k be a nonnegative integer.

Definition 2.11. The Sobolev space $W^{k,p}(\Omega)$ consists of all functions $u \in L^1_{\mathbf{loc}}(\Omega)$ such that for each multi-index α with $|\alpha| \leq k$, $D^\alpha u$ exists and $D^\alpha u \in L^p(\Omega)$.

If $u \in W^{k,p}(\Omega)$, its norm is defined as

$$\|u\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha|\leq k} \int_\Omega |D^\alpha u|^p\, dx\right)^{1/p}, & \text{if } 1 \leq p < \infty, \\ \max_{|\alpha|\leq k} \|D^\alpha u\|_{L^\infty(\Omega)}, & \text{if } p = \infty. \end{cases}$$

The fact is well-known that $W^{k,p}(\Omega)$ is Banach space. We also mention about the seminorm of the Sobolev space $W^{k,p}(\Omega)$, which is defined as

$$|u|_{W^{k,p}} = \begin{cases} \left(\sum_{|\alpha|=k} \int_\Omega |D^\alpha u|\, dx\right)^{1/p}, & \text{if } 1 \leq p < \infty \\ \max_{|\alpha|=k} \|D^\alpha u\|_{L^\infty(\Omega)}, & \text{if } p = \infty. \end{cases}$$

Since the function space $C_0^\infty(\Omega)$ is not dense in $W^{k,p}(\Omega)$ in general, we denote by $W_0^{k,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p}$. We interpret $W_0^{k,p}(\Omega)$ as the space of function $u \in W^{k,p}(\Omega)$ such that $D^\alpha u = 0$ on $\partial\Omega$ for all $|\alpha| \le k-1$.

For the special case $p = 2$, we usually write $H^k(\Omega) \equiv W^{k,2}(\Omega)$ and $H_0^k(\Omega) \equiv W_0^{k,2}(\Omega)$. Note that $H_0^1(\Omega)$ is a subspace $H^1(\Omega)$ and is defined in terms of trace zero function, i.e.,

$$H_0^1(\Omega) = \{u \in H^1(\Omega) \mid u = 0 \text{ on } \partial\Omega\}.$$

Then $H^k(\Omega)$ is Hilbert space equipped with inner product

$$(u, w)_{H^k(\Omega)} = \sum_{|\alpha|\le k} \int_\Omega D^\alpha u(\mathbf{x}) D^\alpha w(\mathbf{x})\, dx \quad \text{for } u, w \in H^k(\Omega).$$

For negative integer, we define the Sobolev space which is dual space of order k. We denote by $H^{-k}(\Omega)$ the dual space of $H_0^k(\Omega)$. However since the dual space of $H^k(\Omega)$ is subspace of the dual space of $H_0^k(\Omega)$, it is frequently useful to denote $H^{-k}(\Omega)$ as the dual space of $H^k(\Omega)$. For order $k = 1$, assume that $f \in H^{-1}(\Omega)$. Then we define its norm as

$$\|f\|_{H^{-1}(\Omega)} = \sup_{w \in H^1(\Omega)} \frac{|\langle f, w\rangle|}{\|w\|_{H^1(\Omega)}}.$$

This norm can equivalently be expressed by

$$\|f\|_{H^{-1}(\Omega)} = \sup\{\langle f, w\rangle \mid w \in H^1(\Omega), \|w\|_{H^1(\Omega)} \le 1\}.$$

On the Banach space $L^1(0, T; X)$, we give the definition of a weak derivative in the following way.

Definition 2.12. Let $u \in L^1(0,T;X)$. The function $w \in L^1(0,T;X)$ is said to be the weak derivative of u, if

$$\int_0^T \phi'(t)u(t)\,dt = -\int_0^T \phi(t)w(t)\,dt \quad \text{for } \phi \in C_0^\infty(0,T).$$

Then we write $w = u_t$.

The integrals which is used in Definition 2.12 are called Bochner integrals. See the details in [59]. Then Sobolev space $W^{1,p}(0,T;X)$ is space of all measurable $u \in L^p(0,T;X)$ such that u_t exists in the weak sense and $u_t \in L^p(0,T;X)$, equipped with norm

$$\|u\|_{W^{1,p}(0,T;X)} = \begin{cases} \left(\int_0^T (\|u_t(t)\|_X^p + \|u(t)\|_X^p)\,dt\right)^{1/p} & \text{for } 1 \le p < \infty \\ \text{ess sup}_{0\le t\le T}(\|u_t(t)\|_X + \|u(t)\|_X) & \text{for } p = \infty. \end{cases}$$

We also consider the function space of vector function. If for vector valued function $\mathbf{u}$, each component u_i is in the Sobolev space $\mathbf{H}^k(\Omega)$ we write $\mathbf{u} \in \mathbf{H}^k(\Omega) = \left(H^k(\Omega)\right)^d$; its inner product has the form

$$(\mathbf{u}, \mathbf{w})_{\mathbf{H}^k(\Omega)} = \sum_{i=1}^d \int_\Omega \sum_{|\alpha|\le k} D^\alpha u_i D^\alpha w_i dx$$

and the associated norm is

$$\|\mathbf{u}\|_{H^k(\Omega)} = \left(\sum_{i=1}^d \int_\Omega \sum_{|\alpha|\le k} |D^\alpha u_i|^2 dx\right)^{1/2},$$

where $\left(H^k(\Omega)\right)^d = \{(u_1, u_2, \cdots, u_d) \mid u_i \in H^k(\Omega), 1 \le i \le d\}$. In the case of Hilbert space of the vector-valued function that we will mainly deal with, we will use the notation

$$\begin{aligned}\mathbf{H}^1(\Omega) &= \left(H^1(\Omega)\right)^d,\\ \mathbf{H}^{1/2}(\partial\Omega) &= \left(H^{1/2}(\partial\Omega)\right)^d,\\ \mathbf{H}^{-1/2}(\partial\Omega) &= \left(H^{-1/2}(\partial\Omega)\right)^d,\\ \mathbf{L}^2(\Omega) &= \left(L^2(\Omega)\right)^d.\end{aligned}$$

Note that $\mathbf{L}^2(\Omega) = \mathbf{H}^0(\Omega)$. Similarly the inner product of $\mathbf{L}^2(\Omega)$ is defined as

$$(\mathbf{u}, \mathbf{w})_{\mathbf{L}^2(\Omega)} = \sum_{i=1}^{d} \int_\Omega u_i v_i dx$$

and the associate norm is

$$\|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} = \left(\sum_{i=1}^{d} \int_\Omega |u_i|^2 dx\right)^{1/2}.$$

2.2.5 Sobolev spaces on $\mathbf{R}^d$ and Fourier transform

Before we present the definition of Sobolev space $H^s(\mathbf{R}^d)$ for all $s \in \mathbf{R}$, we need to talk about the tempered distribution. In particular case $\Omega = \mathbf{R}^d$, the requirement of test function with compact support is naturally replaced by rapidly decreasing function at infinity. So this assertion makes us to consider the following definition. See the detail in [47].

Let $\mathcal{S}(\mathbf{R}^d)$ be the space of all functions on $\mathbf{R}^d$ which are smooth and such that $|x|^k|D^\alpha\phi(x)|$ is bounded for every $k \in \mathbf{N}$ and every muti index α. A tempered distribution on $\mathbf{R}^d$ is a linear mapping $\phi \longmapsto (f, \phi)$ from $\mathcal{S}(\mathbf{R}^d)$ to $\mathbf{R}$ such that $(f, \phi_n) \to (f, \phi)$ if $\phi_n \to \phi$ in $\mathcal{S}(\mathbf{R}^d)$. The set of all tempered distribution is denoted by $\mathcal{S}'(\mathbf{R}^d)$.

Now we introduce the Fourier transform which provides extremely powerful tool for converting certain linear PDE into algebraic equations.

Definition 2.13. For every $u \in L^1(\mathbf{R}^d)$, the Fourier transform of g is defined by

$$\mathcal{F}[u](\boldsymbol{\xi}) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} e^{-i\boldsymbol{\xi}\cdot\mathbf{x}} u(\mathbf{x}) dx.$$

The inverse Fourier transform of u is defined by

$$\mathcal{F}^{-1}[u](\boldsymbol{x}) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} e^{i\boldsymbol{\xi}\cdot\mathbf{x}} u(\boldsymbol{\xi}) d\xi.$$

Using the tempered distribution and Fourier transform, we can produce a definition of Sobolev space $H^s(\mathbf{R}^d)$ for all $s \in \mathbf{R}$. Refer to the book [56].

Definition 2.14. For any $s \in \mathbf{R}$, we define

$$H^s(\mathbf{R}^d) = \{u \in S'(\mathbf{R}^d) \mid \langle \xi \rangle^s \mathcal{F}[u] \in L^2(\mathbf{R}^d)\},$$

where $\langle \boldsymbol{\xi} \rangle = (1 + |\boldsymbol{\xi}|^2)^{1/2}$ and $|\boldsymbol{\xi}|^2 = |\xi_1|^2 + |\xi_2^2| + \cdots + |\xi_d|^2$.

2.2.6 Sobolev spaces on manifolds

If $\partial\Omega$ is smooth, it is useful to define Sobolev space on manifold. Let X, Y be subsets of $\mathbf{R}^d$. A smooth map $f : X \to Y$ is called diffeomorphism if f is bijective and the inverse map $f^{-1} : Y \to X$ is also smooth. X an Y are called diffeomorphic if such a map exists. Then X is a $d-$dimensional manifold if $x \in X$ has open neighborhood V in X which is diffeomorphic to open set $U \subset \mathbf{R}^d$. See the details in [24]. So diffeomorphisms provide a tool to make local changes of coordinates, i.e., $\partial\Omega$ can be transformed to a coordinate surface by diffeomorphisms. This local considerations are achieved by partition of unity which is the useful device to prove PDEs.

Definition 2.15. Let S be a closed subset in $\mathbf{R}^d$and let collection $\{U_j\}$ be a covering of S such that U_j are open subsets in $\mathbf{R}^d$ (not in S). A partition of unity subordinate to covering $\{U_j\}$ is a set of test function $\phi_j \in \mathcal{D}(\mathbf{R}^d)$ such that

1. $0 \leq \phi_j \leq 1$,
2. supp $\phi_j \subset U_j$,
3. $\sum_j \phi_j = 1$ in a open neighborhood.

2.3 Review of some theorems

In this section, we present important theorems which are applied throughout this paper.

First, Plancherel's Theorem and some properties related to Fourier transform are presented in the next two theorem. The details argument can found in [20].

Theorem 2.1. *(**Plancherel's Theorem**) Assume that $u \in L^1(\mathbf{R}^d) \bigcap L^2(\mathbf{R}^d)$. Then $\mathcal{F}[u]$, $\mathcal{F}^{-1}[u] \in L^2(\mathbf{R}^d)$ and*

$$\|\mathcal{F}[u]\|_{L^2(\mathbf{R}^d)} = \|\mathcal{F}^{-1}[u]\|_{L^2(\mathbf{R}^d)} = \|u\|_{L^2(\mathbf{R}^d)}.$$

Theorem 2.2. *Assume that $\mathcal{F}[u]$, $\mathcal{F}^{-1}[v] \in L^2(\mathbf{R}^d)$, Then*

1. $\mathcal{F}[D^{\boldsymbol{\alpha}} u](\mathbf{x}) = (i\mathbf{x})^{\boldsymbol{\alpha}} \mathcal{F}[u]$ for multi-index $\boldsymbol{\alpha}$ and $\mathbf{x} \in \mathbf{R}^d$,
2. $\mathcal{F}[u * v] = (2\pi)^{d/2} \mathcal{F}[u]\mathcal{F}[v]$, where $*$ is the convolution of two functions,
3. $\mathcal{F}[u] = v$ if and only if $u = \mathcal{F}^{-1}[v]$.

The Riesz representation theorem enables us to see how a measure is associated to a functional on $C_0(X)$. When we consider the Riesz representation theorem, it is natural to deal with functions in $C_0(X)$ on a locally compact space. In fact, we note that this theorem is expressed in several different versions.

Theorem 2.3. ***(Riesz representation theorem)*** *Let X be a locally compact Hausdorff space. Then to each positive bounded linear functional f on $C_0(X)$, there exists a Borel measure ν determined by f such that*

$$f(u) = \int_X u\,d\nu \quad \text{for } u \in C_0(X).$$

From the Riesz representation theorem, we can also see that the dual space of $C_0(X)$ is identified to (isometrically isomorphic to) the space of all Borel measures on X with the norm defined by $\|\nu\| = |\nu|(X)$. See [35, 49] for the details.

The Banach fixed theorem is one of the most important method for analyzing the solvability for nonlinear operator equations. See the details in [3]. Let X be a Banach space with norm $\|\cdot\|_X$ and $K \subseteq X$.

Definition 2.16. An operator $T : K \to X$ is said to be a contraction with contractivity constant $\alpha \in [0, 1)$ if

$$\|Tx - Ty\|_X \leq \alpha\|x - y\|_X \quad \text{for all } x, y \in K.$$

Based on contraction maps, then the Banach fixed theorem is introduced.

Theorem 2.4. ***(Banach fixed theorem)*** *Assume that K is a nonempty closed subset of X and an operator $T : K \to K$ is a contraction mapping with $0 \leq \alpha < 1$. Then there exists a unique $x \in K$ such that*

$$x = T(x).$$

The Arzela–Ascoli theorem and Alaoglu's theorem are useful to show the existence of solutions. To state the Arzela-Ascoli theorem, we need the following definition.

Definition 2.17. Let (f_n) be a sequence of real-valued functions defined in $D \subset \mathbf{R}^d$ and $\mathbf{x} \in D$. The sequence (f_n) is said to be equicontinuous at $\mathbf{x}$ if for every $\epsilon > 0$ there exists a $\delta > 0$, independent of n, such that

$$|f_n(\mathbf{y}) - f_n(\mathbf{x})| < \epsilon \quad \text{for } \mathbf{y} \in D \text{ with } |\mathbf{y} - \mathbf{x}| < \delta.$$

Theorem 2.5. ***(Arzela–Ascoli theorem)*** *Let (f_n) be a sequence of real-valued functions defined on a compact set $S \subset \mathbf{R}^d$. Assume that there is a constant C such that $|f_n(\mathbf{x})| < C$ for every $n \in \mathbf{N}$ and every $\mathbf{x} \in S$ and (f_n) is equicontinuous at every $\mathbf{x} \in S$. Then there exists a subsequence which converges uniformly on S.*

Next we introduce Alaoglu's theorem, recalling the definition of weak-* convergence.

Theorem 2.6. ***(Alaoglu's theorem)*** *Let X be a separable Banach space and (f_n) be a bounded sequence in X^*. Then the sequence (f_n) has a weakly* convergent subsequence.*

The trace theorem is presented below. The detailed arguments can founded in [32].

Theorem 2.7. ***(Trace theorem)*** *Let Ω be a Lipschitzian domain and let tr be the operator defined by*

$$\mathrm{tr}(w) = w|_{\partial\Omega} \quad \textit{for } w \in C^\infty(\overline{\Omega}).$$

Then tr *can be extended to a bounded linear surjective operator, also denoted* tr, *from* $H^1(\Omega)$ *onto* $H^{1/2}(\partial\Omega)$.

The operator tr is called the trace operator. The important property of operator tr is that tr is surjective map from $H^1(\Omega)$ onto $H^{1/2}(\partial\Omega)$. Thus the operator tr has bounded right inverse. See the details in [47].

Korn's inequalities are crucial in the investigation of the existence of solutions to variational problems. One of Korn's inequalities is introduced below. See [32] for the details.

Theorem 2.8. *Let* Ω *be a bounded Lipschitzian domain in* $\mathbf{R}^d$. *Then there is a positive constant, independent of* $\mathbf{w}$, *such that*

$$\int_\Omega |w_{i,j}w_{i,j}|^{p/2}dx \leq C\left(\int_\Omega |\varepsilon_{ij}[\mathbf{w}]\varepsilon_{ij}[\mathbf{w}]|^{p/2}dx + \int_\Omega |w_i w_i|^{p/2}dx\right)$$

for every $\mathbf{w} \in \mathbf{W}^{1,p}(\Omega)$ *and* $1 < p < \infty$.

The implicit function theorem provides many important results on local convergence theory of optimization techniques. We present a brief outline based on the discussion in Lang [34].

Theorem 2.9. ***(Implicit Function Theorem)*** *Assume that* $\mathbf{F} : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ *a function such that*

1. $\mathbf{F}(\mathbf{x}_0, \mathbf{0}) = \mathbf{0}$ *for* $\mathbf{x}_0 \in \mathbf{R}^n$,

2. *The function* $\mathbf{F}(\cdot,\cdot)$ *is a Lipschitz continuously differentiable in some neighborhood of* $(\mathbf{x}_0, \mathbf{0})$,

3. $\nabla_{\mathbf{x}}\mathbf{F}(\mathbf{x},\mathbf{y})$ *is non-singular at* $(\mathbf{x},\mathbf{y}) = (\mathbf{x}_0, \mathbf{0})$.

Then the function $\mathbf{x} : \mathbf{R}^m \to \mathbf{R}^n$ *defined implicitly by* $\mathbf{F}(\mathbf{x}(\mathbf{y}),\mathbf{y}) = \mathbf{0}$ *is well defined and Lipschitz continuous for* $\mathbf{y} \in \mathbf{R}^m$ *in some neighborhood of the origin.*

2.4 Some methods and definitions

2.4.1 Penalty methods

Penalty methods provide an alternative approach to constrained problems. These remove the constraint of the original problem and leads to an unconstrained problem. It also avoids the necessity of introducing additional unknowns in the form of Lagrange multipliers. See [32] for the detailed arguments.

Let V be a Banach space and K be a closed convex subset of V. Then penalty functional $P : V \to \mathbf{R}$ satisfies the following conditions

1. $P : V \to \mathbf{R}$ is weakly lower semicontinuous,

2. $P(v) \geq 0$ and $P(v) = 0$ if and only if $v \in K$.

Condition 2 implies that if solution v violate constraint, $P(v) > 0$. Otherwise, $P(v) = 0$. Now we introduce the notation which is used in penalty formulation:

$$
\begin{aligned}
s_+ &= \max(s,0) \quad \text{in } L^2(\Omega), \quad \text{i.e.,} \\
s_+(x) &= \begin{cases} s(x) & \text{if } s(x) \geq 0, \\ 0 & \text{if s}(x) < 0. \end{cases}
\end{aligned}
$$

The penalty method plays a crucial role in showing existence of solutions to the Euler–Bernoulli beam equation, as we shall see later on.

2.4.2 Linear complementarity problems

Definition 2.18. Given a vector $\mathbf{q} \in \mathbf{R}^n$ and a matrix $\mathbf{M} \in \mathbf{R}^{n\times n}$, the linear complementarity problems (LCP) is to find a vector $\mathbf{z} \in \mathbf{R}^n$ such that

$$\begin{aligned} \mathbf{z} &\geq 0, \\ \mathbf{q} + \mathbf{M}\mathbf{z} &\geq 0, \\ \mathbf{z}^T \cdot (\mathbf{q} + \mathbf{M}\mathbf{z}) &= 0 \end{aligned}$$

or to show that no such vector z exists.

We mention a notation related to linear complementarity problem: in general, for vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{0} \leq \mathbf{a} \perp \mathbf{b} \geq \mathbf{0}$ means that $\mathbf{a}, \mathbf{b} \geq \mathbf{0}$ component-wise and $\mathbf{a}^T \cdot \mathbf{b} = 0$. In special case that a, b are scalar, $a \perp b$ means that both are non-negative and either a or b is zero.

This problem is the subject of a number of books, including the encyclopedic reference [16]. Indeed, LCP has been applied to many fields in applied sciences and technology since it was proposed in the mid 1960's. We shall see how our contact problem leads to LCPs.

2.4.3 Semi-smooth functions

We introduce the definition of semi-smooth function. The detailed argument can found in [21].

Definition 2.19. Let $\mathbf{G} : \Omega \subseteq \mathbf{R}^n \to \mathbf{R}^m$ be a locally Lipschitz continuous function on Ω, where Ω is open. Then $\mathbf{G}$ is said to be semi-smooth at a point $\overline{\mathbf{x}}$ if $\mathbf{G}$ is directional differentiable near $\overline{\mathbf{x}}$ and there exist a neighborhood $\Omega' \subseteq \Omega$ of $\overline{\mathbf{x}}$ and function $f : (0, \infty) \to [0, \infty)$ with $\lim_{t \downarrow 0} f(t) = 0$, such that for any $\mathbf{x} \in \Omega'$ different from $\overline{\mathbf{x}}$, we have

$$\frac{\|\mathbf{G}'(\mathbf{x}; \mathbf{x} - \overline{\mathbf{x}}) - \mathbf{G}'(\overline{\mathbf{x}}; \mathbf{x} - \overline{\mathbf{x}})\|}{\|\mathbf{x} - \overline{\mathbf{x}}\|} \leq f(\|\mathbf{x} - \overline{\mathbf{x}}\|).$$

CHAPTER 3
ELASTIC BODIES : CONTINUOUS FORMULATION OF DYNAMIC FRICTIONLESS CONTACT

3.1 Contact conditions

In this Section, we will derive contact conditions for our formulation, based on Signorini's contact condition. Let $\mathbf{n}(\mathbf{x}) = (n_1, n_2, n_3)$ be the outward normal vector at $\mathbf{x}$ to the material surface $\partial\Omega$. Note that we consider three dimension case, i.e., $\Omega \subset \mathbf{R}^3$.

Since we want to focus on dynamic frictionless contact, for our dynamic contact problem we set $\Gamma_F = \Gamma_D = \emptyset$. Thus Γ_c becomes the whole boundary $\partial\Omega$. Note that there may occur a contact force on some parts of $\partial\Omega$, or not on other parts.

For a stress tensor $\boldsymbol{\sigma}$, we denote by σ_n and $\boldsymbol{\sigma}_T$ the normal and tangential components of $\boldsymbol{\sigma}$, respectively, and define

$$\sigma_n = \sigma_{ij} n_i n_j \text{ and } (\sigma_T)_i = \sigma_{ij} n_j - \sigma_n n_i.$$

Since the kinematic contact condition must be compatible with stress on $\partial\Omega$, we can have the following contact condition: if a contact force $N\mathbf{n}$ is applied on surface $\partial\Omega$, the stress vector must satisfy

$$\sigma_{ij} n_j = N n_i \text{ on } \partial\Omega.$$

Then due to Newton's third law (action and reaction), the contact force is a opposite to the direction that a elastic body moves downward. So in the physical situation, we regard the downward direction as negative.

For other contact conditions, Signorini's problem has a contact constraint which will induce a convex closed subset in a Banach space X in terms of the mathematical framework:

$$\mathbf{u} \cdot \mathbf{n} + g \geq 0.$$

From the physical point of view, Signorini contact conditions can be interpreted as the following way: when the elastic body does not reach to the rigid foundation, i.e., $\mathbf{u}\cdot\mathbf{n}+g > 0$, the contact force $N\mathbf{n}$ must be equal to zero, since no contact occurs and when there is a contact force, i.e., $N > 0$, the elastic body touches to the rigid foundation, i.e., $\mathbf{u} \cdot \mathbf{n} + g = 0$. Thus Signorini contact conditions result in linear complementary boundary conditions

$$0 \leq \mathbf{u} \cdot \mathbf{n} + g \ \perp \ N \geq 0 \text{ on } \partial\Omega.$$

In order to see frictionless contact condition, from the contact condtion $\sigma_{ij} n_j = N n_i$ on $\partial\Omega$ we have

$$\begin{aligned} (\sigma_T)_i &= \sigma_{ij} n_j - \sigma_n n_i \\ &= N n_i - \sigma_{ij} n_i n_j n_i = N n_i - (N n_j n_j) n_i = 0. \end{aligned}$$

Thus the tangential components $\boldsymbol{\sigma}_T$ of stress tensor $\boldsymbol{\sigma}$ must be equal to zero. This implies that we arrive at frictionless contact condtions.

3.2 Dynamic frictionless contact problem

The dynamic contact problem comes from the following physical situation. See the Figure 3.1 for the illustration.

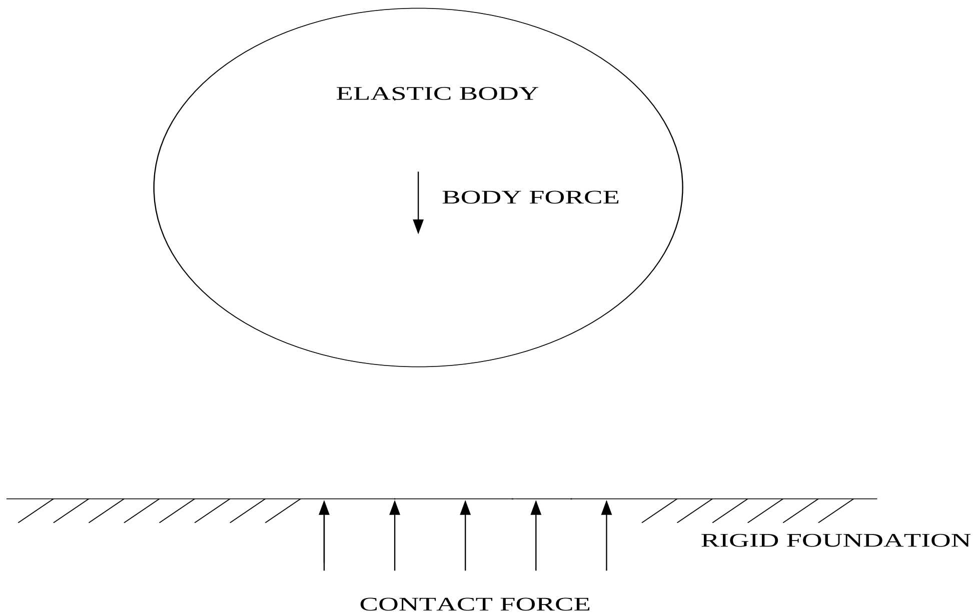

Figure 3.1: Dynamic frictionless contact problem with elastic body.

In order to derive the dynamic contact continuous formulation, we need to consider a equation of motion inside a elastic body Ω. From the physical point of view, this equation is obtained, by applying linear momentum principles based on Newton's second law and Newton's third law. This equation of motion is expressed by

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla \cdot \boldsymbol{\sigma}[\mathbf{u}] + \mathbf{f} \quad \text{in } \Omega.$$

The above expression has the form of a hyperbolic second order equation. Indeed, a hyperbolic equation is naturally a generalized expression of a wave equation. Therefore it may sometimes be helpful to interpret this dynamic contact formulation in comparison with wave equation.

Before we present the dynamic contact continuous formulation, we introduce the notations: we write $\mathbf{u}(\mathbf{x},t)$ as $\mathbf{u}$ and $\mathbf{f}(\mathbf{x})$ as $\mathbf{f}$ and $N(\mathbf{x},t)$ as N, and $g(\mathbf{x})$ as g, for the purpose of a simplified notation.

Finally we formulate the dynamic contact continuous formulation for time interval $[0,T]$:

$$\begin{aligned}
\rho\frac{\partial^2\mathbf{u}}{\partial t^2} &= \nabla\cdot\boldsymbol{\sigma}[\mathbf{u}]+\mathbf{f} \quad \text{in } \Omega\times(0,T], && (3.1)\\
\boldsymbol{\sigma}[\mathbf{u}]\cdot\mathbf{n} &= N\cdot\mathbf{n} \quad \text{on } \partial\Omega\times(0,T], && (3.2)\\
0\le \mathbf{u}\cdot\mathbf{n}+g &\perp N\ge 0 \quad \text{on } \partial\Omega\times(0,T], && (3.3)\\
\mathbf{u}(\mathbf{x},0) &= \mathbf{u}^0 \quad \text{in } \Omega, && (3.4)\\
\dot{\mathbf{u}}(\mathbf{x},0) &= \mathbf{v}^0 \quad \text{in } \Omega, && (3.5)
\end{aligned}$$

where $\dot{\mathbf{u}}(\mathbf{x},0)=\partial\mathbf{u}/\partial t(\mathbf{x},0)=\mathbf{v}(\mathbf{x},0)$. We denote the velocity $\mathbf{v}(\mathbf{x},t)$ by $\mathbf{v}$. Equations (3.5) and (3.5) are called the initial values for the displacement and velocity, respectively. We also assume that $\mathbf{u}^0\in\mathbf{H}^1(\Omega)$, $\mathbf{v}^0\in\mathbf{L}^2(\Omega)$. Throughout this thesis, we assume that $\mathbf{f}$ and g do not depend on time t and that $\mathbf{f}\in\mathbf{L}^2(\Omega)$, $g\in C^\infty(\partial\Omega)$.

In the next chapter, we will discuss how we approach the continuous dynamic contact formulation. Variational inequalities and time discretizations will play an important role in solving the dynamic contact problem.

3.3 Time discretization

For a dynamical problem, time discretization is one of the most useful numerical methods. First, we partition time $[0,T]$:

$$0=t_0<t_1<t_2<\cdots<t_l<t_{l+1}<\cdots<T,$$

where T is the end of time. Then we partition time space so that we establish numerical formulations for dynamic continuous contact problem.

We denote by $\mathbf{u}^l$ approximate displacement $\mathbf{u}(\mathbf{x}, t_l)$ and by $\mathbf{v}^l$ approximate velocity $\mathbf{v}(\mathbf{x}, t_l)$ at each instant time t_l, respectively. Also $N(\mathbf{x}, t_l)$ is denoted by N^l. Then we have the same time step size $h = t_{l+1} - t_l$ for $l \geq 0$ and so $l = T/h$.

The dynamic contact continuous problem will be replaced with the following approximation formulas:

- Acceleration relation

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = \frac{1}{h}(\mathbf{v}^{l+1} - \mathbf{v}^l) \tag{3.6}$$

- Velocity relation

$$\frac{1}{h}(\mathbf{u}^{l+1} - \mathbf{u}^l) = \frac{1}{2}(\mathbf{v}^{l+1} + \mathbf{v}^l). \tag{3.7}$$

3.4 Numerical formulas

Using the different numerical methods, we set up numerical formulations for motion equation. Thus

1. If we use the midpoint rule,

$$\frac{\rho}{h}(\mathbf{v}^{l+1} - \mathbf{v}^l) = \nabla \cdot \boldsymbol{\sigma}[\frac{1}{2}(\mathbf{u}^{l+1} + \mathbf{u}^l)] + \mathbf{f}^l \quad \text{in } \Omega, \tag{3.8}$$

2. If we use the implicit Euler method,

$$\frac{\rho}{h}(\mathbf{v}^{l+1} - \mathbf{v}^l) = \nabla \cdot \boldsymbol{\sigma}[\mathbf{u}^{l+1}] + \mathbf{f}^l \quad \text{in } \Omega, \tag{3.9}$$

3. If we use the explicit Euler method,

$$\frac{\rho}{h}(\mathbf{v}^{l+1} - \mathbf{v}^l) = \nabla \cdot \boldsymbol{\sigma}[\mathbf{u}^l] + \mathbf{f}^l \quad \text{in } \Omega. \tag{3.10}$$

For contact conditions, we consider implicit Euler method

$$\boldsymbol{\sigma}[\mathbf{u}^{l+1}] = N^l \mathbf{n} \quad \text{on } \partial\Omega, \tag{3.11}$$

$$0 \leq N^l \perp \mathbf{u}^{l+1} \cdot \mathbf{n} + g \geq 0 \quad \text{on } \partial\Omega \tag{3.12}$$

or explicit Euler method

$$\boldsymbol{\sigma}[\mathbf{u}^l] = N^{l+1} \mathbf{n} \quad \text{on } \partial\Omega, \tag{3.13}$$

$$0 \leq N^{l+1} \perp \mathbf{u}^l \cdot \mathbf{n} + g \geq 0 \quad \text{on } \partial\Omega. \tag{3.14}$$

From (3.7), we have

$$\mathbf{v}^{l+1} = \frac{2}{h}(\mathbf{u}^{l+1} - \mathbf{u}^l) - \mathbf{v}^l. \tag{3.15}$$

Using (3.15), we can better express the numerical formulation of the equations of motion.

1. For the midpoint rule

$$\mathbf{u}^{l+1} = \frac{h^2}{4\rho}(\nabla \cdot \boldsymbol{\sigma}[\mathbf{u}^{l+1}] + \boldsymbol{\sigma}[\mathbf{u}^l]) + \mathbf{u}^l + h\mathbf{v}^l + \frac{h^2}{2\rho}\mathbf{f} \quad \text{in } \Omega. \tag{3.16}$$

2. For the implicit Euler method

$$\mathbf{u}^{l+1} = \frac{h^2}{2\rho}\nabla \cdot \boldsymbol{\sigma}[\mathbf{u}^{l+1}] + \mathbf{u}^l + h\mathbf{v}^l + \frac{h^2}{2\rho}\mathbf{f} \quad \text{in } \Omega. \tag{3.17}$$

3. For the explicit Euler method

$$\mathbf{u}^{l+1} = \frac{h^2}{2\rho}\nabla \cdot \boldsymbol{\sigma}[\mathbf{u}^l] + \mathbf{u}^l + h\mathbf{v}^l + \frac{h^2}{2\rho}\mathbf{f} \quad \text{in } \Omega. \tag{3.18}$$

3.5 Existence of a solution for one time step

Employing the implicit Euler method for equation of motion and contact condition, we show existence of solution for the time stepping problem in this Section. Now we impose a constraint condition on the next step solution so that the variational inequality equivalent to the numerical formulas (3.17), (3.11), and (3.12) is derived.

Suppose that $\mathbf{w} \cdot \mathbf{n}|_{\partial\Omega}$ is well defined for $\mathbf{w} \in (H^1(\Omega))^3$. We define the set of admissible displacements as $\mathbf{K} = \{\mathbf{w} \in (H^1(\Omega))^3 \mid \mathbf{w} \cdot \mathbf{n} + g \geq 0 \text{ a.e. on } \partial\Omega\}$. Indeed, if $\mathbf{w} \in (H^1(\Omega))^d$, $\mathbf{w} \cdot \mathbf{n} = \mathrm{tr}(w_i) \cdot n_i$ a.e. on $\partial\Omega$, where tr is a trace operator from $H^1(\Omega)$ onto $H^{1/2}(\partial\Omega)$.

Lemma 3.1. *Let* $\mathbf{\Phi}^l = \frac{h^2}{2\rho}\mathbf{f}^l + \frac{h}{2}\mathbf{v}^l + \mathbf{u}^l$. *The next step solution* $\mathbf{u}^{l+1}$ *satisfies (3.17), (3.11), and (3.12) if and only if* $\mathbf{u}^{l+1}$ *is a sufficiently smooth solution of the variational inequality: find* $\mathbf{u}^{l+1} \in \mathbf{K}$ *such that*

$$\int_\Omega \left(\mathbf{u}^{l+1} \cdot (\mathbf{w} - \mathbf{u}^{l+1}) + \frac{h^2}{2\rho}\boldsymbol{\sigma}[\mathbf{u}^{l+1}] : \nabla(\mathbf{w} - \mathbf{u}^{l+1})\right) dx \geq \int \mathbf{\Phi}^l \cdot (\mathbf{w} - \mathbf{u}^{l+1})\, dx \ \ \forall \mathbf{w} \in \mathbf{K}. \tag{3.19}$$

Proof. Suppose that the next step solution $\mathbf{u}^{l+1} \in \mathbf{K}$ of (3.17), (3.11), and (3.12) is a sufficiently smooth. From (3.17),

$$\mathbf{u}^{l+1} - \frac{h^2}{2\rho}\nabla \cdot \boldsymbol{\sigma}[\mathbf{u}^{l+1}] = \mathbf{\Phi}^l. \tag{3.20}$$

Choose $\mathbf{w} \in \mathbf{K}$. Multiplying both sides of (3.20) by $\mathbf{w} - \mathbf{u}^{l+1}$ gives

$$\int_\Omega \mathbf{u}^{l+1} \cdot (\mathbf{w} - \mathbf{u}^{l+1})\, dx - \frac{h^2}{2\rho}\int_\Omega \nabla \cdot \boldsymbol{\sigma}[\mathbf{u}^{l+1}] \cdot (\mathbf{w} - \mathbf{u}^{l+1})\, dx = \int_\Omega \mathbf{\Phi}^l \cdot (\mathbf{w} - \mathbf{u}^{l+1})\, dx.$$

Using integration by parts, we obtain

$$\int_\Omega \mathbf{u}^{l+1} \cdot (\mathbf{w} - \mathbf{u}^{l+1})\, dx + \frac{h^2}{2\rho}\int_\Omega \boldsymbol{\sigma}[\mathbf{u}^{l+1}] : \nabla(\mathbf{w} - \mathbf{u}^{l+1})\, dx$$

$$= \int_\Omega \mathbf{\Phi}^l \cdot (\mathbf{w} - \mathbf{u}^{l+1})\,dx + \frac{h^2}{2\rho}\int_{\partial\Omega} \mathbf{n}^T \cdot \boldsymbol{\sigma}[\mathbf{u}^{l+1}] \cdot (\mathbf{w} - \mathbf{u}^{l+1})\,ds.$$

By the boundary condition (3.11) and symmetry of stress tensor $\boldsymbol{\sigma}[\mathbf{u}^{l+1}]$,

$$\begin{aligned}\int_\Omega \mathbf{u}^{l+1} \cdot (\mathbf{w} - \mathbf{u}^{l+1})\,dx + \frac{h^2}{2\rho}\int_\Omega \boldsymbol{\sigma}[\mathbf{u}^{l+1}] : \nabla(\mathbf{w} - \mathbf{u}^{l+1})\,dx \\ = \int_\Omega \mathbf{\Phi}^l \cdot (\mathbf{w} - \mathbf{u}^{l+1})\,dx + \frac{h^2}{2\rho}\int_{\partial\Omega} \mathbf{N}^l \mathbf{n} \cdot (\mathbf{w} - \mathbf{u}^{l+1})\,ds.\end{aligned}$$

On the boundary $\partial\Omega$, we have

$$N^l \mathbf{n} \cdot (\mathbf{w} - \mathbf{u}^{l+1}) = N^l(\mathbf{w} \cdot \mathbf{n} + g) - N^l(\mathbf{u}^{l+1} \cdot \mathbf{n} + g).$$

Thus by the linear complementary boundary conditions (3.12),

$$\begin{aligned}\int_\Omega \mathbf{u}^{l+1} \cdot (\mathbf{w} - \mathbf{u}^{l+1})\,dx + \frac{h^2}{2\rho}\int_\Omega \boldsymbol{\sigma}[\mathbf{u}^{l+1}] : \nabla(\mathbf{w} - \mathbf{u}^{l+1})\,dx \\ = \int_\Omega \mathbf{\Phi}^l \cdot (\mathbf{w} - \mathbf{u}^{l+1})\,dx + \frac{h^2}{2\rho}\int_{\partial\Omega} \mathbf{N}^l(\mathbf{w} \cdot \mathbf{n} + g)\,ds.\end{aligned}$$

Since $N^l \geq 0$ and $\mathbf{w} \cdot \mathbf{n} + g \geq 0$, $\mathbf{u}^{l+1}$ is a solution which satisfies the variational inequality (3.19).

Suppose that $\mathbf{u}^{l+1} \in \mathbf{K}$ satisfies variational inequality (3.19). First, we claim that $\mathbf{u}^{l+1}$ satisfies (3.17). Notice that $(H_0^1(\Omega))^3 \subset \mathbf{K}$, since that gap function $g \geq 0$. We choose $\mathbf{w} = \mathbf{u}^{l+1} \pm \mathbf{z}$ with arbitrary $\mathbf{z} \in (H_0^1(\Omega))^3$. From variational inequality (3.19),

$$\pm\left(\int_\Omega \mathbf{u}^{l+1} \cdot \mathbf{z}\,dx + \frac{h^2}{2\rho}\int_\Omega \boldsymbol{\sigma}[\mathbf{u}^{l+1}] : \nabla \cdot \mathbf{z}\,dx - \int_\Omega \mathbf{\Phi}^l \cdot \mathbf{z}\,dx\right) \geq 0.$$

So we have

$$\int_\Omega \mathbf{u}^{l+1} \cdot \mathbf{z}\,dx + \frac{h^2}{2\rho}\int_\Omega \boldsymbol{\sigma}[\mathbf{u}^{l+1}] : \nabla \cdot \mathbf{z}\,dx - \int_\Omega \mathbf{\Phi}^l \cdot \mathbf{z}\,dx = 0.$$

Using integration by parts,

$$\int_\Omega \mathbf{u}^{l+1}\cdot\mathbf{z}\,dx + \frac{h^2}{2\rho}\int_{\partial\Omega}\mathbf{n}^T\cdot\boldsymbol{\sigma}[\mathbf{u}^{l+1}]\cdot\mathbf{z}\,ds - \frac{h^2}{2\rho}\int_\Omega\big(\nabla\cdot\boldsymbol{\sigma}[\mathbf{u}^{l+1}]\big)\cdot\mathbf{z}\,dx - \int_\Omega \boldsymbol{\Phi}^l\cdot\mathbf{z}\,dx = 0.$$

Since $\mathbf{z}|_{\partial\Omega} = 0$,

$$\int_\Omega(\mathbf{u}^{l+1} - \frac{h^2}{2\rho}\nabla\cdot\boldsymbol{\sigma}[\mathbf{u}^{l+1}] - \boldsymbol{\Phi}^l)\cdot\mathbf{z}\,dx = 0 \quad \forall\mathbf{z}\in(H_0^1(\Omega))^3.$$

Therefore, we obtain the numerical formulation (3.17). Next, we claim that the boundary conditions (3.11–3.12) are satisfied. We first want to show that $\boldsymbol{\sigma}[\mathbf{u}^{l+1}]\cdot\mathbf{n} = N^l\mathbf{n}$ and $N^l \geq 0$ on $\partial\Omega$. Let $\mathbf{K}_1$ be a subset of $\mathbf{K}$ such that

$$\mathbf{K}_1 = \{\mathbf{z}\in(H^1(\Omega))^3 \mid \mathbf{z}\cdot\mathbf{n}\geq 0 \text{ on } \partial\Omega\}.$$

In the variational inequality (3.19), we choose $\mathbf{w} = \mathbf{u}^{l+1} + \mathbf{z}$ with arbitrary $\mathbf{z}\in\mathbf{K}_1$ and obtain

$$\int_\Omega \mathbf{u}^{l+1}\cdot\mathbf{z}\,dx + \frac{h^2}{2\rho}\int_\Omega\boldsymbol{\sigma}[\mathbf{u}^{l+1}] : \nabla\cdot\mathbf{z}\,dx - \int_\Omega\boldsymbol{\Phi}^l\cdot\mathbf{z}\,dx \geq 0.$$

Using integration by parts,

$$\int_\Omega \mathbf{u}^{l+1}\cdot\mathbf{z}\,dx + \frac{h^2}{2\rho}\int_{\partial\Omega}\mathbf{n}^T\cdot\boldsymbol{\sigma}[\mathbf{u}^{l+1}]\cdot\mathbf{z}\,ds - \int_\Omega(\frac{h^2}{2\rho}\nabla\cdot\boldsymbol{\sigma}[\mathbf{u}^{l+1}] + \boldsymbol{\Phi}^l)\cdot\mathbf{z}\,dx \geq 0.$$

By (3.17), we have

$$\int_{\partial\Omega}\mathbf{n}^T\cdot\boldsymbol{\sigma}[\mathbf{u}^{l+1}]\cdot\mathbf{z}\,ds \geq 0. \tag{3.21}$$

Now, we claim that $\mathbf{n}^T\cdot\boldsymbol{\sigma}[\mathbf{u}^{l+1}]\cdot\mathbf{n}\geq 0$ on $\partial\Omega$. Assume that $\mathbf{n}^T\cdot\boldsymbol{\sigma}[\mathbf{u}^{l+1}]\cdot\mathbf{n} < 0$. Let $\mathbf{z} = \alpha\mathbf{n}$ with $\alpha > 0$. Then

$$\int_{\partial\Omega}\mathbf{n}^T\cdot\boldsymbol{\sigma}[\mathbf{u}^{l+1}]\cdot\mathbf{z}\,ds < 0.$$

This contradicts (3.21). So we have

$$\mathbf{n}^T \cdot \boldsymbol{\sigma}[\mathbf{u}^{l+1}] \cdot \mathbf{n} \geq 0 \text{ on } \partial\Omega. \tag{3.22}$$

Applying (3.22), we want to show that $\boldsymbol{\sigma}[\mathbf{u}^{l+1}] \cdot \mathbf{n} = N^l \mathbf{n}$ and $N^l \geq 0$. Pick any tangent vector $\mathbf{y}$ on $\partial\Omega$. Then $\mathbf{y} \cdot \mathbf{n} = 0$. So the set $\mathbf{K}_1$ can contain any tangent vector on $\partial\Omega$. Let $\mathbf{z} = \alpha\mathbf{y}$ with $\alpha \neq 0$. From (3.21),

$$\int_{\partial\Omega} \mathbf{n}^T \cdot \boldsymbol{\sigma}[\mathbf{u}^{l+1}] \cdot \mathbf{z}\, ds = \int_{\partial\Omega} \alpha \mathbf{n}^T \cdot \boldsymbol{\sigma}[\mathbf{u}^{l+1}] \cdot \mathbf{y}\, ds \geq 0.$$

Assume that $\mathbf{n}^T \cdot \boldsymbol{\sigma}[\mathbf{u}^{l+1}] \cdot \mathbf{y} > 0$ on $\partial\Omega$. For $\alpha < 0$,

$$\int_{\partial\Omega} \alpha \mathbf{n}^T \cdot \boldsymbol{\sigma}[\mathbf{u}^{l+1}] \cdot \mathbf{y}\, ds < 0.$$

This contradicts (3.21). Assume that $\mathbf{n}^T \cdot \boldsymbol{\sigma}[\mathbf{u}^{l+1}] \cdot \mathbf{y} < 0$. For $\alpha > 0$,

$$\int_{\partial\Omega} \alpha \mathbf{n}^T \cdot \boldsymbol{\sigma}[\mathbf{u}^{l+1}] \cdot \mathbf{y}\, ds < 0.$$

This contradicts (3.21). So $\mathbf{n}^T \cdot \boldsymbol{\sigma}[\mathbf{u}^{l+1}] \cdot \mathbf{y} = 0$ on $\partial\Omega$, which means that $\boldsymbol{\sigma}[\mathbf{u}^{l+1}] \cdot \mathbf{n}$ must be represented as $N\mathbf{n}$ with some scalar function N. However, from (3.22) $\mathbf{n}^T \cdot \boldsymbol{\sigma}[\mathbf{u}^{l+1}] \cdot \mathbf{n} = N^l \mathbf{n} \cdot \mathbf{n} = N^l \geq 0$ on $\partial\Omega$.

Secondly, we want to show that $N^l(\mathbf{u}^{l+1} \cdot \mathbf{n} + g) = 0$ on $\partial\Omega$. Pick any $\mathbf{z} \in \mathbf{K}_1$ so that $\mathbf{z} = (\mathbf{u}^{l+1} \cdot \mathbf{n} + g)\mathbf{n}$ on $\partial\Omega$. We choose $\mathbf{w} = \mathbf{u}^{l+1} - \mathbf{z}$. So $\mathbf{w} = \mathbf{u}^{l+1} - (\mathbf{u}^{l+1} \cdot \mathbf{n} + g)\mathbf{n}$ on $\partial\Omega$. Then $\mathbf{w} \cdot \mathbf{n} + g = \mathbf{u}^{l+1} \cdot \mathbf{n} - (\mathbf{u}^{l+1} \cdot \mathbf{n} + g) + g = 0$ on $\partial\Omega$. This implies that $\mathbf{w} \in \mathbf{K}$. Therefore using integration by parts, from (3.19) we have

$$\begin{aligned} 0 &\leq \int_\Omega \left(\mathbf{u}^{l+1} \cdot (\mathbf{w} - \mathbf{u}^{l+1}) + \frac{h^2}{2\rho} \boldsymbol{\sigma}[\mathbf{u}^{l+1}] : \nabla \cdot (\mathbf{w} - \mathbf{u}^{l+1}) \right) dx - \int_\Omega \boldsymbol{\Phi}^l \cdot (\mathbf{w} - \mathbf{u}^{l+1})\, dx \\ &= \int_{\partial\Omega} \mathbf{n}^T \cdot \boldsymbol{\sigma}[\mathbf{u}^{l+1}] \cdot (\mathbf{w} - \mathbf{u}^{l+1})\, ds + \int_\Omega \left(\mathbf{u}^{+1} - \frac{h^2}{2\rho} \nabla \cdot \boldsymbol{\sigma}[\mathbf{u}^{l+1}] - \boldsymbol{\Phi}^l \right) \cdot (\mathbf{w} - \mathbf{u}^{l+1})\, dx \end{aligned}$$

$$= \int_{\partial\Omega} N^l(\mathbf{w}\cdot\mathbf{n} + g - \mathbf{u}^{l+1}\cdot\mathbf{n} - g)\,ds$$
$$= -\int_{\partial\Omega} N^l(\mathbf{u}^{l+1} + g)\,ds.$$

Therefore

$$\int_{\partial\Omega} N^l(\mathbf{u}^{l+1}\cdot\mathbf{n} + g)\,ds \le 0. \tag{3.23}$$

Suppose that $N^l > 0$. Then $\mathbf{u}^{l+1}\cdot\mathbf{n} + g = 0$, since if $\mathbf{u}^{l+1}\cdot\mathbf{n} + g > 0$ it contradicts (3.23). Similarly, if $\mathbf{u}^{l+1}\cdot\mathbf{n} + g > 0$, then $N^l = 0$. Thus we have

$$N^l(\mathbf{u}^{l+1}\cdot\mathbf{n} + g) = 0 \text{ on } \partial\Omega,$$

as required. □

In order to see that there is a unique solution $\mathbf{u}^{l+1}$ to the variational inequality (3.19), we generalize it to abstract setting.

Definition 3.1. Define the functional

$$F(\mathbf{w}) = \int_\Omega \left(\frac{1}{2}|\mathbf{w}|^2 + \frac{h^2}{2\rho}\boldsymbol{\sigma}[\mathbf{w}] : \boldsymbol{\varepsilon}[\mathbf{w}]\right) dx - \int_\Omega \boldsymbol{\Phi}^l\cdot\mathbf{w}\,dx \quad \text{for } \mathbf{w}\in\mathbf{K}.$$

In this case

$$a(\mathbf{w},\mathbf{w}) = \int_\Omega |\mathbf{w}|^2 + \frac{h^2}{\rho}\boldsymbol{\sigma}[\mathbf{w}] : \boldsymbol{\varepsilon}[\mathbf{w}]\,dx$$

and

$$f(\mathbf{w}) = \int_\Omega \boldsymbol{\Phi}^l\cdot\mathbf{w}\,dx.$$

Note that this functional is different from total energy functional. We can easily see that $a(\cdot,\cdot)$ is symmetric and V-elliptic and bounded using the properties of Hooke's tensor E_{ijkl}. As we shall see in the next Section, the total energy functional is decreased, if we use the implicit Euler method. Thus the initial conditions and $\mathbf{f} \in \mathbf{L}^2(\Omega)$ imply that $\mathbf{u}^l \in \mathbf{H}^1(\Omega)$ and $\mathbf{v}^l \in \mathbf{L}^2(\Omega)$ for all $l \geq 1$ and $h > 0$. So $f(\mathbf{w})$ is bounded linear functional. From definition 3.1 we set (3.19) into the generalized variational inequality:

$$\text{Find } \mathbf{u} \in \mathbf{K} : a(\mathbf{u}, \mathbf{w} - \mathbf{u}) \geq f(\mathbf{w} - \mathbf{u}) \quad \forall \mathbf{w} \in \mathbf{K}. \tag{3.24}$$

It has been known that (3.24) is equivalent to minimization problem:

$$\text{Find } \mathbf{u} \in \mathbf{K} : F(\mathbf{u}) \leq F(\mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{K}.$$

Also these are uniquely solvable. See [32] for the details. Therefore we can conclude that there exists the next step solution $\mathbf{u}^{l+1}$ of (3.19) uniquely.

3.6 Discussion of the implementation

We consider numerical methods to obtain the approximated solution. According to numerical analysis, the solution of the implicit Euler method is stable. So this is another reason that the implicit Euler method is employed to implement a numerical results. In order to obtain the next solution $\mathbf{u}^{l+1}$, we use minimization problem:

$$F(\mathbf{u}^{l+1}) \leq F(\mathbf{w}) \quad \forall w \in \mathbf{K}.$$

From the minimization problem, we would obtain solution $\mathbf{u}^{l+1}$, by using Finite Element Method and Karush-Kuhn-Tucker condition, called KKT condition. See the details in [3] and [43]. The actual and detailed implementation of the elastic body with frictionless dynamic contact condition will be a future work. It is expected that the procedure to implement numerical results would be very complicated. However, we intrinsically need the boundedness of the contact force and finer regularity properties, before we start computing numerical solutions.

3.7 Total energy functional

We define the total energy functional of dynamic contact continuous problem, which plays a fundamental role in showing the boundedness of the approximate solutions.

Definition 3.2. For $\mathbf{u}, \mathbf{v}$ total energy functional is defined by

$$E(\mathbf{u}, \mathbf{v}) = \frac{1}{2} \int_\Omega \big(\rho |\mathbf{v}|^2 + \boldsymbol{\sigma}[\mathbf{u}] : \boldsymbol{\varepsilon}[\mathbf{u}]\big)\, dx - \int_\Omega \mathbf{f} \cdot \mathbf{u}\, dx.$$

In Definition 3.2, the first term is the kinetic energy, the second term is the elastic energy, and the last term is the potential energy.

Lemma 3.2. *Suppose that* $\mathbf{f}$ *does not depend on time* t *and contact does not occur. If the midpoint rule is applied to equations of motion, we have*

$$E(\mathbf{u}^{l+1}, \mathbf{v}^{l+1}) = E(\mathbf{u}^{l}, \mathbf{v}^{l}) \quad \textit{for any } l \geq 0.$$

Proof. Using (3.8) and (3.15),

$$\frac{\rho}{2h} \int_\Omega (\mathbf{v}^{l+1} - \mathbf{v}^{l}) \cdot (\mathbf{v}^{l+1} + \mathbf{v}^{l})\, dx = \frac{1}{h} \int_\Omega (\nabla \cdot \boldsymbol{\sigma}[\tfrac{1}{2}(\mathbf{u}^{l+1} + \mathbf{u}^{l})] + \mathbf{f}) \cdot (\mathbf{u}^{l+1} - \mathbf{u}^{l})\, dx.$$

Applying integration by parts,

$$\begin{aligned}\frac{\rho}{2h}\int_\Omega \left(|\mathbf{v}^{l+1}|^2 - |\mathbf{v}^l|^2\right)\,dx &= \frac{1}{2h}\int_{\partial\Omega} \mathbf{n}^T\cdot\left(\boldsymbol{\sigma}[\mathbf{u}^{l+1}]+\boldsymbol{\sigma}[\mathbf{u}^l]\right)\cdot\left(\mathbf{u}^{l+1}-\mathbf{u}^l\right)\,ds\\ &\quad -\frac{1}{2h}\int_\Omega \left(\boldsymbol{\sigma}[\mathbf{u}^{l+1}]+\boldsymbol{\sigma}[\mathbf{u}^l]\right):\nabla\cdot\left(\mathbf{u}^{l+1}-\mathbf{u}^l\right)\,dx\\ &\quad +\frac{1}{h}\int_\Omega \mathbf{f}\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\,dx.\end{aligned}$$

Since $\boldsymbol{\sigma}[\mathbf{u}^{l+1}]:\nabla\cdot[\mathbf{u}^l] = \boldsymbol{\sigma}[\mathbf{u}^l]:\nabla\cdot[\mathbf{u}^{l+1}]$ and contact force is zero,

$$\begin{aligned}&\frac{1}{2}\int_\Omega \left(\rho|\mathbf{v}^{l+1}|^2 + \boldsymbol{\sigma}[\mathbf{u}^{l+1}]:\nabla\cdot\mathbf{u}^{l+1}\right)dx - \int_\Omega \mathbf{f}\cdot\mathbf{u}^{l+1}\,dx\\ &\quad -\left[\frac{1}{2}\int_\Omega \left(\rho|\mathbf{v}^l|^2+\boldsymbol{\sigma}[\mathbf{u}^l]:\nabla\cdot\mathbf{u}^l\right)dx - \int_\Omega \mathbf{f}\cdot\mathbf{u}^l\,dx\right] = 0.\end{aligned}$$

Note that $\boldsymbol{\sigma}[\mathbf{u}]:\nabla\cdot[\mathbf{u}] = \boldsymbol{\sigma}[\mathbf{u}]:\boldsymbol{\varepsilon}[\mathbf{u}]$. By Definition 3.2,

$$E(\mathbf{u}^{l+1},\mathbf{v}^{l+1}) = E(\mathbf{u}^l,\mathbf{v}^l) \quad \text{for any } l\geq 0.$$

□

From Lemma 3.2, using the midpoint rule for motion equation enables the numerical formulation to satisfy the conservation law when contact force does not apply to elastic body.

Lemma 3.3. *Suppose that* $\mathbf{f}$ *does not depend on time* t *and implicit Euler method is used in equation of motion and on boundary condition. Then*

$$E(\mathbf{u}^{l+1},\mathbf{v}^{l+1}) \leq E(\mathbf{u}^l,\mathbf{v}^l) \quad \textit{for } l\geq 0.$$

Proof. Using (3.9) and by the same argument as Lemma 3.2,

$$\frac{\rho}{2h}\int_\Omega (\mathbf{v}^{l+1}-\mathbf{v}^l)\cdot(\mathbf{v}^{l+1}+\mathbf{v}^l)\,dx = \frac{1}{h}\int_\Omega \left(\nabla\cdot\boldsymbol{\sigma}[\mathbf{u}^{l+1}]+\mathbf{f}\right)\cdot\left(\mathbf{u}^{l+1}-\mathbf{u}^l\right)\,dx.$$

So we have

$$\begin{aligned}\frac{\rho}{2h}\int_\Omega \left(|\mathbf{v}^{l+1}|^2 - |\mathbf{v}^l|^2\right)\,dx &= \frac{1}{h}\int_\Omega \nabla\cdot\boldsymbol{\sigma}[\mathbf{u}^{l+1}]\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l) + \mathbf{f}\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\,dx \\ &= \frac{1}{h}\int_{\partial\Omega} \mathbf{n}^T\cdot\boldsymbol{\sigma}[\mathbf{u}^{l+1}]\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\,ds - \\ &\quad \frac{1}{h}\int_\Omega \boldsymbol{\sigma}[\mathbf{u}^{l+1}] : \nabla\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\,dx + \frac{1}{h}\int_\Omega \mathbf{f}\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\,dx.\end{aligned}$$

Since $\boldsymbol{\sigma}[\mathbf{u}^{l+1}]\cdot\mathbf{n} = N^l\mathbf{n} \quad \text{on } \partial\Omega$,

$$\begin{aligned}\frac{\rho}{2}\int_\Omega \left(|\mathbf{v}^{l+1}|^2 - |\mathbf{v}^l|^2\right)dx &= \int_{\partial\Omega} N^l\mathbf{n}\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\,ds - \int_{\partial\Omega}\boldsymbol{\sigma}[\mathbf{u}^{l+1}] : \nabla\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\,dx \\ &\quad + \int_\Omega \mathbf{f}\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\,dx \\ &= \int_{\partial\Omega}[N^l\mathbf{n}\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l) + N^l(g-g)]\,ds + \int_\Omega \mathbf{f}\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\,dx \\ &\quad - \frac{1}{2}\int_\Omega \boldsymbol{\sigma}[\mathbf{u}^{l+1}+\mathbf{u}^l] : \nabla\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\,dx \\ &\quad - \frac{1}{2}\int_\Omega \boldsymbol{\sigma}[\mathbf{u}^{l+1}-\mathbf{u}^l] : \nabla\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\,dx \\ &= \int_{\partial\Omega} N^l(\mathbf{u}^{l+1}\cdot\mathbf{n}+g)\,ds - \int_{\partial\Omega} N^l(\mathbf{u}^l\cdot\mathbf{n}+g)\,ds \\ &\quad - \frac{1}{2}\int_\Omega \boldsymbol{\sigma}[\mathbf{u}^{l+1}+\mathbf{u}^l] : \nabla\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\,dx \\ &\quad - \frac{1}{2}\int_\Omega \boldsymbol{\sigma}[\mathbf{u}^{l+1}-\mathbf{u}^l] : \nabla\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\,dx \\ &\quad + \int_\Omega \mathbf{f}\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\,dx.\end{aligned}$$

Since $0 \le \mathbf{u}^{l+1}\cdot\mathbf{n} + g \perp N^l \ge 0 \quad \text{on } \partial\Omega$,

$$\begin{aligned}\frac{\rho}{2}\int_\Omega \left(|\mathbf{v}^{l+1}|^2 - |\mathbf{v}^l|^2\right)dx &\le -\frac{1}{2}\int_\Omega \boldsymbol{\sigma}[\mathbf{u}^{l+1}+\mathbf{u}^l] : \nabla\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\,dx - \\ &\quad \frac{1}{2}\int_\Omega \boldsymbol{\sigma}[\mathbf{u}^{l+1}-\mathbf{u}^l] : \nabla\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\,dx + \int_\Omega \mathbf{f}\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\,dx.\end{aligned}$$

From the condition of Hooke's tensor E_{ijkl},

$$\frac{\rho}{2}\int_\Omega \left(|\mathbf{v}^{l+1}|^2 - |\mathbf{v}^l|^2\right)dx \le -\frac{1}{2}\int_\Omega \boldsymbol{\sigma}[\mathbf{u}^{l+1}+\mathbf{u}^l] : \nabla\cdot[\mathbf{u}^{l+1}-\mathbf{u}^l]\,dx + \int_\Omega \mathbf{f}\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\,dx$$

$$= -\frac{1}{2}\int_\Omega \left(\boldsymbol{\sigma}[\mathbf{u}^{l+1}] : \nabla\cdot[\mathbf{u}^{l+1}] - \boldsymbol{\sigma}[\mathbf{u}^l] : \nabla\cdot[\mathbf{u}^l]\right)\, dx + \int_\Omega \mathbf{f}\cdot(\mathbf{u}^{l+1} - \mathbf{u}^l)\, dx.$$

Therefore we obtain

$$\frac{1}{2}\int_\Omega \left(\rho|\mathbf{v}^{l+1}|^2 + \boldsymbol{\sigma}[\mathbf{u}^{l+1}] : \boldsymbol{\varepsilon}[\mathbf{u}^{l+1}]\right) dx - \int_\Omega \mathbf{f}\cdot\mathbf{u}^{l+1}\, dx$$

$$- \left[\frac{1}{2}\int_\Omega \left(\rho|\mathbf{v}^{l}|^2 + \boldsymbol{\sigma}[\mathbf{u}^{l}] : \boldsymbol{\varepsilon}[\mathbf{u}^{l}]\right) dx - \int_\Omega \mathbf{f}\cdot\mathbf{u}^{l} dx\right] \le 0.$$

By Definition 3.2,

$$E(\mathbf{u}^{l+1}, \mathbf{v}^{l+1}) \le E(\mathbf{u}^l, \mathbf{v}^l) \quad \text{for any } l \ge 0.$$

□

If we employ the implicit Euler method for the equations of motion and the contact conditions, energy is dissipated. From the initial conditions, the initial energy is finite. This enables us to show the boundedness of $\mathbf{u}^l$, $\mathbf{v}^l$ at each time t_l for any $l \ge 1$.

Lemma 3.4. *Suppose that f does not depend on time t and the explicit Euler method is used in equation of motion and on boundary condition. Then*

$$E(\mathbf{u}^{l+1}, \mathbf{v}^{l+1}) \ge E(\mathbf{u}^l, \mathbf{v}^l) \quad \textit{for any } l \ge 0.$$

Proof. By (3.10) and the same argument as Lemma 3.3,

$$\frac{\rho}{2h}\int_\Omega (\mathbf{v}^{l+1} - \mathbf{v}^l)\cdot(\mathbf{v}^{l+1} + \mathbf{v}^l)\, dx = \frac{1}{h}\int_\Omega \left(\nabla\cdot\boldsymbol{\sigma}[\mathbf{u}^l] + \mathbf{f}\right)\cdot\left(\mathbf{u}^{l+1} - \mathbf{u}^l\right)\, dx.$$

Since $\boldsymbol{\sigma}[\mathbf{u}^l] \cdot \mathbf{n} = N^{l+1}\mathbf{n} \quad \text{on } \partial\Omega$,

$$\begin{aligned}
\frac{\rho}{2}\int_\Omega \left(|\mathbf{v}^{l+1}|^2 - |\mathbf{v}^l|^2\right)\, dx \;=\;& \int_{\partial\Omega} N^{l+1}\mathbf{n}\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\, ds - \int_{\partial\Omega} \boldsymbol{\sigma}[\mathbf{u}^l] : \nabla\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\, dx \\
&+ \int_\Omega \mathbf{f}\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\, dx \\
=\;& \int_{\partial\Omega} [N^{l+1}\mathbf{n}\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l) + N^{l+1}(g-g)]\, ds \\
&- \frac{1}{2}\int_\Omega \boldsymbol{\sigma}[\mathbf{u}^{l+1}+\mathbf{u}^l] : \nabla\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\, dx \\
&- \frac{1}{2}\int_\Omega \boldsymbol{\sigma}[\mathbf{u}^{l+1}-\mathbf{u}^l] : \nabla\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\, dx \\
&+ \int_\Omega \mathbf{f}\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\, dx \\
=\;& \int_{\partial\Omega} N^{l+1}(\mathbf{u}^{l+1}\cdot\mathbf{n}+g)\, ds - \int_{\partial\Omega} N^{l+1}(\mathbf{u}^l\cdot\mathbf{n}+g)\, ds \\
&- \frac{1}{2}\int_\Omega \boldsymbol{\sigma}[\mathbf{u}^{l+1}+\mathbf{u}^l] : \nabla\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\, dx \\
&- \frac{1}{2}\int_\Omega \boldsymbol{\sigma}[\mathbf{u}^{l+1}-\mathbf{u}^l] : \nabla\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\, dx \\
&+ \int_\Omega \mathbf{f}\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\, dx.
\end{aligned}$$

Since $0 \le \mathbf{u}^l\cdot\mathbf{n} + g \perp N^{l+1} \ge 0 \quad \text{on } \partial\Omega$, we have

$$\begin{aligned}
\frac{\rho}{2}\int_\Omega \left(|\mathbf{v}^{l+1}|^2 - |\mathbf{v}^l|^2\right)\, dx \;\ge\;& -\frac{1}{2}\int_\Omega \boldsymbol{\sigma}[\mathbf{u}^{l+1}+\mathbf{u}^l] : \nabla\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\, dx \\
&+ \frac{1}{2}\int_\Omega \boldsymbol{\sigma}[\mathbf{u}^{l+1}-\mathbf{u}^l] : \nabla\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\, dx \\
&+ \int_\Omega \mathbf{f}\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\, dx.
\end{aligned}$$

By the properties of Hooke's tensor E_{ijkl},

$$\begin{aligned}
\frac{\rho}{2}\int_\Omega |\left(\mathbf{v}^{l+1}|^2 - |\mathbf{v}^l|^2\right)\, dx \;\ge\;& -\frac{1}{2}\int_\Omega \boldsymbol{\sigma}[\mathbf{u}^{l+1}+\mathbf{u}^l] : \nabla\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\, dx \\
&+ \int_\Omega \mathbf{f}\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\, dx \\
=\;& -\frac{1}{2}\int_\Omega \left(\boldsymbol{\sigma}[\mathbf{u}^{l+1}] : \nabla\cdot\mathbf{u}^{l+1} - \boldsymbol{\sigma}[\mathbf{u}^l] : \nabla\cdot\mathbf{u}^l\right)\, dx \\
&+ \int_\Omega \mathbf{f}\cdot(\mathbf{u}^{l+1}-\mathbf{u}^l)\, dx.
\end{aligned}$$

Therefore we have

$$\frac{1}{2}\int_{\Omega}\left(\rho|\mathbf{v}^{l+1}|^2+\boldsymbol{\sigma}[\mathbf{u}^{l+1}]:\boldsymbol{\varepsilon}[\mathbf{u}^{l+1}]\right)\,dx-\int_{\Omega}\mathbf{f}\cdot\mathbf{u}^{l+1}\,dx \\ -\left[\frac{1}{2}\int_{\Omega}\left(\rho|\mathbf{v}^{l}|^2+\boldsymbol{\sigma}[\mathbf{u}^{l}]:\boldsymbol{\varepsilon}[\mathbf{u}^{l}]\right)\,dx-\int_{\Omega}\mathbf{f}\cdot\mathbf{u}^{l}\,dx\right]\geq 0.$$

By Definition 3.2,

$$E(\mathbf{u}^{l+1},\mathbf{v}^{l+1})\geq E(\mathbf{u}^{l},\mathbf{v}^{l})\quad\text{for any } l\geq 0.$$

□

Compared to the result of Lemma 3.3, the explicit Euler method is not suitable to obtain boundedness and would not be reasonable in terms of a physical point of view. Due to these reasons, we will avoid considering the explicit Euler method in this thesis.

CHAPTER 4
ELASTIC BODIES : COVERGENCE

4.1 Standard results for frictionless contact

We recall that if we use the implicit Euler method in equation of motion and on boundary condition, total energy is dissipated, i.e.,

$$E(\mathbf{u}^{l+1}, \mathbf{v}^{l+1}) \leq E(\mathbf{u}^{l}, \mathbf{v}^{l}).$$

In this Section, we require the numerical formulation of motion equation and boundary (contact) condition made by the implicit Euler method. Since the total initial energy functional is finite,

$$E(\mathbf{u}^{l}, \mathbf{v}^{l}) \leq E(\mathbf{u}^{0}, \mathbf{v}^{0}) < \infty \quad \text{for any } l \geq 1.$$

According to bound of total energy for discrete time t_l, we will see that the numerical solutions $\mathbf{u}^l, \mathbf{v}^l$ are bounded in some spaces, independent of the time step size h. In order to show this result, we begin with a discrete nonlinear version of the *Grownall Lemma*. See the detail in [53].

In this Section, instead of notations $\mathbf{u}^l$, $\mathbf{v}^l$, we write those as $\mathbf{u}^{l;h}, \mathbf{v}^{l;h}$ to show the dependence of h more explicitly. We recall the partition of time $[0, T]$ used in Section 3.3. Then $0 \leq l \leq T/h$ and $t_l = lh$ for $t_l \in [0, T]$. Also we note that as $h \downarrow 0$, $lh \to t \in [0, T]$.

Lemma 4.1. *Suppose that $y^{0:h} = y^0 \geq 0$ for all $h > 0$ and*

$$y^{n+1:h} \leq y^{n:h} + hG(y^{n:h}, h), \quad n = 0, 1, 2, \cdots, \tag{4.1}$$

where $G(y,h)$ is nonnegative, locally Lipschitz continuous and monotone increasing in y with Lipschitz constant independent of h and $G(\cdot,h) \to g(\cdot)$ uniformly on compact sets as $h \to 0$. Suppose also that the initial value problem

$$\frac{dk(t)}{dt} = g(k), \quad k(0) = y^0$$

has a unique solution. Then

$$\limsup_{h\downarrow 0} y^{l:h} \le k(t) \quad \textit{for all } t \ge 0 \textit{ and some } l \ge 1,$$

where $k(t) < +\infty$ and $t = lh$.

Applying Lemma 4.1, we will derive the bound of virtual work by external forces, not depending on h.

Lemma 4.2. *Assume that $\mathbf{f}$ does not depend on time and neither $E(\mathbf{u}^0, \mathbf{v}^0)$ nor $\int_\Omega \mathbf{f} \cdot \mathbf{u}^{l;h}\,dx$ is not zero for any $l \ge 0$. Then as $h \to 0$, we have a function $k(t)$ such that*

$$\limsup_{h\downarrow 0, lh\to t} \int_\Omega \mathbf{f} \cdot \mathbf{u}^{l;h}\,dx \le k(t) \quad \textit{for } t \in [0,T].$$

Proof. For any $l \ge 1$ we put

$$y^{l:h} = \left| \int_\Omega \mathbf{f} \cdot \mathbf{u}^{l;h}\,dx \right|. \tag{4.2}$$

Obviously, $y^{l:h} \ge 0$ for $0 \le l \le T/h$. Now we need this form

$$y^{l+1:h} \le y^{l:h} + hG(y^{l:h}, h).$$

From (4.2), for $h > 0$ we have

$$\begin{aligned} y^{l+1:h} - y^{l:h} &\leq \left| \int_\Omega \mathbf{f} \cdot (\mathbf{u}^{l+1} - \mathbf{u}^l) \, dx \right| \\ &= \frac{h}{2} \left| \int_\Omega \mathbf{f} \cdot (\mathbf{v}^{l+1} + \mathbf{v}^l) dx \right|. \end{aligned} \tag{4.3}$$

From the total energy functional and the property of Hooke's tensor E_{ijkl},

$$\begin{aligned} E(\mathbf{u}^{l;h}, \mathbf{v}^{l;h}) &= \frac{1}{2} \int_\Omega \left(\rho |\mathbf{v}^{l;h}|^2 + \boldsymbol{\sigma}[\mathbf{u}^{l;h}] : \boldsymbol{\varepsilon}[\mathbf{u}^{l;h}] \right) \, dx - \int_\Omega \mathbf{f} \cdot \mathbf{u}^{l;h} \, dx \\ &\geq \frac{1}{2} \rho \|\mathbf{v}^{l;h}\|^2_{\mathbf{L}^2(\Omega)} - \int_\Omega \mathbf{f} \cdot \mathbf{u}^{l;h} \, dx. \end{aligned}$$

Thus

$$\begin{aligned} \|\mathbf{v}^{l;h}\|_{\mathbf{L}^2(\Omega)} &\leq \sqrt{\frac{2}{\rho} \left(E(\mathbf{u}^{l;h}, \mathbf{v}^{l;h}) + \int_\Omega \mathbf{f} \cdot \mathbf{u}^{l;h} \, dx \right)} \\ &\leq \sqrt{\frac{2}{\rho}} \sqrt{E(\mathbf{u}^0, \mathbf{v}^0) + \int_\Omega \mathbf{f} \cdot \mathbf{u}^{l;h} \, dx}. \end{aligned}$$

Simply, we put $E(\mathbf{u}^0, \mathbf{v}^0) = E^0$. Then we have

$$\begin{aligned} \|\mathbf{v}^{l;h}\|_{\mathbf{L}^2(\Omega)} &\leq \sqrt{\frac{2}{\rho}} \sqrt{E^0 + y^{l:h}}, \\ \|\mathbf{v}^{l+1;h}\|_{\mathbf{L}^2(\Omega)} &\leq \sqrt{\frac{2}{\rho}} \sqrt{E^0 + y^{l+1:h}}. \end{aligned}$$

Thus by (4.3),

$$\begin{aligned} y^{l+1:h} - y^{l:h} &\leq \frac{h}{2} \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \left(\|\mathbf{v}^{l+1;h}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{v}^{l;h}\|_{\mathbf{L}^2(\Omega)} \right) \\ &\leq \frac{h}{\sqrt{2\rho}} \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \left(\sqrt{E^0 + y^{l+1:h}} + \sqrt{E^0 + y^{l:h}} \right). \end{aligned}$$

So we obtain

$$y^{l+1:h} - \frac{h}{\sqrt{2\rho}} \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \sqrt{E^0 + y^{l+1:h}} \leq y^{l:h} + \frac{h}{\sqrt{2\rho}} \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \sqrt{E^0 + y^{l:h}}.$$

Let

$$\varphi(y^{l+1:h},h) = y^{l+1:h} - \frac{h}{\sqrt{2\rho}}\|\mathbf{f}\|_{(L^2(\Omega))^d}\sqrt{E^0 + y^{l+1:h}} \tag{4.4}$$

and

$$\psi(y^{l:h},h) = y^{l:h} + \frac{h}{\sqrt{2\rho}}\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}\sqrt{E^0 + y^{l:h}}. \tag{4.5}$$

Thus

$$\varphi(y^{l+1:h},h) \le \psi(y^{l:h},h). \tag{4.6}$$

Consider continuous function in terms of only y. Then

$$\varphi(y,h) = y - \frac{h}{\sqrt{2\rho}}\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}\sqrt{E^0 + y} \quad \text{for any } y \ge 0.$$

Putting $h > 0$ be sufficiently small, we have

$$\frac{\partial\varphi(y,h)}{\partial y} = 1 - \frac{h}{2\sqrt{2}}\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}\left(E^0 + y\right)^{-1/2}.$$

Then $\varphi(\cdot,h)$ is strictly increasing for the fixed $h > 0$. So inverse function $\varphi^{-1}(\cdot,h)$ exists and is strictly increasing. Now taking inverse function φ^{-1} on both side in (4.6), we have $y^{l+1:h} \le \varphi^{-1}(\psi(y^{l:h},h),h)$. Let $\varphi^{-1}(\psi(y^{l:h},h),h) = y^{l;h} + hG(y^{l;h},h)$. Then,

$$G(y^{l;h},h) = \frac{\varphi^{-1}(\psi(y^{l:h},h),h) - y^{l;h}}{h}. \tag{4.7}$$

First, we claim that $G(y,h) \ge 0$ and is monotonically increasing in y. In order to show that $G(y^{l;h},h)$ is monotonically increasing in y, it is sufficient to show that $\partial G/\partial y \ge 0$. Now take a derivative with respect to only y. Then we have

$$\frac{\partial G(y,h)}{\partial y} = \frac{1}{h}\left(\frac{\partial\varphi^{-1}(\psi(y,h),h)}{\partial y} - 1\right). \tag{4.8}$$

By implicit function theorem, we can simplify $\varphi^{-1}(\psi(y,h),h)$ into $\varphi^{-1}(y^*,h)$, where $y^* = \psi(y,0)$ for some $y^* \geq 0$ and $y^* = \psi(y,h)$ is well defined in some neighborhood of $h = 0$. Thus from (4.8)

$$\frac{\partial G(y,h)}{\partial y} = \frac{1}{h}\left(\frac{\partial \varphi^{-1}(y^*,h)}{\partial y} - 1\right).$$

Since $\partial\varphi^{-1}(y,h)/\partial y = [\partial\varphi(y,h)/\partial y]^{-1}$, for sufficiently small $h > 0$ we obtain

$$\begin{aligned}\frac{\partial \varphi^{-1}(y^*,h)}{\partial y} &= \left[1 - \frac{h}{2\sqrt{2\rho}}\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}(E^0 + y^*)^{-1/2}\right]^{-1} \\ &= \left[1 - \frac{h}{2\sqrt{2\rho}}\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}(E^0 + \psi(y,h))^{-1/2}\right]^{-1}.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\partial G(y,h)}{\partial y} &= \frac{1}{h}\left(\left[1 - \frac{h}{2\sqrt{2\rho}}\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}(E^0 + \psi(y,h))^{-1/2}\right]^{-1} - 1\right) \\ &= \frac{\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}/\sqrt{8\rho(E^0 + \psi(y,h))}}{1 - h\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}/\sqrt{8\rho(E^0 + \psi(y,h))}} > 0.\end{aligned} \tag{4.9}$$

Next , we want to show that $G(0,h) \geq 0$. From (4.7) we obtain

$$G(0,h) = \frac{\varphi^{-1}(h\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}\sqrt{E^0}/\sqrt{2\rho},\, h)}{h}.$$

Since $\varphi^{-1}(0,h) \geq 0$ and φ^{-1} is strictly increasing, $G(0,h) \geq 0$.

From(4.9), $\partial G(y,h)/\partial y$ is bounded for $y < \infty$. Since $\varphi(y,h)$ and $\psi(y,h)$ are continuous in $y \geq 0$, $G(y,h)$ is locally Lipschitz continuous.

We want to find $g(y)$ such that $G(y,h) \to g(y)$ uniformly on compact set as $h \to 0$. By implicit function theorem,

$$\lim_{h\to 0} G(y,h) = \lim_{h\to 0}\frac{\varphi^{-1}(y^*,h) - y}{h}.$$

Since $\varphi(y,0) = y$ and $\varphi^{-1}(y,0) = y$,

$$\lim_{h\to 0} G(y,h) = \lim_{h\to 0} \frac{\varphi^{-1}(y^*,h) - \varphi^{-1}(y,0)}{h}. \tag{4.10}$$

Notice that $\psi(y^*,0) = y^*$ from (4.5) and recall that we put $\psi(y,0) = y^*$. Note that for any $h > 0$

$$\frac{\partial \psi(y,h)}{\partial y} = 1 + \frac{h}{2\sqrt{2}} \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \left(E^0 + y\right)^{-1/2} > 0.$$

Since $\psi(y,h)$ is bijective in y, $y^* = y(0)$ by implicit function theorem. Therefore from (4.10),

$$\begin{aligned} \lim_{h\to 0} G(y,h) &= \lim_{h\to 0} \frac{\varphi^{-1}(y^*,h) - \varphi^{-1}(y^*,0)}{h} \\ &= \frac{\partial \varphi^{-1}}{\partial h}(y^*,0). \end{aligned}$$

Since $\varphi^{-1}(\varphi(y,h),h), = y$, taking $\varphi(y,h) = z$,

$$\begin{aligned} 0 &= \frac{\partial \varphi^{-1}(z,h)}{\partial h} \\ &= \frac{\partial \varphi^{-1}(z,h)}{\partial z} \frac{\partial z}{\partial h} + \frac{\partial \varphi^{-1}(z,h)}{\partial h}. \end{aligned}$$

Then putting $h = 0$,

$$\begin{aligned} 0 &= \frac{\partial \varphi^{-1}(y,0)}{\partial y} \frac{\partial \varphi(y,0)}{\partial h} + \frac{\partial \varphi^{-1}(y,0)}{\partial h} \\ &= \frac{\partial \varphi(y,0)}{\partial h} + \frac{\partial \varphi^{-1}(y,0)}{\partial h}. \end{aligned} \tag{4.11}$$

Therefore using (4.11),

$$\begin{aligned} \lim_{h\to 0} G(y,h) &= \frac{\partial \varphi^{-1}(y^*,0)}{\partial h} \\ &= -\frac{\partial \varphi(y^*,0)}{\partial h} \\ &= \frac{1}{\sqrt{2\rho}} \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \sqrt{E^0 + y^*} \end{aligned}$$

$$= \frac{1}{\sqrt{2\rho}} \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \sqrt{E^0 + \psi(y,0)}$$
$$= \frac{1}{\sqrt{2\rho}} \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \sqrt{E^0 + y}.$$

So we have

$$g(y) = \frac{1}{\sqrt{2\rho}} \|\mathbf{f}\|_{(L^2(\Omega))^d} \sqrt{E^0 + y} \quad \text{as } h \to 0.$$

We claim that the initial value problem

$$\frac{dk}{dt} = g(k) = \frac{1}{\sqrt{2\rho}} \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \sqrt{E^0 + k}, \quad k(0) = y^0$$

has unique solution. This ordinary differential equation has unique solution

$$k(t) = \frac{1}{4} \left(\frac{t}{\sqrt{2\rho}} \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + 2(E^0 + y^0)^{1/2} \right)^2 - E^0.$$

Therefore by Lemma 4.1, the result follows. □

Before we verify the next Lemma 4.3, we mention a continuous linear interpolant, denoted by $\mathbf{u}^h(\mathbf{x}, t)$. The value $\mathbf{u}^h(\mathbf{x}, t)$ is the continuous linear interpolant of $\mathbf{u}^{l;h} = \mathbf{u}^h(\mathbf{x}, lh)$ and $\mathbf{u}^{l+1;h} = \mathbf{u}^h(\mathbf{x}, (l+1)h)$ for $t \in [lh, (l+1)h]$. Similarly let $\mathbf{v}^h(\mathbf{x}, t)$ be a continuous linear interpolant of $\mathbf{v}^{l;h} = \mathbf{v}^h(\mathbf{x}, lh)$ and $\mathbf{v}^{l+1;h} = \mathbf{v}^h(\mathbf{x}, (l+1)h)$ for $t \in [lh, (l+1)h]$.

Lemma 4.3. *Suppose that*

$$\limsup_{h \downarrow 0, lh \to t} \int_\Omega \mathbf{f} \cdot \mathbf{u}^{l;h}(x, t)\, dx \le k(t), \quad \forall t \in [0, T].$$

Then $\mathbf{v}^h$ *are uniformly bounded in* $L^\infty(0, T; \mathbf{L}^2(\Omega))$ *and* $\mathbf{u}^h$ *are uniformly bounded in* $W^{1,\infty}(0, T; \mathbf{L}^2(\Omega))$ *and* $L^\infty(0, T; \mathbf{H}^1(\Omega))$*, as* $h \to 0, lh \to t$.

Proof. Since $E(\mathbf{u}^l, \mathbf{v}^l) \leq E(\mathbf{u}^0, \mathbf{v}^0)$ for any $l \geq 1$,

$$\frac{1}{2}\int_\Omega \left(\rho|\mathbf{v}^{l;h}|^2 + \boldsymbol{\sigma}[\mathbf{u}^{l;h}] : \boldsymbol{\varepsilon}[\mathbf{u}^{l;h}]\right)\, dx - \int_\Omega \mathbf{f}\cdot\mathbf{u}^{l;h}\, dx \leq E(u^0, v^0).$$

Then by our assumption, we have

$$\frac{1}{2}\int_\Omega \left(\rho|\mathbf{v}^{l;h}|^2 + \boldsymbol{\sigma}[\mathbf{u}^{l;h}] : \boldsymbol{\varepsilon}[\mathbf{u}^{l;h}]\right)\, dx \leq E(u^0, v^0) + k(t).$$

From the property of Hooke's tensor E_{ijkl}, there exists a constant $m > 0$ such that

$$\frac{1}{2}\int_\Omega \left(\rho|\mathbf{v}^{l;h}|^2 + m|\boldsymbol{\varepsilon}[\mathbf{u}^{l;h}]|^2\right)\, dx \leq E(u^0, v^0) + k(t). \tag{4.12}$$

Thus we have

$$\limsup_{h\downarrow 0, lh\to t} \|\mathbf{v}^{l;h}\|_{\mathbf{L}^2(\Omega)} < \infty \quad \text{for any } l \geq 1. \tag{4.13}$$

Since $\mathbf{v}^h(\mathbf{x}, t)$ are the linear continuous interpolant, $\mathbf{v}^h$ are uniformly bounded in $L^\infty(0, T; \mathbf{L}^2(\Omega))$.

Next, we claim that $\mathbf{u}^h$ are uniformly bounded in $W^{1,\infty}(0, T; \mathbf{L}^2(\Omega))$. Now we consider

$$\mathbf{u}(\mathbf{x}, t_l) = \mathbf{u}(\mathbf{x}, 0) + \int_0^{t_l} \mathbf{v}(\mathbf{x}, \tau)\, d\tau \quad \text{for any } l \geq 1.$$

Then we obtain

$$\begin{aligned}
\|\mathbf{u}^{l;h}\|_{\mathbf{L}^2(\Omega)} &\leq \|\mathbf{u}^0\|_{\mathbf{L}^2(\Omega)} + \int_0^{t_l} \|\mathbf{v}(\tau)\|_{\mathbf{L}^2(\Omega)}\, d\tau \\
&\leq \|\mathbf{u}^0\|_{\mathbf{L}^2(\Omega)} + \int_0^{T} \max_{0\leq\tau\leq T} \|\mathbf{v}(\tau)\|_{\mathbf{L}^2(\Omega)}\, d\tau \\
&\leq \|\mathbf{u}^0\|_{\mathbf{L}^2(\Omega)} + \int_0^{T} k(\tau)\, d\tau.
\end{aligned}$$

Since $\mathbf{u}^0 \in \mathbf{H}^1(\Omega)$ and $k(t)$ is bounded,

$$\limsup_{h\downarrow 0, lh\to 0} \|\mathbf{u}^{l;h}\|_{\mathbf{L}^2(\Omega)} < \infty. \tag{4.14}$$

Since $\mathbf{v}^h(\mathbf{x},t)$ and $\mathbf{u}^h(\mathbf{x},t)$ are the linear continuous interpolants, by (4.13) and (4.14),

$$\|\mathbf{u}^h(t)\|_{W^{1,\infty}(0,T;\mathbf{L}^2(\Omega))} = \text{ess sup}_{0\le t\le T}\left(\|\mathbf{v}^h(t)\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}^h(t)\|_{\mathbf{L}^2(\Omega)}\right) < \infty.$$

Also we claim that $\mathbf{u}^h$ are uniformly bounded in $L^\infty(0,T;\mathbf{H}^1(\Omega))$. Using Korn's inequality (Theorem 2.8) with (4.12) and by (4.14), we obtain

$$\|\mathbf{u}^h(t)\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))} < \infty,$$

as required. □

Note that a continuous function $k(t)$ used in the Lemma 4.3 may be different for each occurrence. In order to achieve the boundedness of N^l in the Sobolev space $H^{-1/2}(\partial\Omega)$, we could need nicer spaces for $\mathbf{u}^h$ and $\mathbf{v}^h$. In the next Section we will see how to derive an estimate of N^l which is depending on time step size h, using Fourier transform and Extension operators. Basically we will use the spaces discussed in the previous Lemma, in order to do so.

4.2 Sharper estimate for frictionless contact

In this Section, we obtain a uniform estimate of the contact force N^l in the Sobolev space $H^{-1/2}(\partial\Omega)$ at each time t_l, employing the implicit Euler method. We note that the linear complementarity problem condition will not be used to derive the estimate. Other numerical schemes with the linear complementarity problem condition would be employed to obtain better estimates.

4.2.1 Implicit Euler method

We recall the numerical formulation of motion and the boundary condition applied by the implicit Euler method:

$$\mathbf{u}^{l+1} = \frac{h^2}{2\rho}\nabla\cdot\boldsymbol{\sigma}[\mathbf{u}^{l+1}] + \frac{h^2}{2\rho}\mathbf{f} + h\mathbf{v}^l + \mathbf{u}^l, \tag{4.15}$$

$$\boldsymbol{\sigma}[\mathbf{u}^{l+1}] = N^l\mathbf{n} \quad \text{on } \partial\Omega. \tag{4.16}$$

In this Section our aim is to derive an estimate of N^l on $H^{-1/2}(\partial\Omega)$, based on (4.15) and (4.16).

4.2.2 Extension operators $\text{ext}_k : H^{1/2}(\partial\Omega) \to H^1(\Omega)$

According to Trace theorem 2.7, there is a continuous linear operator tr from $H^1(\Omega)$ onto $H^{1/2}(\partial\Omega)$. Then the operator tr has bounded right inverse. See, for example, [47]. In the next Subsection, we will construct a family of the bounded right inverse operators $\text{ext}_k : H^{1/2}(\partial\Omega) \to H^1(\Omega)$, i.e., $\text{tr} \circ \text{ext}_k = I_{H^{1/2}(\partial\Omega)}$, where I is identity map and k is a positive number.

4.2.3 Results on the half space $\mathbf{R}^d_+$

We start by obtaining estimates for the geometrically simple case $\Omega = \mathbf{R}^d_+ = \mathbf{R}^{d-1} \times \mathbf{R}_+$. Since $\partial\Omega = \mathbf{R}^{d-1} \times \{0\}$ can be identified with $\mathbf{R}^{d-1}$, we write $\mathbf{x} = (\widetilde{\mathbf{x}}, x_d)$ for any $\mathbf{x} \in \mathbf{R}^d$ so that $\Omega = \{(\widetilde{\mathbf{x}}, x_d) \mid x_d > 0\}$ and $\partial\Omega = \{(\widetilde{\mathbf{x}}, x_d) \mid x_d = 0\}$. Now we want to construct the concrete extension operator of the form $\text{ext}_k(w) = u$. Before doing this, we introduce some useful functions; let $\mu : \mathbf{R}^{d-1} \to \mathbf{R}$ be a test function of class C_0^∞ whose compact support is $\overline{B} = \{\widetilde{\mathbf{x}} \mid \widetilde{\mathbf{x}} \in \mathbf{R}^{d-1}, \mid \widetilde{\mathbf{x}} \mid \leq 1\}$, $\mu \geq 0$

and $\|\mu\|_{L^1(\mathbf{R}^{d-1})} = 1$. For any $\alpha > 0$, let $\mu_\alpha(\widetilde{\mathbf{x}}) = (1/\alpha^{d-1})\mu(\alpha^{-1}\widetilde{\mathbf{x}})$. When we derive the estimate, we will have a natural choice of $\alpha(x_d) = x_d$. As we shall see in Lemma 4.7, $\alpha(x_d) = (x_d)^\eta$ for $\eta < 1$ will turn out to be inappropriate in the process of deriving estimates.

For $w \in H^{1/2}(\partial\Omega)$ and $u \in H^1(\Omega)$, we want to use the extension operator $\mathrm{ext}_k(w) = u$ which has the form

$$u(\widetilde{\mathbf{x}}, x_d) = \int_{\mathbf{R}^{d-1}} w(\widetilde{\mathbf{x}} - \widetilde{\mathbf{y}})\mu_{\alpha(x_d)}(\widetilde{\mathbf{y}})\, d\widetilde{\mathbf{y}} \cdot (1 - kx_d)_+, \tag{4.17}$$

where $\widetilde{\mathbf{y}} = (y_1, y_2, \cdots, y_{d-1})$ and $d\widetilde{\mathbf{y}} = dy_1 dy_2 \cdots dy_{d-1}$. This idea is based on Lars-Erik Andersson's paper [1].

Lemma 4.4. *If* $\lim_{x_d \downarrow 0} \alpha(x_d) = 0$, *then* $\lim_{x_d \downarrow 0} u(\widetilde{\mathbf{x}}, x_d) = w(\widetilde{\mathbf{x}})$.

Proof. From (4.17), the Fourier transform of u is

$$\mathcal{F}[u]\left(\widetilde{\boldsymbol{\xi}}, x_d\right) = (2\pi)^{-(d-1)}(1 - kx_d)_+ \int_{\mathbf{R}^{d-1}} \left[e^{-\widetilde{\mathbf{x}}\cdot\widetilde{\boldsymbol{\xi}}} \int_{\mathbf{R}^{d-1}} w(\widetilde{\mathbf{x}} - \widetilde{\mathbf{y}})\mu_{\alpha(x_d)}(\widetilde{\mathbf{y}})\, d\widetilde{\mathbf{y}}\right] d\widetilde{\mathbf{x}}.$$

Using substitution $\widetilde{\mathbf{z}} = \alpha^{-1}\widetilde{\mathbf{y}}$ and putting $B = (2\pi)^{-(d-1)}(1 - kx_d)_+$,

$$\begin{aligned}
\mathcal{F}[u]\left(\widetilde{\boldsymbol{\xi}}, x_d\right) &= B \int_{\mathbf{R}^{d-1}} \left[e^{-\widetilde{\mathbf{x}}\cdot\widetilde{\boldsymbol{\xi}}} \int_{\mathbf{R}^{d-1}} w(\widetilde{\mathbf{x}} - \widetilde{\mathbf{y}})\frac{1}{\alpha^{d-1}}\mu(\alpha^{-1}\widetilde{\mathbf{y}})\, d\widetilde{\mathbf{y}}\right] d\widetilde{\mathbf{x}} \\
&= B \int_{\mathbf{R}^{d-1}} \left[e^{-\widetilde{\mathbf{x}}\cdot\widetilde{\boldsymbol{\xi}}} \int_{\mathbf{R}^{d-1}} w(\widetilde{\mathbf{x}} - \alpha(x_d)\widetilde{\mathbf{z}})\frac{1}{\alpha^{d-1}}\mu(\widetilde{\mathbf{z}})\alpha^{d-1}\, d\widetilde{\mathbf{z}}\right] d\widetilde{\mathbf{x}} \\
&= B \int_{\mathbf{R}^{d-1}} \left[e^{-\widetilde{\mathbf{x}}\cdot\widetilde{\boldsymbol{\xi}}} w(\widetilde{\mathbf{x}} - \alpha(x_d)\widetilde{\mathbf{z}}) \int_{\mathbf{R}^{d-1}} \mu(\widetilde{\mathbf{z}})\, d\widetilde{\mathbf{z}}\right] d\widetilde{\mathbf{x}}.
\end{aligned}$$

Recalling that $\|\mu\|_{L^1(\mathbf{R}^{d-1})} = 1$, we have

$$\mathcal{F}[u]\left(\widetilde{\boldsymbol{\xi}}, x_d\right) = (2\pi)^{-(d-1)}(1 - kx_d)_+ \int_{\mathbf{R}^{d-1}} e^{-\widetilde{\mathbf{x}}\cdot\widetilde{\boldsymbol{\xi}}} w(\widetilde{\mathbf{x}} - \alpha(x_d)\widetilde{\mathbf{z}})\, d\widetilde{\mathbf{x}}.$$

Thus if $\lim_{x_d \downarrow 0} \alpha(x_d) = 0$, $\mathcal{F}[u]\left(\widetilde{\boldsymbol{\xi}}, x_d\right) = \mathcal{F}[w]\left(\widetilde{\boldsymbol{\xi}}\right)$. Therefore if $\lim_{x_d \downarrow 0} \alpha(x_d) = 0$,

$$\lim_{x_d \downarrow 0} u(\widetilde{\mathbf{x}}, x_d) = w(\widetilde{\mathbf{x}}).$$

□

Lemma 4.5. *From (4.17), we have Fourier transform*

$$\mathcal{F}[u](\widetilde{\boldsymbol{\xi}}, x_d) = (2\pi)^{(d-1)/2}(1 - kx_d)_+ \cdot \mathcal{F}[w](\widetilde{\boldsymbol{\xi}})\mathcal{F}[\mu_{\alpha(x_d)}](\widetilde{\boldsymbol{\xi}}).$$

Proof. Putting $B = (2\pi)^{-(d-1)/2}(1 - kx_d)_+$,

$$\begin{aligned} \mathcal{F}[u](\widetilde{\boldsymbol{\xi}}, x_d) &= B \int_{\mathbf{R}^{d-1}} \left[e^{-i\widetilde{\boldsymbol{\xi}}\cdot\widetilde{\mathbf{x}}} \int_{\mathbf{R}^{d-1}} w(\widetilde{\mathbf{x}} - \widetilde{\mathbf{y}})\mu_{\alpha(x_d)}(\widetilde{\mathbf{y}})\, d\widetilde{\mathbf{y}} \right] d\widetilde{\mathbf{x}} \\ &= B \int_{\mathbf{R}^{d-1}} e^{-i\widetilde{\boldsymbol{\xi}}\cdot(\widetilde{\mathbf{x}}-\widetilde{\mathbf{y}})} w(\widetilde{\mathbf{x}} - \widetilde{\mathbf{y}})\, d\widetilde{\mathbf{x}} \left[\int_{\mathbf{R}^{d-1}} e^{-i\widetilde{\boldsymbol{\xi}}\cdot\widetilde{\mathbf{y}}} \mu_{\alpha(x_d)}(\widetilde{\mathbf{y}})\, d\widetilde{\mathbf{y}} \right]. \end{aligned}$$

Using substitution $\widetilde{\mathbf{x}} - \widetilde{\mathbf{y}} = \widetilde{\mathbf{z}}$,

$$\begin{aligned} \mathcal{F}[u](\widetilde{\boldsymbol{\xi}, x_d}) &= B \int_{\mathbf{R}^{d-1}} e^{-i\widetilde{\boldsymbol{\xi}}\cdot\widetilde{\mathbf{z}}} w(\widetilde{\mathbf{z}})\, d\widetilde{\mathbf{z}} \left[\int_{\mathbf{R}^{d-1}} e^{-i\widetilde{\boldsymbol{\xi}}\cdot\widetilde{\mathbf{y}}} \mu_{\alpha(x_d)}(\widetilde{\mathbf{y}})\, d\widetilde{\mathbf{y}} \right] \\ &= (2\pi)^{(d-1)/2}(1 - kx_d)_+ \cdot \mathcal{F}[w](\widetilde{\boldsymbol{\xi}})\mathcal{F}[\mu_{\alpha(x_d)}](\widetilde{\boldsymbol{\xi}}). \end{aligned}$$

□

Lemma 4.6. *From the definition of* $\mu_{\alpha(x_d)}$,

$$\mathcal{F}[\mu_\alpha](\widetilde{\boldsymbol{\xi}}) = \mathcal{F}[\mu](\alpha\widetilde{\boldsymbol{\xi}}).$$

Proof. The Fourier transformation of $\mu_{\alpha(x_d)}$ has

$$\mathcal{F}[\mu_\alpha](\widetilde{\boldsymbol{\xi}}) = (2\pi)^{-(d-1)/2}\alpha^{-(d-1)} \int_{\mathbf{R}^{d-1}} e^{-i\widetilde{\boldsymbol{\xi}}\cdot\widetilde{\mathbf{x}}} \mu(\alpha^{-1}\widetilde{\mathbf{x}})\, d\widetilde{\mathbf{x}}.$$

Using substitute $\alpha^{-1}\widetilde{\mathbf{x}} = \widetilde{\mathbf{y}}$,

$$\begin{aligned}\mathcal{F}[\mu_\alpha](\widetilde{\boldsymbol{\xi}}) &= (2\pi)^{-(d-1)/2}\alpha^{-(d-1)}\int_{\mathbf{R}^{d-1}} e^{-i\alpha\widetilde{\boldsymbol{\xi}}\cdot\widetilde{\mathbf{y}}}\mu(\widetilde{\mathbf{y}})\alpha^{(d-1)}d\widetilde{\mathbf{y}} \\ &= (2\pi)^{-(d-1)/2}\int_{\mathbf{R}^{d-1}} e^{-i\alpha\widetilde{\boldsymbol{\xi}}\cdot\widetilde{\mathbf{y}}}\mu(\widetilde{\mathbf{y}})\,d\widetilde{\mathbf{y}} \\ &= \mathcal{F}[\mu](\alpha\widetilde{\boldsymbol{\xi}}).\end{aligned}$$

□

Lemma 4.7. *From (4.17), we obtain a estimate*

$$|u|_{H^1(\Omega)} \le C\sqrt{1+k}\|w\|_{H^{1/2}(\partial\Omega)}$$

Proof. The $H^1(\Omega)$ semi norm is written as

$$|u|^2_{H^1(\Omega)} = \int_{\mathbf{R}^{d-1}}\int_0^\infty \left[\sum_{i=1}^{d-1}\left|\frac{\partial u(\widetilde{\mathbf{x}},x_d)}{\partial x_i}\right|^2 + \left|\frac{\partial u(\widetilde{\mathbf{x}},x_d)}{\partial x_d}\right|^2\right] dx_d\,d\widetilde{\mathbf{x}}. \tag{4.18}$$

Let $|\nabla_{\widetilde{\mathbf{x}}}u(\widetilde{\mathbf{x}},x_d)|^2 = \sum_{i=1}^{d-1}|\partial u(\widetilde{\mathbf{x}},x_d)/\partial x_i|^2$. In first term of the right side (4.18), by Plancherel's Theorem (2.1), Theorem 2.2, and Lemma 4.5,

$$\begin{aligned}\int_{\mathbf{R}^{d-1}}|\nabla_{\widetilde{\mathbf{x}}}u(\widetilde{\mathbf{x}},x_d)|^2\,d\widetilde{\mathbf{x}} &= \int_{\mathbf{R}^{d-1}}\left|\widetilde{\boldsymbol{\xi}}\right|^2\left|\mathcal{F}[u](\widetilde{\boldsymbol{\xi}},x_d)\right|^2 d\widetilde{\boldsymbol{\xi}} &(4.19)\\ &= B\int_{\mathbf{R}^{d-1}}\left|\widetilde{\boldsymbol{\xi}}\right|^2\left|\mathcal{F}[w](\widetilde{\boldsymbol{\xi}})\right|^2\left|\mathcal{F}[\mu_{\alpha(x_d)}](\widetilde{\boldsymbol{\xi}})\right|^2 d\widetilde{\boldsymbol{\xi}}, &(4.20)\end{aligned}$$

where $B = (2\pi)^{d-1}|1-kx_d|^2$. Note that

$$\frac{\partial\mathcal{F}[u]\left(\widetilde{\boldsymbol{\xi}},x_d\right)}{\partial x_d} = \mathcal{F}\left[\frac{\partial u}{\partial x_d}\right](\widetilde{\boldsymbol{\xi}},x_d). \tag{4.21}$$

In the second term of the right side (4.18), by Plancherel's Theorem and (4.21), we have

$$\int_{\mathbf{R}^{d-1}}\left|\frac{\partial u(\widetilde{\mathbf{x}},x_d)}{\partial x_d}\right|^2 d\widetilde{\mathbf{x}} = \int_{\mathbf{R}^{d-1}}\left|\frac{\partial\mathcal{F}[u]\left(\widetilde{\boldsymbol{\xi}},x_d\right)}{\partial x_d}\right|^2 d\widetilde{\boldsymbol{\xi}}. \tag{4.22}$$

Using Lemmas 4.5, 4.6 and chain rile, for $0 \le x_d \le 1/k$ we have

$$
\begin{aligned}
&\frac{\partial \mathcal{F}[u](\widetilde{\boldsymbol{\xi}}, x_d)}{\partial x_d} \\
&= (2\pi)^{\frac{d-1}{2}} \mathcal{F}[w]\left(\widetilde{\boldsymbol{\xi}}\right)\left[-k\mathcal{F}[\mu]\left(\alpha(x_d)\widetilde{\boldsymbol{\xi}}\right) + (1-kx_d)\frac{\partial \mathcal{F}[\mu_{\alpha(x_d)}](\widetilde{\boldsymbol{\xi}})}{\partial x_d}\right] \\
&= (2\pi)^{\frac{d-1}{2}} \mathcal{F}[w]\left(\widetilde{\boldsymbol{\xi}}\right)\left[-k\mathcal{F}[\mu]\left(\alpha(x_d)\widetilde{\boldsymbol{\xi}}\right) + (1-kx_d)\frac{\partial \mathcal{F}[\mu](\alpha(x_d)\widetilde{\boldsymbol{\xi}})}{\partial x_d}\right] \\
&= (2\pi)^{\frac{d-1}{2}} \mathcal{F}[w]\left(\widetilde{\boldsymbol{\xi}}\right)\left[-k\mathcal{F}[\mu]\left(\alpha(x_d)\widetilde{\boldsymbol{\xi}}\right) + (1-kx_d)\frac{\partial \alpha(x_d)}{\partial x_d}\left(\widetilde{\boldsymbol{\xi}}\right)^T \cdot \nabla\mathcal{F}[u](\alpha(x_d)\widetilde{\boldsymbol{\xi}})\right].
\end{aligned}
$$

Thus using (4.22), and applying Cauchy–Schwartz inequality and the fact that $(a-b)^2 \le 2(a^2+b^2)$ for a, $b \in \mathbf{R}$, we obtain

$$
\begin{aligned}
&\int_{\mathbf{R}^{d-1}} \left|\frac{\partial u(\widetilde{\mathbf{x}}, x_d)}{\partial x_d}\right|^2 d\widetilde{\mathbf{x}} \qquad (4.23)\\
&\le 2(2\pi)^{d-1}\left|\frac{d\alpha(x_d)}{dx_d}\right|^2 (1-kx_d)^2 \int_{\mathbf{R}^{d-1}} \left|\mathcal{F}[w]\left(\widetilde{\boldsymbol{\xi}}\right)\right|^2 \left|\widetilde{\boldsymbol{\xi}}\right|^2 \left|\nabla\mathcal{F}[\mu]\left(\alpha(x_d)\widetilde{\boldsymbol{\xi}}\right)\right|^2 d\widetilde{\boldsymbol{\xi}} \\
&\quad + 2(2\pi)^{d-1}k^2 \int_{\mathbf{R}^{d-1}} \left|\mathcal{F}[w]\left(\widetilde{\boldsymbol{\xi}}\right)\right|^2 \left|\nabla\mathcal{F}[\mu]\left(\alpha(x_d)\widetilde{\boldsymbol{\xi}}\right)\right|^2 d\widetilde{\boldsymbol{\xi}}.
\end{aligned}
$$

Note that since $0 \le x_d \le 1/k$, $|1-kx_d| \le 1$, the first term of right side will be

$$
\begin{aligned}
&2(2\pi)^{d-1}\left|\frac{d\alpha(x_d)}{dx_d}\right|^2 (1-kx_d)^2 \int_{\mathbf{R}^{d-1}} \left|\mathcal{F}[w]\left(\widetilde{\boldsymbol{\xi}}\right)\right|^2 \left|\widetilde{\boldsymbol{\xi}}\right|^2 \left|\nabla\mathcal{F}[\mu]\left(\alpha(x_d)\widetilde{\boldsymbol{\xi}}\right)\right|^2 d\widetilde{\boldsymbol{\xi}} \\
&\quad \le (2\pi)^{d-1} 2\left|\frac{d\alpha(x_d)}{dx_d}\right|^2 \int_{\mathbf{R}^{d-1}} \left|\mathcal{F}[w]\left(\widetilde{\boldsymbol{\xi}}\right)\right|^2 \left|\widetilde{\boldsymbol{\xi}}\right|^2 \left|\nabla\mathcal{F}[\mu]\left(\alpha(x_d)\widetilde{\boldsymbol{\xi}}\right)\right|^2 d\widetilde{\boldsymbol{\xi}}.
\end{aligned}
$$

Since $\mu \in C_0^\infty(\mathbf{R}^{d-1})$ has compact support, $\mathcal{F}[\mu](\widetilde{\boldsymbol{\xi}})$ goes to zero faster than any rational function of $\left|\widetilde{\boldsymbol{\xi}}\right|$. Also note that since $\nabla\mathcal{F}[\mu]\left(\widetilde{\boldsymbol{\xi}}\right) = i\widetilde{\mathbf{x}}\mu(\widetilde{\mathbf{x}})$, $\nabla\mathcal{F}[\mu](\widetilde{\boldsymbol{\xi}})$ decays faster than any rational function of $\left|\widetilde{\boldsymbol{\xi}}\right|$. So we can choose m to be a sufficiently large integer and choose a constant C so that $\left|\mathcal{F}[\mu]\left(\widetilde{\boldsymbol{\xi}}\right)\right|, \left|\nabla\mathcal{F}[\mu]\left(\widetilde{\boldsymbol{\xi}}\right)\right| \le C\left(1+\left|\widetilde{\boldsymbol{\xi}}\right|\right)^{-m}$.

Therefore from (4.20) and the first term of inequality (4.23),

$$
\begin{aligned}
&(2\pi)^{d-1}(1-kx_d)^2 \int_{\mathbf{R}^{d-1}} \left|\widetilde{\boldsymbol{\xi}}\right|^2 \left|\mathcal{F}[w](\widetilde{\boldsymbol{\xi}})\right| \left|\mathcal{F}[\mu_{\alpha(x_d)}](\widetilde{\boldsymbol{\xi}})\right|^2 d\widetilde{\boldsymbol{\xi}} \\
&+2(2\pi)^{d-1}\left|\frac{d\alpha(x_d)}{dx_d}\right|^2 \int_{\mathbf{R}^{d-1}} \left|\mathcal{F}[w]\left(\widetilde{\boldsymbol{\xi}}\right)\right| \left|\widetilde{\boldsymbol{\xi}}\right|^2 \left|\nabla\mathcal{F}[\mu]\left(\alpha(x_d)\widetilde{\boldsymbol{\xi}}\right)\right|^2 d\widetilde{\boldsymbol{\xi}} \\
&\leq (2\pi)^{d-1} C \left(1+2\left|\frac{d\alpha(x_d)}{dx_d}\right|^2\right) \int_{\mathbf{R}^{d-1}} \frac{\left|\mathcal{F}[w]\left(\widetilde{\boldsymbol{\xi}}\right)\right|^2 \left|\widetilde{\boldsymbol{\xi}}\right|^2}{\left(1+\alpha(x_d)^2\left|\widetilde{\boldsymbol{\xi}}\right|^2\right)^m} d\widetilde{\boldsymbol{\xi}}
\end{aligned}
$$

Then integrating the above inequality with respect to x_d,

$$
\begin{aligned}
&(2\pi)^{d-1}\int_0^{1/k}(1-kx_d)^2\left[\int_{\mathbf{R}^{d-1}} \left|\widetilde{\boldsymbol{\xi}}\right|^2 \left|\mathcal{F}[w](\widetilde{\boldsymbol{\xi}})\right| \left|\mathcal{F}[\mu_{\alpha(x_d)}](\widetilde{\boldsymbol{\xi}})\right|^2 d\widetilde{\boldsymbol{\xi}}\right] dx_d \\
&+2(2\pi)^{d-1}\int_0^{1/k}\left|\frac{d\alpha(x_d)}{dx_d}\right|\left[^2\int_{\mathbf{R}^{d-1}} \left|\mathcal{F}[w]\left(\widetilde{\boldsymbol{\xi}}\right)\right| \left|\widetilde{\boldsymbol{\xi}}\right|^2 \left|\nabla\mathcal{F}[\mu]\left(\alpha(x_d)\widetilde{\boldsymbol{\xi}}\right)\right|^2 d\widetilde{\boldsymbol{\xi}}\right] dx_d \\
&\leq C\int_{\mathbf{R}^{d-1}} \left|\widetilde{\boldsymbol{\xi}}\right|^2 \left|\mathcal{F}[w](\widetilde{\boldsymbol{\xi}})\right|^2 \left[\int_0^{1/k} \frac{1+2\left|\frac{d\alpha(x_d)}{dx_d}\right|^2}{\left(1+\alpha(x_d)^2\left|\widetilde{\boldsymbol{\xi}}\right|^2\right)^m} dx_d\right] d\widetilde{\boldsymbol{\xi}}.
\end{aligned}
$$

Suppose that $\alpha(x_d) = (x_d)^\eta$. If $\eta < 1$, $\frac{d\alpha(x_d)}{dx_d}$ is unbounded as $x_d \downarrow 0$. So let $\eta \geq 1$. Now taking substitution $s = x_d \left|\widetilde{\boldsymbol{\xi}}\right|^{1/\eta}$, we consider

$$
\begin{aligned}
\int_0^{1/k} \frac{dx_d}{\left(1+(x_d)^{2\eta}\left|\widetilde{\boldsymbol{\xi}}\right|^2\right)^m} &= |\widetilde{\boldsymbol{\xi}}|^{-1/\eta}\int_0^{\left|\widetilde{\boldsymbol{\xi}}\right|^{1/\eta}/k} \frac{ds}{(1+s^{2\eta})^m} \\
&\leq |\widetilde{\boldsymbol{\xi}}|^{-1/\eta}\int_0^{\infty} \frac{ds}{(1+s^{2\eta})^m}, \qquad (4.24)
\end{aligned}
$$

provided that $2\eta m > 1$. In (4.24), we can take the natural choice $\eta = 1$ so that the integrand is bounded. So since m is sufficiently large, we have

$$
\int_0^\infty (1+s^2)^{-m} ds \leq \int_0^\infty (1+s^2)^{-1} ds = \frac{\pi}{2}.
$$

Thus taking $\eta = 1$ so that $\alpha(x_d) = x_d$, we have

$$
\begin{aligned}
&(2\pi)^{d-1}\int_0^{1/k}(1-kx_d)^2\left[\int_{\mathbf{R}^{d-1}}\left|\widetilde{\boldsymbol{\xi}}\right|^2\left|\mathcal{F}[w](\widetilde{\boldsymbol{\xi}})\right|\left|\mathcal{F}[\mu_{\alpha(x_d)}](\widetilde{\boldsymbol{\xi}})\right|^2 d\widetilde{\boldsymbol{\xi}}\right]dx_d\\
&+2(2\pi)^{d-1}\int_0^{1/k}\left|\frac{d\alpha(x_d)}{dx_d}\right|\left[{}^2\int_{\mathbf{R}^{d-1}}\left|\mathcal{F}[w]\left(\widetilde{\boldsymbol{\xi}}\right)\right|\left|\widetilde{\boldsymbol{\xi}}\right|^2\left|\nabla\mathcal{F}[\mu]\left(\alpha(x_d)\widetilde{\boldsymbol{\xi}}\right)\right|^2 d\widetilde{\boldsymbol{\xi}}\right]dx_d\\
&\le C\int_{\mathbf{R}^{d-1}}\left|\widetilde{\boldsymbol{\xi}}\right|^{2-1}\left|\mathcal{F}[w](\widetilde{\boldsymbol{\xi}})\right|^2 d\widetilde{\boldsymbol{\xi}}\\
&\le C\int_{\mathbf{R}^{d-1}}\left|\mathcal{F}[w]\left(\widetilde{\boldsymbol{\xi}}\right)\right|^2\left(1+\left|\widetilde{\boldsymbol{\xi}}\right|^2\right)^{1/2} d\widetilde{\boldsymbol{\xi}}\\
&= C\|w\|^2_{H^{1/2}(\mathbf{R}^{d-1})}.
\end{aligned}
\tag{4.25}
$$

Finally in the second term of inequality (4.23), using Lemma 4.6 and the fact that $\left|\mathcal{F}[\mu]\left(\widetilde{\boldsymbol{\xi}}\right)\right| \le \|\mu\|_{L^1(\mathbf{R}^{\mathbf{d}-1})} = 1$, we have

$$
\begin{aligned}
&(2\pi)^{d-1}2k^2\int_{\mathbf{R}^{d-1}}\left|\mathcal{F}[w]\left(\widetilde{\boldsymbol{\xi}}\right)\right|^2\left|\mathcal{F}[\mu]\left(\alpha(x_d)\widetilde{\boldsymbol{\xi}}\right)\right|^2 d\widetilde{\boldsymbol{\xi}}\,dx_d\\
&\quad\le 2k^2\int_{\mathbf{R}^{d-1}}\left|\mathcal{F}[w]\left(\widetilde{\boldsymbol{\xi}}\right)\right|^2 d\widetilde{\boldsymbol{\xi}}\,dx_d\\
&\quad\le 2k^2\int_{\mathbf{R}^{d-1}}\left|\mathcal{F}[w]\left(\widetilde{\boldsymbol{\xi}}\right)\right|^2\left(1+\left|\widetilde{\boldsymbol{\xi}}\right|^2\right)^{1/2} d\widetilde{\boldsymbol{\xi}}\,dx_d\\
&\quad= 2k^2\|w\|^2_{H^{1/2}(\mathbf{R}^{d-1}).}
\end{aligned}
\tag{4.26}
$$

Therefore combining (4.25) with (4.26), we have

$$
|u|^2_{H^1(\mathbf{R}^d_+)} \le C\|w\|^2_{H^{1/2}(\mathbf{R}^{d-1})} + 2k\|w\|^2_{H^{1/2}(\mathbf{R}^{d-1})}.
$$

Taking $C_1 = \max\{C, 2\}$, we obtain

$$
|u|^2_{H^1(\mathbf{R}^d_+)} \le C_1(1+k)\|w\|^2_{H^{1/2}(\mathbf{R}^{d-1})},
$$

as required. □

Note that C used in the Lemma 4.7 is independent of $w \in H^{1/2}(\mathbf{R}^{d-1})$ and k.

Lemma 4.8. *For all $w \in H^{1/2}(\Omega)$,*

$$\|ext_k(w)\|_{L^2(\Omega)} \leq C\frac{1}{\sqrt{k}}\|w\|_{H^{1/2}(\partial\Omega)}.$$

Proof. There is a $w \in H^{1/2}(\partial\Omega)$ such that $\text{ext}_k(w) = u$ for any $u \in H^1(\Omega) \subset L^2(\Omega)$. Using Plancherel's Theorem and Lemma 4.5,

$$\begin{aligned}
\|u\|^2_{L^2(\Omega)} &= \|\mathcal{F}[u]\|^2_{L^2(\Omega)} \\
&= (2\pi)^{(d-1)}\int_0^{1/k}(1-kx_d)^2\left[\int_{\mathbf{R}^{d-1}}\left|\mathcal{F}[w]\left(\widetilde{\boldsymbol{\xi}}\right)\right|^2\left|\mathcal{F}[\mu_{\alpha(x_d)}]\left(\widetilde{\boldsymbol{\xi}}\right)\right|^2 d\widetilde{\boldsymbol{\xi}}\right]dx_d \\
&= C\int_0^{1/k}\int_{\mathbf{R}^{d-1}}\left|\mathcal{F}[w]\left(\widetilde{\boldsymbol{\xi}}\right)\right|^2 d\widetilde{\boldsymbol{\xi}}dx_d \\
&\leq C\int_0^{1/k}\int_{\mathbf{R}^{d-1}}\left|\mathcal{F}[w]\left(\widetilde{\boldsymbol{\xi}}\right)\right|^2\left(1+\left|\widetilde{\boldsymbol{\xi}}\right|^2\right)^{1/2} d\widetilde{\boldsymbol{\xi}}dx_d \\
&= C\frac{1}{k}\|w\|^2_{H^{1/2}(\partial\Omega)}.
\end{aligned}$$

Therefore the result follows. □

Lemma 4.9. *From Lemmas 4.7 and 4.8, we obtain*

$$\|ext_k(w)\|_{H^1(\Omega)} \leq C\sqrt{k}\|w\|_{H^{1/2}(\partial\Omega)}.$$

Proof. By Lemmas 4.7 and 4.8,

$$\begin{aligned}
\|\text{ext}_k(w)\|^2_{H^{1/2}(\partial\Omega)} &= |u|^2_{H^1(\Omega)} + \|u\|^2_{L^2(\Omega)} \\
&\leq C(1+k)\|w\|^2_{H^{1/2}(\partial\Omega)} + C\frac{1}{k}\|w\|^2_{H^{1/2}(\partial\Omega)} \\
&\leq Ck\|w\|^2_{H^{1/2}(\partial\Omega)},
\end{aligned}$$

as required. □

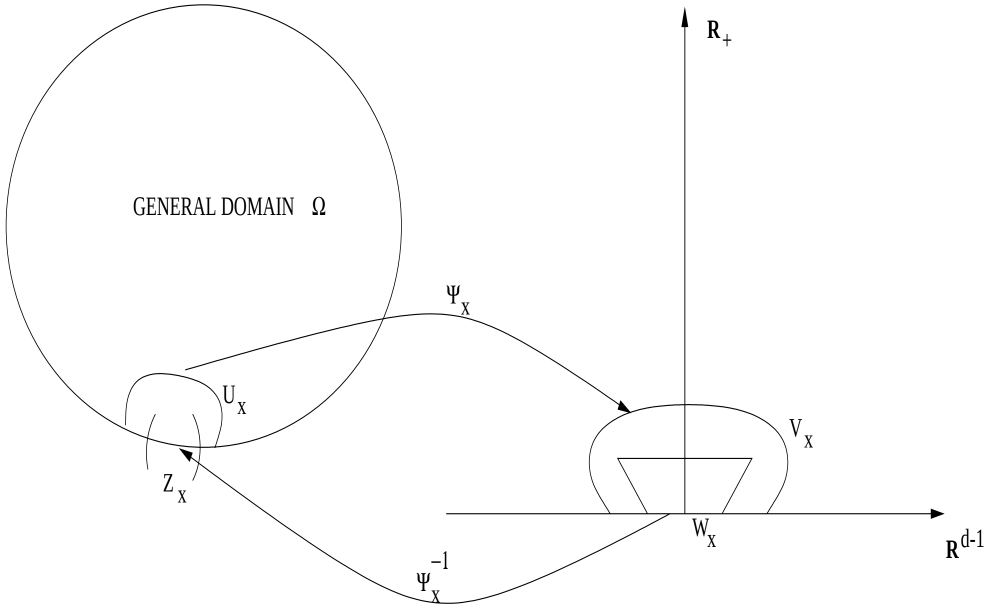

Figure 4.1: The diffeomorphism $\Psi_x : U_x \to V_x$.

4.2.4 General domain Ω

Consider a bounded domain $\Omega \subset \mathbf{R}^d$ with smooth boundary. In fact, it is enough that the boundary is C^1. For any $x \in \partial\Omega$, we can have diffeomorphism $\Psi_x : U_x \to V_x$, where U_x is neighborhood of x and V_x is neighborhood of 0 in $\mathbf{R}^d_+$. See Figure 4.1 for the illustration. In V_x, we find a set $W_x \subset \mathbf{R}^{d-1} \times \{0\}$ which is containing origin and relatively open in $\mathbf{R}^{d-1} \times \{0\}$. Consider the closure of $\{(x, x_d) \mid 0 \le d(\widetilde{x}, W_x) < x_d,\ 0 < x_d < 1/k\}$ which is subset of V_x. Then for sufficiently small $1/k$, we can find such a set W_x. Let $\psi_x^{-1}(W_x) = Z_x \subset \partial\Omega$ so that Z_x contains x and is relatively open in $\partial\Omega$. Thus Z_x is open covering of $\partial\Omega$. Since $\partial\Omega$ is compact, there is a finite subcovering $\{Z_{x_1}, Z_{x_2}, \cdots, Z_{x_p}\}$. Use the partition of unity $\{\phi_1, \phi_2, \cdots, \phi_p\}$ subordinate to this finite covering. Choose a sufficiently large number $k > 0$ such that the closure of $\{(\widetilde{x}, x_d) \mid 0 \le d(\widetilde{x}, W_{x_j}) < x_d < 1/k\} \subset V_{x_j}$, for $1 \le j \le p$. Now for $w \in H^{1/2}(\partial\Omega)$, define

$$\text{ext}_{k,j}(w) = \text{ext}_k(\phi_j w \circ \psi_{x_j}^{-1}) \circ \psi_{x_j}.$$

Note that $\phi_j w \in H^{1/2}(\partial\Omega)$. Then by partition of unity, we can set

$$\widetilde{\text{ext}_k}(w) = \sum_{j=1}^{p} \text{ext}_{k,j}(w).$$

Lemma 4.10. *$\widetilde{ext_k}$ is right inverse of trace operator $tr : H^1(\Omega) \to H^{1/2}(\partial\Omega)$.*

Proof. For any $w \in H^{1/2}(\partial\Omega)$, we claim that $\text{tr} \circ \widetilde{\text{ext}_k}(w) = w$.

$$\begin{aligned}
\text{tr} \circ \widetilde{\text{ext}_k}(w) &= \text{tr}(\widetilde{\text{ext}_k}(w)) \\
&= \sum_{j=1}^{p} \text{tr}(\text{ext}_{k,j}(w)) \\
&= \sum_{j=1}^{p} \text{tr}(\text{ext}_k(\phi_j w \circ \psi_{x_j}^{-1}) \circ \psi_{x_j}) \\
&= \sum_{j=1}^{p} (\phi_j w \circ \psi_{x_j}^{-1}) \circ \psi_{x_j} \\
&= \sum_{j=1}^{p} \phi_j w = w.
\end{aligned}$$

Therefore $\text{tr} \circ \widetilde{\text{ext}_k}(w) = I_{H^{1/2}(\partial\Omega)}$. □

Lemma 4.11. *For general bounded domain $\Omega \subset R^d$ with Lipschitz boundary, we have*

$$\begin{aligned}
\|\widetilde{ext_k}(w)\|_{H^1(\Omega)} &\leq C\sqrt{k}\|w\|_{H^{1/2}(\partial\Omega)} \text{ and} && (4.27) \\
\|\widetilde{ext_k}(w)\|_{L^2(\Omega)} &\leq C\frac{1}{\sqrt{k}}\|w\|_{H^{1/2}(\partial\Omega)} \quad \text{for all } w \in H^{1/2}(\partial\Omega). && (4.28)
\end{aligned}$$

Proof. For general bounded domain $\Omega \subset R^d$,

$$\begin{aligned}
\|\widetilde{\text{ext}}_k(w)\|_{H^1(\Omega)} &= \|\sum_{j=1}^{p} \text{ext}_{k,j}(w)\|_{H^1(\Omega)} \\
&\leq \sum_{j=1}^{p} \|\text{ext}_{k,j}(w)\|_{H^1(\Omega)} \\
&= \sum_{j=1}^{p} \|\text{ext}_k((\phi_j w \circ \psi_{x_j}^{-1}) \circ \psi_{x_j})\|_{H^1(\Omega).}
\end{aligned}$$

Then for finite open covering $\{Z_{x_j} | 1 \leq j \leq p\}$, take $U_{x_j} \supset Z_{x_j}$ such that $\Omega = \bigcup U_{x_j}$, and let ζ_j be associate partition of unity. Thus for $\zeta_j u$ corresponding to $\phi_j w$, we have

$$\begin{aligned}
\|\widetilde{\text{ext}}_k(w)\|_{H^1(\Omega)} &\leq \sum_{j=1}^{p} \|\zeta_j \cdot u\|_{H^1(\Omega)} \\
&\leq \sum_{j=1}^{p} \|\zeta_j u\|_{H^1(\mathbf{R}_+^d)} \\
&= \sum_{j=1}^{p} C\sqrt{k}\|\phi_j w\|_{H^{1/2}(\mathbf{R}^{d-1})} \\
&= \sum_{j=1}^{p} C\sqrt{k}\|w\|_{H^{1/2}(Z_{x_j})} \\
&= \sum_{j=1}^{p} C\sqrt{k}\|\phi_j w\|_{H^{1/2}(\partial\Omega)} \\
&\leq C\sqrt{k}\|\phi_j w\|_{H^{1/2}(\partial\Omega).}
\end{aligned}$$

Therefore

$$\|\widetilde{\text{ext}}_k(w)\|_{H^1(\Omega)} \leq C\sqrt{k}\|w\|_{H^{1/2}(\partial\Omega)}.$$

Similarly, we have

$$\|\widetilde{\text{ext}}_k(w)\|_{L^2(\Omega)} \leq C\frac{1}{\sqrt{k}}\|w\|_{H^{1/2}(\partial\Omega)}.$$

□

Now using the estimates (4.27), (4.28), we derive estimate of the contact force N^l in $H^{-1/2}(\partial\Omega)$ at each time t_l.

Remark 4.12. *We have dealt with the trace operator tr and extension operators ext_k on scalar functions. We can extend these on vector functions; we use the notation of the trace operator as* $\mathbf{tr} : \mathbf{H}^1(\Omega) \to \mathbf{H}^{1/2}(\partial\Omega)$ *and extension operators as* $\widetilde{\mathbf{ext}}_k(\mathbf{w}) : \mathbf{H}^{1/2}(\partial\Omega) \to \mathbf{H}^1(\Omega)$. *Indeed, for vector valued functions* $\mathbf{w} \in \mathbf{H}^{1/2}(\partial\Omega)$ *the trace theorem has to be replaced by the decomposed trace theorem. See [32]. However, notice that we do not consider the tangential components due to frictionless contact conditions.*

Lemma 4.13. *For general bounded domain $\Omega \subset \mathbf{R}^d$ with Lipschitzian domain, we have*

$$\|N^l\|_{\mathbf{H}^{-1/2}(\partial\Omega)} = O\left(\frac{1}{\sqrt{h}}\right), \quad \textit{as } h \to 0. \tag{4.29}$$

Proof. We apply the extension operator $\widetilde{\mathbf{ext}}_k$ described in Remark 4.12. There is a $\mathbf{w} \in \mathbf{H}^{1/2}(\partial\Omega)$ such that $\widetilde{\mathbf{ext}}_k(\mathbf{w}) = \boldsymbol{\omega}$ for $\boldsymbol{\omega} \in \mathbf{H}^1(\Omega)$. Then take the dot product of $\boldsymbol{\omega}$ with (4.15). Then

$$\frac{h^2}{2\rho}\int_\Omega \left(\nabla \cdot \boldsymbol{\sigma}[\mathbf{u}^{l+1}]\right) \cdot \boldsymbol{\omega}\, dx = \int_\Omega (\boldsymbol{\Phi}^l - \mathbf{u}^{l+1}) \cdot \boldsymbol{\omega}\, dx \quad \text{in } \Omega, \tag{4.30}$$

where $\boldsymbol{\Phi}^l = (h^2/2\rho)\mathbf{f} + h\mathbf{v}^l + \mathbf{u}^l$. Recall that we used the implicit method for the contact condition. By integration by parts on the left side of (4.30),

$$\frac{h^2}{2\rho}\left[\int_{\partial\Omega} \mathbf{n}^T \cdot \boldsymbol{\sigma}[\mathbf{u}^{l+1}] \cdot \boldsymbol{\omega}\, ds - \int_\Omega \boldsymbol{\sigma}[\mathbf{u}^{l+1}] : \nabla\boldsymbol{\omega}\, dx\right] = \int_\Omega (\boldsymbol{\Phi}^l - \mathbf{u}^{l+1}) \cdot \boldsymbol{\omega}\, dx.$$

Thus we have

$$\begin{aligned}\frac{h^2}{2\rho}\int_{\partial\Omega} N^l \mathbf{n}\cdot\mathbf{w}\,ds &= \frac{h^2}{2\rho}\int_\Omega \boldsymbol{\sigma}[\mathbf{u}^{l+1}] : \nabla\boldsymbol{\omega}\,dx + \int_\Omega (\boldsymbol{\Phi}^l - \mathbf{u}^{l+1})\cdot\boldsymbol{\omega}\,dx \\ &= \frac{h^2}{2\rho}\int_\Omega \boldsymbol{\sigma}[\mathbf{u}^{l+1}] : \nabla\boldsymbol{\omega}\,dx + \frac{h^2}{2\rho}\int_\Omega \mathbf{f}\cdot\boldsymbol{\omega}\,dx \\ &\quad + h\int_\Omega \mathbf{v}^l\cdot\boldsymbol{\omega}\,dx - \int_\Omega (\mathbf{u}^{l+1} - \mathbf{u}^l)\cdot\boldsymbol{\omega}\,dx.\end{aligned}$$

So using (3.7),

$$\begin{aligned}\int_{\partial\Omega} N^l \mathbf{n}\cdot\mathbf{w}\,ds &= \int_\Omega \boldsymbol{\sigma}[\mathbf{u}^{l+1}] : \nabla\boldsymbol{\omega}\,dx + \int_\Omega \mathbf{f}\cdot\boldsymbol{\omega}\,dx \\ &\quad + \frac{2\rho}{h}\int_\Omega \mathbf{v}^l\cdot\boldsymbol{\omega}\,dx - \frac{\rho}{h}\int_\Omega (\mathbf{v}^{l+1} + \mathbf{v}^l)\cdot\boldsymbol{\omega}\,dx. \\ &= \int_\Omega \boldsymbol{\sigma}[\mathbf{u}^{l+1}] : \nabla\boldsymbol{\omega}\,dx + \int_\Omega \mathbf{f}\cdot\boldsymbol{\omega}\,dx \\ &\quad + \frac{\rho}{h}\int_\Omega \left(\mathbf{v}^l - \mathbf{v}^{l+1}\right)\cdot\boldsymbol{\omega}\,dx.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\left|\int_{\partial\Omega} N^l \mathbf{n}\cdot\mathbf{w}\,ds\right| &\leq \left|\int_\Omega \boldsymbol{\sigma}[\mathbf{u}^{l+1}] : \nabla\boldsymbol{\omega}\,dx\right| + \left|\int_\Omega \mathbf{f}\cdot\boldsymbol{\omega}\,dx\right| \\ &\quad + \frac{\rho}{h}\left|\int_\Omega \left(\mathbf{v}^l - \mathbf{v}^{l+1}\right)\cdot\boldsymbol{\omega}\,dx\right|.\end{aligned}$$

Since $\boldsymbol{\sigma}[\mathbf{u}^{l+1}] : \nabla\boldsymbol{\omega} = E_{ijkl}u_{k,l}^{l+1}\omega_{i,j}$, we obtain

$$\begin{aligned}\left|\int_{\partial\Omega} N^l \mathbf{n}\cdot\mathbf{w}\,ds\right| &\leq C\|\mathbf{u}^{l+1}\|_{\mathbf{H}^1(\Omega)}\|\widetilde{\mathbf{ext}_k}(\mathbf{w})\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}\|\widetilde{\mathbf{ext}_k}(\mathbf{w})\|_{\mathbf{L}^2(\Omega)} \\ &\quad + \frac{\rho}{h}\left(\|\mathbf{v}^l\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{v}^{l+1}\|_{\mathbf{L}^2(\Omega)}\right)\|\widetilde{\mathbf{ext}_k}(\mathbf{w})\|_{\mathbf{L}^2(\Omega)}.\end{aligned}$$

By Lemma 4.11,

$$\begin{aligned}\left|\int_{\partial\Omega} N^l \mathbf{n}\cdot\mathbf{w}\,ds\right| &\leq Ck^{1/2}\|\mathbf{u}^{l+1}\|_{\mathbf{H}^1(\Omega)}\|\mathbf{w}\|_{\mathbf{H}^{1/2}(\partial\Omega)} + Ck^{-1/2}\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}\|\mathbf{w}\|_{\mathbf{H}^{1/2}(\partial\Omega)} \\ &\quad + \left(C\frac{k^{-1/2}}{h}\|\mathbf{v}^l\|_{\mathbf{L}^2(\Omega)} + C\frac{k^{-1/2}}{h}\|\mathbf{v}^{l+1}\|_{\mathbf{L}^2(\Omega)}\right)\|\mathbf{w}\|_{\mathbf{H}^{1/2}(\Omega)}.\end{aligned}$$

Therefore,

$$\sup_{\|\mathbf{w}\|_{\mathbf{H}^{1/2}(\partial\Omega)}\leq 1}\left|\int_{\partial\Omega} N^l \mathbf{n}\cdot\mathbf{w}\,ds\right| \;\leq\; Ck^{1/2}\|\mathbf{u}^{l+1}\|_{\mathbf{H}^1(\Omega)} + Ck^{-1/2}\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + C\frac{k^{-1/2}}{h}\|\mathbf{v}^l\|_{\mathbf{L}^2(\Omega)} + C\frac{k^{-1/2}}{h}\|\mathbf{v}^{l+1}\|_{\mathbf{L}^2(\Omega)}.$$

Put $k = 1/h$. Then we have

$$\|N^l\|_{\mathbf{H}^{-1/2}(\partial\Omega)} \;\leq\; C\frac{1}{\sqrt{h}}\|\mathbf{u}^{l+1}\|_{\mathbf{H}^1(\Omega)} + C\sqrt{h}\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + C\frac{1}{\sqrt{h}}\|\mathbf{v}^l\|_{\mathbf{L}^2(\Omega)} + C\frac{1}{\sqrt{h}}\|\mathbf{v}^{l+1}\|_{\mathbf{L}^2(\Omega)}.$$

Thus by Lemma 4.3, we have

$$\|N^l\|_{\mathbf{H}^{-1/2}(\partial\Omega)} = O\left(\frac{1}{\sqrt{h}}\right), \quad \text{as } h \to 0.$$

□

CHAPTER 5
EULER–BERNOULLI BEAM IN DYNAMIC CONTACT : PENALTY METHOD

5.1 Formulation of Euler–Bernoulli beam with Signorini's contact condition

We recall Section 1.3. Then the function $f(x,t)$ is the body force applied to the rod; and time t is in between initial time and some fixed time T. We will assume that ρ, A, and E and I are constants. Note that we use the right end $x = l$ instead of $x = L$ in this Chapter.

The Euler–Bernoulli equation with Signorini's contact condition comes from the following physical situation illustrated in Figure 5.1.

If we impose frictionless Signorini's contact conditions along the length of the rod, we represent the equation of motion

$$\rho A \frac{\partial^2 u}{\partial t^2} = -EI \frac{\partial^4 u}{\partial x^4} + f(x,t) + N(x,t), \tag{5.1}$$

where the magnitude of the vertical contact forces (pressures), $N(x,t)$ satisfies the linear complementary condition

$$0 \leq N(x,t) \quad \perp \quad u(x,t) + g(x) \geq 0. \tag{5.2}$$

Note that $g(x)$, called the *gap function*, displays a measure of the "the initial normalized gap" between the rod and the rigid foundation. We assume that applied body force $f(x,t) = f(x)$. So body force f and gap function g do not depend on time t. We also assume that the gap function $g(x) \geq 0$. Note that we can scale t and x to

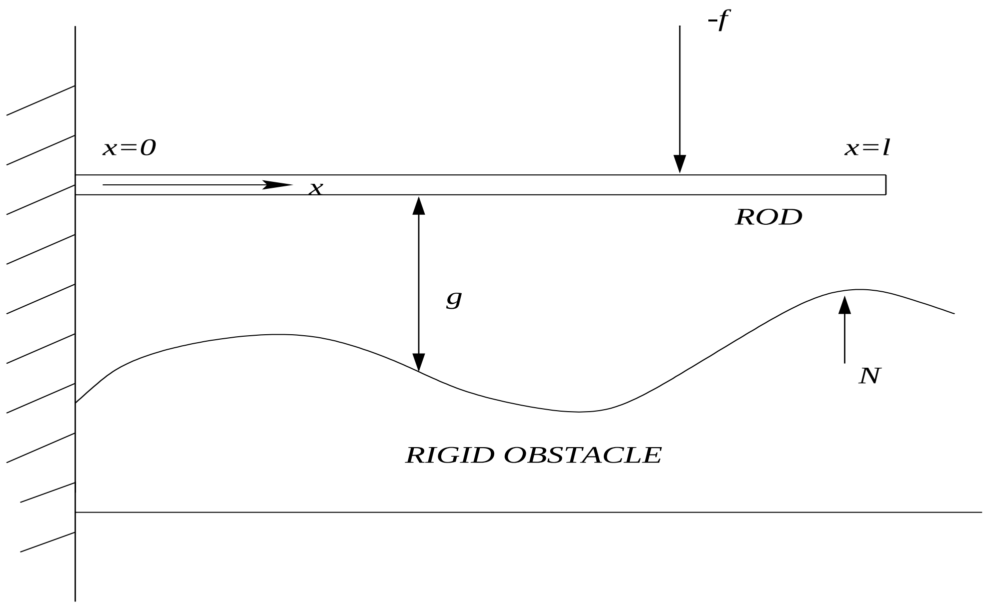

Figure 5.1: Euler–Bernoulli beam with frictionless contact.

get $\rho A = 1$ and $EI = 1$. (However, we cannot simultaneously scale $l = 1$.) From the physical point of view, LCP condition can be interpreted as the same way as contact conditions of elastic body.

Thus we are lead to consider solving the following PDE:

$$u_{tt} = -u_{xxxx} + f(x) + N(x,t) \quad \text{in } (0,l) \times (0,T], \tag{5.3}$$

$$0 \le N(x,t) \perp u + g(x) \ge 0 \quad \text{in } (0,l) \times (0,T], \tag{5.4}$$

$$u(0,t) = u_x(0,t) = 0 \quad \text{on } (0,T], \tag{5.5}$$

$$u_{xx}(l,t) = u_{xxx}(l,t) = 0 \quad \text{on } (0,T], \tag{5.6}$$

$$u(x,0) = u^0(x) \quad \text{in } (0,l), \tag{5.7}$$

$$u_t(x,0) = v^0(x) \quad \text{in } (0,l). \tag{5.8}$$

We assume that $f \in L^2(0,l)$, $u^0 \in H^2_{cf}(0,l)$, $v^0 \in L^2(0,l)$. Indeed this implies that we can assume that the initial energy is finite. We also assume that $g \in C^\infty[0,l]$, and $g(0) > 0$. Equation (5.5) gives the essential boundary conditions for the clamped end at $x = 0$, while (5.6) gives the natural boundary conditions for a free end at $x = l$. Note that the last two equations (5.7, 5.8) are the initial conditions. The solution u that we seek is in the space $L^\infty(0,T;H^2_{cf}(0,l)) \cap W^{1,\infty}(0,T;L^2(0,l)) \cap C([0,l]\times[0,T])$ where $H^2_{cf}(0,l)$ is the subset of $H^2(0,l)$ which satisfies the clamped end conditions at $x = 0$ ($u(0) = 0$, $u'(0) = 0$) with the same norm. Note that the subscript "c" denotes "clamped" and "f" denotes "free". Let $H^\alpha_{cf}(0,l)$ denote the subspace of $H^\alpha(0,l)$ that is the closure in H^α of the set of all $C^\infty[0,l]$ functions satisfying the clamped end conditions at $x = 0$. The normal contact force $N(x,t)$ is a Borel measure on $[0,l]\times[0,T]$.

Note that to interpret (5.4), we require that N is a non-negative measure on $[0,l]\times[0,T]$, $u(x,t)+g(x) \geq 0$ for all $(x,t) \in [0,l]\times[0,T]$, and that

$$\int_0^T \int_0^l N(x,t)\,[u(x,t)+g(x)]\,dx\,dt = 0. \tag{5.9}$$

We will set up an approximate penalty formulation with a penalty parameter $\epsilon > 0$. Then we will show that the approximate solution u_ϵ exists for a fixed penalty parameter ϵ, and that this solution conserves energy (including the energy associated with the penalty). Furthermore, the integral of the normal contact force over space and time $\int_0^T \int_0^l N_\epsilon(x,t)\,dx\,dt$ will be shown to be uniformly bounded as $\epsilon \downarrow 0$, and so there is a weakly* convergent subsequence in the space of measures. However, to establish convergence of a subsequence we need still more regularity; we will prove a uniform bound on u_ϵ in $C^p(0,T;H^{1/2+\sigma}(0,l))$ for suitable values of p, $\sigma > 0$.

5.2 Existence theory

The existence theory that we develop here is based on eigenfunction decompositions for the homogeneous Euler–Bernoulli equations which are studied in Subsection 5.2.1. A penalty approximation is then described in Subsection 5.2.2 where it is shown that u_ϵ exist for penalty approximation via fundamental solutions. Energy conservation is shown for the penalty approximation in Subsection 5.2.3, which is used to obtain uniform bounds on the H^2 norm of u_ϵ and the L^2 norm of $\dot{u}_\epsilon$ in space. In Subsection 5.2.4, bounds are obtained for the integral of the normal contact force over both space and time; this is used to uniformly bound the normal contact force N_ϵ in the space of measures. In this Section, it is shown that any limit of u_ϵ must satisfy the constraint $u + g \geq 0$ mentioned above. To complete the proof, we need stronger regularity that can be obtained from energy bounds. This is done in Subsection 5.2.5. Finally, the proof of existence is completed.

5.2.1 Decomposition into eigenfunctions

Since the fourth order differential operator $K = \partial^4/\partial x^4$ is an elliptic self-adjoint operator with our boundary conditions, we have a sequence of real eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$ and $\lim_{i\to\infty} \lambda_i = \infty$, and the eigenfunctions ϕ_i are orthonormal basis in $L^2(0, l)$ with $\partial^4\phi_i/\partial x^4 = \lambda_i\phi_i$. Then for the penalized PDE system, we can write

$$u_\epsilon(x, t) = \sum_{i=1}^{\infty} u_\epsilon^i(t)\phi_i(x). \tag{5.10}$$

Before we solve the PDE (5.25)-(5.29), we first will need to consider the fundamental solution w for Linear operator $\partial^2/\partial t^2 + K$, where K is the above operator. We will solve the equation

$$w_{tt} = -w_{xxxx} + \delta(t) \cdot \delta(x - x^*) \quad \text{for a fixed point } x^* \in (0, l) \tag{5.11}$$

with the initial conditions ($w(x, 0) = 0$, $w_t(x, 0^+) = \delta(x - x^*)$) and the the same boundary conditions as (5.26) and (5.27): $w(0, t) = w_x(0, t) = 0$ and $w_{xx}(l, t) = w_{xxx}(l, t) = 0$. The solution $w(x, t)$ can be solved by means of the eigenfunctions ϕ_i. Thus suppose that

$$\begin{aligned} \phi_i'''' &= \lambda_i \phi_i, & (5.12) \\ \phi_i(0) &= \phi_i'(0) = 0, & (5.13) \\ \phi_i''(l) &= \phi_i'''(l) = 0. & (5.14) \end{aligned}$$

Then we can write

$$w(x, t) = \sum_{i=1}^{\infty} w_i(t) \phi_i(x). \tag{5.15}$$

Since for $t > 0$, $w_{tt}(x, t) = -w_{xxxx}(x, t)$ from (5.11), using (5.12) and (5.15)

$$\sum_{i=1}^{\infty} (w_i)_{tt}(t) \phi_i(x) = -\sum_{i=1}^{\infty} w_i(t) (\phi_i)_{xxxx}(x) = -\sum_{i=1}^{\infty} \lambda_i w_i(t) \phi_i(x).$$

Thus we obtain

$$(w_i)_{tt}(t) = -\lambda_i w_i(t) \quad \text{for } t > 0. \tag{5.16}$$

We can also extend $w(x, t; x^*) = 0$ for $t < 0$.

Lemma 5.1. *The fundamental solution of equation (5.11) can be represented in terms of the eigenfunctions as*

$$w(x,t;x^*) = \sum_{i=1}^{\infty} \frac{\sin(\lambda_i^{1/2} t)}{\lambda_i^{1/2}} \phi_i(x)\phi_i(x^*) \quad \textit{for the fixed point } x^* \in (0,l).$$

Proof. From the ordinary differential equation (ODE) (5.16), we have

$$w_i(t) = A_i \sin(\lambda_i^{1/2} t) + B_i \cos(\lambda_i^{1/2} t).$$

By the initial condition $w(x,0) = 0$, $B_i = 0$ for all $i \geq 1$, and thus $w_i(t) = A_i \sin \lambda_i^{1/2} t$. From (5.15),

$$w_t(x,t) = \sum_{i=1}^{\infty} A_i \lambda_i^{1/2} \cos(\lambda_i^{1/2} t)\phi_i(x) \quad \text{for } t > 0.$$

Applying the initial condition $w_t(x,0^+) = \delta(x - x^*)$,

$$w_t(x,0^+) = \sum_{i=1}^{\infty} A_i \lambda_i^{1/2} \phi_i(x) = \delta(x - x^*). \tag{5.17}$$

Multiplying by ϕ_j for each $j \geq 1$ and taking a integral over $(0,l)$ on the both of last two equations of (5.17),

$$\int_0^l \sum_{i=1}^{\infty} A_i \lambda_i^{1/2} \phi_i(x)\phi_j(x)dx = \int_0^l \delta(x - x^*)\phi_j(x)dx.$$

Thus we find $A_j = \lambda_j^{-1/2} \phi_j(x^*)$ for each $j \geq 1$. Therefore from (5.15), we have

$$w(x,t;x^*) = \sum_{i=1}^{\infty} \frac{\sin(\lambda_i^{1/2} t)}{\lambda_i^{1/2}} \phi_i(x)\phi_i(x^*) \quad \text{for } t > 0.$$

□

Lemma 5.2. *Under the assumption (5.12)-(5.14), we have a constant M such that*

$$\max_{0 \leq x \leq l} |\phi_i(x)| \leq M < \infty \quad \textit{for each } i \geq 1.$$

Proof. From ODE (5.12), we have the solution

$$\phi_i(x) = A_i e^{\lambda_i^{1/4}x} + B_i e^{-\lambda_i^{1/4}x} + C_i \sin(\lambda_i^{1/4}x) + D_i \cos(\lambda_i^{1/4}x) \quad \text{for each } i \geq 1.$$

Using the clamped boundary conditions (5.13), we have better form of the solution

$$\phi_i(x) = -D_i[\cosh(\lambda_i^{1/4}x) - \cos(\lambda_i^{1/4}x)] - C_i[\sinh(\lambda_i^{1/4}x) - \sin(\lambda_i^{1/4}x)]. \tag{5.18}$$

Using the boundary condition (5.14), we have a homogeneous linear system

$$\begin{bmatrix} \cosh(\lambda_i^{1/4}l) + \cos(\lambda_i^{1/4}l) & \sinh(\lambda_i^{1/4}l) + \sin(\lambda_i^{1/4}l) \\ \sinh(\lambda_i^{1/4}l) - \sin(\lambda_i^{1/4}l) & \cosh(\lambda_i^{1/4}l) + \cos(\lambda_i^{1/4}l) \end{bmatrix} \begin{bmatrix} D_i \\ C_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{5.19}$$

In order to obtain no trivial solutions $C_i,\ D_i$, the determinant of system (5.19) has to be zero:

$$[\cosh(\lambda_i^{1/4}l) + \cos(\lambda_i^{1/4}l)]^2 - [\sinh^2(\lambda_i^{1/4}l) - \sin^2(\lambda_i^{1/4}l)] = 0.$$

That is,

$$\cosh^2(\lambda_i^{1/4}l) + 2\cosh(\lambda_i^{1/4}l)\cos(\lambda_i^{1/4}l) + \cos^2(\lambda_i^{1/4}l) - \sinh^2(\lambda_i^{1/4}l) + \sin^2(\lambda_i^{1/4}l) = 0.$$

Using the well-known facts that $\cosh^2 z - \sinh^2 z = 1$ and $\cos^2 z + \sin^2 z = 1$ we get $2 + 2\cosh(\lambda_i^{1/4}l)\cos(\lambda_i^{1/4}l) = 0$. So the eigenvalues λ_i satisfy the equation

$$-1/\cosh(\lambda_i^{1/4}l) = \cos(\lambda_i^{1/4}l). \tag{5.20}$$

Note that as i becomes large, $\lambda_i^{1/4} \cong (2i+1)\pi/2l$. From the homogeneous system (5.19), let $-D_i = c[\sinh(\lambda_i^{1/4}l) + \sin(\lambda_i^{1/4}l)]$ and $-C_i = -c[\cosh(\lambda_i^{1/4}l) + \cos(\lambda_i^{1/4}l)]$. Then plugging C_i and D_i into (5.18), we have eigenfunction

$$\begin{aligned} \phi_i(x) &= c[\sinh(\lambda_i^{1/4}l) + \sin(\lambda_i^{1/4}l)][\cosh(\lambda_i^{1/4}x) - \cos(\lambda_i^{1/4}x)] \\ &\quad - c[\cosh(\lambda_i^{1/4}l) + \cos(\lambda_i^{1/4}l)][\sinh(\lambda_i^{1/4}x) - \sin(\lambda_i^{1/4}x)]. \end{aligned} \tag{5.21}$$

So using the orthonormal property of the eigenfunction ϕ_i ($\|\phi_i\|_{L^2} = 1$), we can find c. Put $\lambda_i^{1/4} = a$ and $\phi_i = \phi$. Then

$$\begin{aligned}
\|\phi\|_{L^2(0,l)} &= \frac{c^2}{4a}\left\{[al + \cos^2(al) - 3\sin(al)\cos(al)]e^{2al}\right. \\
&\quad + [4al\sin(al) - 6\sin(al) - \cos(al)]e^{al} \\
&\quad + [2al - 4al\cos^2(al) - 6\sin(al)\cos(al)] \\
&\quad - [4al\sin(al) + 6\sin(al) + 6\cos(al)]e^{-al} \\
&\quad \left. + [al - 3\sin(al)\cos(al) - 3\cos^2(al)]e^{-2al}\right\}. \qquad (5.22)
\end{aligned}$$

For sufficiently large al ($\lambda_i \gg l^{-4}$), the last three terms of (5.22) are bounded. So focusing on the dominant term, we have

$$1 = \|\phi\|^2_{L^2(0,l)} = c^2\left[\frac{al}{4a}e^{2al} + O(e^{al})\right] = c^2 e^{2al}\left[(l/4) + O(e^{-al})\right]$$

Taking the positive sign, $c \sim \left[(2/l^{1/2})e^{-al} + O(e^{-2al})\right]$. Plugging this into (5.21) we obtain

$$\begin{aligned}
\phi(x) &\sim \frac{2e^{-al}}{l^{1/2}}\{[\sinh(al) + \sin(al)][\cosh(ax) - \cos(ax)] \\
&\qquad - [\cosh(al) + \cos(al)][\sinh(ax) - \sin(ax)]\} \\
&= \frac{2e^{-al}}{l^{1/2}}\left\{[\frac{e^{al}}{2} + O(1)][\cosh(ax) - \cos(ax)] - [\frac{e^{al}}{2} + O(1)][\sinh(ax) - \sin(ax)]\right\} \\
&= \frac{1}{l^{1/2}}\left\{[1 + O(e^{-al})][\cosh(ax) - \cos(ax)] - [1 + O(e^{-al})][\sinh(ax) - \sin(ax)]\right\}.
\end{aligned}$$

Since $e^{-al}\cosh(ax) \le e^{-a(l-x)} \le 1$ and $e^{-al}\sinh(ax) \le e^{-a(l-x)}/2 + O(1) = O(1)$, we have

$$\phi_i(x) \sim \frac{1}{l^{1/2}}[\cosh(\lambda_i^{1/4}x) - \sinh(\lambda_i^{1/4}x) + O(1)] = O(1), \quad \text{for sufficiently large } \lambda_i.$$

Since $\cosh z - \sinh z \leq 1$ for $z \geq 0$, $\max_x \sup_i |\phi_i(x)|$ is bounded and the result follows. □

The next Proposition is useful to prove Lemma 5.16.

Proposition 5.3. *Assume that* $w \in H_{cf}^{(\sigma+1/2)/4}(0,l)$ *for any* $\sigma > 0$*. Then for the elliptic self-adjoint fourth order partial differential operator* $K = \partial^4/\partial x^4$*, the norms* $w \mapsto \|w\|_{H^{\sigma+1/2}(0,l)}$ *and* $w \mapsto \left(\|K^{(\sigma+1/2)/4}w\|^2_{L^2(0,l)} + \|w\|^2_{L^2(0,l)}\right)^{1/2}$ *are equivalent.*

5.2.2 Penalty method

As an alternative to the original Euler–Bernoulli equation with the Signorini's condition, we consider a penalty formulation which provides a more satisfactory information for an approximate solution u_ϵ. See the details in [32] for the detailed arguments. We define the sequence of contact force, N_ϵ as a penalty function for the constraint $u + g \geq 0$,

$$N_\epsilon = \frac{1}{\epsilon}\varphi \circ (-g - u_\epsilon), \quad \epsilon > 0, \tag{5.23}$$

where

$$\varphi(s) = \sqrt{1 + (s_+)^2} - 1, \qquad \text{with } s_+ = \max(s, 0).$$

Note that φ is C^1 with bounded 2nd derivatives everywhere except at zero. Now taking the penalty functional N_ϵ in (5.1) instead of the contact force N, we obtain the penalty formulation

$$(u_\epsilon)_{tt} = -(u_\epsilon)_{xxxx} + f(x) + \frac{1}{\epsilon}\varphi(-g - u_\epsilon) \quad \text{for } \epsilon > 0. \tag{5.24}$$

We have approximated the linear complementary condition (5.4) with a penalty term. Now that we have the penalty formulation, we will begin with solving the penalized boundary value problem:

$$\begin{aligned} (u_\epsilon)_{tt} &= -(u_\epsilon)_{xxxx} + f(x) + \frac{1}{\epsilon}\varphi \circ (-g - u_\epsilon) \quad \text{in } (0,l) \times (0,T], && (5.25) \\ u_\epsilon(0,t) &= (u_\epsilon)_x(0,t) = 0 \quad \text{on } (0,T], && (5.26) \\ (u_\epsilon)_{xx}(l,t) &= (u_\epsilon)_{xxx}(l,t) = 0 \quad \text{on } (0,T], && (5.27) \\ u_\epsilon(x,0) &= u_1(x) \quad \text{in } (0,l), && (5.28) \\ (u_\epsilon)_t(x,0) &= v_1(x) \quad \text{in } (0,l). && (5.29) \end{aligned}$$

We will assume that $f \in L^2(0,l)$, $u_1 \in H^2_{cf}(0,l)$, and $v_1 \in L^2(0,l)$. Note that by scaling x and t appropriately we can put $\rho A = EI = 1$ in order to simplify our computations. In Lemma 5.5, we will show that the approximate solution u_ϵ exists using the Banach fixed-point theorem.

Lemma 5.4. *The homogeneous system* $(u_\epsilon)_{tt} + (u_\epsilon)_{xxxx} = 0$ *has a solution*

$$u_{\epsilon,hom}(x,t) = \int_0^l w(x,t;x^*)v_1(x^*)\,dx^* + \int_0^l \frac{\partial w}{\partial t}(x,t;x^*)u_1(x^*)\,dx^*.$$

Proof. Since $\phi_i''''(x) = \lambda_i \phi_i(x)$ for each $i \geq 1$, using (5.10) we have ODE:

$$\frac{\partial^2 u_\epsilon^i(t)}{\partial t^2} + \lambda_i u_\epsilon^i(t) = 0. \tag{5.30}$$

Thus the solution of (5.30) is $u_\epsilon^i(t) = A\sin(\lambda_i^{1/2}t) + B\cos(\lambda_i^{1/2}t)$ for each $i \geq 1$. So

$$u_\epsilon(x,t) = \sum_{i=1}^{\infty}[A\sin(\lambda_i^{1/2}t) + B\cos(\lambda_i^{1/2}t)]\phi_i(x). \tag{5.31}$$

From the initial condition (5.28), we have

$$u_\epsilon(x,0) = \sum_{i=1}^{\infty} B\phi_i(x) = u_1(x).$$

Similarly, for $x^* \in [0,l]$ we obtain

$$u_\epsilon(x^*,0) = \sum_{i=1}^{\infty} B\phi_i(x^*) = u_1(x^*).$$

Since $\|\phi_i\|_{L^2(0,l)} = 1$, $B = \int_0^l u_1(x^*)\phi_i(x^*)\,dx^*$. From the initial condition (5.29), we have

$$(u_\epsilon)_t(x,0) = \sum_{i=1}^{\infty} A\lambda_i^{1/2}\phi_i(x) = v_1(x).$$

Similarly, we can obtain $A = \lambda_i^{-1/2}\int_0^l v_1(x^*)\phi(x^*)\,dx^*$. Therefore plugging A, B into (5.31),

$$\begin{aligned}
u_{\epsilon,hom}(x,t) &= \sum_{i=1}^{\infty}\left[\frac{\sin(\lambda_i^{1/2}t)}{\lambda_i^{1/2}}\int_0^l v_1(x^*)\phi_i(x^*)\,dx^* + \cos(\lambda_i^{1/2}t)\int_0^l u_1(x^*)\phi_i(x^*)\,dx^*\right]\phi_i(x) \\
&= \int_0^l \sum_{i=1}^{\infty}\frac{\sin(\lambda_i^{1/2}t)}{\lambda_i^{1/2}}\phi_i(x)v_1(x^*)\phi_i(x^*)\,dx^* \\
&\quad + \int_0^l \sum_{i=1}^{\infty}\cos(\lambda_i^{1/2}t)\phi_i(x)u_1(x^*)\phi_i(x^*)dx^* \\
&= \int_0^l w(x,t;x^*)v_1(x^*)\,dx^* + \int_0^l \frac{\partial w}{\partial t}(x,t;x^*)u_1(x^*)\,dx^*,
\end{aligned}$$

as required. □

Lemma 5.5. *There exists a unique solution u_ϵ of the penalty equations (5.25)–(5.29).*

Proof. In order to show that the solution u_ϵ satisfying (5.25), (5.28), (5.29) is a solution of the integral equation, we first need to solve the homogeneous system of

(5.25), i.e, $(u_\epsilon)_{tt} + (u_\epsilon)_{xxxx} = 0$. By Lemma 5.4, we have a solution $u_{\epsilon,hom}$ of the homogeneous system:

$$u_{\epsilon,hom}(x,t) = \int_0^l w(x,t;x^*)v_1(x^*)dx^* + \int_0^l \frac{\partial w}{\partial t}(x,t;x^*)u_1(x^*)\,dx^*.$$

In the inhomogeneous system $(u_\epsilon)_{tt} + (u_\epsilon)_{xxxx} = f(x) + \frac{1}{\epsilon}\varphi \circ (-u_\epsilon - g)$, the particular solution $u_{\epsilon,par}$ is given by the integral equation:

$$u_{\epsilon,par}(x,t) = \int_0^t \int_0^l w(x,t-s;x^*)\left[f(x^*) + \frac{1}{\epsilon}\varphi(-u_\epsilon(x^*,s) - g(x^*))\right] dx^*ds.$$

So the penalized solution is given by

$$\begin{aligned} u_\epsilon(x,t) \;\;&=\;\; \int_0^l w(x,t;x^*)v_1(x^*)dx^* + \int_0^l \frac{\partial w}{\partial t}(x,t;x^*)u_1(x^*)\,dx^* + \\ &\int_0^t \int_0^l w(x,t-s;x^*)\left[f(x^*) + \frac{1}{\epsilon}\varphi(-u_\epsilon(x^*,s) - g(x^*))\right] dx^*ds \end{aligned} \tag{5.32}$$

Let the first two terms of (5.32) be

$$r(x,t) = \int_0^l w(x,t;x^*)v_1(x^*)dx^* + \int_0^l \frac{\partial w}{\partial t}(x,t;x^*)u_1(x^*)\,dx^* \text{ in } C(0,T;L^2(0,l)).$$

Note that $\|u\|_{C(0,T;L^2(0,l))} = \sup_{t\in[0,T]}\left\{\int_0^l |u(x,t)|^2dx\right\}^{1/2}$. Now we define a nonlinear integral operator $\Gamma : C(0,T;L^2(0,l)) \to C(0,T;L^2(0,l))$ by

$$\Gamma u(x,t) := \int_0^t \int_0^l w(x,t-s;x^*)\left[f(x^*) + \frac{1}{\epsilon}\varphi(-u(x^*,s) - g(x^*))\right] dx^*ds. \tag{5.33}$$

We claim that Γ is a contraction mapping on $C(0,T;L^2(0,l))$ for sufficiently small T. First we define the bounded operator as $\mathcal{L}_t : L^2(0,l) \to L^2(0,l)$ for the fixed $t \in [0,T]$

$$(\mathcal{L}_t z(\cdot,t))(x) = \int_0^l w(x,t;x^*)z(x^*,t)\,dx^*. \tag{5.34}$$

If we express $z(x,t)$ as $z(x,t) = \sum_{i=1}^{\infty} z_i(t)\phi_i(x)$, by Lemma 5.1 and (5.34), we obtain

$$\begin{aligned}
(\mathcal{L}_t z(\cdot,t))(x) &= \int_0^l \left[\sum_{i=1}^{\infty} \frac{\sin(\lambda_i^{1/2}t)}{\lambda_i^{1/2}} \phi_i(x)\phi_i(x^*) \cdot \sum_{j=1}^{\infty} z_j(t)\phi_j(x^*)\right] dx^* \\
&= \sum_{i=1}^{\infty} \frac{\sin(\lambda_i^{1/2}t)}{\lambda_i^{1/2}} z_i(t)\phi_i(x)\|\phi_i\|^2_{L^2(0,l)} \\
&= \sum_{i=1}^{\infty} \frac{\sin(\lambda_i^{1/2}t)}{\lambda_i^{1/2}} z_i(t)\phi_i(x).
\end{aligned}$$

Now, we define the norm of a bounded operator as

$$\|\mathcal{L}_t\| = \sup_{\|z\|_{L^2(0,l)} \neq 0} \frac{\|\mathcal{L}_t z\|_{L^2(0,l)}}{\|z\|_{L^2(0,l)}} = \sup_{\|z\|_{L^2(0,l)} = 1} \|\mathcal{L}_t z\|_{L^2(0,l)}.$$

Notice that $\sum_{i=1}^{\infty} z_i(t)^2 = 1$, since $\|z(\cdot,t)\|_{L^2(0,l)} = 1$. This leads us to the estimate:

$$\begin{aligned}
\|\mathcal{L}_t z(\cdot,t)\|^2_{L^2(0,l)} &= (\mathcal{L}_t z(\cdot,t), \mathcal{L}_t z(\cdot,t))_{L^2(0,l)} \\
&= \left(\sum_{i=1}^{\infty} \frac{\sin(\lambda_i^{1/2}t)}{\lambda_i^{1/2}} z_i(t)\phi_i(\cdot),\ \sum_{j=1}^{\infty} \frac{\sin(\lambda_j^{1/2}t)}{\lambda_j^{1/2}} z_j(t)\phi_j(\cdot)\right)_{L^2(0,l)} \\
&= \sum_{i=1}^{\infty} \left[\frac{\sin(\lambda_i^{1/2}t)}{\lambda_i^{1/2}}\right]^2 z_i(t)^2 \leq \left[\sup_{i\geq 1} \frac{\sin(\lambda_i^{1/2}t)}{\lambda_i^{1/2}}\right]^2.
\end{aligned}$$

Therefore the norm $\|\mathcal{L}_t\|$ of the operator $\mathcal{L}_t : L^2(0,l) \to L^2(0,l)$ is

$$\begin{aligned}
\|\mathcal{L}_t\| &= \sup_{\|z\|_{L^2}=1} \|\mathcal{L}_t z\|_{L^2(0,l)} \\
&\leq \sup_{i\geq 1} \frac{|\sin(\lambda_i^{1/2}t)|}{\lambda_i^{1/2}} \leq \sup_{\lambda \geq \lambda_1} \left|\frac{\sin(\lambda^{1/2}t)}{\lambda^{1/2}}\right| \\
&= t \sup_{\lambda\geq\lambda_1} \left|\frac{\sin(\lambda^{1/2}t)}{\lambda^{1/2}t}\right| = t \sup_{\theta \geq \lambda_1^{1/2}t} \left|\frac{\sin\theta}{\theta}\right| \\
&\leq t \min\left(\frac{1}{\lambda_1^{1/2}t}, 1\right) = \min(\lambda_1^{-1/2}, t).
\end{aligned}$$

From (5.33), take $G(x,w) = f(x) + \frac{1}{\epsilon}\varphi(-w - g(x))$ for $w \in \mathbf{R}$. Note that $G(\cdot, u(\cdot,t)) \in L^2(0,l)$. Then operator $w(\cdot) \mapsto G(\cdot, w(\cdot))$ is a Lipschitz operator on $L^2(0,l)$ with constant $1/\epsilon$, i.e., for the fixed $\epsilon > 0$

$$\|G(\cdot, u_1(\cdot,t)) - G(\cdot, u_2(\cdot,t))\|_{L^2(0,l)} \leq \frac{1}{\epsilon}\|u_1(\cdot,t) - u_2(\cdot,t)\|_{L^2(0,l)}.$$

From (5.33) we have

$$\Gamma u_\epsilon(\cdot,t) = \int_0^t \mathcal{L}_{t-s}(G(\cdot, u_\epsilon(\cdot,s)))\, ds.$$

Thus for $(u_\epsilon)_1,\ (u_\epsilon)_2 \in C(0,T;L^2(0,l))$,

$$\begin{aligned}
&\|\Gamma(u_\epsilon)_1(x,t) - \Gamma(u_\epsilon)_2(x,t)\|_{L^2(0,l)} \\
&\quad \leq \int_0^t \|\mathcal{L}_{t-s}\| \|G(\cdot,(u_\epsilon)_1(\cdot,s)) - G(\cdot,(u_\epsilon)_2(\cdot,s))\|_{L^2(0,l)} ds \\
&\quad \leq \int_0^t \min(\lambda_1^{-1/2}, t-s)\frac{M}{\epsilon}\|(u_\epsilon)_1(\cdot,s) - (u_\epsilon)_2(\cdot,s)\|_{L^2(0,l)} ds \\
&\quad \leq \frac{1}{\lambda_1^{1/2}}\frac{1}{\epsilon}\int_0^t \|(u_\epsilon)_1(\cdot,s) - (u_\epsilon)_2(\cdot,s)\|_{L^2(0,l)} ds \\
&\quad \leq \frac{1}{\lambda_1^{1/2}}\frac{t}{\epsilon}\sup_{s\in[0,t]} \|(u_\epsilon)_1(\cdot,s) - (u_\epsilon)_2(\cdot,s)\|_{L^2(0,l)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sup_{t\in[0,T]} \|\Gamma(u_\epsilon)_1(\cdot,t) - \Gamma(u_\epsilon)_2(\cdot,t)\|_{L^2(0,l)} \\
&\qquad \leq \frac{T}{\lambda_1^{1/2}\epsilon}\sup_{t\in[0,T]} \|(u_\epsilon)_1(\cdot,t) - (u_\epsilon)_2(\cdot,t)\|_{L^2(0,l)}.
\end{aligned}$$

So if T is small enough that $\lambda_1^{-1/2}T < \epsilon$ for fixed $\epsilon > 0$, $u \mapsto \Gamma u + r$ is a contraction mapping on $C(0,T;L^2(0,l))$, for $r \in C(0,T;L^2(0,l))$. By the Banach Fixed point theorem (see, e.g., [55, Ex. 3.19, p. 113]), there exist a unique solution $u_\epsilon(\cdot,t)$ of the penalty formulation for $t \in [0,T]$. By using continuation arguments as for ordinary differential equations [15, §4, pp. 13-15], there is a unique solution u_ϵ in $C(0,T;L^2(0,l))$ for *any* $T>0$. □

5.2.3 Conservation of energy and energy bounds

In order to establish conservation of energy, we need to establish some stronger regularity results.

Lemma 5.6. *If $v_1 \in H^2_{cf}(0,l)$, $u_1 \in H^4_{cf}(0,l)$, and $f \in H^2(0,l)$, then the solution u_ϵ of the penalty equations (5.25), (5.26), (5.27) is in $C^1(0,T;H^2_{cf}(0,l))$.*

Proof. Since we have established that we have a solution $u_\epsilon \in C(0,T;L^2(0,l))$ from the previous Section, we note that the penalty equation can be written as

$$(u_\epsilon)_{tt} + (u_\epsilon)_{xxxx} = f + \frac{1}{\epsilon}\varphi \circ (-u_\epsilon - g)$$

and the right-hand side is in $C(0,T;L^2(0,l))$. We can then apply standard regularity theory for linear hyperbolic PDEs (e.g., [47, Thm. 10.8]) to conclude that $u_\epsilon \in C^1(0,T;L^2(0,l)) \cap C(0,T;H^2_{cf}(0,l))$. This means that $\varphi \circ (-u_\epsilon - g) \in C(0,T;H^2_{cf}(0,l))$. Putting $\widetilde{f}_\epsilon = f + \varphi \circ (-u_\epsilon - g)$ and expanding in terms of eigenfunctions, the solution can be written as $u_\epsilon(x,t) = \sum_i u_{\epsilon,i}(t)\,\phi_i(x)$ where

$$u_{\epsilon,i}(t) = \cos(\lambda_i^{1/2}t)(u_1)_i + \lambda_i^{-1/2}\sin(\lambda_i^{1/2}t)(v_1)_i + \int_0^t \lambda_i^{-1/2}\sin(\lambda_i^{1/2}(t-\tau))\,\widetilde{f}_{\epsilon,i}(\tau)\,d\tau.$$

We can easily see that

$$\dot{u}_{\epsilon,i}(t) = -\lambda_i^{1/2}\sin(\lambda_i^{1/2}t)(u_1)_i + \cos(\lambda_i^{1/2}t)(v_1)_i + \int_0^t \cos(\lambda_i^{1/2}(t-\tau))\,\widetilde{f}_{\epsilon,i}(\tau)\,d\tau$$

and using the equivalence of the norms, $u_\epsilon \in C^1(0,T;H^2_{cf}(0,l))$ as desired. □

For the penalty approximation, we define the energy functional as

$$\mathcal{E}_\epsilon[u_\epsilon] := \int_0^l \left[\frac{1}{2}[(u_\epsilon)_t]^2 + \frac{1}{2}[(u_\epsilon)_{xx}]^2 + \frac{1}{\epsilon}\Phi \circ (-g - u_\epsilon) - f(x) \cdot u_\epsilon\right] dx, \tag{5.35}$$

where $\Phi(s) = \int_0^s \varphi(\sigma)\, d\sigma$.

Lemma 5.7. *Suppose that the approximate solution u_ϵ satisfies (5.25), (5.26), (5.27). Then energy is conserved for u_ϵ. That is, $\mathcal{E}_\epsilon[u_\epsilon(\cdot, t)]$ is independent of t.*

Proof. We first assume that u_1, v_1 and f are all as smooth as required for Lemma 5.6. We claim that for any $0 \le t_1 \le t_2 \le T$ and fixed $\epsilon > 0$,

$$\begin{aligned}
&\int_0^l \left[\frac{1}{2}(u_\epsilon)_t^2(x,t_1) + \frac{1}{2}(u_\epsilon)_{xx}^2(x,t_1) + \frac{1}{\epsilon}\Phi(-g(x) - u_\epsilon(x,t_1)) - f(x) \cdot u_\epsilon(x,t_1)\right] dx \\
&\quad = \int_0^l \left[\frac{1}{2}(u_\epsilon)_t^2(x,t_2) + \frac{1}{2}(u_\epsilon)_{xx}^2(x,t_2) + \frac{1}{\epsilon}\Phi(-g(x) - u_\epsilon(x,t_2)) - f(x) \cdot u_\epsilon(x,t_2)\right].
\end{aligned}$$

From the penalized formulation (5.25),

$$\begin{aligned}
0 &= \left[(u_\epsilon)_{tt} + (u_\epsilon)_{xxxx} - \frac{1}{\epsilon}\varphi(-g - u_\epsilon) - f\right](u_\epsilon)_t \\
&= \frac{1}{2}\frac{d}{dt}\left[(u_\epsilon)_t^2 + (u_\epsilon)_{xx}^2 + \frac{2}{\epsilon}\Phi(-g - u_\epsilon) - 2f \cdot u_\epsilon\right] \\
&\qquad + \frac{d}{dx}((u_\epsilon)_{xxx}(u_\epsilon)_t) - \frac{d}{dx}((u_\epsilon)_{xx}(u_\epsilon)_{tx}).
\end{aligned}$$

Thus,

$$\frac{1}{2}\frac{\partial}{\partial t}\left[(u_\epsilon)_t^2 + (u_\epsilon)_{xx}^2 - \frac{2}{\epsilon}\Phi(-g - u_\epsilon) - 2f \cdot u_\epsilon\right] = \frac{\partial}{\partial x}((u_\epsilon)_{xx}(u_\epsilon)_{tx}) - \frac{\partial}{\partial x}((u_\epsilon)_{xxx}(u_\epsilon)_t). \tag{5.36}$$

For any $0 \le t_1 \le t_2 \le T$, taking an integration over the rectangle $(x,t) \in [0,l] \times [t_1,t_2]$ on the both side of (5.36),

$$\begin{aligned}
&\frac{1}{2}\int_{t_1}^{t_2}\int_0^l \frac{\partial}{\partial t}\left[(u_\epsilon)_t^2 + (u_\epsilon)_{xx}^2 + \frac{2}{\epsilon}\Phi(-g - u_\epsilon) - 2f \cdot u_\epsilon\right] dx\, dt \\
&= \int_{t_1}^{t_2}\int_0^l \left[\frac{\partial}{\partial x}((u_\epsilon)_{xx}(u_\epsilon)_{tx}) - \frac{\partial}{\partial x}((u_\epsilon)_{xxx}(u_\epsilon)_t)\right] dx\, dt.
\end{aligned}$$

So we have

$$\frac{1}{2}\int_0^l \left[(u_\epsilon)_t^2(x,t_2) - (u_\epsilon)_t^2(x,t_1) + (u_\epsilon)_{xx}^2(x,t_2) - (u_\epsilon)_{xx}^2(x,t_1)\right] dx$$

$$\begin{aligned}
&+\frac{1}{\epsilon}\int_0^t \left[\Phi(-g(x)-u_\epsilon(x,t_2)) - \Phi(-g(x)-u_\epsilon(x,t_1))\right] dx \\
&-\int_0^l \left[f(x)u_\epsilon(x,t_2)dx - f(x)u_\epsilon(x,t_1)\right] dx \\
=&\int_{t_1}^{t_2} \left[(u_\epsilon)_{xx}(l,t)\cdot(u_\epsilon)_{tx}(l,t) - (u_\epsilon)_{xx}(0,t)\cdot(u_\epsilon)_{tx}(0,t)\right] dt \\
&+\int_{t_1}^{t_2} \left[(u_\epsilon)_{xxx}(l,t)\cdot(u_\epsilon)_t(l,t) - (u_\epsilon)_{xxx}(0,t)\cdot(u_\epsilon)_t(0,t)\right] dt.
\end{aligned}$$

From the boundary conditions (5.26), (5.27),

$$(u_\epsilon)_t(0,t) = 0 \text{ and } (u_\epsilon)_{tx}(0,t) = (u_\epsilon)_{xt}(0,t) = 0.$$

Therefore the result follows for sufficiently smooth u_1, v_1 and f.

For the general case, we note that if $u_1^k \to u_1$ in $H_{cf}^2(0,l)$, $v_1^k \to v_1$ in $L^2(0,l)$, and $f^k \to f$ in $L^2(0,l)$, then the corresponding solution $u_\epsilon^k \to u_\epsilon$ in $C(0,T;H_{cf}^2(0,l)) \cap C^1(0,T;L^2(0,l))$ and we obtain energy conservation in the limit. □

Proposition 5.8. *Assume that $Y(t,\cdot)$ is monotone increasing and Lipschitz continuous for all t. If $dy(t)/dt \le Y(t,y(t))$ and $dz(t)/dt = Y(t,z(t))$ and $y(0) = z(0) = y_0$, then $y(t) \le z(t)$ for all t.*

Now we will assume that the initial energy is finite in terms of the physical point of view. In Lemma 5.9, it is shown that $\int_0^l f(x)u_\epsilon(x,t)dx$ is bounded by a function of time t only.

Lemma 5.9. *For any time $t \in [0,T]$, we have*

$$\left| \int_0^l f(x) u_\epsilon(x,t) dx \right| \leq C(t).$$

Proof. Since f does not depend on time t,

$$\frac{d}{dt}\int_0^l f(x)u_\epsilon(x,t)dx = \int_0^l f(x)\cdot \frac{d}{dt}u_\epsilon(x,t)dx \leq \|f\|_{L^2(0,l)}\|(u_\epsilon)_t(t)\|_{L^2(0,l)}. \tag{5.37}$$

Let $(u_\epsilon)_t(t)$ be a velocity function $v(t)$. Define the energy function as

$$\begin{aligned} E(t) &:= \mathcal{E}_\epsilon[u_\epsilon(\cdot,t)] \\ &= \int_0^l \left[\frac{1}{2}(u_\epsilon)_t^2(x,t) + \frac{1}{2}(u_\epsilon)_{xx}^2(x,t) + \frac{1}{\epsilon}\Phi(-u_\epsilon(x,t) - g(x)) - f(x)\cdot u_\epsilon(x,t)\right] dx. \end{aligned}$$

By the conservation of energy in the penalized formulation, we have

$$\begin{aligned} &\int_0^l \frac{1}{2}\left[(u_\epsilon)_t^2(x,t) + (u_\epsilon)_{xx}^2(x,t) + \frac{1}{\epsilon}\Phi(-u_\epsilon(x,t) - g(x))\right] dx \\ &\qquad = E(0) + \int_0^l f(x)\cdot u_\epsilon(x,t)dx. \end{aligned}$$

Thus since we put $(u_\epsilon)_t(\cdot,t) = v(\cdot,t)$ for any $t \in [0,T]$,

$$\|v(\cdot,t)\|_{L^2(0,l)} \leq \sqrt{2\left(E(0) + \int_0^l f(x)\cdot u_\epsilon(x,t)dx\right)}. \tag{5.38}$$

From (5.37), we obtain

$$\begin{aligned} \frac{d}{dt}\int_0^l f(x)u_\epsilon(x,t)dx &\leq \|f\|_{L^2(0,l)}\|v(t)\|_{L^2(0,l)} \\ &\leq \|f\|_{L^2(0,l)}\sqrt{2\left(E(0) + \int_0^l f(x)\cdot u_\epsilon(x,t)dx\right)}. \end{aligned}$$

Take $y(t) = \int_0^l f(x)u_\epsilon(x,t)dx$ and

$$Y(t,y) = \|f\|_{L^2(0,l)}\sqrt{2\left(E(0) + y\right)},$$

we have $dy/dt \leq Y(t, y)$. Thus by Proposition 5.8, the result follows where $C(t)$ solves the differential equation $dC/dt = \|f\|_{L^2(0,l)}\sqrt{2\left(E(0) + C(t)\right)}$ and $C(0) = \int_0^l f(x)u_1(x)\,dx$, provided $E(0) > 0$.

Note that we have a continuous function $C(t)$ of time only t:

$$C(t) = \frac{1}{2}\left\{\left[\|f\|_{L^2(0,l)} \cdot t + \sqrt{2(E(0) + C(0))}\right]^2 - E(0)\right\}.$$

□

Lemma 5.10. *The approximate solutions u_ϵ are bounded in $L^\infty(0, T; H^2(0, l))$, as $\epsilon \to 0$.*

Proof. By the conservation of energy, for any $t \in [0, T]$ we have

$$\begin{aligned} E(0) &= \int_0^l \frac{1}{2}\left[(u_\epsilon)_t^2(x,t) + (u_\epsilon)_{xx}^2(x,t) + \frac{2}{\epsilon}\Phi(-g(x) - u_\epsilon(x,t))\right] dx \\ &\quad - \int_0^l f(x) \cdot u_\epsilon(x,t)dx. \end{aligned}$$

Using Lemma 5.7,

$$\begin{aligned} &\frac{1}{2}\int_0^l \left[(u_\epsilon)_t^2(x,t) + (u_\epsilon)_{xx}^2(x,t) + \frac{2}{\epsilon}\Phi(-g(x) - u_\epsilon(x,t))\right] dx \\ &\quad = E(0) + \int_0^l f(x) \cdot u_\epsilon(x,t)dx \leq E(0) + C(t). \end{aligned} \tag{5.39}$$

Thus we obtain

$$\int_0^l (u_\epsilon)_{xx}^2(x,t)dx \leq 2(E(0) + C(t)), \quad \text{for each } \epsilon > 0. \tag{5.40}$$

Using the Dirichlet boundary conditions (5.26), indefinite integrals and Hölder's inequality, we can show that

$$\int_0^l (u_\epsilon)_x^2(x,t)dx \text{ and } \int_0^l (u_\epsilon)^2(x,t)dx \text{ are both bounded by function of time only.}$$

Therefore for any $t \in [0,T]$, $\|u_\epsilon(t)\|_{H^2(0,l)} < \infty$. This means that as $\epsilon \to 0$,

$$\|u_\epsilon\|_{L^\infty(0,T;H^2(0,l))} = \sup_{0\le t\le T} \|u_\epsilon(t)\|_{H^2(0,l)} < \infty.$$

□

Lemma 5.11. *The approximate solutions u_ϵ are bounded in $W^{1,\infty}(0,T;L^2(0,l))$, as $\epsilon \to 0$.*

Proof. From (5.39), we have

$$\int_0^l (u_\epsilon)_t^2(x,t)dx \le 2(E(0)+C(t)) \quad \text{for each } \epsilon > 0.$$

Since for any $t \in [0,T]$, $\|u_\epsilon(\cdot,t)\|_{H^2(0,l)}$ is bounded by a continuous function of t only,

$$\sup_{0\le t\le T} \left\{ \|u_\epsilon(t)\|_{L^2(0,l)} + \|(u_\epsilon)_t(t)\|_{L^2(0,l)} \right\} < \infty.$$

Therefore u_ϵ is bounded in $W^{1,\infty}(0,T,L^2(0,l))$, as $\epsilon \to 0$. □

5.2.4 Bounds on the contact force and constraint violation

In this Section we will first bound the integral of the normal contact force N_ϵ over space and time (which uniformly bounds N_ϵ in the space of measures), and then bound a measure of the constraint violation: $\int_0^l \Phi \circ (-u_\epsilon - g)\, dx$.

Lemma 5.12. *$\int_0^T \int_0^l N_\epsilon \, dx\, dt$ is bounded as $\epsilon \to 0$.*

Proof. We multiply by $x^2/2$ on both side of (5.25) and take an integral over $[0,l] \times$

$[0, T]$. Then we have

$$\begin{aligned}\int_0^T \int_0^l \frac{x^2}{2}(u_\epsilon)_{tt} dx\, dt &= -\int_0^T \int_0^l \frac{x^2}{2}(u_\epsilon)_{xxxx} dx\, dt + \int_0^T \int_0^l \frac{x^2}{2} f(x) dx\, dt \\ &\quad + \int_0^T \int_0^l \frac{x^2}{2} N_\epsilon dx\, dt. \end{aligned} \tag{5.41}$$

Changing the order of integration on the left side (5.41) and using integration by part, and applying the boundary condition (5.26), (5.27),

$$\begin{aligned} &\int_0^l \frac{x^2}{2}((u_\epsilon)_t(T) - (u_\epsilon)_t(0)) dx \\ &= -\int_0^T \left[\left[\frac{x^2}{2}(u_\epsilon)_{xxx} \right]_0^l - \int_0^l x \cdot (u_\epsilon)_{xxx} \right] dt \\ &\quad + \int_0^T \int_0^l \frac{x^2}{2} f(x)\, dx\, dt + \int_0^T \int_0^l \frac{x^2}{2} N_\epsilon\, dx\, dt \\ &= \int_0^T [x \cdot (u_\epsilon)_{xx}]_0^l dt - \int_0^T \int_0^l (u_\epsilon)_{xx}\, dx\, dt \\ &\quad + \int_0^T \int_0^l \frac{x^2}{2} f(x)\, dx\, dt + \int_0^T \int_0^l \frac{x^2}{2} N_\epsilon\, dx\, dt. \\ &= -\int_0^T \int_0^l (u_\epsilon)_{xx}\, dx\, dt + \int_0^T \int_0^l \frac{x^2}{2} f(x)\, dx\, dt \\ &\quad + \int_0^T \int_0^l \frac{x^2}{2} N_\epsilon\, dx\, dt. \end{aligned}$$

By (5.38) and Lemmas 5.9 and 5.11, $\int_0^T \int_0^l \frac{x^2}{2} N_\epsilon\, dx\, dt$ is bounded. Now since $g(0) > 0$, using the energy bound on $\|u_\epsilon\|_{H^2(0,l)}$ and as $u_\epsilon(0,t) = \partial u_\epsilon / \partial x(0,t) = 0$, we can show that there is an $\eta > 0$ (independent of $\epsilon > 0$) where $u_\epsilon(x,t) > -g(x)$ for all $x \in [0, \eta]$. This implies that there is no contact force between rod and rigid obstacle in $[0, \eta]$. Thus for $0 \le x \le \eta$, $N_\epsilon = 0$. Since $N_\epsilon \ge 0$ and $x^2/2 \ge \eta^2/2 \ge 0$ in $[\eta, l]$, we have

$$\int_0^T \int_0^l \frac{x^2}{2} N_\epsilon\, dx\, dt = \int_0^T \int_\eta^l \frac{x^2}{2} N_\epsilon\, dx\, dt \ge \frac{\eta^2}{2} \int_0^T \int_\eta^l N_\epsilon\, dx\, dt = \frac{\eta^2}{2} \int_0^T \int_0^l N_\epsilon\, dx\, dt$$

Therefore the result follows. □

Proposition 5.13. *Weak* convergence in $L^\infty(0,T;H)$ implies weak convergence in $L^2(0,T;H)$, where H is a separable real Hilbert space.*

The next Lemma 5.14 indicates that the solution u satisfies the constraint $u+g\geq 0$.

Lemma 5.14. *If $u_\epsilon \rightharpoonup u$ in $L^2(0,T;L^2(0,l))$ in some subsequence (and there are converging subsequences), then $u+g\geq 0$.*

Proof. From (5.39), for all $0\leq t\leq T$ we have

$$\int_0^l \Phi(-u_\epsilon(x,t)-g(x))dx \leq C\cdot\epsilon.$$

Then $\|\Phi\circ(u_\epsilon(\cdot,t)-g)\|_{L^1(0,l)}\to 0,$ as $\epsilon\to 0$. Thus we obtain

$$\int_0^T\int_0^l \Phi\circ(-u_\epsilon-g)\,dx\,dt\to 0.$$

According to Lemma 5.11, there exists u such that $u_\epsilon \rightharpoonup^* u$ in a subsequence in $L^\infty(0,T,L^2(0,l))$. By Proposition 5.13, we have the solution u such that $u_\epsilon\rightharpoonup u$ in $L^2(0,T,L^2(0,l))$. Define the functional $F[u]:=\int_0^T\int_0^l\Phi\circ(-u-g)\,dx\,dt$. Then $F[u]$ is a convex lower semicontinuous function. So $F[u]$ is a weakly lower semicontinuous by Mazur's Lemma [35, Thm. IV.2.1]. Thus we have

$$0\leq F[u]\leq \liminf F[u_\epsilon]=\int_0^T\int_0^l\Phi\circ(-u_\epsilon-g)\,dx\,dt\to 0 \qquad \text{as } \epsilon\to 0$$

in the subsequence. This implies that $\|\Phi\circ(-u-g)\|_{L^1(0,l)}=0$ and so $\Phi\circ(-u-g)=0$ a.e. By the definition of Φ, $u+g\geq 0$ a.e. Since u and g are in $C(0,T;H^2(0,l))$ it follows that $u+g\geq 0$ everywhere on $[0,l]\times[0,T]$. □

5.2.5 Hölder regularity of the penalty solution and convergence

We will show that the solution u_ϵ is uniformly Hölder continuous with exponent p from $[0,T]$ into $H^{1/2+\sigma}(0,l)$, and then u_ϵ is uniformly bounded in $C^p(0,T;H^{1/2+\sigma}(0,l))$ for some $p,\sigma>0$, where $\sigma/2+p<1/2$. This combined with the weak* convergence of N_ϵ in the space of measures will establish the complementarity conditions for the limit as $h\downarrow 0$.

Lemma 5.15. *Define $f_\alpha(t)=\sin(\alpha t)/\alpha^p$ and $g_\alpha(t)=\cos(\alpha t)/\alpha^p$ with $\alpha>0$ and $0<p\le 1$. Then f_α and g_α are Hölder continuous with exponent p with a Hölder constant that is independent of α.*

Proof. We want to show that for all $t_1,\ t_2$

$$|f_\alpha(t_2)-f_\alpha(t_1)|\le C_p|t_2-t_1|^p,$$

where C_p depends only on p and not on α. By the definition of $f_\alpha(t)$, we have

$$\begin{aligned}|f_\alpha(t_2)-f_\alpha(t_1)| &= \frac{1}{\alpha^p}|\sin(\alpha t_2)-\sin(\alpha t_1)|=\frac{2}{\alpha^p}\left|\cos\left(\frac{\alpha(t_2+t_1)}{2}\right)\sin\left(\frac{\alpha(t_2-t_1)}{2}\right)\right|\\ &\le \frac{2}{\alpha^p}\left|\sin\left(\frac{\alpha(t_2-t_1)}{2}\right)\right|\le\frac{2}{\alpha^p}\left|\sin\left(\frac{\alpha(t_2-t_1)}{2}\right)\right|^p \qquad (5.42)\end{aligned}$$

Note that

$$\sin\left(\frac{\alpha(t_2-t_1)}{2}\right)=\frac{(t_2-t_1)}{2}\int_0^\alpha\cos\left(\frac{(t_2-t_1)x}{2}\right)dx$$

Therefore from (5.42), we obtain

$$|f_\alpha(t_2)-f_\alpha(t_1)| \le \frac{2}{\alpha^p}\left|\sin\left(\frac{\alpha(t_2-t_1)}{2}\right)\right|^p$$

$$
\begin{aligned}
&\leq \frac{2}{\alpha^p}\left[\frac{|t_2-t_1|}{2}\left|\int_0^\alpha \cos\left(\frac{(t_2-t_1)x}{2}\right)dx\right|\right]^p \\
&\leq \frac{2}{\alpha^p}\frac{|t_2-t_1|^p}{2^p}\left[\int_0^\alpha dx\right]^p \\
&= \frac{2}{\alpha^p}\frac{|t_2-t_1|^p}{2^p}\alpha^p \\
&= 2^{1-p}|t_2-t_1|^p = C_p|t_2-t_1|^p,
\end{aligned}
$$

where $C_p = 2^{1-p}$. So the result follows for f_α.

The result for g_α follows since $g_\alpha(t) = \cos(\alpha t)/\alpha^p = \sin(\alpha t + \pi/2)/\alpha^p = f_\alpha(t+\pi/(2\alpha))$. □

Lemma 5.16. *The fundamental solution* $t \longmapsto w(\cdot, t; x^*)$ *is Hölder continuous from* $[0,T]$ *into* $H^{\sigma+1/2}(0,l)$ *with exponent* $0 < p \leq 1$ *and* $\sigma > 0$*, where* $\sigma/2 + p < 1/2$ *uniformly in* x^**.*

Proof. Applying Lemma 5.1,

$$
\begin{aligned}
&\|L^{(\sigma+1/2)/4}w\|^2_{L^2(0,l)} \\
&\quad = \left(\sum_{i=1}^\infty \frac{\sin(\lambda_i^{1/2}t)}{\lambda_i^{1/2}}\lambda_i^{(\sigma+1/2)/4}\phi_i(\cdot)\phi_i(x^*),\ \sum_{j=1}^\infty \frac{\sin(\lambda_j^{1/2}t)}{\lambda_j^{1/2}}\lambda_j^{(\sigma+1/2)/4}\phi_j(\cdot)\phi_j(x^*)\right) \\
&\quad = \sum_{i=1}^\infty \frac{\sin^2(\lambda_i^{1/2}t)}{\lambda_i}\lambda_i^{(\sigma+1/2)/2}\phi_i(x^*)^2.
\end{aligned}
$$

We claim that $\|w(t_2)-w(t_1)\|^2_{H^{\sigma+1/2}(0,l)} \leq C|t_2-t_1|^p$ for some constant C. By Proposition 5.3, we have

$$
\begin{aligned}
&\|w(t_2)-w(t_1)\|^2_{H^{\sigma+1/2}(0,l)} \\
&\quad = \sum_{i=1}^\infty \phi_i(x^*)^2\frac{[\sin(\lambda_i^{1/2}t_2)-\sin(\lambda_i^{1/2}t_1)]^2}{\lambda_i}\lambda_i^{(\sigma+1/2)/2} \\
&\qquad + \sum_{j=1}^\infty \phi_j(x^*)^2\frac{[\sin(\lambda_j^{1/2}t_2)-\sin(\lambda_i^{1/2}t_1)]^2}{\lambda_j}
\end{aligned}
$$

$$= \sum_{i=1}^{\infty} \left[\frac{\sin(\lambda_i^{1/2} t_2) - \sin(\lambda_i^{1/2} t_1)}{(\lambda_i^{1/2})^p} \right] \left(\lambda_i^{\sigma/2+1/4-1+p} + \lambda_i^{-1+p} \right).$$

We claim that $\|w(t_2) - w(t_1)\|^2_{H^{\sigma+1/2}(0,l)} \leq C|t_2 - t_1|^p$ for some constant.

$$\begin{aligned} \|w(\cdot, t_2; x^*) - w(\cdot, t_1; x^*)\|^2_{H^{\sigma+1/2}(0,l)} &\leq C \sum_{i=1}^{\infty} |t_2 - t_1|^{2p} \left(\lambda_i^{\sigma/2-3/4+p} + \lambda_i^{-1+p} \right) \\ &\leq C|t_2 - t_1|^{2p} \sum_{i=1}^{\infty} \left(\lambda_i^{\sigma/2-3/4+p} + \lambda_i^{-1+p} \right) \end{aligned} \quad (5.43)$$

Applying (5.20), for large λ_i, we have $\lambda_i^{1/4} \sim (2i+1)\pi/(2l)$. Thus for sufficiently large i, $\lambda_i \sim Ci^4$. This implies that we can choose another $C > 0$ such that $\lambda_i \geq Ci^4$ for sufficiently large i.

Since the exponents of λ_i satisfy $\sigma/2 - 3/4 + p > -1 + p$, from (5.43), we have

$$\begin{aligned} &\|w(\cdot, t_2; x^*) - w(\cdot, t_1; x^*)\|^2_{H^{\sigma+1/2}(0,l)} \\ &\leq C|t_2 - t_1|^{2p} \left[\sum_{i=1}^{m} \left(\lambda_i^{\sigma/2-3/4+p} + \lambda_i^{-1+p} \right) + \sum_{i=m+1}^{\infty} \left(\lambda_i^{\sigma/2-3/4+p} + \lambda_i^{-1+p} \right) \right] \\ &\leq C|t_2 - t_1|^{2p} \left[\sum_{i=1}^{m} \left(\lambda_i^{\sigma/2-3/4+p} + \lambda_i^{-1+p} \right) + 2 \sum_{i=m+1}^{\infty} \left(\lambda_i^{\sigma/2-3/4+p} \right) \right] \\ &\leq C|t_2 - t_1|^{2p} \left[\sum_{i=1}^{m} \left(\lambda_i^{\sigma/2-3/4+p} + \lambda_i^{-1+p} \right) + 2C \sum_{i=m+1}^{\infty} \left(i^{4(\sigma/2-3/4+p)} \right) \right]. \end{aligned} \quad (5.44)$$

where m is an appropriate large number. So we have

$$\begin{aligned} &\|w(\cdot, t_2; x^*) - w(\cdot, t_1; x^*)\|_{H^{\sigma+1/2}(0,l)} \\ &\leq C|t_2 - t_1|^{p} \sqrt{\sum_{i=1}^{m} \left(\lambda_i^{\sigma/2-3/4+p} + \lambda_i^{-1+p} \right) + 2C \sum_{i=m+1}^{\infty} \left(i^{4(\sigma/2-3/4+p)} \right)}. \end{aligned} \quad (5.45)$$

By the integral test, the second term of inside of square root of (5.45) will be bounded if $4(\sigma/2 - 3/4 + p) < -1$. Therefore the fundamental solution w is Hölder continuous with exponent $0 < p \leq 1$ for some $\sigma > 0$, where $\sigma/2 + p < 1/2$. □

We define the space of Hölder continuous functions $C^p(0,T;H^{1/2+\sigma}(0,l))$ with the norm

$$\|u\|_{C^p(0,T;H^{1/2+\sigma})} = \|u\|_{L^2(0,T;L^2(0,l))} + \sup_{t_1 \neq t_2} \left\{ \frac{\|u(t_1) - u(t_2)\|_{H^{1/2+\sigma}(0,l)}}{|t_2 - t_1|^p} \right\}.$$

Lemma 5.17. *The approximate solution u_ϵ is uniformly bounded in $C^p(0,T;H^{1/2+\sigma}(0,l))$.*

Proof. Recall that the solution u_ϵ can be expressed as the integral equation (5.32) with $N_\epsilon = \varphi \circ (u_\epsilon - g)/\epsilon$:

$$\begin{aligned} u_\epsilon(x,t) &= \int_0^l w(x,t;x^*)v_1(x^*)\,dx^* + \int_0^l \frac{\partial w}{\partial t}(x,t;x^*)u_1(x^*)\,dx^* \\ &\quad + \int_0^t \int_0^l w(x,t-s;x^*)\left[f(x^*) + N_\epsilon(x^*,s)\right] dx^*\,ds. \end{aligned} \tag{5.46}$$

So to bound $\|u_\epsilon(\cdot,t_2) - u_\epsilon(\cdot,t_1)\|_{H^{1/2+\sigma}(0,l)}$ we bound the corresponding differences of each of the terms in (5.46). Suppose $t_1 \leq t_2$.

In the first term of right side of (5.24), applying Lemma 5.16,

$$\begin{aligned} &\left\| \int_0^l w(\cdot,t_2;x^*)v_1(x^*)dx^* - \int_0^l w(\cdot,t_1;x^*)v_1(x^*)dx^* \right\|_{H^{1/2+\sigma}(0,l)} \\ &\leq \int_0^l \|w(\cdot,t_2;x^*) - w(\cdot,t_1;x^*)\|_{H^{1/2+\sigma}(0,l)} |v_1(x^*)| dx^* \\ &\leq C|t_2 - t_1|^p \|v_1\|_{L^2(0,l)} \leq C|t_2 - t_1|^p, \end{aligned}$$

since $v_1 \in L^2(0,l)$. In the second term of right side of (5.46), we want to obtain a bound $C|t_2 - t_1|^p$ on $\|u_\epsilon(\cdot,t_2) - u_\epsilon(\cdot,t_1)\|_{H^{1/2+\sigma}(0,l)}$. Let $u_1(\cdot) = \sum_{i=1}(u_1)_i\phi_i(\cdot)$. Since $u_1 \in H^2(0,l)$,

$$|u_1|^2_{H^2(0,l)} = \sum_{i=1}^{\infty} \lambda_i (u_1)_i^2 < \infty. \tag{5.47}$$

Using Lemma 5.16,

$$\begin{aligned}
&\left\| \int_0^l \frac{\partial w}{\partial t}(\cdot, t_2; x^*) u_1(x^*)\, dx^* - \int_0^l \frac{\partial w}{\partial t}(\cdot, t_1; x^*) u_1(x^*)\, dx^* \right\|_{H^{1/2+\sigma}(0,l)} \\
&= \left\| \int_0^l \left(\frac{\partial w}{\partial t}(\cdot, t_2; x^*) - \frac{\partial w}{\partial t}(\cdot, t_1; x^*) \right) u_1(x^*) dx^* \right\|_{H^{1/2+\sigma}(0,l)} \\
&= \left\| \int_0^l \sum_{i=1}^{\infty} \left(\cos(\lambda_i^{1/2} t_2) - \cos(\lambda_i^{1/2} t_1) \right) \phi_i(\cdot)\phi_i(x^*) \cdot \sum_{j=1}^{\infty} (u_1)_j \phi_j(x^*) dx^* \right\|_{H^{1/2+\sigma}(0,l)} \\
&= \left\| \int_0^l \left(\frac{\partial w}{\partial t}(\cdot, t_2; x^*) - \frac{\partial w}{\partial t}(\cdot, t_1; x^*) \right) u_1(x^*) dx^* \right\|_{H^{1/2+\sigma}(0,l)}.
\end{aligned}$$

By the similar argument to Lemma 5.16, we have

$$\begin{aligned}
&\left\| L^{(\sigma+1/2)/4} \sum_{i=1}^{\infty} \left(\cos(\lambda_i^{1/2} t_2) - \cos(\lambda_i^{1/2} t_1) \right) \phi_i(\cdot)(u_1)_i \right\|_{L^2(0,l)}^2 \\
&= \sum_{i=1}^{\infty} \left(\cos(\lambda_i^{1/2} t_2) - \cos(\lambda_i^{1/2} t_1) \right)^2 \lambda_i^{(\sigma+1/2)/2} (u_1)_i^2 \\
&= \sum_{i=1}^{\infty} \left[\frac{\cos(\lambda_i^{1/2} t_2) - \cos(\lambda_i^{1/2} t_1)}{(\lambda_i^{1/2})^p} \right]^2 \lambda_i^{\sigma/2+1/4-1+p} \cdot \lambda_i (u_1)_i^2 \\
&\leq C|t_2 - t_1|^{2p} \left(\sum_{i=1}^{\infty} \lambda_i^{\sigma/2-3/4+p} \cdot \lambda_i (u_1)_i^2 \right).
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \sum_{i=1}^{\infty} \left(\cos(\lambda_i^{1/2} t_2) - \cos(\lambda_i^{1/2} t_1) \right) \phi_i(\cdot)(u_1)_i \right\|_{L^2(0,l)}^2 \\
&= \sum_{i=1}^{\infty} \left(\cos(\lambda_i^{1/2} t_2) - \cos(\lambda_i^{1/2} t_1) \right)^2 (u_1)_i^2 \\
&= \sum_{i=1}^{\infty} \left[\frac{\cos(\lambda_i^{1/2} t_2) - \cos(\lambda_i^{1/2} t_1)}{(\lambda_i^{1/2})^p} \right]^2 \lambda_i^{-1+p} \cdot \lambda_i (u_1)_i^2 \\
&\leq C|t_2 - t_1|^{2p} \left(\sum_{i=1}^{\infty} \lambda_i^{-1+p} \cdot \lambda_i (u_1)_i^2 \right).
\end{aligned}$$

Applying Proposition 5.3 and Lemma 5.15

$$\left\| \int_0^l \frac{\partial w}{\partial t}(\cdot, t_2; x^*) v_1(x^*)\, dx^* - \int_0^l \frac{\partial w}{\partial t}(\cdot, t_1; x^*) v_1(x^*)\, dx^* \right\|_{H^{1/2+\sigma}(0,l)} \leq C\, |t_2 - t_1|^p.$$

In the third and last term of (5.46), we have

$$\begin{aligned}
&\left\| \int_0^{t_2}\int_0^l w(\cdot,t_2-s;x^*)N_\epsilon(x^*,s)\,dx^*\,ds - \int_0^{t_1}\int_0^l w(\cdot,t_1-s;x^*)N_\epsilon(x^*,s)\,dx^*\,ds \right\|_{H^{1/2+\sigma}(0,l)}\\
&\quad\le \int_{t_1}^{t_2}\int_0^l \|w(\cdot,t_2-s;x^*)-w(\cdot,0;x^*)\|_{H^{1/2+\sigma}(0,l)}|N_\epsilon(x^*,s)|\,dx^*\,ds\\
&\qquad+\int_0^{t_1}\int_0^l \|w(\cdot,t_2-s;x^*)-w(\cdot,t_1-s;x^*)\|_{H^{1/2+\sigma}(0,l)}|N_\epsilon(x^*,s)|\,dx^*\,ds\\
&\quad\le \int_{t_1}^{t_2}\int_0^l C|t_2-s|^p\,|N_\epsilon(x^*,s)|\,dx^*\,ds+\int_0^{t_1}\int_0^l C|t_2-t_1|^p\,|N_\epsilon(x^*,s)|\,dx^*\,ds\\
&\quad\le C|t_2-t_1|^p\int_0^{t_2}\int_0^l |N_\epsilon(x^*,s)|\,dx^*\,ds \le C|t_2-t_1|^p\int_0^{T}\int_0^l |N_\epsilon(x^*,s)|\,dx^*\,ds,
\end{aligned}$$

and since $\int_0^T\int_0^l |N_\epsilon(x^*,s)|\,dx^*\,ds$ is bounded independently of ϵ, we have a bound $C|t_2-t_1|^p$ on $\|u_\epsilon(\cdot,t_2)-u_\epsilon(\cdot,t_1)\|_{H^{1/2+\sigma}(0,l)}$ that is independent of ϵ. Similarly, we have

$$\begin{aligned}
&\left\| \int_0^{t_2}\int_0^l w(\cdot,t_2-s;x^*)f(x^*)\,dx^*\,ds - \int_0^{t_1}\int_0^l w(\cdot,t_1-s;x^*)f(x^*)\,dx^*\,ds \right\|_{H^{1/2+\sigma}(0,l)}\\
&\quad= C|t_2-t_1|^p\|f\|_{L^2(0,l)} = C|t_2-t_1|^p,
\end{aligned}$$

since $f\in L^2(0,l)$. So using the definition of the norm of $C^p(0,T;H^{1/2+\sigma}(0,l))$, $\|u_\epsilon\|_{C^p(0,T;H^{1/2+\sigma}(0,l))}\le M$, where M does not depend on ϵ. □

In the previous Lemma 5.17, u_ϵ is uniformly bounded in $C^p(0,T;H^{1/2+\sigma}(0,l)$. As this space is compactly imbedded in $C([0,l]\times[0,T])$ (this follows directly from the Ascoli theorem [35, p. 57]), there exists a subsequence of u_ϵ that converges u strongly in $C([0,l]\times[0,T])$.

Lemma 5.18. *For any weakly converging subsequence of $(u_\epsilon)_{\epsilon>0}$ in $C^p(0,T;H^{1/2+\sigma}(0,l))$ (and there is at least one such subsequence) we have*

$$\int_0^T\int_0^l N_\epsilon\,(u_\epsilon+g)\,dx\,dt \to \int_0^T\int_0^l N\,(u+g)\,dx\,dt.$$

Thus there is a solution $u \in C(0,T;H^2_{cf}(0,l)) \cap W^{1,\infty}(0,T;L^2(0,l)) \cap C([0,l]\times[0,T])$ *of (5.3–5.8).*

Proof. Note that since $N_\epsilon \geq 0$, subsequence of $N_\epsilon \rightharpoonup^* N$ as measures, so $N \geq 0$ in the sense of measures. Also, the constraint condition $u+g \geq 0$ follows from Lemma 5.14 that $N_\epsilon(u_\epsilon + g) \leq 0$ since $N_\epsilon(x,t) > 0$ only when $u_\epsilon(x,t) + g(x) < 0$. Thus, we have

$$0 \geq \int_0^T \int_0^l N_\epsilon\,(u_\epsilon + g)\,dx\,dt \text{ and } \int_0^T \int_0^l N\,(u+g)\,dx\,dt \geq 0.$$

Note that $C^p(0,T;H^{1/2+\sigma}(0,l))$ is compactly imbedded in $C([0,T]\times[0,l])$ for $p,\,\sigma > 0$. To see this, suppose that B is a bounded subset of $C^p(0,T;H^{1/2+\sigma}(0,l))$. Then for each $s,\,t \in [0,T]$ there is a bound $\|z(t)\|_{H^{1/2+\sigma}(0,l)} \leq M$ and $\|z(t) - z(s)\|_{H^{1/2+\sigma}(0,l)} \leq M\,|t-s|^p$ for each $z \in B$. The set B is an equicontinuous set of functions into $H^{1/2+\sigma}(0,l)$ by the Hölder bound: for any $\epsilon > 0$ we can set $\delta = (\epsilon/M)^{1/p}$ so that $|s-t| < \delta$ implies that $\|z(t)-z(s)\|_{H^{1/2+\sigma}(0,l)} < \epsilon$. Furthermore, for each $t \in [0,T]$, the set $\{\, z(t) \mid z \in B \,\}$ is bounded in $H^{1/2+\sigma}(0,l)$ and therefore is compact in $C[0,l]$. Thus by the Ascoli theorem [35, pp.57–59] there is a uniformly convergent subsequence in $C(0,T;\,C(0,l))$ of any bounded sequence u_ϵ in $C^p(0,T;\,H^{1/2+\sigma}(0,l))$. Denote the subsequence which converges strongly in $C(0,T;\,C(0,l)) = C([0,T]\times[0,l])$, u_ϵ. Call this limit u so that $\|u_\epsilon - u\|_{C([0,T]\times[0,l])} \to 0$ as $\epsilon \downarrow 0$ in this subsequence. Thus applying Lemma 5.12,

$$\int_0^T \int_0^l N_\epsilon\, u_\epsilon\,dx\,dt \to \int_0^T \int_0^l N\,u\,dx\,dt,$$

as $\epsilon \downarrow 0$ in the subsequence. Since $g \in C[0,l]$, and N_ϵ converges weak* to N,

$$\int_0^T \int_0^l N_\epsilon(x,t)\,g(x)\,dx\,dt \to \int_0^T \int_0^l N(x,t)\,g(x)\,dx\,dt.$$

Combining these results gives

$$0 \geq \int_0^T \int_0^l N_\epsilon(u_\epsilon + g)\, dx\, dt \to \int_0^T \int_0^l N(u+g)\, dx\, dt \geq 0.$$

Now we see that the integral $\int_0^T \int_0^l N(u+g)\, dx\, dt = 0$ as it is the only non-negative number that is a limit of non-positive numbers. Thus there is a solution $u \in C(0,T; H^2_{cf}(0,l)) \cap W^{1,\infty}(0,T; L^2(0,l))$ of our problem (5.3–5.8). The proof is complete. $\square$

CHAPTER 6
EULER–BERNOULLI BEAM IN DYNAMIC CONTACT : TIME DISCRETIZATION

6.1 Formulation of the discrete-time problem

For a dynamical problem, time discretization is one of the useful numerical methods. In order to obtain a numerical formulation, we will employ the two numerical schemes on the time space:

- Elasticity (u_{xxxx}) - Midpoint rule is used
- Contact condition - Implicit Euler is used.

First we consider a partition of time:

$$0 = t_0 < t_1 < t_2 < \cdots < t_l < t_{l+1} < \cdots < T.$$

We denote by $u^l(x)$ numerical solution of displacement $u(x, t_l)$ and by $v^l(x)$ numerical solution of velocity $v(x, t_l)$ and $N^l(x)$ numerical solution of magnitude of contact force, $N(x, t_l)$, respectively at each discretized time $t_l = lh$. Then the time step size is $h = t_{l+1} - t_l$, for $l \geq 0$. From (5.3), we take $\rho A = EI = 1$ by proper scaling.

Using our numerical scheme we establish numerical formulation:

$$\frac{v^{l+1} - v^l}{h} = -\left(\frac{u^{l+1}_{xxxx} + u^l_{xxxx}}{2}\right) + f(x) + N^l, \tag{6.1}$$

$$\frac{u^{l+1} - u^l}{h} = \frac{v^{l+1} + v^l}{2}, \tag{6.2}$$

$$0 \leq N^l \quad \perp \quad u^{l+1} + g \geq 0, \tag{6.3}$$

where $u^l = u^l(x)$, $v^l = v^l(x)$, $N^l = N^l(x)$ for each $l \geq 0$.

6.2 Energy dissipation in semi-discrete case

In this Section, we will see that numerical formulations (6.1–6.3) cause energy dissipation. Indeed, energy is conserved when beam does not touch a rigid foundation and energy is dissipated when beam reaches to the rigid foundation as we shall see in the next Lemma. Now we define energy functional which is dependent on displacement u and velocity v:

$$E(u,v) = \frac{1}{2}\int_0^L \left(|v|^2 + |u_{xx}|^2\right)\,dx - \int_0^L f\cdot u\,dx. \tag{6.4}$$

The first term $\int_0^L |v|^2 dx$ is the kinetic energy, the second term $\int_0^L |u_{xx}|^2 dx$ is the elastic energy and the last term $-\int_0^L f\cdot u\,dx$ is the external potential energy. We will derive energy dissipation for our time-discretization.

Lemma 6.1. *In the semi-discrete case, energy is dissipated.*

Proof. We want to show that $E(u^{l+1}, v^{l+1}) \le E(u^l, v^l)$. Using (6.1–6.2), we have

$$\begin{aligned}\int_0^L \frac{|v^{l+1}|^2 - |v^l|^2}{2h}\,dx &= -\int_0^L \frac{(u^{l+1}_{xxxx} + u^l_{xxxx})(u^{l+1}-u^l)}{2h}\,dx \\ &\quad + \int_0^L \frac{f\cdot(u^{l+1}-u^l)}{h}\,dx + \int_0^L \frac{N^l\cdot(u^{l+1}-u^l)}{h}\,dx. \end{aligned}\tag{6.5}$$

Multiplying by h on the both side of (6.5) and using integration by parts and the boundary conditions, we obtain

$$\begin{aligned}\int_0^L \frac{|v^{l+1}|^2 - |v^l|^2}{2}\,dx &= -\int_0^L \frac{|u^{l+1}_{xx}|^2 - |u^l_{xx}|^2}{2}\,dx \\ &\quad + \int_0^L f\cdot(u^{l+1}-u^l)\,dx + \int_0^L N^l\cdot(u^{l+1}-u^l)\,dx.\end{aligned}$$

Thus from the LCP condition (6.3),

$$\begin{aligned}
&\frac{1}{2}\int_0^L \left(|v^{l+1}|^2 - |v^l|^2\right)\,dx \\
&\quad = -\frac{1}{2}\int_0^L \left(|u_{xx}^{l+1}|^2 - |u_{xx}^l|^2\right)\,dx + \int_0^L f\cdot(u^{l+1}-u^l)\,dx \\
&\qquad + \int_0^L N^l\cdot(u^{l+1}+g)\,dx - \int_0^L N^l\cdot(u^l+g)\,dx \\
&\quad \le -\frac{1}{2}\int_0^L \left(|u_{xx}^{l+1}|^2 - |u_{xx}^l|^2\right)\,dx + \int_0^L f\cdot(u^{l+1}-u^l)\,dx,
\end{aligned} \tag{6.6}$$

as $\int_0^L N^l\cdot(u^{l+1}+g)\,dx = 0$ by (6.3), but N^l and $u^l+g\ge 0$ so $\int_0^L N^l\cdot(u^l+g)\,dx \ge 0$. Therefore we have

$$\begin{aligned}
&\left(\frac{1}{2}\int_0^L \left(|v^{l+1}|^2 + |u_{xx}^{l+1}|^2\right)\,dx\right) - \int_0^L f\cdot u^{l+1}\,dx \\
&\quad \le \left(\frac{1}{2}\int_0^L |v^l|^2 + |u_{xx}^l|^2\,dx\right) - \int_0^L f\cdot u^l\,dx
\end{aligned}$$

as desired. □

From (6.6), we note that the energy E is conserved if $N^l = 0$, and energy is dissipated by the LCP condition (6.3) if $N^l(x) > 0$ for some $x \in (0, L)$. Assume that the initial energy is finite. Then Lemma 6.1 implies that $v^l \in L^2(0, L)$ and $u^l \in H^2_{cf}(0, L)$ for all $l \ge 1$ and $h > 0$, and that they are bounded in these spaces independently of l and $h > 0$.

6.3 Convergence of the time discretization

In this Section one time step solution for the numerical formulation is obtained algebraically. Also the convergence for our numerical scheme is investigated. Those are considered in the semi-discrete case: only time space is discretized. .

6.3.1 Convergence of the numerical scheme

As mentioned in Chapter 6, the fourth order differential operator $K = \partial^4/\partial x^4$ has the orthonormal basis ϕ_i with $\partial^4\phi_i/\partial x^4 = \lambda_i\phi_i$ and satisfying the homogeneous boundary conditions ($\phi_i(0) = \phi_i'(0) = \phi_i''(L) = \phi_i'''(L) = 0$). We order the eigenvalues λ_i so that $0 < \lambda_1 \leq \lambda_2 \leq, \cdots \leq \lambda_i \leq \cdots$ and $\lim_{i\to\infty}\lambda_i = \infty$. Properties of these eigenfunctions are discussed in Subsection 5.2.1, we can write the discrete-time solution quantities as

$$u^l(x) = \sum_{i=1}^{\infty} u_i^l\phi_i(x), \quad v^l(x) = \sum_{i=1}^{\infty} v_i^l\phi_i(x), \quad \text{and} \quad N^l(x) = \sum_{i=1}^{\infty} N_i^l\phi_i(x).$$

So using the above numerical solution expressions and the numerical formulation (6.1–6.2), we have

$$\frac{v_i^{l+1} - v_i^l}{h} = -\lambda_i\left(\frac{u_i^{l+1} + u_i^l}{2}\right) + N_i^l, \tag{6.7}$$

$$\frac{u_i^{l+1} - u_i^l}{h} = \frac{v_i^{l+1} + v_i^l}{2}. \tag{6.8}$$

Note that when we investigate the convergence of our numerical scheme, we will ignore the external body force $f(x)$.

Lemma 6.2. *From (6.7) and (6.8), u_i^{l+1} and v_i^{l+1} are expressed in terms of u_i^l and v_i^l for each $i \geq 1$ and each $l \geq 0$ in the following way:*

$$\begin{bmatrix} u_i^{l+1} \\ v_i^{l+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \lambda_i^{1/2} \end{bmatrix} \begin{bmatrix} \cos\chi_i & \sin\chi_i \\ -\sin\chi_i & \cos\chi_i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \lambda_i^{-1/2} \end{bmatrix} \begin{bmatrix} u_i^l \\ v_i^l \end{bmatrix} + \frac{hN_i^l}{1 + h^2\lambda_i/4}\begin{bmatrix} \frac{h}{2} \\ 1 \end{bmatrix}, \tag{6.9}$$

where $\chi_i = \chi(h\lambda_i^{1/2})$, i.e., function χ_i depends only on $h\lambda_i^{1/2}$.

Proof. From (6.8), we have

$$v_i^{l+1} = \frac{2}{h}(u_i^{l+1} - u_i^l) - v_i^l \tag{6.10}$$

Multiplying by h on (6.7) and plugging (6.10) into (6.7), we obtain

$$\frac{2}{h}(u_i^{l+1} - u_i^l) - 2v_i^l = -\frac{h}{2}(\lambda_i u_i^{l+1} + \lambda_i u_i^l) + hN_i^l. \tag{6.11}$$

Thus multiplying by $h/2$ on (6.11) gives

$$(1 + h^2\lambda_i/4)u_i^{l+1} = (1 - h^2\lambda_i/4)u_i^l + hv_i^l + h^2N_i^l/2.$$

So the discrete-time solution at the next step is

$$u_i^{l+1} = (1 + h^2\lambda_i/4)^{-1}\left[(1 - h^2\lambda_i/4)u_i^l + hv_i^l + h^2N_i^l/2\right].$$

In order to obtain the next step's velocity, we use (6.10):

$$\begin{aligned}
v_i^{l+1} &= \frac{2}{h}\left[(1 + h^2\lambda_i/4)^{-1}\left((1 - h^2\lambda_i/4)u_i^l + hv_i^l + h^2N_i^l/2\right) - u_i^l\right] - v_i^l \\
&= \frac{2/h}{1 + h^2\lambda_i/4}\left[(1 - h^2\lambda_i/4)u_i^l + hv_i^l + h^2N_i^l/2 - (1 + h^2\lambda_i/4)u_i^l\right] - v_i^l \\
&= \frac{2/h}{1 + h^2\lambda_i/4}\left[-\frac{h^2}{2}\lambda_i u_i^l + hv_i^l + \frac{h^2}{2}N_i^l - \frac{h}{2}(1 + h^2\lambda_i/4)v_i^l\right] \\
&= \frac{1}{1 + h^2\lambda_i/4}\left[-h\lambda_i u_i^l + (1 - h^2\lambda_i/4)v_i^l + hN_i^l\right].
\end{aligned}$$

Therefore solving the equations for u^{l+1} and v^{l+1} in terms of u^l and v^l gives:

$$\begin{bmatrix} u_i^{l+1} \\ v_i^{l+1} \end{bmatrix} = \frac{1}{1 + h^2\lambda_i/4}\begin{bmatrix} 1 - h^2\lambda_i/4 & h \\ -h\lambda_i & 1 - h^2\lambda_i/4 \end{bmatrix}\begin{bmatrix} u_i^l \\ v_i^l \end{bmatrix} + \frac{hN_i^l}{1 + h^2\lambda_i/4}\begin{bmatrix} \frac{h}{2} \\ 1 \end{bmatrix}.$$

The above system can be written as:

$$\begin{bmatrix} u_i^{l+1} \\ v_i^{l+1} \end{bmatrix} = \frac{1}{1 + h^2\lambda_i/4}\begin{bmatrix} 1 & 0 \\ 0 & \lambda_i^{\frac{1}{2}} \end{bmatrix}\begin{bmatrix} 1 - \frac{h^2}{4}\lambda_i & h\lambda_i^{1/2} \\ -h\lambda_i^{1/2} & 1 - \frac{h^2}{4}\lambda_i \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \lambda_i^{-\frac{1}{2}} \end{bmatrix}\begin{bmatrix} u_i^l \\ v_i^l \end{bmatrix}$$

$$+\frac{hN_i^l}{1+h^2\lambda_i/4}\begin{bmatrix} \frac{h}{2} \\ \\ 1 \end{bmatrix}.$$

Note that we have

$$\begin{aligned}\left(\frac{1-h^2\lambda_i/4}{1+h^2\lambda_i/4}\right)^2+\left(\frac{h\lambda_i^{1/2}}{1+h^2\lambda_i/4}\right)^2 &= \frac{h^4\lambda_i^2/16+h^2\lambda_i/2+1}{(1+h^2\lambda_i/4)^2} \\ &= \left(\frac{1+h^2\lambda_i/4}{1+h^2\lambda_i/4}\right)^2=1.\end{aligned}$$

So we can write

$$\sin\chi_i=\frac{h\lambda_i^{1/2}}{1+h^2\lambda_i/4}, \qquad \cos\chi_i=\frac{1-h^2\lambda_i/4}{1+h^2\lambda_i/4},$$

where $\chi_i=\chi(h\lambda_i^{1/2})$. Hence the result follows. □

Indeed, we can require that χ_i be restricted to $[0,\pi]$.

Remark 6.3. *Consider a sequence of vectors* $\mathbf{z}_{l+1}=\mathbf{C}\mathbf{z}_l+\mathbf{b}_l$, *for* $l\in\mathbf{N}$. *Then we have*

$$\mathbf{z}_l=\mathbf{C}^l\mathbf{z}_0+\sum_{j=0}^{l-1}\mathbf{C}^{l-1-j}\mathbf{b}_j.$$

It is easy to prove this formula using mathematical induction.

Lemma 6.4. *From (6.7) and (6.8),* u_i^l *for each* $l\geq 1$ *can be expressed as*

$$\begin{aligned}u_i^l &= u_i^0\cos(l\chi_i)+v_i^0\sin(l\chi_i)/\lambda_i^{1/2} \\ &\quad +\frac{h}{1+h^2\lambda_i/4}\sum_{j=0}^{l-1}\left(\frac{h\cos\{(l-1-j)\chi_i\}}{2}+\frac{\sin\{(l-1-j)\chi_i\}}{\lambda_i^{1/2}}\right)N_i^j,\end{aligned} \tag{6.12}$$

where u_i^0 *and* v_i^0 *are coefficients for the initial displacement and velocity, respectively.*

Proof. In order to apply Remark 6.3, we set

$$\mathbf{z}_l = \begin{bmatrix} u_i^l \\ v_i^l \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & \lambda_i^{1/2} \end{bmatrix} \begin{bmatrix} \cos\chi_i & \sin\chi_i \\ -\sin\chi_i & \cos\chi_i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \lambda_i^{-1/2} \end{bmatrix}, \mathbf{b}_l = \begin{bmatrix} \frac{h}{2} N_i^l \\ N_i^l \end{bmatrix}.$$

So from Lemma 6.2, we have

$$\begin{bmatrix} u_i^l \\ v_i^l \end{bmatrix} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & \lambda_i^{1/2} \end{bmatrix} \begin{bmatrix} \cos\chi_i & \sin\chi_i \\ -\sin\chi_i & \cos\chi_i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \lambda_i^{-1/2} \end{bmatrix} \right\}^l \begin{bmatrix} u_i^0 \\ v_i^0 \end{bmatrix} +$$

$$\frac{h}{1 + h^2\lambda_i/4} \sum_{j=0}^{l-1} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & \lambda_i^{1/2} \end{bmatrix} \begin{bmatrix} \cos\chi_i & \sin\chi_i \\ -\sin\chi_i & \cos\chi_i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \lambda_i^{-1/2} \end{bmatrix} \right\}^{l-1-j} \begin{bmatrix} \frac{h}{2} N_i^j \\ N_i^j \end{bmatrix}.$$

Note that

$$\begin{bmatrix} \cos\chi_i & \sin\chi_i \\ -\sin\chi_i & \cos\chi_i \end{bmatrix}$$

is a transformation matrix called the clockwise rotation through χ_i. By mathematical induction,

$$\begin{bmatrix} u_i^l \\ v_i^l \end{bmatrix} = \begin{bmatrix} \cos(l\chi_i) & \lambda_i^{-1/2}\sin(l\chi_i) \\ -\lambda_i^{1/2}\sin(l\chi_i) & \cos(l\chi_i) \end{bmatrix} \begin{bmatrix} u_i^0 \\ v_i^0 \end{bmatrix} +$$

$$\frac{h}{1 + h^2\lambda_i/4} \sum_{j=0}^{l-1} \begin{bmatrix} \cos\{(l-1-j)\chi_i\} & \lambda_i^{-1/2}\sin\{(l-1-j)\chi_i\} \\ -\lambda_i^{1/2}\sin\{(l-1-j)\chi_i\} & \cos\{(l-1-j)\chi_i\} \end{bmatrix} \begin{bmatrix} \frac{h}{2} N_i^j \\ N_i^j \end{bmatrix}.$$

Multiplying by row vector [1, 0] on the both side of the above system, the coefficient u_i^l of $u^l(x)$ is obtained as desired. □

We define the impulse response function (or fundamental solution of the time-discretization) for fixed $x^* \in (0, L)$ to be

$$w^l(x) = \sum_{i=1}^{\infty} w_i^l \phi_i(x),$$

where $w_i^l = (h\cos(l\chi_i)/2 + \sin(l\chi_i)/\lambda_i^{1/2})/(1 + h^2\lambda_i/4)$. Similar to the fundamental solution of the PDE system, we extend $w_i^l = 0$ for $l < 0$. Thus using this form of impulse response function with Lemma 6.4, we have

$$u_i^l = u_i^0 \cos(l\chi_i) + v_i^0 \frac{\sin(l\chi_i)}{\lambda_i^{1/2}} + h \sum_{j=0}^{l-1} w_i^{l-j-1} N_i^j. \tag{6.13}$$

Recalling the fundamental solution of the PDE system, we define impulse response function for fixed $x^* \in (0, L)$,

$$w^l(\cdot, x^*) = \sum_{i=1}^{\infty} w_i^l \phi_i(x^*)\phi_i(\cdot). \tag{6.14}$$

Lemma 6.5. *Using the impulse response function $w^l(\cdot)$, the discrete-time solution $u^l(\cdot)$ can be expressed as:*

$$u^l(\cdot) = \sum_{i=1}^{\infty} u_i^0 \cos(l\chi_i) \cdot \phi_i(\cdot) + \sum_{i=1}^{\infty} v_i^0 \frac{\sin(l\chi_i)}{\lambda_i^{1/2}} \phi_i(\cdot) + h \sum_{j=0}^{l-1} \int_0^L w^{l-j-1}(\cdot, x^*) N^j(x^*) dx^*.$$

Proof. Employing (6.13) for fixed $x^* \in (0, L)$, we have

$$\sum_{i=1}^{\infty} u_i^l \phi_i(\cdot) = \sum_{i=1}^{\infty} u_i^0 \cos(l\chi_i) \cdot \phi_i(\cdot) + \sum_{i=1}^{\infty} v_i^0 \frac{\sin(l\chi_i)}{\lambda_i^{1/2}} \phi_i(\cdot) + h \sum_{j=0}^{l-1} \sum_{i=1}^{\infty} w_i^{l-j-1} N_i^j \phi_i(\cdot).$$

Since $N^j(\cdot) = \sum_{r=1}^{\infty} N_r^j \phi_r(\cdot)$ and ϕ_i is orthonormal basis in $L^2(0, L)$, we have

$$N_i^j = \int_0^L \sum_{r=1}^{\infty} N_r^j \phi_r(x^*)\phi_i(x^*) dx^* = \int_0^L N^j(x^*)\phi_i(x^*) dx^*.$$

Thus

$$\begin{aligned} u^l(\cdot) &= \sum_{i=1}^{\infty} u_i^0 \cos(l\chi_i) \cdot \phi_i(\cdot) + \sum_{i=1}^{\infty} v_i^0 \frac{\sin(l\chi_i)}{\lambda_i^{1/2}} \phi_i(\cdot) \\ &\quad + h \sum_{j=0}^{l-1} \sum_{i=1}^{\infty} w_i^{l-j-1} \phi_i(\cdot) \int_0^L N^j(x^*)\phi_i(x^*) dx^* \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{\infty} u_i^0 \cos(l\chi_i) \cdot \phi_i(\cdot) + \sum_{i=1}^{\infty} v_i^0 \frac{\sin(l\chi_i)}{\lambda_i^{1/2}} \phi_i(\cdot) \\
&\quad + h \sum_{j=0}^{l-1} \int_0^L \sum_{i=1}^{\infty} w_i^{l-j-1} \phi_i(x^*) \phi_i(\cdot) N^j(x^*) dx^* \\
&= \sum_{i=1}^{\infty} u_i^0 \cos(l\chi_i) \cdot \phi_i(\cdot) + \sum_{i=1}^{\infty} v_i^0 \frac{\sin(l\chi_i)}{\lambda_i^{1/2}} \phi_i(\cdot) \\
&\quad + h \sum_{j=0}^{l-1} \int_0^L w^{l-j-1}(\cdot, x^*) N^j(x^*) dx^*.
\end{aligned}
$$

as required. □

We now need a Lemma giving some basic bounds on the function $\chi(s)$. These basic bounds will be used to establish a uniform Hölder continuity result for the discrete fundamental solution w, and then for the solution u_h of the discrete-time problem.

Lemma 6.6. *If* $\cos\chi(s) = (1 - s^2/4)/(1 + s^2/4)$ *and* $\sin\chi(s) = s/(1 + s^2/4)$*, then* $\chi(s) \leq s$ *for* $s \geq 0$*.*

Proof. Taking a derivative $\sin\chi(s)$ with respect to s,

$$
\frac{d\,\sin\chi(s)}{d\,\chi} \frac{d\,\chi}{d\,s} = \frac{d}{d\,s}\left(\frac{s}{1 + s^2/4}\right).
$$

Thus we have

$$
\begin{aligned}
\cos\chi \cdot \frac{d\,\chi}{d\,s} &= \frac{1 - s^2/4}{(1 + s^2/4)^2} \\
&= \frac{1}{1 + s^2/4} \cdot \frac{1 - s^2/4}{1 + s^2/4} = \frac{1}{1 + s^2/4} \cos\chi < \cos\chi.
\end{aligned}
$$

So if $s \neq 2$, $d\,\chi/d\,s < 1$. Since $\chi(0) = 0$, we have $\chi(s) \leq s$ for $s \geq 0$ by Proposition 5.8. If $s = 2$, $\cos\chi(2) = 0$. So $\chi(2) = \pi/2 < 2$. Therefore the result follows. □

Lemma 6.7. *The following uniform Hölder continuity property holds for $p = 2\gamma$, $0 < p \leq 1$:*

$$\left| \frac{\sin\{(l+r)\chi(h\lambda^{1/2})\} - \sin\{l\chi(h\lambda^{1/2})\}}{\lambda^\gamma} \right| \leq C_p \cdot (rh)^p,$$

where C_p is independent of h and λ.

Proof. Suppose $r \geq 1$. We have

$$\begin{aligned} \left| \frac{\sin\{(l+r)\chi(h\lambda^{1/2})\} - \sin\{l\chi(h\lambda^{1/2})\}}{\lambda^\gamma} \right| &= \frac{2}{\lambda^\gamma} \left| \cos\left\{ \frac{(2l+r)\chi}{2} \right\} \right| \left| \sin\left(\frac{r\chi}{2} \right) \right| \\ &\leq \frac{2}{\lambda^\gamma} \left| \sin\left(\frac{r\chi}{2} \right) \right|. \end{aligned}$$

Since $r\chi - \sin(r\chi/2) \geq 0$ for $r\chi \geq 0$, we have

$$\lambda^{-\gamma} \left| \sin\{(l+r)\chi\} - \sin(l\chi) \right| \leq 2\lambda^{-\gamma} \min(r\chi, 1).$$

Applying Lemma 6.6, for $h\lambda^{1/2} \leq 2$

$$\lambda^{-\gamma} \left| \sin\{(l+r)\chi\} - \sin(l\chi) \right| \leq 2\lambda^{-\gamma} \min(rh\lambda^{1/2}, 1), \tag{6.15}$$

and for $h\lambda^{1/2} \geq 2$, (6.15) also holds by inspection as $rh\lambda^{1/2} > 1$. Dividing by $(rh)^p$ on the both side of (6.15),

$$\frac{\lambda^{-\gamma} \left| \sin\{(l+r)\chi\} - \sin\{l\chi\} \right|}{(rh)^p} \leq 2\lambda^{-\gamma}(rh)^{-p} \min(rh\lambda^{1/2}, 1).$$

If $rh\lambda^{1/2} \leq 1$,

$$\begin{aligned} \lambda^{-\gamma} \left| \sin\{(l+r)\chi\} - \sin\{l\chi\} \right| (rh)^{-p} &\leq 2\lambda^{-\gamma}(rh)^{-p}(rh\lambda^{1/2}) \\ &= 2\lambda^{-\gamma+1/2}(rh)^{1-p} \\ &\leq 2\lambda^{-\gamma+1/2}\lambda^{p/2-1/2} = 2\lambda^{p/2-\gamma}. \end{aligned}$$

If $rh\lambda^{1/2} \geq 1$, $|\lambda^{-\gamma}\sin\{(l+r)\chi\} - \lambda^{-\gamma}\sin\{l\chi\}|\,(rh)^{-p} \leq 2\lambda^{-\gamma}(rh)^{-p} \leq 2\lambda^{p/2-\gamma}$. Thus putting $p = 2\gamma$, we have

$$\lambda^{-\gamma}\left|\sin\{(l+r)\chi(h\lambda^{1/2})\} - \sin\{l\chi(h\lambda^{1/2})\}\right| \leq 2(hr)^p,$$

as required. □

Let the value $u_h(\cdot,t)$ be a continuous piecewise linear interpolant of $u_h(\cdot,lh) = u^l$ and $u_h(\cdot,(l+1)h) = u^{l+1}$ for $t \in [lh,(l+1)h]$. Then recalling Lemma 6.4, $u_h(\cdot,lh)$ computed at step l is expressed as

$$\begin{aligned} u_h(\cdot,lh) &= \sum_{i=1}^{\infty} u_i^0 \cos(l\chi_i)\cdot\phi_i(\cdot) + \sum_{i=1}^{\infty} v_i^0 \frac{\sin(l\chi_i)}{\lambda_i^{1/2}}\phi_i(\cdot) \\ &\quad + h\sum_{j=0}^{l-1}\int_0^L w_h(\cdot,(l-j-1)h,x^*)N^j(x^*)dx^*, \end{aligned} \tag{6.16}$$

where $w_h(\cdot,lh,x^*) = \sum_{i=1}^{\infty}(h\cos(l\chi_i)/2 + \sin(l\chi_i)/\lambda_i^{1/2})\phi_i(\cdot)\phi_i(x^*)/(1+h^2\lambda_i/4)$. Now we define the discrete-time contact force $N_h(x,t)$ as

$$N_h(x,t) = h\sum_{j=0}^{\lfloor T/h\rfloor-1}\delta(t-(j+1)h)N^j(x),$$

where δ is the Dirac-δ function and $\lfloor T/h\rfloor$ is the number of time-steps. We also identify N_h with a non-negative Borel measure on $[0,L]\times[0,T]$ by

$$N_h(B) = \int_B N_h(x,t)\,dx\,dt,$$

where B is any Borel set in $[0,L]\times[0,T]$. The next Lemma shows that the Borel measures N_h can be expressed in another way. Indeed, it will play an important role in bounding the measure N_h.

Lemma 6.8. *From the definition of $N_h(x,t)$, we have*

$$\int_0^T \int_0^L N_h(x,t)\,dx\,dt = h \sum_{l=0}^{\lfloor T/h \rfloor -1} \int_0^L N^l(x)\,dx.$$

Proof. Using the definition N_h, we have

$$\begin{aligned}
\int_0^T \int_0^L N_h(x,t)\,dx\,dt &= \lim_{\epsilon\downarrow 0} \int_0^{T+\epsilon} \int_0^L N_h(x,t)\,dx\,dt \\
&= h \lim_{\epsilon\downarrow 0} \int_0^{T+\epsilon} \int_0^L \sum_{l=0}^{\lfloor T/h \rfloor -1} \delta(t-(l+1)h) N^l(x)\,dx\,dt \\
&= h \int_0^L \sum_{l=0}^{\lfloor T/h \rfloor -1} \lim_{\epsilon\downarrow 0} \left(\int_0^{T+\epsilon} \delta(t-(l+1)h)\,dt \right) N^l(x)\,dx \\
&= h \sum_{l=0}^{\lfloor T/h \rfloor -1} \int_0^L N^l(x)\,dx, \qquad (6.17)
\end{aligned}$$

as required. Note that ϵ is not dependent on h. □

Lemma 6.9. *The Borel measures N_h are uniformly bounded as measures on $[0,L] \times [0,T]$ as $h \downarrow 0$ for $v^l \in L^2(0,L)$ and $u^l \in H^2_{cf}(0,L)$.*

Proof. Multiplying (6.1) by h and ignoring the body force $f(x)$,

$$v^{l+1} - v^l = -\frac{h}{2}(u^{l+1}_{xxxx} + u^l_{xxxx}) + hN^l.$$

Then multiplying $x^2/2$ on the both side in (6.1) and taking integral on the both side in (6.1),

$$\int_0^L \frac{x^2}{2}(v^{l+1} - v^l)dx = -\frac{h}{2}\int_0^L \frac{x^2}{2}(u^{l+1}_{xxxx} + u^l_{xxxx})dx + h\int_0^L \frac{x^2}{2}N^l dx.$$

Thus taking sum over $l \geq 0$ and using an integration by parts,

$$h \sum_{l=0}^{\lfloor T/h \rfloor -1} \int_0^L \frac{x^2}{2} N^l dx = \sum_{l=0}^{\lfloor T/h \rfloor -1} \int_0^L \frac{x^2}{2}(v^{l+1} - v^l)dx + \frac{h}{2} \sum_{l=0}^{\lfloor T/h \rfloor -1} \int_0^L \frac{x^2}{2}(u^{l+1}_{xxxx} + u^l_{xxxx})dx$$

$$
\begin{aligned}
&= \sum_{l=0}^{\lfloor T/h \rfloor -1} \int_0^L \frac{x^2}{2}(v^{l+1} - v^l)dx + \frac{h}{2} \sum_{l=0}^{\lfloor T/h \rfloor -1} \int_0^L (u_{xx}^{l+1} + u_{xx}^l)dx \\
&\leq \frac{L^2}{2}(\|v^{\lfloor T/h \rfloor}\|_{L^2(0,L)} + \|v^0\|_{L^2(0,L)}) \\
&\qquad + CL^{1/2} \cdot \max_{0 \leq l \leq \lfloor T/h \rfloor} \|u_{xx}^l\|_{L^2(0,L)},
\end{aligned}
$$

where $C = T$ does not depend on time step size h.

We want to show that $\int_0^L N^l dx$ is bounded by $\int_0^L \frac{x^2}{2} N^l dx$ for all $l \geq 0$. Since $g(0) > 0$ and $u(0, t_{l+1}) = \partial u / \partial x(0, t_{l+1}) = 0$ and $u^{l+1} \in H_{cf}^2(0, L)$, there is an $\eta > 0$ such that $u^{l+1}(x) > -g(x)$ for all $x \in [0, \eta]$. So by LCP condition of the numerical formulation, $N^l = 0$ for $0 \leq x \leq \eta$. Since $N^l \geq 0$ and $x^2/2 \geq \eta^2/2 > 0$ for $[\eta, L]$, we have

$$
\int_0^L \frac{x^2}{2} N^l dx = \int_\eta^L \frac{x^2}{2} N^l dx \geq \frac{\eta^2}{2} \int_\eta^L N^l dx = \frac{\eta^2}{2} \int_0^L N^l dx.
$$

So by Lemma 6.8, the Borel measure N_h is bounded, independent of h as $h \downarrow 0$. The proof is complete. □

Lemma 6.10. *The discrete-time solution $t \mapsto u_h(\cdot, t)$ is uniformly Hölder continuous into $H^\beta(0, L)$ as $h \downarrow 0$ with an exponent $0 < p \leq 1$ and $\beta > 0$ in the following sense:*

$$
\|u_h(\cdot, (l + r)h) - u_h(\cdot, lh)\|_{H^\beta(0,L)} < C_p(rh)^p,
$$

for integers l and r, where $\beta/2 + p < 3/4$ and C_p is independent of h.

Proof. Applying (6.16), the last term of $u_h(\cdot, (l + r)h)$ becomes

$$
h \sum_{j=0}^{l+k-1} \int_0^L w_h(\cdot, (l + r - j - 1)h, x^*) N^j(x^*) dx^*. \tag{6.18}
$$

Similarly, last term of $u_h(\cdot, lh)$ becomes

$$\sum_{j=0}^{l-1} \int_0^L w_h(\cdot, (l-j-1)h, x^*) N^j(x^*) dx^*. \tag{6.19}$$

We denote (6.18) by (I) and (6.19) by (II). Thus using Lemma 6.7, we have

$$\begin{aligned}
&\|u_h(\cdot, (l+r)h) - u_h(\cdot, lh)\|_{H^\beta(0,L)} \\
&\quad \le \left\| \sum_{i=1}^{\infty} u_i^0 [\cos\{(l+r)\chi_i\} - \cos(l\chi_i)] \phi_i(x) \right\|_{H^\beta(0,L)} \\
&\qquad + \left\| \sum_{i=1}^{\infty} v_i^0 \frac{\sin\{(l+r)\chi_i\} - \sin(l\chi_i)}{\lambda_i^{1/2}} \phi_i(x) \right\|_{H^\beta(0,L)} \\
&\qquad + h \left\| \int_0^L [(\mathrm{I}) - (\mathrm{II})] dx^* \right\|_{H^\beta(0,L)}.
\end{aligned} \tag{6.20}$$

Since $u^0 \in H^2_{cf}(0, L)$, $|u^0|^2_{H^2(0,L)} = \sum_i \lambda_i (u_i^0)^2 < \infty$. Using Proposition 5.3, in the first term of (6.20) we have

$$\begin{aligned}
&\| \sum_{i=1}^{\infty} u_i^0 [\cos\{(l+r)\chi_i\} - \cos(l\chi_i)] \phi_i(x) \|^2_{H^\beta(0,L)} \\
&\quad = \sum_{i=1}^{\infty} [\cos\{(l+r)\chi_i\} - \cos(l\chi_i)]^2 \lambda_i^{\beta/2} (u_i^0)^2 \\
&\qquad + \sum_{i=1}^{\infty} [\cos\{(l+r)\chi_i\} - \cos(l\chi_i)]^2 (u_i^0)^2 \\
&\quad = \sum_{i=1}^{\infty} \left[\frac{\cos\{(l+r)\chi_i\} - \cos(l\chi_i)}{\lambda_i^{p/2}} \right]^2 \lambda_i^{\beta/2+p-1} \cdot \lambda_i (u_i^0)^2 \\
&\qquad + \sum_{i=1}^{\infty} \left[\frac{\cos\{(l+r)\chi_i\} - \cos(l\chi_i)}{\lambda_i^{p/2}} \right]^2 \lambda_i^{p-1} \cdot \lambda_i (u_i^0)^2 \\
&\quad = (rh)^{2p} \left[\sum_{i=1}^{\infty} \lambda_i (u_i^0)^2 \lambda_i^{\beta/2+p-1} + \sum_{i=1}^{\infty} \lambda_i (u_i^0)^2 \lambda_i^{p-1} \right] \\
&\quad = (rh)^{2p} \sum_{i=1}^{\infty} \lambda_i (u_i^0)^2 \left[\lambda_i^{\beta/2+p-1} + \lambda_i^{p-1} \right].
\end{aligned}$$

Similarly, in the second term of (6.20), we have

$$\left\|\sum_{i=1}^{\infty} v_i^0 \frac{\sin\{(l+)\chi_i\} - \sin(l\chi_i)}{\lambda_i^{1/2}}\phi_i(x)\right\|^2_{H^\beta(0,L)}$$
$$= \sum_{i=1}^{\infty}\left[v_i^0 \frac{\sin\{(l+r)\chi_i\} - \sin(l\chi_i)}{\lambda_i^{p/2}}\right]^2 \lambda_i^{\beta/2+p-1}$$
$$+ \sum_{i=1}^{\infty}\left[v_i^0 \frac{\sin\{(l+r)\chi_i\} - \sin(l\chi_i)}{\lambda_i^{p/2}}\right]^2 \lambda_i^{-1+p}$$
$$= (rh)^{2p}\left[\sum_{i=1}^{\infty}(v_i^0)^2\lambda_i^{\beta/2+p-1} + \sum_{i=1}^{\infty}(v_i^0)^2\lambda_i^{p-1}\right]$$
$$= (rh)^{2p}\sum_{i=1}^{\infty}\lambda_i(v_i^0)^2\left[\lambda_i^{\beta/2+p-1} + \lambda_i^{p-1}\right].$$

Note that since $v^0 \in L^2(0,L)$, $\|v^0\|^2_{L^2(0.L)} = \sum_i (v_i^0)^2 < \infty$. Similarly in the third term of (6.20),

$$h\left\|\int_0^L [(\mathrm{I}) - (\mathrm{II})]dx^*\right\|_{H^\beta(0,L)}$$
$$\leq h\int_0^L \|(\mathrm{I}) - (\mathrm{II})\|_{H^\beta(0,L)}dx^*$$
$$\leq h\int_0^L \left\|\sum_{j=0}^{l-1}[w_h(\cdot,(l+r-j-1)h,x^*) - w_h(\cdot,(l-j-1)h,x^*)]N^j(x^*)\right\|_{H^\beta(0,L)} dx^*$$
$$+h\int_0^L \left\|\sum_{j=l}^{l+k-1} w_h(\cdot,(l+r-j-1)h,x^*)N^j(x^*)\right\|_{H^\beta(0,L)} dx^*. \tag{6.21}$$

Recall that $w_h(\cdot,lh,x^*) = \sum_{i=1}^{\infty}(h\cos(l\chi_i)/2 + \sin(l\chi_i)/\lambda_i^{1/2})\phi_i(\cdot)\phi_i(x^*)/(1+h^2\lambda_i/4)$ and $\max_{0\leq x\leq L}|\phi_i(x)| \leq M$. Note that $\lambda_i^{1/2}h/(1+\lambda_i h^2/4) \leq 1$ and $1/(1+\lambda_i h^2/4) \leq 1$ for $\lambda_i^{1/2}h \geq 0$. In the first term of (6.21) for $0 \leq j \leq l-1$, we have

$$\|[w_h(\cdot,(l+r-j-1)\chi_i,x^*) - w_h(\cdot,(l-j-1)\chi_i,x^*)]\, N^j(x^*)\|^2_{H^\beta(0,L)}$$
$$\leq \sum_{i=1}^{\infty}(\lambda_i^{\beta/2}+1)\left[\frac{\lambda_i h^2\cdot A_i^2}{4\lambda_i(1+h^2\lambda_i/4)^2} + \frac{A_iB_i}{\lambda_i(1+h^2\lambda_i/4)^2}\right]|\phi_i(x^*)|^2|N^j(x^*)|^2$$

$$
\begin{aligned}
&+\sum_{i=1}^{\infty}(\lambda_i^{\beta/2}+1)\frac{B_i^2}{\lambda_i(1+h^2\lambda_i/4)^2}|\phi_i(x^*)|^2|N^j(x^*)|^2 \\
&\le \frac{M^2}{4}|N^j(x^*)|^2\sum_{i=1}^{\infty}\left[\frac{A_i}{\lambda_i^{p/2}}\right]^2(\lambda_i^{\beta/2+p-1}+\lambda_i^{p-1}) \\
&+\sqrt{2}M^2|N^j(x^*)|^2\sum_{i=1}^{\infty}\left[\frac{B_i}{\lambda_i^{p/2}}\right]^2(\lambda_i^{\beta/2+p-1}+\lambda_i^{p-1}) \\
&+M^2|N^j(x^*)|^2\sum_{i=1}^{\infty}\left[\frac{B_i}{\lambda_i^{p/2}}\right]^2(\lambda_i^{\beta/2+p-1}+\lambda_i^{p-1}) \\
&\le (\frac{5}{4}+\sqrt{2})M^2|N^j(x^*)|^2(kh)^{2p}\sum_{i=1}^{\infty}(\lambda_i^{\beta/2+p-1}+\lambda_i^{p-1}). \qquad (6.22)
\end{aligned}
$$

where $A_i = |\cos\{(l+r-j-1)\chi_i\} - \cos\{(l-j-1)\chi_i\}|$ and $B_i = |\sin\{(l+r-j-1)\chi_i\} - \sin\{(l-j-1)\chi_i\}|$. Similarly in the second term of (6.21), for $l \le j \le l+k-1$ we have

$$
\begin{aligned}
&\|w_h(\cdot,(l+k-r-1)\chi_i,x^*)N^j(x^*)\|^2_{H^\beta(0,L)} \\
&\le C(rh)^{2p}\sum_{i=1}^{\infty}(\lambda_i^{\beta/2+p-1}+\lambda_i^{p-1}). \qquad (6.23)
\end{aligned}
$$

Note that for sufficiently large i, there exist $C>0$ such that $\lambda_i \ge Ci^4$. This was shown in Lemma 5.2 and by the integral test, the result follows, provided that $\beta/2+p < 3/4$. □

Note that the condition of Lemma 6.10 is the same case as for the penalty method.

Note that the Borel measures N_h on $[0,L]\times[0,T]$ are bounded. By *Riesz Representation Theorem* [49, 35], the dual space of $C([0,L]\times[0,T])$ is isometrically isomorphic to the space of Borel measures on $[0,L]\times[0,T]$. Then by *Alaoglu's Theorem* [47, p. 203], there is a subsequence of N_h which is weakly* convergent to N. We wish to show a corresponding result for u. As we saw in Lemma 6.10, u_h is bounded in $C^p(0,T;H^\beta(0,L))$. By the *Arzela–Ascoli Theorem* [35, pp. 57–59], that space is compactly imbedded in $C([0,L]\times[0,T])$. Thus there is a subsequence (of the subsequence in which $N^h \rightharpoonup^* N$) in which $u_h \to u$ in $C([0,L]\times[0,T])$. Since $u_h+g \geq 0$ for each $h>0$, it follows that $u+g\geq 0$.

We now want to show that the complementarity conditions $0 \leq N \perp u + g \geq 0$ hold in the weak sense. Now for any continuous $\Phi \geq 0$ on $[0,L]\times[0,T]$, $\int_{[0,L]\times[0,T]} \Phi\, N_h\, dx\, dt \geq 0$ since N_h is a non-negative measure. Since a subsequence of N_h converges weak* to N, $\int_{[0,L]\times[0,T]} \Phi\, N\, dx\, dt \geq 0$. Thus $N \geq 0$ in the sense of Borel measures. The condition that $u(x,t)+g(x)\geq 0$ holds as noted in the previous paragraph, and taking limits in the subsequence gives

$$0 = \int_0^L \int_0^T N_h(x,t)(u_h(x,t)+g(x))\, dx\, dt \to \int_0^T \int_0^L N(x,t)(u(x,t)+g(x))\, dx\, dt, \tag{6.24}$$

and so the LCP condition holds.

Lemma 6.11. *In a certain subsequence with $h \downarrow 0$ the time-discretized functions u_h, v_h, and N_h converge to a solution, u_h uniformly in $C([0,L]\times[0,T])$, v_h weak* in $L^\infty(0,T;L^2(0,L))$ and N_h weak* in the space of measures on $[0,L]\times[0,T]$.*

Proof. By Lemma 6.9, $N_h \rightharpoonup^* N$ as measure. Since $N_h \geq 0$, $N \geq 0$. Then since $C^p(0,T;H^\beta(0,L))$ is compactly imbedded in $C([0,L]\times[0,T])$, by the Arzela–Ascoli Theorem [35, pp. 57–59] there exists a suitable subsequence of u_h such that $u_h \to u$ in $C([0,L]\times[0,T])$. We also denote this subsequence by u_h, and restrict our attention to this subsequence.

Since v_h is uniformly bounded in $L^\infty(0,T;L^2(0,L))$ and $L^\infty(0,T;L^2(0,L))$ is identified with the dual space of $L^1(0,T;L^2(0,L))$, by Alaoglu's theorem there is a weak* converging subsequence, also denoted v_h and restrict attention to this subsequence.

We want to show that for such N and u, (6.24) is satisfied. Since $u_h(\cdot,t)$ is an interpolant of $u_h(\cdot,lh)$ and $u_h(\cdot,(l+1)h)$, and $N_h(x,t) = h\sum_{j=0}^{\lfloor T/h\rfloor-1}\delta(t-(j+1)h)N^j(x)$ for $t\in[lh,\,(l+1)h]$, we have

$$\begin{aligned}
&\int_0^T\int_0^L N_h(x,t)(u_h(x,t)+g(x))\,dx\,dt\\
&\quad= h\int_0^T\int_0^L\left[\sum_{j=0}^{\lfloor T/h\rfloor-1}\delta(t-(j+1)h)N^j(x)\right](u_h(x,t)+g(x))\,dxdt\\
&\quad= h\int_0^L\left[\sum_{j=0}^{\lfloor T/h\rfloor-1}\int_0^T\delta(t-(j+1)h)(u_h(x,t)+g(x))\,dt\right]N^j(x)\,dx\\
&\quad= h\int_0^L\sum_{j=0}^{\lfloor T/h\rfloor-1}N^j(x)(u^{j+1}(x)+g(x))\,dx = 0.
\end{aligned}$$

So we obtain

$$0=\int_0^T\int_0^L N_h(x,t)(u_h(x,t)+g(x))\,dxdt \to \int_0^T\int_0^L N(x,t)(u(x,t)+g(x))\,dxdt = 0.$$

The proof is completed. □

6.3.2 Do the discrete-time solutions converge strongly?

While we cannot fully answer this question at this time, we will lay the groundwork in this Subsection for the numerical evidence to be presented later for strong convergence.

We recall the numerical solution $u^l(x)$ at each discretized time t_l:

$$u_l(x) = \sum_{i=1}^{\infty} \widehat{u}_i^l \phi_i(x). \tag{6.25}$$

In this Section we want to use $\widehat{u}_i^l$ to indicate coefficients of the eigenfunctions, in contrast to u_i^l which indicates coefficients of the FEM basis functions (which will be described in Section 7.1). Similarly we can write the velocity as $v^l(x) = \sum_{i=1}^{\infty} \widehat{v}_i^l \phi_i(x)$, and also force $f(x) = \sum_{i=1}^{\infty} \widehat{f}_i \phi_i(x)$. Note that we write $\widehat{u}_i^{l;h}$ and $\widehat{v}_i^{l;h}$ instead of $\widehat{u}_i^l$ and $\widehat{v}_i^l$, respectively, in order to show the dependence on $h > 0$ more explicitly. Then we consider numerical trajectories $u_h(x,t)$ by piecewise continuous linear interpolation of $u_h(x,t_l) = u^{l;h}(x)$ and $v_h(x,t)$ by piecewise continuous linear interpolation of $v_h(x,t_l) = v^{l;h}(x)$ for each $l \geq 0$. So we express these as

$$u_h(x,t) = \sum_{i=1}^{\infty} \widehat{u}_i^h(t)\phi_i(x) \text{ and } v_h(x,t) = \sum_{i=1}^{\infty} \widehat{v}_i^h(t)\phi_i(x).$$

Then the value of $\widehat{u}_i^h(t)$ is the linear interpolant of $\widehat{u}_i^h(lh) = \widehat{u}_i^{l;h}$ and $\widehat{u}_i^h((l+1)h) = \widehat{u}_i^{l+1;h}$ for $t \in [lh, (l+1)h]$.

Let $\mathbf{u}^{l;h} = (\widehat{u}_1^{l;h}, \widehat{u}_2^{l;h}, \widehat{u}_3^{l;h}, \cdots)$ and $\mathbf{v}^{l;h} = (\widehat{v}_1^{l;h}, \widehat{v}_2^{l;h}, \widehat{v}_3^{l;h}, \cdots)$ and $\mathbf{f} = (\widehat{f}_1, \widehat{f}_2, \widehat{f}_3, \cdots)$ and $\boldsymbol{\omega}^{l;h} = (\omega_1^{l;h}, \omega_2^{l;h}, \omega_3^{l;h}, \cdots)$, where $\omega_i^{l;h} = \lambda_i^{1/2} \widehat{u}_i^{l;h}$ for $i \geq 1$. We use notation ℓ^2 as the Hilbert space of sequences $\mathbf{x} = (x_1, x_2, x_3, \cdots)$, where $\|\mathbf{x}\|_{\ell^2} = \sqrt{\sum_i^{\infty} |x_i|^2} < \infty$.

Lemma 6.12. *The energy is expressed in the discrete form:*

$$E(u^l, v^l) = \sum_{i=1}^{\infty} \left(\left(\widehat{v}_i^{l;h} \right)^2 + \lambda_i \left(\widehat{u}_i^{l;h} \right)^2 - f_i \widehat{u}_i^{l;h} \right). \tag{6.26}$$

Furthermore, $\boldsymbol{\omega}^{l;h}$, $\mathbf{v}^{l;h} \in \ell^2$ *and those are uniformly bounded in* ℓ^2.

Proof. Recalling the energy (6.4), we have the energy functional

$$E(u^l, v^l) = \frac{1}{2}\int_0^L \left(|v^l|^2 + |u^l_{xx}|^2 - f \cdot u^l\right) dx.$$

Then the kinetic energy becomes

$$\frac{1}{2}\int_0^L |v^l|^2 dx = \frac{1}{2}\int_0^L \left[\sum_{i=1}^{\infty} \widehat{v}_i^{l;h}\phi_i(x) \sum_{j=1}^{\infty} \widehat{v}_j^{l;h}\phi_j(x)\right] dx = \frac{1}{2}\sum_{i=1}^{\infty}\left(\widehat{v}_i^{l;h}\right)^2.$$

Using integration by parts and the boundary condition and recalling decomposition into eigenfunctions, the elastic energy becomes

$$\frac{1}{2}\int_0^L |u^l_{xx}|^2 dx = \frac{1}{2}\int_0^L \left[\sum_{i=1}^{\infty} \widehat{u}_i^{l;h}\phi_i''(x) \sum_{j=1}^{\infty} \widehat{u}_j^{l;h}\phi_j''(x)\right] dx = \frac{1}{2}\sum_{i=1}^{\infty}\lambda_i\left(\widehat{u}_i^{l;h}\right)^2.$$

Similarly, we have the potential energy

$$\int_0^l f \cdot u^l \, dx = \sum_{i=1}^{\infty} \widehat{f}_i \widehat{u}_i^{l;h}.$$

Thus (6.26) is obtained. Since the initial energy is bounded and $f \in L^2(0, L)$, $\boldsymbol{\omega}^{l;h}$, $\mathbf{v}^{l;h} \in \ell^2$ for $l \geq 1$ and those are uniformly bounded in ℓ^2. □

Now suppose that we do not consider body force f in the energy function. Since for $t_l \leq t \leq t_{l+1}$, $u_h(x,t)$ is the interpolant of $\widehat{u}^{l;h}$ and $\widehat{u}^{l+1;h}$ and $v_h(x,t)$ is the interpolant of $\widehat{v}^{l;h}$ and $\widehat{v}^{l+1;h}$, by the energy boundness we have

$$\sum_{i=1}^{\infty}\lambda_i\left(\widehat{u}_i^h(t)\right)^2 \text{ and } \sum_{i=1}^{\infty}\left(\widehat{v}_i^h(t)\right)^2 < \infty.$$

So $\boldsymbol{\omega}^h$, $\mathbf{v}^h \in \ell^2$ and are uniformly bounded in ℓ^2, where $\boldsymbol{\omega}^h = (\omega_1^h(t), \omega_2^h(t), \omega_3^h(t), \cdots)$ for $\omega_i^h(t) = \lambda_i \widehat{u}^h(t)$ and $\mathbf{v}^h = (\widehat{v}_1^h(t), \widehat{v}_2^h(t), \widehat{v}_3^h(t), \cdots)$. Thus there are a subsequence of $\mathbf{v}^h$ and a subsequence of $\boldsymbol{\omega}^h$ which are convergent to $\mathbf{v}(t)$ and $\boldsymbol{\omega}(t)$, respectively in ℓ^2, as $h \downarrow 0$. These facts induce the next Lemma 6.13.

By inspection of the eigenfunctions, the frequency of oscillation is proportional on $\lambda^{1/4}$. So high frequency modes correspond to large eigenvalues and low frequency modes correspond to small eigenvalues. Also, only the elastic energy defines the modes, since they are eigenfunctions of the fourth order operator $K = \partial^4/\partial x^4$ in the in the continuous case or eigenvectors of $\mathbf{M}^{-1}\mathbf{K}$ in the fully discretized case, which will be considered in the Section 7.4. In the next Lemma, it is shown that the amount of energy in the high frequency modes is almost zero under the assumption of the strong convergence. In the physical point of view, this implies that high frequency modes would be converted to heat. In the Section 7.5, Lemma 6.13 will be supported by numerical evidence. The detailed arguments will be presented in the Section 7.5.

Lemma 6.13. *Let $t \in [lh, (l+1)h]$ for any $l \geq 1$. Suppose that $\boldsymbol{\omega}^{l;h} \to \boldsymbol{\omega}(t)$ and $\mathbf{v}^{l;h} \to \mathbf{v}(t)$ (strongly) in ℓ^2, as $h \downarrow 0$, $lh \to t$. Then we have*

$$\lim_{c\to\infty} \limsup_{h\downarrow 0} \frac{1}{2} \sum_{i;i\geq c} \left(|\widehat{v}_i^{l;h}|^2 + \lambda_i |\widehat{u}_i^{l;h}|^2 \right) = 0.$$

Proof. For the fixed $l \geq 1$ and any $c \geq 1$, we obtain

$$\begin{aligned}
\left(\sum_{i=c}^{\infty} |\omega_i^{l;h}|^2 \right)^{1/2} &= \left(\sum_{i=c}^{\infty} |\omega_i^{l;h} - \omega_i(t) + \omega_i(t)|^2 \right)^{1/2} \\
&\leq \left(\sum_{i=c}^{\infty} |\omega_i^{l;h} - \omega_i(t)|^2 \right)^{1/2} + \left(\sum_{i=c}^{\infty} |\omega_i(t)|^2 \right)^{1/2}.
\end{aligned}$$

Since $\|\boldsymbol{\omega}^{l;h} - \boldsymbol{\omega}(t)\|_{\ell^2} \to 0$ as $h \downarrow 0$, $lh \to t$,

$$\begin{aligned}
\limsup_{h\downarrow 0} \left(\sum_{i=c}^{\infty} |\omega_i^{l;h}|^2 \right)^{1/2} &\leq \limsup_{h\downarrow 0} \left[\left(\sum_{i=c}^{\infty} |\omega_i^{l;h} - \omega_i(t)|^2 \right)^{1/2} + \left(\sum_{i=c}^{\infty} |\omega_i(t)|^2 \right)^{1/2} \right] \\
&= \limsup_{h\downarrow 0} \left(\sum_{i=c}^{\infty} |\omega_i(t)|^2 \right)^{1/2}. \qquad (6.27)
\end{aligned}$$

Since $\sum_{i=c}^{\infty} |\omega_i(t)|^2 = \|\boldsymbol{\omega}(t)\|_{\ell^2}^2 - \sum_{i=1}^{c-1} |\omega_i(t)|^2$, we have

$$\lim_{c\to\infty} \sum_{i=c}^{\infty} |\omega_i(t)|^2 = \|\boldsymbol{\omega}(t)\|_{\ell^2}^2 - \lim_{c\to\infty} \sum_{i=1}^{c-1} |\omega_i(t)|^2 = \|\boldsymbol{\omega}(t)\|_{\ell^{2-}}^2 - \|\boldsymbol{\omega}(t)\|_{\ell^2}^2 = 0. \tag{6.28}$$

Thus combining (6.27) with (6.28),

$$\lim_{c\to\infty} \limsup_{h\downarrow 0} \left(\sum_{i=c}^{\infty} |\omega_i^{l;h}|^2 \right)^{1/2} \leq 0.$$

Since $|\omega_i^{l;h}|^2 = \lambda_i |\widehat{u}_i^{l;h}|^2 \geq 0$ for each $i \geq 1$, we have for elastic energy

$$\lim_{c\to\infty} \limsup_{h\downarrow 0} \sum_{i;i\geq c} |\omega_i^{l;h}|^2 = \lim_{c\to\infty} \limsup_{h\downarrow 0} \sum_{i;i\geq c} \lambda_i |\widehat{u}_i^{l;h}|^2 = 0.$$

Similar to the above argument, we have for kinetic energy

$$\lim_{c\to\infty} \limsup_{h\downarrow 0} \sum_{i;i\geq c} |\widehat{v}_i^{l;h}|^2 = 0.$$

Therefore the result follows. □

We note that in general, $\mathbf{u}^l \rightharpoonup \mathbf{u}$ in ℓ^p with $1 < p < \infty$ if and only if $\lim_{l\to\infty} u_i^l = u_i$, for $i \geq 1$ and $\sup_{1\leq l<\infty} \|\mathbf{u}^l\|_{\ell^p} < \infty$.

CHAPTER 7
EULER–BERNOULLI BEAM IN DYNAMIC CONTACT : DISCRETIZATION IN TIME AND SPACE

7.1 Finite element method with B-splines

The Finite Element Method is one of the most popular numerical methods for solving a static elliptic boundary value problems. So we will approximate the solution in the spatial domain $[0, L]$, using the Finite Element Method [6, 19]. We partition the domain $[0, L]$ into

$$0 = x_0 < x_1 < x_2 < x_3 < x_4 < \cdots < x_{m+1} = L.$$

We denote $k = x_{i+1} - x_i$ as size of subinterval $[x_{i+1}, x_i]$ for $i \geq 1$. Let

$$V = H_{cf}(0, L) = \{u \in H^2(0, L) \mid u(0) = u'(0) = 0\},$$

where $H_{cf}(0, L)$ is a subset of Sobolev space $H^2(0, L)$, using the same norm. We choose B-spline functions $\psi_i(x)$, $1 \leq i \leq m + 1$ for the basis functions. The B-spline will be a cubic spline [4, pp. 166–176] with nodes x_i, $i = 1, 2, 3, \ldots, m + 1$. Note that unlike the usual piecewise continuous linear basis function, we need $m + 1$ basis functions from the construction of B-spline. Thus the finite element space becomes

$$V_k = \text{span}\{\, \psi_i \mid 1 \leq i \leq m + 1 \,\}.$$

These basis function will need to be in H^2. Thus we can construct the standard B-spline function $B(s)$, according to the property of B-splines and the condition that

$B(0) = 1$, $B(s) = B(-s)$, and $B'(0) = 0$:

$$B(s) = \frac{2}{3} \begin{cases} 1 + \frac{3}{4}|s|^3 - \frac{3}{2}|s|^2 & \text{if } |s| \le 1, \\ \frac{1}{4}(2 - |s|)^3 & \text{if } 1 \le |s| \le 2, \\ 0 & \text{if } |s| \ge 2. \end{cases}$$

Thus $B(s)$ is piecewise cubic on interval $[i,\ i+1]$ for $i \in \mathbf{Z}$. We set each basis function, based on shifted B-splines, to be:

$$\psi_i(x) = B(\frac{x - x_i}{k}),$$

where $x_i = i\,k$, $1 \le i \le m + 1$. Especially, in order to satisfy essential boundary condition, we need to change the first basis function into

$$\psi_1(x) = 2\{B(\frac{x}{k} + 1) + B(\frac{x}{k} - 1)\} - B(\frac{x}{k}).$$

For other basis functions $i = 2, 3, \cdots, m + 1$, we use usual shifted B-splines:

$$\psi_i(x) = B(\frac{x}{k} - i).$$

See Figure 7.1 for the construction of basis functions with B-splines.

Employing the finite element method, we write a approximate solution u^l, v^l as

$$u^l(x) = \sum_{i=1}^{m+1} u_i^l \psi_i(x) \text{ and } v^l(x) = \sum_{i=1}^{m+1} v_i^l \psi_i(x). \tag{7.1}$$

Using (6.2), we have numerical motion equation

$$\frac{2}{h^2}u^{l+1} + \frac{1}{2}u_{xxxx}^{l+1} = \frac{2}{h^2}u^l - \frac{1}{2}u_{xxxx}^l + \frac{2}{h}v^l + f(x) + N^l. \tag{7.2}$$

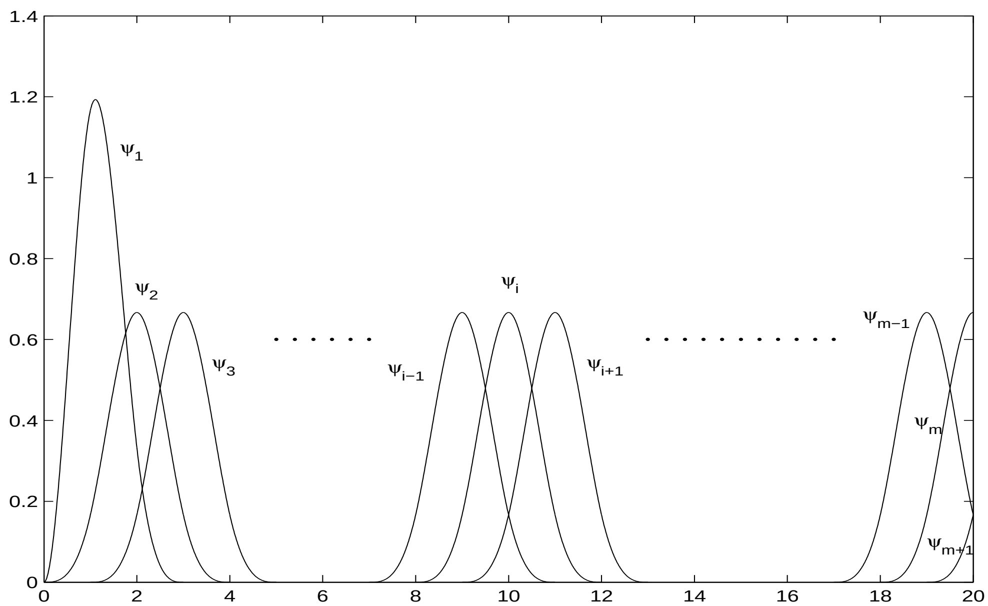

Figure 7.1: Construction of basis functions with B-splines.

Then we will put approximate solutions $u^l = \sum_{j=1}^{m+1} u_j^l \psi_j(x)$, $v^l = \sum_{j=1}^{m+1} v_j^l \psi_j(x)$, $N^l = \sum_{j=1}^{m+1} N_j^l \psi_j(x)$. First multiplying by basis function $\psi_i(x)$ on both side of (7.2) and by integration by part, we have

$$\begin{aligned}
&\frac{2}{h^2}\sum_{j=1}^{m+1} u_j^{l+1}\int_0^L \psi_i\psi_j\,dx + \frac{1}{2}\sum_{j=1}^{m+1} u_j^{l+1}\int_0^L \psi_j''\psi_j''\,dx \\
&\quad = \frac{2}{h^2}\sum_{j=1}^{m+1} u_j^{l}\int_0^L \psi_i\psi_j\,dx - \frac{1}{2}\sum_{j=1}^{m+1} u_j^{l}\int_0^L \psi_j''\psi_j''\,dx \\
&\qquad - \frac{2}{h}\sum_{j=1}^{m+1} v_j^{l}\int_0^L \psi_i\psi_j\,dx + \int_0^L f\psi_i dx + \sum_{j=1}^{m+1} N_j^l\int_0^L \psi_i\psi_j\,dx.
\end{aligned}$$

Therefore we obtain a linear system for one time step:

$$\begin{aligned}
\left(\mathbf{M} + \frac{h^2}{4}\mathbf{K}\right)\mathbf{u}^{l+1} &= \left(\mathbf{M} - \frac{h^2}{4}\mathbf{K}\right)\mathbf{u}^{l} + h\mathbf{M}\mathbf{v}^l + \frac{h^2}{2}\left(\mathbf{f} + \mathbf{M}\mathbf{N}^l\right), \\
\mathbf{v}^{l+1} &= \frac{2}{h}(\mathbf{u}^{l+1} - \mathbf{u}^l) - \mathbf{v},
\end{aligned} \tag{7.3}$$

where mass and stiffness matrix have the following forms, respectively:

$$M_{ij} = \int_0^L \psi_i \psi_j \, dx \text{ and } K_{ij} = \int_0^L \psi_i'' \psi_j'' \, dx.$$

At each discretized time t_l, we will obtain numerical solutions $\mathbf{u}^l = (u_1^l, u_2^l, \cdots, u_{m+1}^l)$, $\mathbf{v}^l = (v_1^l, v_2^l, \cdots, v_{m+1}^l)$ and $\mathbf{N}^l = (N_1^l, N_2^l, \cdots, N_{m+1}^l)$. Also note that $\mathbf{f} = (b_1, b_2, \cdots, b_{m+1})$ is load vector, where $b_i = \int_0^l f(x)\psi_i(x) \, dx$. Recalling that each basis function is $\psi_i(x) = B((x - x_i)/k)$, the mass and stiffness matrices are banded matrix with 3 sub-diagonals and 3 super-diagonals. Note that these matrices are symmetric positive definite:

$$\mathbf{M} = \frac{4}{9}k \begin{bmatrix}
\frac{1647}{560} & \frac{283}{280} & \frac{239}{2240} & \frac{1}{1120} & & & & & & & \\
\frac{283}{280} & \frac{151}{140} & \frac{1191}{2240} & \frac{3}{56} & \frac{1}{2240} & & & & & & \\
\frac{239}{2240} & \frac{1191}{2240} & \frac{151}{140} & \frac{1191}{2240} & \frac{3}{56} & \ddots & & & & & \\
\frac{1}{1120} & \frac{3}{56} & \frac{1191}{2240} & \frac{151}{140} & \frac{1191}{2240} & \ddots & \ddots & & & & \\
 & \frac{1}{2240} & \frac{3}{56} & \frac{1191}{2240} & \frac{151}{140} & \ddots & \ddots & \ddots & & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & & \ddots & \ddots & \ddots & \frac{151}{140} & \frac{1191}{2240} & \frac{3}{56} & \frac{1}{2240} & \\
 & & & & \ddots & \ddots & \frac{1191}{2240} & \frac{151}{140} & \frac{1191}{2240} & \frac{3}{56} & \frac{1}{2240} \\
 & & & & & \ddots & \frac{3}{56} & \frac{1191}{2240} & \frac{599}{500} & \frac{531}{1120} & \frac{3}{112} \\
 & & & & & & \frac{1}{2240} & \frac{3}{56} & \frac{531}{120} & \frac{151}{280} & \frac{129}{2240} \\
 & & & & & & & \frac{1}{2240} & \frac{3}{112} & \frac{129}{2240} & \frac{1}{112}
\end{bmatrix}$$

$$
\mathbf{K} = \frac{4}{9k^3}
\begin{bmatrix}
\frac{81}{2} & -6 & -\frac{3}{8} & \frac{3}{4} & & & & & & & \\
-6 & 6 & -\frac{27}{8} & 0 & \frac{3}{8} & & & & & & \\
-\frac{3}{8} & -\frac{27}{8} & 6 & -\frac{27}{8} & 0 & \ddots & & & & & \\
\frac{3}{4} & 0 & -\frac{27}{8} & 6 & -\frac{27}{8} & \ddots & \ddots & & & & \\
 & \frac{3}{8} & 0 & -\frac{27}{8} & 6 & \ddots & \ddots & \ddots & & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & & \ddots & \ddots & \ddots & 6 & -\frac{27}{8} & 0 & \frac{3}{8} & \\
 & & & & \ddots & \ddots & -\frac{27}{8} & 6 & -\frac{27}{8} & 0 & \frac{3}{8} \\
 & & & & & \ddots & 0 & -\frac{27}{8} & \frac{21}{4} & -\frac{9}{4} & 0 \\
 & & & & & & \frac{3}{8} & 0 & -\frac{9}{4} & 3 & -\frac{9}{8} \\
 & & & & & & & \frac{3}{8} & 0 & -\frac{9}{8} & \frac{3}{4}
\end{bmatrix}
$$

7.2 Energy dissipation in fully-discrete case

If the fully-discrete scheme has the same linear complementary condition as semi-discrete case, energy dissipation can fail to hold. This was indeed observed in some preliminary computations. So in the fully-discrete case, we need to modify the LCP condition in order to guarantee energy dissipation. Following the definition of energy functional (6.4) and (7.1), we can define energy functional in the fully-discrete case:

$$
E(\mathbf{u}^l, \mathbf{v}^l) = \frac{1}{2}\left((\mathbf{v}^l)^T\mathbf{M}\mathbf{v}^l + (\mathbf{u}^l)^T\mathbf{K}\mathbf{u}^l\right) - \mathbf{f}\cdot\mathbf{u}^l. \tag{7.4}
$$

Lemma 7.1. *If we have LCP condition*

$$0 \leq \mathbf{MN}^l \quad \perp \quad \mathbf{u}^{l+1} + \mathbf{g} \geq 0, \tag{7.5}$$

where $\mathbf{g} = (g_1, g_2, \cdots, g_{m+1})$ *and* $g_i = g(x_i)$*, then energy is dissipated.*

Proof. Using numerical formulation (6.1–6.2) and (7.1), we have

$$\begin{aligned}
&\frac{1}{2h}\left(\sum_{i=1}^{m+1} v_i^{l+1}\psi_i(x) - \sum_{i=1}^{m+1} v_i^l\psi_i(x)\right)\left(\sum_{j=1}^{m+1} v_j^{l+1}\psi_j(x) + \sum_{j=1}^{m+1} v_i^l\psi_j(x)\right) \\
&\quad = -\frac{1}{2h}\left(\sum_{i=1}^{m+1} u_i^{l+1}\psi_i^{''''}(x) + \sum_{i=1}^{m+1} u_i^l\psi_i^{''''}(x)\right)\left(\sum_{j=1}^{m+1} u_j^{l+1}\psi_j(x) - \sum_{j=1}^{m+1} u_j^l\psi_j(x)\right) \\
&\qquad + \frac{1}{h}\sum_{i=1}^{m+1} f_i\psi_i(x)\left(\sum_{j=1}^{m+1} u_j^{l+1}\psi_j(x) - \sum_{j=1}^{m+1} u_j^l\psi_j(x)\right) \\
&\qquad + \frac{1}{h}\sum_{i=1}^{m+1} N_i\psi_i(x)\left(\sum_{j=1}^{m+1} u_j^{l+1}\psi_j(x) - \sum_{j=1}^{m+1} u_j^l\psi_j(x)\right).
\end{aligned}$$

Then taking integral with respect to x and using integration by parts,

$$\begin{aligned}
&\frac{1}{2}\left(\sum_{i,j} v_i^{l+1}\int_0^L \psi_i\psi_j\,dx \cdot v_j^{l+1} - \sum_{i,j} v_i^l \int_0^L \psi_i\psi_j\,dx \cdot v_j^l\right) \\
&\quad = -\frac{1}{2}\left(\sum_{i,j} u_i^{l+1}\int_0^L \psi_i\psi_j\,dx \cdot u_j^{l+1} - \sum_{i,j} u_i^l \int_0^L \psi_i\psi_j\,dx \cdot u_j^l\right) \\
&\qquad + \sum_{i,j} f_i \int_0^L \psi_i\,dx \cdot (u_j^{l+1} - u_j^l) + \sum_{i,j} N_i^l \int_0^L \psi_i\psi_j\,dx \cdot (u_j^{l+1} - u_j^l).
\end{aligned}$$

Using $M_{ij} = \int_0^L \psi_i\psi_j\,dx$, and $K_{ij} = \int_0^L \psi_i''\psi_j''\,dx$,

$$\begin{aligned}
\frac{1}{2}\left((\mathbf{v}^{l+1})^T\mathbf{M}\mathbf{v}^l - (\mathbf{v}^l)^T\mathbf{M}\mathbf{v}^l\right) &= -\frac{1}{2}\left((\mathbf{u}^{l+1})^T\mathbf{K}\mathbf{u}^{l+1} - (\mathbf{u}^l)^T\mathbf{K}\mathbf{u}^l\right) \\
&\quad + \mathbf{f}^T(\mathbf{u}^{l+1} - \mathbf{u}^l) + (\mathbf{N}^l)^T\mathbf{M}(\mathbf{u}^{l+1} - \mathbf{u}^l) \\
&= -\frac{1}{2}((\mathbf{u}^{l+1})^T\mathbf{K}\mathbf{u}^{l+1} - (\mathbf{u}^l)^T\mathbf{K}\mathbf{u}^l) \\
&\quad + \mathbf{f}^T(\mathbf{u}^{l+1} - \mathbf{u}^l) + (\mathbf{N}^l)^T\mathbf{M}(\mathbf{u}^{l+1} + \mathbf{g} - \mathbf{u}^l - \mathbf{g}).
\end{aligned}$$

By the LCP condition (7.5), we have

$$\frac{1}{2}\left((\mathbf{v}^{l+1})^T\mathbf{M}\mathbf{v}^l - (\mathbf{v}^l)^T\mathbf{M}\mathbf{v}^l\right) \leq -\frac{1}{2}\left((\mathbf{u}^{l+1})^T\mathbf{K}\mathbf{u}^{l+1} - (\mathbf{u}^l)^T\mathbf{K}\mathbf{u}^l\right) + \mathbf{f}^T(\mathbf{u}^{l+1} - \mathbf{u}^l).$$

Thus

$$\begin{aligned}\frac{1}{2}&\left((\mathbf{v}^{l+1})^T\mathbf{M}\mathbf{v}^l + (\mathbf{u}^{l+1})^T\mathbf{K}\mathbf{u}^{l+1}\right) - \mathbf{f}^T\cdot\mathbf{u}^{l+1}\\ &\leq \frac{1}{2}\left((\mathbf{v}^l)^T\mathbf{M}\mathbf{v}^{l+1} + (\mathbf{u}^l)^T\mathbf{K}\mathbf{u}^l\right) - \mathbf{f}^T\cdot\mathbf{u}^l.\end{aligned}$$

Therefore we have

$$E(\mathbf{u}^{l+1}, \mathbf{v}^{l+1}) \leq E(\mathbf{u}^l, \mathbf{v}^l),$$

as required. □

Notice that we apply the LCP condition in Lemma 7.1, when we compute numerical solutions.

7.3 Solution techniques for the linear complementarity problems

7.3.1 Non-smooth Newton method

To solve the linear system (7.3) for one time step with the linear complementarity condition (7.5), we consider using the non-smooth Newton method (see [45] for details). In order to find the next step solution $\mathbf{u}^{l+1}$ from the linear system (7.3) and the LCP condition (7.5), we consider the mapping $\mathbf{F} : \mathbf{R}^{m+1} \to \mathbf{R}^{m+1}$:

$$\mathbf{F} : \mathbf{u}^{l+1} \mapsto \min(\mathbf{M}\mathbf{N}^l, \mathbf{u}^{l+1} + \mathbf{g}). \tag{7.6}$$

Note that $\min(\mathbf{a},\mathbf{b})$ is meant component-wise for vectors $\mathbf{a}$ and $\mathbf{b}$, and so $\min(\mathbf{a},\mathbf{b}) = \mathbf{0}$ is equivalent to $0 \le \mathbf{a} \perp \mathbf{b} \ge 0$. Thus the LCP condition (7.5) is equivalent to $\mathbf{F}(\mathbf{u}^{l+1}) = \mathbf{0}$. Since $\mathbf{MN}^l$ is implicitly a function of $\mathbf{u}^{l+1}$ via the linear system (7.3), we can express $\mathbf{MN}^l$ as:

$$\mathbf{MN}^l = \frac{2}{h^2}\left[\left(\mathbf{M} + \frac{h^2}{4}\mathbf{K}\right)\mathbf{u}^{l+1} - \left(\mathbf{M} - \frac{h^2}{4}\mathbf{K}\right)\mathbf{u}^l - h\mathbf{M}\mathbf{v}^l\right] - \mathbf{f}. \tag{7.7}$$

So for each i we have two cases:

1. $\sum_j M_{ij}N_j^l \le u_i^{l+1} + g_i$ so $F_i(\mathbf{u}^{l+1}) = \sum_j M_{ij}N_j^l$,
2. $\sum_j M_{ij}N_j^l \ge u_i^{l+1} + g_i$ so $F_i(\mathbf{u}^{l+1}) = u_i^{l+1} + g_i$.

We can find the next step solution $\mathbf{u}^{l+1}$, using the non-smooth Newton method:

$$\mathbf{u}_{n+1}^{l+1} = \mathbf{u}_n^{l+1} - \nabla\mathbf{F}(\mathbf{u}_n^{l+1})^{-1}F(\mathbf{u}_n^{l+1}) \text{ for } n \ge 0.$$

This is Newton method for solving the nonlinear system $\mathbf{F}(\mathbf{u}^{l+1}) = 0$. Even though $\mathbf{F}$ is a non-smooth function [8], Newton method still converges super-linearly since $\mathbf{F}$ is a *semi-smooth* function [40, 39]. This is because max, and min are semi-smooth functions and Newton method method for semi-smooth function still converges locally at super-linear rate provided $\mathbf{F}$ is "BD regular" [45]. That is, it converges superlinearly provided $\partial\mathbf{F}(\mathbf{u}) := \{\lim_{j\to\infty}\nabla\mathbf{F}(\mathbf{u}^j) \mid \lim_{j\to\infty}\mathbf{u}^j = \mathbf{u}\}$ does not contain any singular matrices.

In practice, in order to obtain computation, we use a smooth function $\theta_\alpha(a,b)$, instead of $\min(a,b)$

$$\theta_\alpha(a,b) = \frac{1}{2}\left((a+b) - h_\alpha(a-b) + \alpha\right),$$

where $h_\alpha(y) = \sqrt{y^2+\alpha^2} - \alpha$ is an approximation to $|y|$. The number $\alpha > 0$ is called a smoothing parameter. Clearly, as $\alpha \to 0$, we have

$$\theta_\alpha(a,b) \to \min(a,b). \tag{7.8}$$

Applying (7.8), we have for each $i, j,\ 1 \le i, j \le m+1$,

$$\theta_\alpha(\sum_j M_{ij}N_j^l,\ u_i^{l+1} + g_i) \to \min(\sum_j M_{ij}N_j^l,\ u_i^{l+1} + g_i),\ \text{as } \alpha \to 0.$$

So let $F_i(u_j^{l+1}) = \left(\sum_j M_{ij}N_j^l + (u_i^{l+1} + g_i) - \sqrt{(\sum_j M_{ij}N_j^l - u_i^{l+1} - g_i)^2 + \alpha^2} + \alpha\right)/2.$

Then from the numerical formulation (7.3),

$$\sum_j M_{ij}N_j^l = \sum_j (\frac{2}{h^2}M_{ij} + \frac{1}{2}K_{ij})u_j^{l+1} + \sum_j (\frac{1}{2}K_{ij} - \frac{2}{h^2}M_{ij})u_j^l - \frac{2}{h}\sum_j M_{ij}v_{j.}^l \tag{7.9}$$

When we want to find the $(n+1)\times(n+1)$ Jacobian matrix of $\mathbf{F}$, we will put (7.9) into vector function $F_i(u_j^{l+1})$. The elements of Jacobian matrix $\nabla\mathbf{F}$ has a different form, depending on $i,\ j$. If $i = j$, we have

$$\frac{\partial F_i}{\partial u_j^{l+1}} = \frac{1}{2}\left(\frac{2}{h^2}M_{ij} + \frac{1}{2}K_{ij} + 1 - \frac{(\sum_j M_{ij}N_j^l - u_i^{l+1} - g_i)(2M_{ij}/h^2 + K_{ij}/2 - 1)}{\sqrt{(\sum_j M_{ij}N_j^l - u_i^{l+1} - g_i)^2 + \alpha^2}}\right).$$

Otherwise, i.e, $i \neq j$, we have

$$\frac{\partial F_i}{\partial u_j^{l+1}} = \frac{1}{2}\left(\frac{2}{h^2}M_{ij} + \frac{1}{2}K_{ij} - \frac{(\sum_j M_{ij}N_j^l - u_i^{l+1} - g_i)(2M_{ij}/h^2 + K_{ij}/2)}{\sqrt{(\sum_j M_{ij}N_j^l - u_i^{l+1} - g_i)^2 + \alpha^2}}\right).$$

Note that this Jacobian matrix $\nabla\mathbf{F}$ is not symmetric, i.e., $\partial F_i/\partial u_j^{l+1} \neq \partial F_j/\partial u_i^{l+1}$, in general.

7.3.2 The smoothed guarded Newton method

In this Subsection, we present in detail how we solve $\mathbf{F}(\mathbf{u}^{l+1}) = 0$. First we introduce the guarded Newton method; that is, Newton method combined with backtracking line search. Since the vector function $\mathbf{F}$ is smooth enough, $\mathbf{F}$ is assumed to be continuously differentiable. Let $p(\mathbf{x}) = \frac{1}{2}\|\mathbf{F}(\mathbf{x})\|_2^2$ for $\mathbf{x} \in \mathbf{R}^{m+1}$. Each iteration of line search method is given by

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \rho_n \mathbf{d}_n,$$

where ρ_n is called step length and $\mathbf{d}_n$ is called the direction. Most line search requires ρ_n to be descent direction. So we take the direction

$$\mathbf{d}_n = -\{\nabla \mathbf{F}(\mathbf{x}_n)\}^{-1}\mathbf{F}(\mathbf{x}_n). \tag{7.10}$$

Then we have

$$\begin{aligned}
\mathbf{d}_n^T \cdot \nabla p(\mathbf{x}) &= -\left(\{\nabla \mathbf{F}(\mathbf{x}_n)\}^{-1}\mathbf{F}(\mathbf{x}_n)\right)^T \{\nabla \mathbf{F}(\mathbf{x}_n)\}\mathbf{F}(\mathbf{x}_n) \\
&= -\mathbf{F}(\mathbf{x}_n)^T \left(\{\nabla \mathbf{F}(\mathbf{x}_n)\}^{-1}\right)^T \{\nabla \mathbf{F}(\mathbf{x}_n)\}\mathbf{F}(\mathbf{x}_n).
\end{aligned}$$

If Jacobian matrix $\nabla \mathbf{F}(\mathbf{x}_n)$ is symmetric,

$$\mathbf{d}_n^T \cdot \nabla p(\mathbf{x}_n) = -\mathbf{F}(\mathbf{x}_n)^T \cdot \mathbf{F}(\mathbf{x}_n) < 0.$$

So p can be reduced along this direction $\mathbf{d}_n$. However since our Jacobian matrix $\nabla \mathbf{F}$ is not symmetric, p may not be reduced. So we want to employ another condition that imposes on the step length ρ_n. The condition is to provide reduction in p, i.e.,

$$p(\mathbf{x}_n + \rho_n \mathbf{d}_n) < p(\mathbf{x}_n).$$

Indeed, this condition is not quite the sufficient decrease criterion used in optimization [42]. But in practice our strategy seems to work well. This is supported

Table 7.1: Average number of linear system solved per step

$h \quad \backslash \quad k$	$k = 1/5$	$k = 1/25$	$k = 1/50$	$k = 1/500$
$h = 1/10$	20.48	28.08	29.36	35.79
$h = 1/20$	19.68	25.95	26.99	27.23
$h = 1/50$	19.20	23.36	23.96	18.28
$h = 1/100$	19.19	22.79	23.01	19.14
$h = 1/1000$	19.28	22.87	19.87	14.60

by Table 7.1. From (7.10), we do not take inverse matrix of Jacobian matrix $\nabla\mathbf{F}(\mathbf{x}_n)$ in the actual computation. Instead, we solve the linear system $\nabla\mathbf{F}(\mathbf{x}_n)\cdot\mathbf{d}_n = -\mathbf{F}(\mathbf{x}_n)$, using LU factorization. This is more efficient in computation. The initial step length ρ_0 is chosen to be 1 in the guarded Newton method. Note that solving $\mathbf{F}(\mathbf{x}) = 0$ is equivalent to solving $\min_{\mathbf{x}} p(\mathbf{x}) = 0$. This gives us Algorithm 7.2 which is called the guarded Newton method.

Algorithm 7.2. *Choose the initial next step solution* $\mathbf{u}_0$ *of (7.3).*

repeat *until* $\|\mathbf{F}(\mathbf{u})\|_2 < \epsilon$

$\mathbf{d} \leftarrow \{\nabla\mathbf{F}(\mathbf{u})\}^{-1}\mathbf{F}(\mathbf{u})$

$\rho \leftarrow 1$

repeat *until* $\|\mathbf{F}(\mathbf{u} + \rho\mathbf{d})\|_2 < \|\mathbf{F}(\mathbf{u})\|_2$

$\rho \leftarrow \rho/2$

end(repeat)

$\mathbf{u} \leftarrow \mathbf{u} + \rho \mathbf{d}$

end(repeat)

A potential disadvantage of the method is that when the initial point is remote from a solution, the method might not be converge or may converge very slowly. To resolve those shortcomings, we will consider a better algorithm. We add a smoothing parameter α. In practical computation, the initial parameter α is chosen to be a large number. Now combining the guarded Newton method, we have the following Algorithm 7.3 which is called the smoothed guarded Newton method.

Algorithm 7.3. *Choose a large number α_0 for α.*

repeat *until $\alpha < \epsilon$*

try the guarded Newton method

if *the guarded Newton method succeed*

$\alpha \leftarrow \alpha/10$

end(if)

else

$\alpha \leftarrow 2\alpha$

end(else)

end(repeat)

7.4 Numerical evidence for strong convergence

In this Section, we present the numerical evidence that our numerical solutions converge strongly (via Lemma 6.13) of obtaining and assessing this evidence. Let $\boldsymbol{\phi}_i$

be the ith eigenvector with eigenvalue λ_i of the generalized eigenproblem (7.11). Then we have

$$\boldsymbol{\phi}_i^T \mathbf{M} \boldsymbol{\phi}_i = 1 \text{ and } \mathbf{K}\boldsymbol{\phi}_i = \lambda_i \mathbf{M}\boldsymbol{\phi}_i, \tag{7.11}$$

where $\boldsymbol{\phi}_i = ((\phi_i)_1, (\phi_i)_2, (\phi_i)_3, \cdots (\phi_i)_{m+1})$. Note that this is the Galerkin discretization of the eigenfunction problem

$$\frac{\partial^4 \phi_i(x)}{\partial x^4} = \lambda_i \phi_i(x), \quad \int_0^L \phi_i(x)^2 = 1$$

with the usual boundary conditions. Also note that $\mathbf{M}^{-1}\mathbf{K}$ is self-adjoint with respect to the inner products $(\mathbf{z}, \mathbf{w})_{\mathbf{M}} = \mathbf{z}^T\mathbf{M}\mathbf{w}$ and $(\mathbf{z}, \mathbf{w})_{\mathbf{K}} = \mathbf{z}^T\mathbf{K}\mathbf{w}$. So for any given function $\vartheta : \mathbf{R} \to \mathbf{R}$ we can define $\vartheta(\mathbf{M}^{-1}\mathbf{K})$ via $\vartheta(\mathbf{M}^{-1}\mathbf{K})\boldsymbol{\phi}_i = \vartheta(\lambda_i)\boldsymbol{\phi}_i$. In particular, let $\kappa^*(\lambda) = 1$ if $\lambda \leq \lambda_c$ and $\kappa^*(\lambda) = 0$ otherwise. The $\kappa^*(\mathbf{M}^{-1}\mathbf{K})\mathbf{z}$ is the projection onto span$\{\boldsymbol{\phi_i} \mid i = 1, 2, \ldots,$ and $\lambda_i \leq \lambda_c\}$ that is orthogonal with respect to both $(\cdot,\cdot)_{\mathbf{M}}$ and $(\cdot,\cdot)_{\mathbf{K}}$. The elastic energy in the modes i with $\lambda_i \leq \lambda_c$ is therefore $\frac{1}{2}(\kappa^*(\mathbf{M}^{-1}\mathbf{K})\mathbf{u})^T\mathbf{K}\,\kappa^*(\mathbf{M}^{-1}\mathbf{K})\mathbf{u}$ and the kinetic energy is $\frac{1}{2}(\kappa^*(\mathbf{M}^{-1}\mathbf{K})\mathbf{v})^T\mathbf{M}\,\kappa^*(\mathbf{M}^{-1}\mathbf{K})\mathbf{v}$. Since $\kappa^*(\mathbf{M}^{-1}\mathbf{K})$ is not easily computable without performing an complete (and expensive) eigenvalue/eigenvector decomposition of $\mathbf{M}^{-1}\mathbf{K}$, we will instead construct a rational approximation to it.

Choosing $\lambda_c > 0$ for any cut-off $c \geq 1$, we have

$$\frac{1}{\lambda_c}\mathbf{M}^{-1}\mathbf{K}\boldsymbol{\phi}_i = \frac{\lambda_i}{\lambda_c}\boldsymbol{\phi}_i.$$

Thus for any large integer $p > 0$

$$\left(\mathbf{I} + (\frac{1}{\lambda_c}\mathbf{M}^{-1}\mathbf{K})^{2p}\right)^{-1}\boldsymbol{\phi}_i = \frac{1}{1 + (\lambda_i/\lambda_c)^{2p}}\boldsymbol{\phi}_i.$$

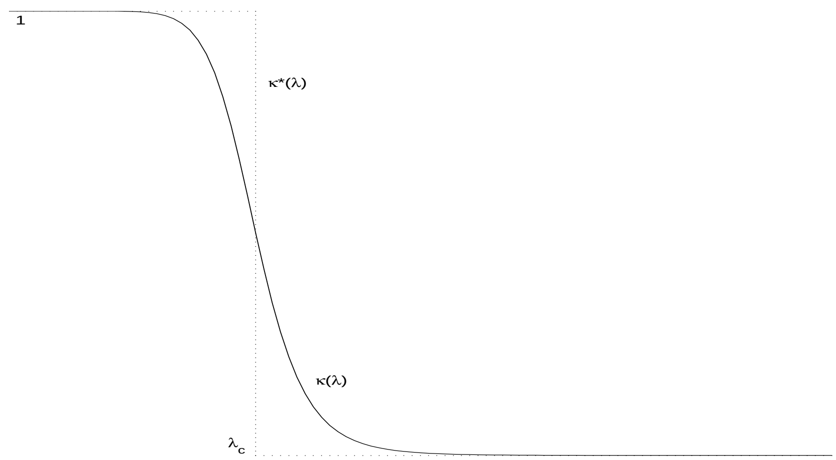

Figure 7.2: The construction of map κ.

Then we fix a continuous map κ of λ

$$\kappa(\lambda) = \frac{1}{(1 + (\lambda/\lambda_c)^{2p})} \text{ and then } \kappa(\mathbf{M}^{-1}\mathbf{K}) = \left(\mathbf{I} + (\frac{1}{\lambda_c}\mathbf{M}^{-1}\mathbf{K})^{2p}\right)^{-1} . \qquad (7.12)$$

Now we have

$$\kappa(\mathbf{M}^{-1}\mathbf{K})\boldsymbol{\phi}_i = \kappa(\lambda)\boldsymbol{\phi}_i.$$

Lemma 7.4. *At each time step $l \geq 1$, the energy in the fully-discrete case with no body force is*

$$\frac{1}{2}\left((\mathbf{v}^l)^T\mathbf{M}\mathbf{v}^l + (\mathbf{u}^l)^T\mathbf{K}\mathbf{u}^l\right) = \frac{1}{2}\sum_{i=1}^{m+1}\left(|\widehat{v}_i^{l;h}|^2 + \lambda_i|\widehat{u}_i^{l;h}|^2\right).$$

Proof. Using (7.11), we have

$$\begin{aligned}
&\frac{1}{2}\left((\mathbf{v}^l)^T\mathbf{M}\mathbf{v}^l + (\mathbf{u}^l)^T\mathbf{K}\mathbf{u}^l\right)\\
&\quad = \frac{1}{2}\left(\sum_{i=1}^{m+1}\widehat{v}_i^{l;h}\boldsymbol{\phi}_i \cdot \mathbf{M} \cdot \sum_{j=1}^{m+1}\widehat{v}_j^{l;h}\boldsymbol{\phi}_j + \sum_{i=1}^{m+1}\widehat{u}_i^{l;h}\boldsymbol{\phi}_i \cdot \mathbf{K} \cdot \sum_{j=1}^{m+1}\widehat{u}_j^{l;h}\boldsymbol{\phi}_j\right)
\end{aligned}$$

$$= \frac{1}{2}\left(\sum_{i=1}^{m+1} |\widehat{v}_i^{l;h}|^2 + \sum_{i=1}^{m+1} \widehat{u}_i^{l;h}\boldsymbol{\phi}_i \cdot \mathbf{M} \cdot \sum_{j=1}^{m+1} \lambda_j \widehat{u}_j^{l;h}\boldsymbol{\phi}_j\right)$$
$$= \frac{1}{2}\sum_{i=1}^{m+1}\left(|\widehat{v}_i^{l;h}|^2 + \lambda_i|\widehat{u}_i^{l;h}|^2\right).$$

□

Using Lemma 7.4, we can demonstrate numerical evidence using Lemma 6.13 that the convergence is strong. The ratio

$$\frac{(\kappa^*(\mathbf{M}^{-1}\mathbf{K})\mathbf{u}^l)^T\mathbf{K}\,\kappa^*(\mathbf{M}^{-1}\mathbf{K})\mathbf{u}^l + (\kappa^*(\mathbf{M}^{-1}\mathbf{K})\mathbf{v}^l)^T\mathbf{M}\,\kappa^*(\mathbf{M}^{-1}\mathbf{K})\mathbf{v}^l}{(\mathbf{u}^l)^T\mathbf{K}\,\mathbf{u}^l + (\mathbf{v}^l)^T\mathbf{M}\,\mathbf{v}^l}$$

is the ratio of the elastic and kinetic energy in the modes with $\lambda_i \leq \lambda_c$ to the total elastic and kinetic energy for the numerical solution at time-step t_l. Following Lemma 6.13, this should go to one as $\lambda_c \uparrow \infty$, uniformly in the numerical parameters $h > 0$, l and $k > 0$. Of course, for *fixed* $k > 0$, this will happen as $\lambda_c \uparrow \infty$ anyway. So we need to first fix λ_c and then compute these ratios for k and h becoming small; from the apparent limits of the energy ratios for several fixed λ_c, we observe the overall trend as $\lambda_c \uparrow \infty$. This will be done in the following Section.

7.5 Computing $\kappa(\mathbf{M}^{-1}\mathbf{K})\mathbf{z}$

In this Section, we discuss how to efficiently compute $\kappa(\mathbf{M}^{-1}\mathbf{K})\mathbf{z}$. Note that we do not compute $\kappa^*(\mathbf{M}^{-1}\mathbf{K})$ directly using an eigendecomposition of $\mathbf{M}^{-1}\mathbf{K}$, as this is computationally expensive. So we choose a rational function $\kappa(\lambda)$ to approximate the step function $\kappa^*(\lambda)$. We can then efficiently compute $\kappa(\mathbf{M}^{-1}\mathbf{K})\mathbf{z}$ for any vector

$\mathbf{z}$. For simplicity we choose

$$\kappa(\lambda) = \frac{1}{1+(\lambda/\lambda_c)^{2p}} \text{ for } p \text{ moderately large.}$$

In fact, we implement this function for $p = 5$. The key to efficient computation of $\kappa(\mathbf{M}^{-1}\mathbf{K})\mathbf{z}$ is the factorization of $\kappa(\lambda)$. The zeros of the denominator (7.12) are solutions of $(\lambda_j/\lambda_c)^{2p} = -1$. The solutions of this equation are

$$\lambda_j/\lambda_c = \zeta_j := \exp((2j+1)\pi i/2p), \quad j = 0, 1, 2, \cdots, 2p-1, \text{ where } i = \sqrt{-1}.$$

Thus we have

$$\begin{aligned} \kappa(\lambda) &= (\lambda/\lambda_c - \zeta_0)^{-1}(\lambda/\lambda_c - \zeta_1)^{-1} \cdots (\lambda/\lambda_c - \zeta_{2p-1})^{-1} \\ &= \lambda_c^{2p}(\lambda - \lambda_c\zeta_0)^{-1}(\lambda - \lambda_c\zeta_1)^{-1} \cdots (\lambda - \lambda_c\zeta_{2p-1})^{-1}. \end{aligned}$$

Therefore

$$\begin{aligned} &\kappa(\mathbf{M}^{-1}\mathbf{K}) \\ &= \lambda_c^{2p}(\mathbf{M}^{-1}\mathbf{K} - \lambda_c\zeta_0\mathbf{I})^{-1}(\mathbf{M}^{-1}\mathbf{K} - \lambda_c\zeta_1\mathbf{I})^{-1} \cdots (\mathbf{M}^{-1}\mathbf{K} - \lambda_c\zeta_{2p-1}\mathbf{I})^{-1} \\ &= \lambda_c^{2p}\left(\mathbf{M}^{-1}(\mathbf{K} - \lambda_c\zeta_0\mathbf{M})\right)^{-1}\left(\mathbf{M}^{-1}(\mathbf{K} - \lambda_c\zeta_1\mathbf{M})\right)^{-1} \cdots \left(\mathbf{M}^{-1}(\mathbf{K} - \lambda_c\zeta_{2p-1}\mathbf{M})\right)^{-1} \\ &= \lambda_c^{2p}(\mathbf{K} - \lambda_c\zeta_0\mathbf{M})^{-1}\mathbf{M}(\mathbf{K} - \lambda_c\zeta_1\mathbf{M})^{-1} \cdots \mathbf{M}(\mathbf{K} - \lambda_c\zeta_{2p-1}\mathbf{M})^{-1}\mathbf{M}. \end{aligned}$$

This gives us Algorithm 7.5 for computing $\kappa(\mathbf{M}^{-1}\mathbf{K})\mathbf{z}$.

Algorithm 7.5. *Computing* $\kappa(\mathbf{M}^{-1}\mathbf{K})\mathbf{z}$

for $j = 0, 1, 2, \cdots, 2p-1$

$\mathbf{w} \leftarrow \mathbf{M}\mathbf{z}$

solve $(\mathbf{K} - \lambda_c\zeta_j\mathbf{M})\mathbf{z} = \mathbf{w}$ *for* $\mathbf{z}$

end(for)

$\mathbf{z} \leftarrow \lambda_c^{2p}\mathbf{z}$

In Algorithm 7.5, $(\mathbf{K} - \lambda_c\zeta_j\mathbf{M})\mathbf{z} = \mathbf{w}$ has matrices over the complex numbers $\mathbf{C}$. In order to compute those matrices, we consider the following linear system:

$$(\mathbf{A} + i\mathbf{B})(\mathbf{x} + i\mathbf{y}) = (\mathbf{u} + i\mathbf{v}), \tag{7.13}$$

where $\mathbf{A}$, $\mathbf{B}$ are real matrices and $\mathbf{x}$, $\mathbf{y}$, $\mathbf{u}$, $\mathbf{v}$ are real vectors. The system (7.13) is equivalent to the following linear system:

$$\begin{bmatrix} \mathbf{A} & -\mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{Ax} - \mathbf{By} \\ \mathbf{Bx} + \mathbf{Ay} \end{bmatrix} = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}, \tag{7.14}$$

since $(\mathbf{A}+i\mathbf{B})(\mathbf{x}+i\mathbf{y}) = (\mathbf{Ax}-\mathbf{By})+i(\mathbf{Ay}+\mathbf{Bx})$. We can change the linear system (7.14) so that we have equivalent real banded matrix with double the bandwidth. Thus the linear system $(\mathbf{K}-\lambda_c\zeta_j\mathbf{M})\mathbf{z} = \mathbf{w}$ can be solved as a banded system with an upper and lower bandwidth of six, which can be done in $O(m+1)$ time. The matrix-vector products $\mathbf{Mz}$ can also be computed in $O(m+1)$ time. Thus Algorithm 7.5 can be executed in just $O(pm+p)$ time.

The ratios contained in Table 7.2 are obtained as follows: let $E(\mathbf{u}^l, \mathbf{v}^l)$ be the total energy in actual computation and let $E_c(\mathbf{u}^l, \mathbf{v}^l)$ be the energy in the low frequency modes. Then the ratio that we use is

$$\tau = \frac{\sum_{l=0}^{\lfloor T/h \rfloor} E_c(\mathbf{u}^l, \mathbf{v}^l)}{\sum_{l=0}^{\lfloor T/h \rfloor} E(\mathbf{u}^l, \mathbf{v}^l)}.$$

Looking across the rows of Table 7.2 we note that there does seem to be some slow convergence of the ration as h goes to zero, and this ratio increases as $m+1$ (the

Table 7.2: The ratio of energy E_c to total energy E

The number of nodes	c	$h = 1/10$	$h = 1/50$	$h = 1/100$
500	10	0.650153	0.407755	0.380260
	30	0.910487	0.812236	0.777214
	100	0.997099	0.986641	0.972011
	300	0.999846	0.999166	0.997870
1000	10	0.653481	0.412869	0.378693
	30	0.917148	0.855211	0.755536
	100	0.997575	0.981944	0.968371
	300	0.999846	0.998196	0.997980

number of grid nodes) increases; this limit seems to be very close to one for large λ_c; picking $c = 100$, for the lowest 100 out of 500 or 1000 possible modes, we can account for about 97% of the total kinetic and elastic energy. This implies that we can account for almost all the energy in the bottom 100 frequency modes, and account for about 75% of the total energy in the bottom 30 modes. So Table 7.2 presents substantial numerical evidence of the applicability of Lemma 6.13 and therefore of strong convergence of the numerical solutions.

7.6 Numerical experiments and results

The package that we used for handling the matrices and vectors is *Meschach* [54], which uses the C programming language. We took particular advantage of the banded matrix routines in that package. Our numerical experiments were performed on a Hewlett–Packard Visualize B2000.

In this Section, we show our numerical simulation results. In our computation, we take the length of rod to be $L = 20$ and the initial displacement $u^0(x) = x^2/4$ which is consistent with the essential boundary condition and the initial velocity $v^0(x) = -2 \cdot x$ and gap function $g(x) = (x - 12)^2$, and the end time $T = 10$. We assume that the rod is moving downward, negative direction in simulation. The gap function g indicates the distance between the rigid foundation and the initial position where the rod is located vertically. Note that the potential energy is not included in our computation, since the body force $f(x)$ is zero.

From the energy functional in (7.4) in the fully-discrete case, we obtain four graphs for the total energy in Figure 7.3. According to those graphs, our numerical implementation supports the energy dissipation that we anticipated theoretically. The first graph shows that the energy function using 100 nodes is erratic. Indeed, we anticipated that the smaller time step size h we used, the higher the energy. This appears to be true for all cases except for $k = 0.2$ and for $h = 0.01$ and for $h = 0.001$. We would conjecture that the reason is that the approximations are not sufficiently refined for this value of k. On the other hand, other graphs show that energy conservation is expected as step size h becomes smaller and smaller.

In Figure 7.4, the motion of the rod is presented. Each curve is the profile of the rod at given time. In this simulation, we used $k = 1/50$ in space and a time step of $h = 1/100$. According to our numerical experiments, that case brings the most

Table 7.3: Computation time(u:user time, s:system time)

$h \quad \backslash \quad k$	2×10^{-1}	4×10^{-2}	2×10^{-2}	2×10^{-3}
1×10^{-1}	0.783u 0.007s	6.460u 0.041s	19.892u 0.035s	1083.158u 2.931s
5×10^{-2}	1.503u 0.003s	11.082u 0.044s	37.642u 0.113s	1295.474u 3.453s
2×10^{-2}	3.632u 0.011s	25.968u 0.033s	81.783u 0.158s	1633.937u 3.597s
1×10^{-2}	7.283u 0.039s	47.621u 0.179s	151.851u 0.255s	3852.925u 7.367s
1×10^{-3}	73.242u 0.390s	477.255u 1.234s	1220.408u 2.054s	29675.837u 65.416s

comfortable and solid result. An interesting point is that the end of the rod touches rigid foundation at some time step, and oscillates very rapidly. See the pictures at the right of the top and the left of the bottom of Figure 7.4.

Figures 7.5 and 7.6 present the velocity of the rod. So we would guess the phenomenon that the rate of deformation of the rod is very fast in some time steps. Figure 7.5 shows the velocity after the rod bounces away form the rigid foundation.

Finally we have 3-dimensional picture showing the contact force in Figure 7.7. According to those picture, when the end of rod touches rigid foundation, it seems that its contact force is the largest among other contact positions. Even though the number of nodes in the two pictures are different and they show different magnitudes for the contact force, the graphs have a similar shape.

Table 7.1 and Table 7.3 are presented to show the speed of the computations. Note that in the case $k = 1/500$ we use different convergence $\|\mathbf{F}(\mathbf{u})\|_2 < \epsilon$. This was necessary because of difficulties with roundoff and ill-conditioning in the stiffness matrix $\mathbf{K}$ particularly. So we instead used $\|\nabla\mathbf{F}(\mathbf{u})^{-1}\mathbf{F}(\mathbf{u})\|_2 < \epsilon$ to avoid these numerical difficulties. So in the Table 7.3, we can see that the ratio of times differs from the other cases.

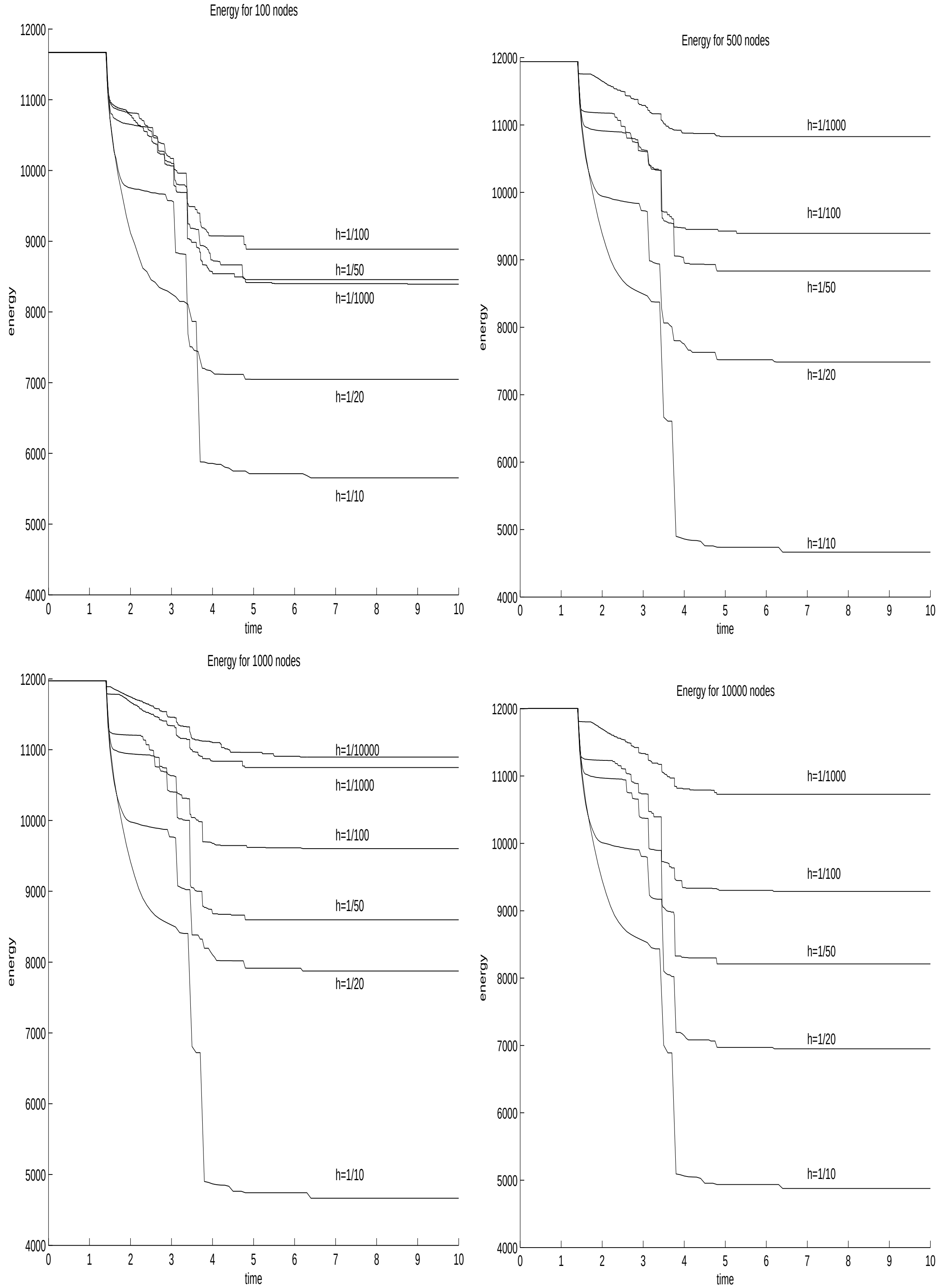

Figure 7.3: Energy function.

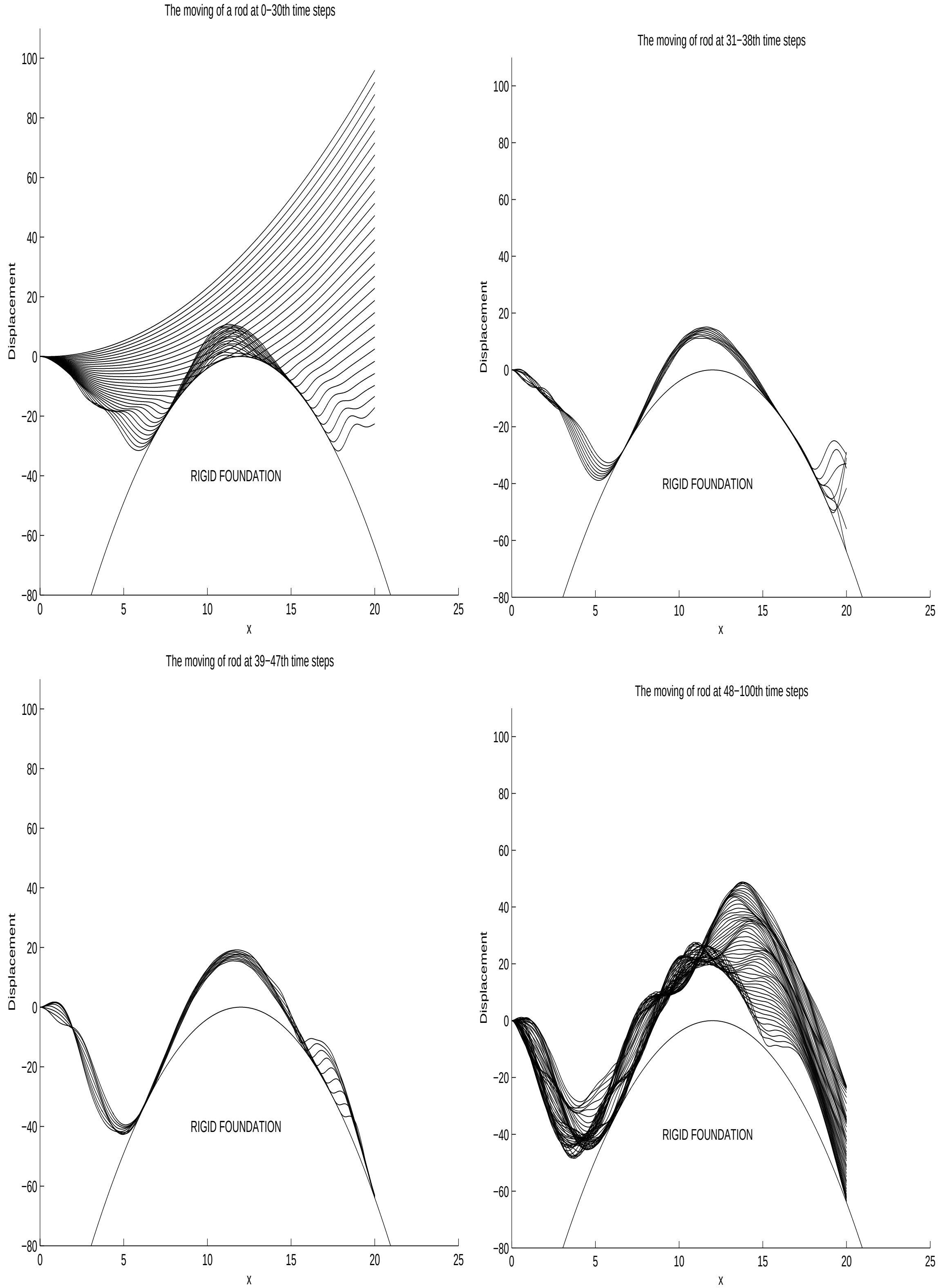

Figure 7.4: Flow of solution.

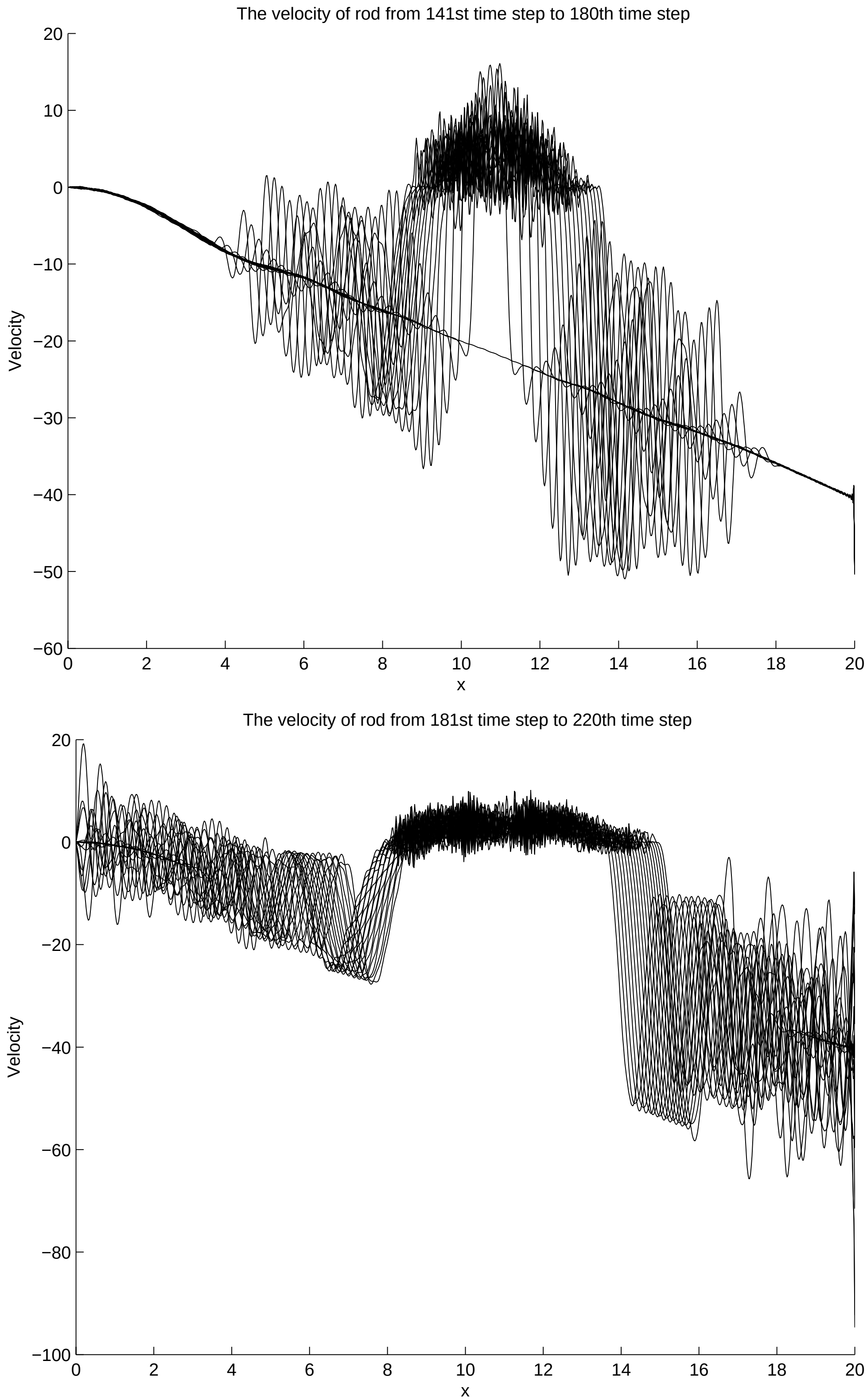

Figure 7.5: The velocity of the rod at each time step 141-220.

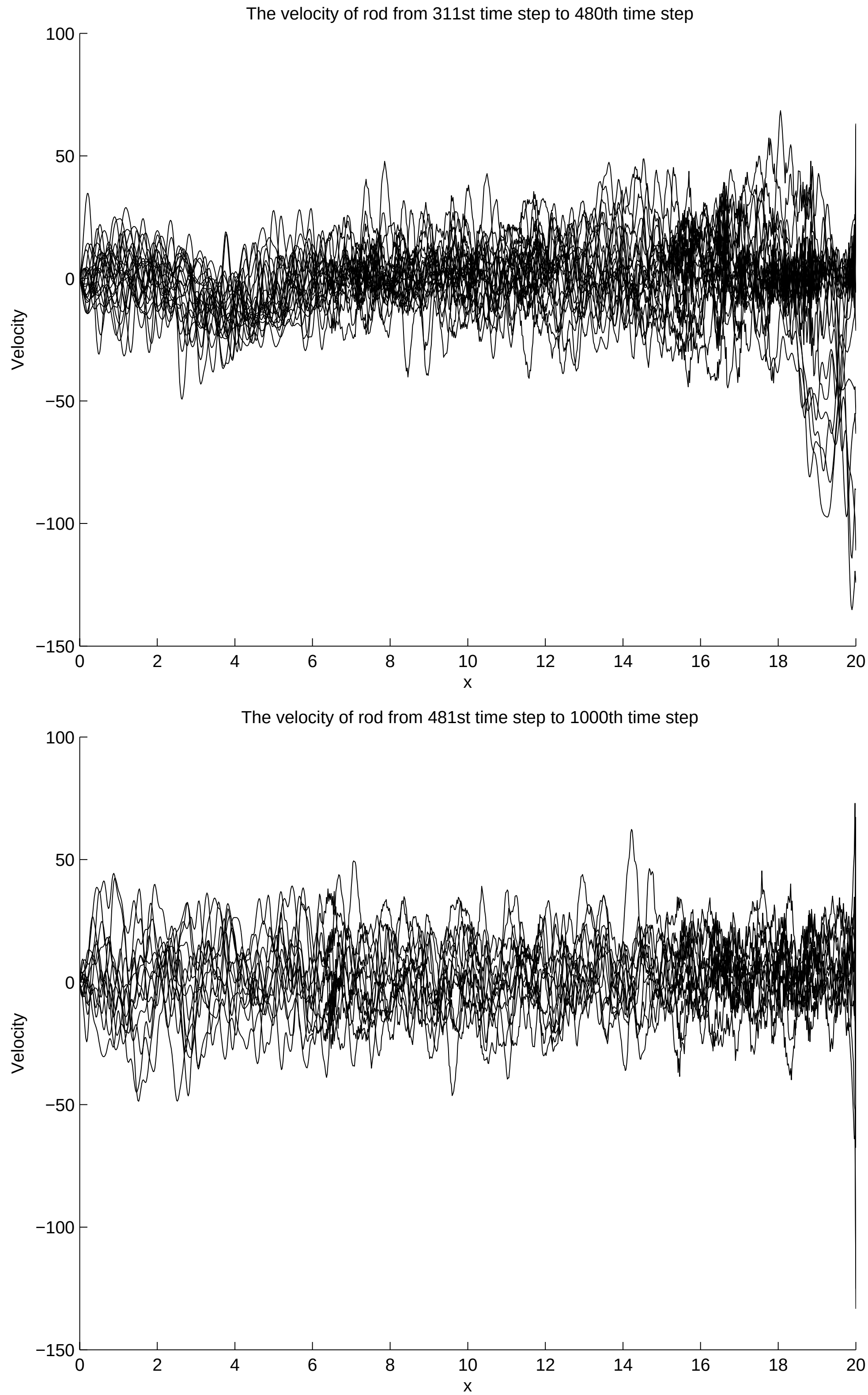

Figure 7.6: The velocity of the rod at each time step 221-1000.

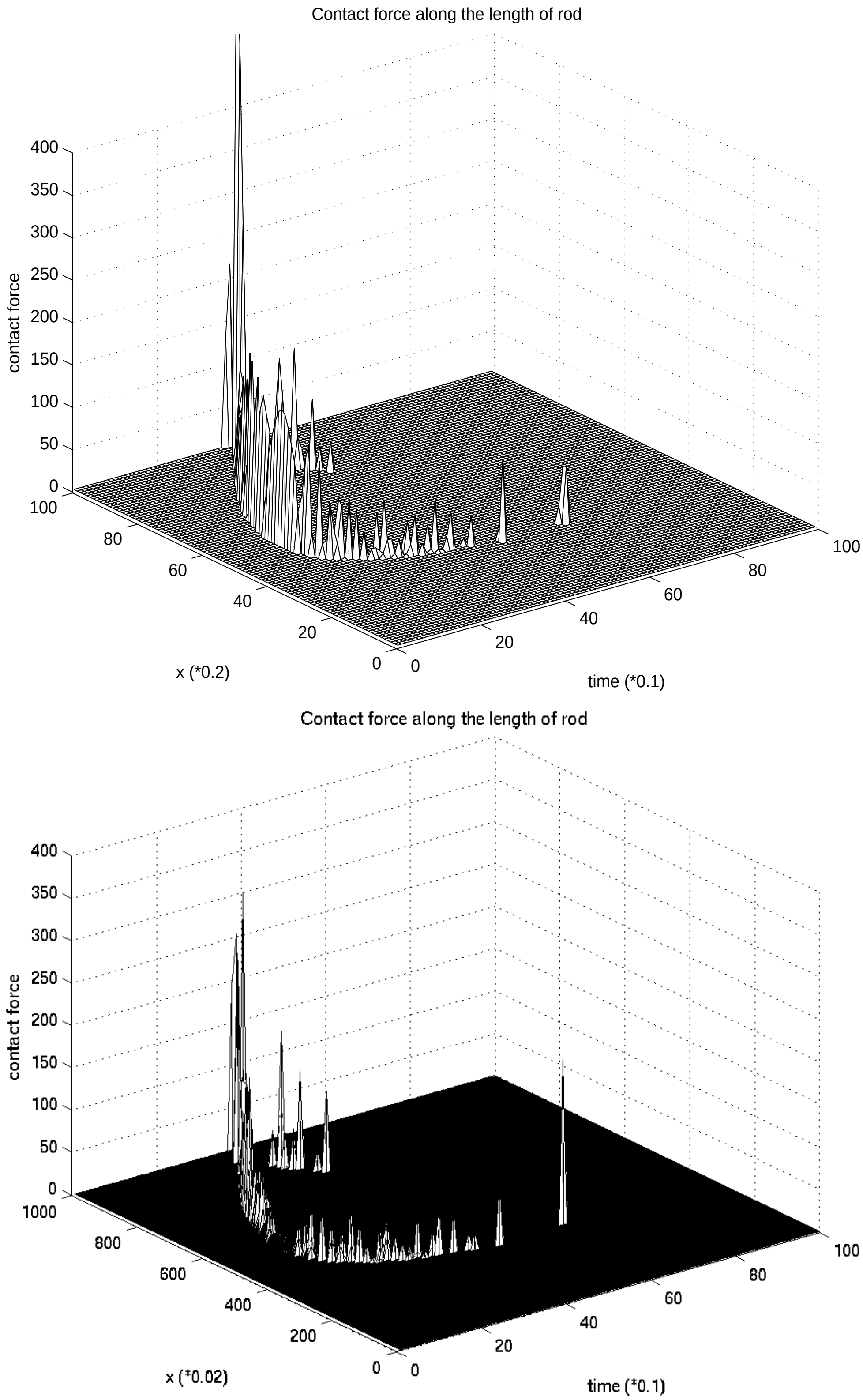

Figure 7.7: Contact force at each time step.

CHAPTER 8
CONCLUSIONS

8.1 Elastic bodies in dynamic contact

It has been shown that continuous piecewise linear interpolant $\mathbf{u}^h$ is uniformly bounded in space $W^{1,\infty}(0,T;\mathbf{L}^2(\Omega))$ and $L^\infty(0,T;\mathbf{H}^1(\Omega))$, and $\mathbf{v}^h$ is uniformly bounded in $L^\infty(0,T;\mathbf{L}^2(\Omega))$. Indeed we could need a nicer space where $\mathbf{u}^h$, $\mathbf{v}^h$ are uniformly bounded. Consequently, our final goal for this part is to show that

$$\|N^l\|_{H^{-1/2}(\Omega)} = O(1).$$

To extend this program, we would need to develop a bound of the form

$$\sum_{l=0}^{T/h} \|N^l\|_{H^{-1/2}(\partial\Omega)} \leq C.$$

Then we could apply Alaoglu's theorem to obtain a weak* convergent subsequence of the time-discretized normal contact forces $N_h(\mathbf{x},t) = \sum_{l=0}^{\infty} N^l(\mathbf{x})\,\delta(t-t_l)$. The limit would belong to the space of $H^{-1/2}(\partial\Omega)$-valued measures.

However, there are a number of obstacles to this program. First, there are reasons to believe that the estimate

$$\|N^l\|_{H^{-1/2}(\partial\Omega)} = O(h^{-1/2})$$

is sharp if we only assume that the initial energy is bounded. For example, in one spatial dimension (where N^l is just a scalar), consider the following solution of the wave equation $u_{xx} = u_{tt}$:

$$u(x,t) = \begin{cases} (x+t)^{-1/2+\epsilon}, & x+t > 0, \\ 0, & x+t \leq 0, \end{cases}$$

with $\epsilon > 0$. This wave has finite energy on any finite interval. If we consider this solution on $\Omega = (0, \infty)$ with Signorini conditions at zero, then $N(t) \sim \text{const}\, t^{-1/2+\epsilon}$ for small $t > 0$.

On the other hand, if we restrict the initial conditions so that $u^0 \in H^{3/2}(\Omega)$ and $v^0 \in H^{1/2}(\Omega)$, then it can be shown that u^l and v^l also belong to these spaces and

$$\|N^l\|_{H^{-1/2}(\partial\Omega)} = O\left(\ln \frac{1}{h}\right).$$

Further work would need to be done to obtain a stronger convergence theory sufficient to establish strong convergence in $H^1(\Omega)$ for the defomation u and in $L^2(\Omega)$ for velocity v, or to get conservation of energy in the limit.

Numerical methods can also be developed based on the time-stepping approach described here and the Finite Element Method. However, we still need the finer regularity properties.

8.2 The Euler–Bernoulli beam in contact

The existence of sequence u_ϵ has been shown, based on penalty method. Also we proved boundness of u_ϵ in $C^p(0,T;H^{1/2+\sigma})$. Since $C^p(0,T;H^{1/2+\sigma})$ is compactly imbedded in $C([0,l] \times [0,T])$, there is a subsequence of u_ϵ that converges to u strongly in $C([0,l] \times [0,T])$. This plays important role to show the existence of solution.

Lemma 5.18 implies that N and $u + g$ satisfy the linear complementarity conditions a.e. by the weak* convergence of N_ϵ and the uniform convergence of u_ϵ. Thus our limits N and u satisfy the desired system of conditions (5.3–5.8), and solutions exist.

Note that our theory does not say anything about uniqueness or about conservation of energy. Indeed it can be easily shown that uniqueness does not hold for the system of conditions (5.3–5.8). Consider the problem with initial conditions $u_0(x) = \phi_1(x)$ and $v_0(x) = 0$ for all x and the gap function is $g(x) = 0$ for all x. Then since $\phi_1(x) \geq 0$ for all $x \in [0, l]$, the solution is $u(x,t) = \cos(\omega_1 t)\phi_1(x)$ for a suitable constant $\omega_1 > 0$ *until* the impact time $\omega_1 t = \pi/2$. After impact any upward velocity proportional to ϕ_1 is possible according to (5.3–5.8): put $u(x,t) = -\gamma \cos(\omega_1 t)\phi_1(x)$ for $\pi/2 \leq \omega_1 t \leq \pi$ with $\gamma > 0$. Then the normal contact force is $(1+\gamma)\omega_1\phi_1(x)\,\delta(t - \pi/(2\omega_1))$ in the neighborhood of $t = \pi/(2\omega_1)$.

In this respect the system of equations and conditions (5.3–5.8) describes a system much like the bouncing of a simple particle where the coefficient of restitution is not specified. But such a solution as described in the previous paragraph with $\gamma > 1$ cannot be a limit of solutions of the penalty problem as $\epsilon \downarrow 0$ because of conservation of energy in the penalty equations. However, more subtle difficulties may arise. Because the convergence theory developed here deals with a weak notion of convergence, we cannot get convergence in $H^2(0,l)$ spatially. Thus it is theoretically possible for the solution of the penalty equations u_ϵ to generate higher and higher frequency components as $\epsilon \downarrow 0$ so that the limiting solution (converging only weakly in $H^2(0,l)$) may actually be dissipative (that is, losing energy). In physical terms this would correspond to elastic energy being converted into heat. Note that heat can be considered as elastic vibrations with a length scale comparable to the inter-atomic or inter-molecular distances in the material.

Another difficulty that could potentially arise is that as the penalty equations are reversible, the "dissipativity" could occur going backwards in time. This would mean that the initial conditions for the penalty problem would actually be dependent on the penalty parameter: $u_{0,\epsilon}(x)$ where $u_{0,\epsilon}$ converges weakly but not strongly in $H^2(0,l)$ to u_0. In physical terms, this would correspond to heat being converted into (useful) elastic energy. As such it would violate the Second Law of Thermodynamics. As we should realize, the Second Law of Thermodynamics is a consequence of analysis of *statistical* systems and is a result that holds with extremely high probability, rather than a certain result for deterministic systems.

This line of thinking leads in several different directions: One is to consider the idea of explicitly incorporating the idea of "coefficient of restitution" into the system of conditions (5.3–5.8) for the beam in contact. This seems a rather problematic task since to do this we would need to separate the post-impact normal velocity at a point due to the impact force from the post-impact normal velocity due to elastic waves. Since we can only guarantee that the velocity is spatially in $L^2(0,l)$, this is not likely to be easy.

Another line of thinking is to consider replacing the Euler–Bernoulli model of a beam with a more realistic model, such as the Timoshenko beam model [57, 58]. This model consists of two coupled 2nd order PDEs, and combining this with a suitable version of the Signorini contact conditions results in a system that bears a number of strong resemblance with the case of a one-dimensional vibrating string (which satisfies the wave equation and the Signorini contact conditions) as analyzed by Schatzman [51].

A numerical method was obtained by first discretizing with respect to time, and then with respect to space. Convergence theory has been developed with respect to the time-discretization. The full discretization has been implemented and numerical results obtained. These numerical results seem to suggest that conservation of energy may hold for generic initial conditions.

We consider semi-discrete and fully discrete approximations to the motion of an Euler–Bernoulli beam with frictionless contact. For the semi-discrete approximation, we are able to show that there is a subsequence of the discrete time approximations that converges (albeit in a sufficiently weak sense) to a (weak) solution of the PDE and the Signorini contact conditions. From there we go on to develop a fully discrete approximation by using the Finite Element Method using B-splines to construct the basis functions. This scheme was implemented, and the Linear Complementarity Problems (LCPs) that arise at each time step were solved using a smoothed guarded Newton method applied to a reformulation of the LCP as a nonsmooth equation. These methods turn out to be quite efficient, especially since the one-dimensional structure of the problem results in banded matrices when handled properly. Furthermore, the number of linear solves carried out per time-step seems not to grow as the discretization parameters (h in time and k in space) go to zero.

Of particular interest in this thesis is the question of strong convergence of the solutions, sufficient to determine if the limiting solution conserves energy or not. A numerical scheme is devised in this thesis to test this question in a computationally efficient manner. The results from the computation give evidence that the numerical solutions for our problem do indeed converge strongly, and that even though the time-discretization is dissipative, the limit solution also conserves energy. No analytical demonstration of energy conservation is given; it can be demonstrated to be false in general, but may be true generically.

8.3 Open questions and future works

Our numerical results give strong evidence of the conservation of energy. However, the question of whether there can be conservation of energy for Signorini contact conditions is an open one. Our future work is to investigate this open problem and complete the dynamic problem with elastic body in the first part of this thesis.

More refined tools are needed for this analysis. Amongst the tools that we would likely need to carry out this analysis, pseudo-differential operators and other Fourier transform based techniques for the analysis of PDEs would be included.

Regarding the work on Euler–Bernoulli beams, it would be interesting to investigate the Timoshenko beam. This work would probably more closely resemble the work of Schatzman on the one spatial dimension wave equation with contact along its length. These problems are a little different from elastic body problems because contact can occur in the interior of the domain for the beam problems, while for elastic bodies, contact can occur only on the boundary. This may be very significant in the study of conservation and dissipation of energy in impacts.

Even though friction has not been considered in this thesis, frictional contact problems with dynamic effects will be our future work.

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