A NEAR-MINIMAX APPROXIMATION METHOD

Let $f(x)$ be continuous on $[a, b] = [-1, 1]$. Consider approximating f by an interpolatory polynomial of degree at most $n = 3$. Let x_0, x_1, x_2, x_3 be interpolation node points in $[-1, 1]$; let $c_3(x)$ be of degree ≤ 3 and interpolate $f(x)$ at $\{x_0, x_1, x_2, x_3\}$. The interpolation error is

$$f(x) - c_3(x) = \frac{\omega(x)}{4!} f^{(4)}(\xi_x), \quad -1 \leq x \leq 1 \tag{1}$$

$$\omega(x) = (x - x_0)(x - x_1)(x - x_2)(x - x_3) \tag{2}$$

with ξ_x in $[-1, 1]$. We want to choose the nodes $\{x_0, x_1, x_2, x_3\}$ so as to minimize the maximum value of $|f(x) - c_3(x)|$ on $[-1, 1]$.

From (1), the only general quantity, independent of f, is $\omega(x)$. Thus we choose $\{x_0, x_1, x_2, x_3\}$ to minimize

$$\max_{-1 \leq x \leq 1} |\omega(x)| \tag{3}$$

Expand to get

$$\omega(x) = x^4 + \text{lower degree terms}$$

This is a monic polynomial of degree 4. From the theorem in the preceding section, the smallest possible value for (3) is obtained with

$$\omega(x) = \widetilde{T}_4(x) = \frac{T_4(x)}{2^3} = \frac{1}{8}(8x^4 - 8x^2 + 1) \qquad (4)$$

and the smallest value of (3) is $1/2^3$ in this case. The equation (4) defines implicitly the nodes $\{x_0, x_1, x_2, x_3\}$: they are the roots of $T_4(x)$.

In our case this means solving

$$T_4(x) = \cos(4\theta) = 0, \qquad x = \cos(\theta)$$

$$\begin{aligned} 4\theta &= \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots \\ \theta &= \pm\frac{\pi}{8}, \pm\frac{3\pi}{8}, \pm\frac{5\pi}{8}, \pm\frac{7\pi}{8}, \ldots \\ x &= \cos\left(\frac{\pi}{8}\right), \cos\left(\frac{3\pi}{8}\right), \cos\left(\frac{5\pi}{8}\right), \ldots \qquad (5) \end{aligned}$$

using $\cos(-\theta) = \cos(\theta)$.

$$x = \cos\left(\frac{\pi}{8}\right), \cos\left(\frac{3\pi}{8}\right), \cos\left(\frac{5\pi}{8}\right), \cos\left(\frac{7\pi}{8}\right), \ldots$$

The first four values are distinct; the following ones are repetitive. For example,

$$\cos\left(\frac{9\pi}{8}\right) = \cos\left(\frac{7\pi}{8}\right)$$

The first four values are

$$\{x_0, x_1, x_2, x_3\} = \{\pm 0.382683, \pm 0.923880\} \qquad (6)$$

Example. Let $f(x) = e^x$ on $[-1, 1]$. Use these nodes to produce the interpolating polynomial $c_3(x)$ of degree 3. From the interpolation error formula and the bound of $1/2^3$ for $|\omega(x)|$ on $[-1, 1]$, we have

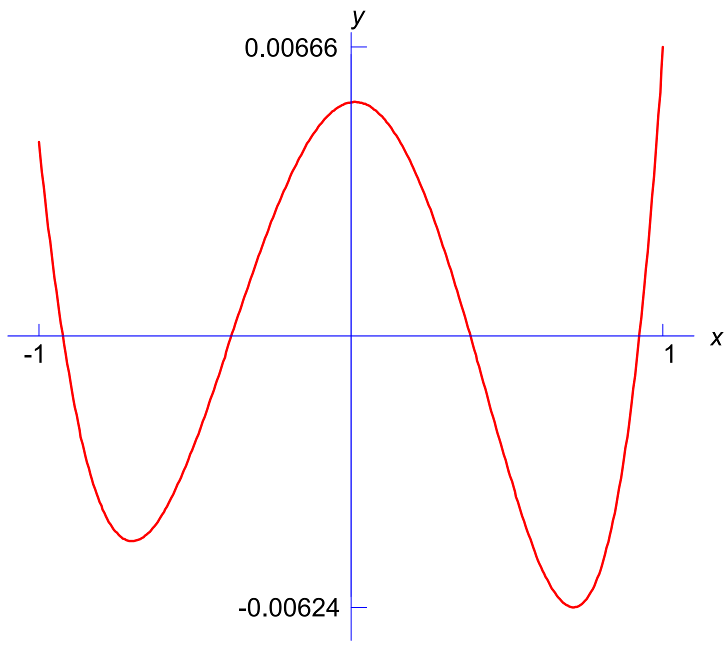
$$\begin{aligned} \max_{-1 \le x \le 1} |f(x) - c_3(x)| &\le \frac{1/2^3}{4!} \max_{-1 \le x \le 1} e^{\xi_x} \\ &\le \frac{e}{192} \doteq 0.014158 \end{aligned}$$

By direct calculation,

$$\max_{-1 \le x \le 1} |e^x - c_3(x)| \doteq 0.00666$$

Interpolation Data: $f(x) = e^x$

i	x_i	$f(x_i)$	$f[x_0, \ldots, x_i]$
0	0.923880	2.5190442	2.5190442
1	0.382683	1.4662138	1.9453769
2	-0.382683	0.6820288	0.7047420
3	-0.923880	0.3969760	0.1751757



The error $e^x - c_3(x)$

For comparison, $E(t_3) \doteq 0.0142$ and $\rho_3(e^x) \doteq 0.00553$.

THE GENERAL CASE

Consider interpolating $f(x)$ on $[-1,1]$ by a polynomial of degree $\leq n$, with the interpolation nodes $\{x_0,\ldots,x_n\}$ in $[-1,1]$. Denote the interpolation polynomial by $c_n(x)$. The interpolation error on $[-1,1]$ is given by

$$\begin{aligned} f(x) - c_n(x) &= \frac{\omega(x)}{(n+1)!} f^{(n+1)}(\xi_x) \qquad (7) \\ \omega(x) &= (x - x_0) \cdots (x - x_n) \end{aligned}$$

with ξ_x and unknown point in $[-1,1]$. In order to minimize the interpolation error, we seek to minimize

$$\max_{-1 \leq x \leq 1} |\omega(x)| \qquad (8)$$

The polynomial being minimized is monic of degree $n+1$,

$$\omega(x) = x^{n+1} + \text{ lower degree terms}$$

From the theorem of the preceding section, this minimum is attained by the monic polynomial

$$\widetilde{T}_{n+1}(x) = \frac{1}{2^n} T_{n+1}(x)$$

Thus the interpolation nodes are the zeros of $T_{n+1}(x)$; and by the procedure that led to (5), they are given by

$$x_j = \cos\left(\frac{2j+1}{2n+2}\pi\right), \qquad j = 0, 1, \ldots, n \tag{9}$$

The near-minimax approximation $c_n(x)$ of degree n is obtained by interpolating to $f(x)$ at these $n+1$ nodes on $[-1, 1]$.

The polynomial $c_n(x)$ is sometimes called a *Chebyshev approximation*.

Example. Let $f(x) = e^x$. the following table contains the maximum errors in $c_n(x)$ on $[-1, 1]$ for varying n. For comparison, we also include the corresponding minimax errors. These figures illustrate that for practical purposes, $c_n(x)$ is a satisfactory replacement for the minimax approximation $m_n(x)$.

n	max $\lvert e^x - c_n(x)\rvert$	$\rho_n(e^x)$
1	3.72E − 1	2.79E − 1
2	5.65E − 2	4.50E − 2
3	6.66E − 3	5.53E − 3
4	6.40E − 4	5.47E − 4
5	5.18E − 5	4.52E − 5
6	3.80E − 6	3.21E − 6

THEORETICAL INTERPOLATION ERROR

For the error

$$f(x) - c_n(x) = \frac{\omega(x)}{(n+1)!} f^{(n+1)}(\xi_x)$$

we have

$$\max_{-1 \le x \le 1} |f(x) - c_n(x)| \le \frac{\max_{-1 \le x \le 1} |\omega(x)|}{(n+1)!} \max_{-1 \le \xi \le 1} |f(\xi)|$$

From the theorem of the preceding section,

$$\max_{-1 \le x \le 1} \left| \widetilde{T}_{n+1}(x) \right| = \max_{-1 \le x \le 1} |\omega(x)| = \frac{1}{2^n}$$

in this case. Thus

$$\max_{-1 \le x \le 1} |f(x) - c_n(x)| \le \frac{1}{(n+1)! 2^n} \max_{-1 \le \xi \le 1} |f(\xi)|$$

OTHER INTERVALS

Consider approximating $f(x)$ on the finite interval $[a, b]$. Introduce the linear change of variables

$$x = \frac{1}{2}\left[(1-t)\,a + (1+t)\,b\right] \tag{10}$$

$$t = \frac{2}{b-a}\left[x - \frac{b+a}{2}\right] \tag{11}$$

Introduce

$$F(t) = f\left(\frac{1}{2}\left[(1-t)\,a + (1+t)\,b\right]\right), \quad -1 \le t \le 1$$

The function $F(t)$ on $[-1, 1]$ is equivalent to $f(x)$ on $[a, b]$, and we can move between them via (10)-(11). We can now proceed to approximate $f(x)$ on $[a, b]$ by instead approximating $F(t)$ on $[-1, 1]$.

Example. Approximating $f(x) = \cos x$ on $[0, \pi/2]$ is equivalent to approximating

$$F(t) = \cos\left(\frac{1+t}{4}\pi\right), \quad -1 \le t \le 1$$