CHEBYSHEV POLYNOMIALS

Chebyshev polynomials are used in many parts of numerical analysis, and more generally, in applications of mathematics. For an integer $n \geq 0$, define the function

$$T_n(x) = \cos\left(n \cos^{-1} x\right), \qquad -1 \leq x \leq 1 \qquad (1)$$

This may not appear to be a polynomial, but we will show it is a polynomial of degree n. To simplify the manipulation of (1), we introduce

$$\theta = \cos^{-1}(x) \quad \text{or} \quad x = \cos(\theta), \qquad 0 \leq \theta \leq \pi \quad (2)$$

Then

$$T_n(x) = \cos(n\theta) \qquad (3)$$

Example. $n = 0$

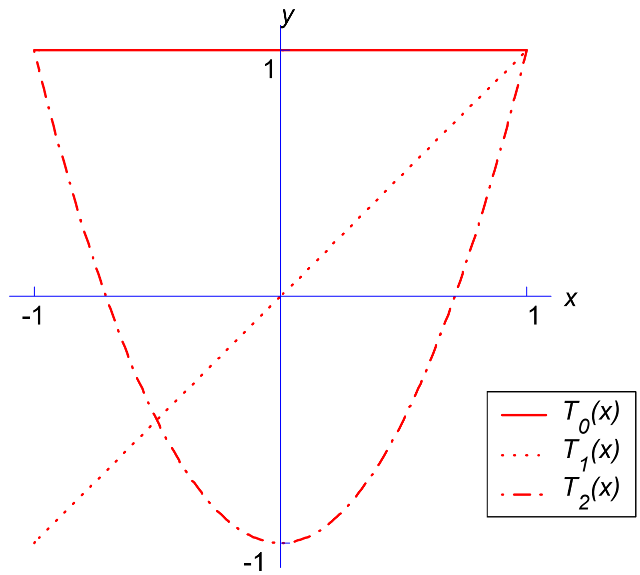
$$T_0(x) = \cos(0 \cdot \theta) = 1$$

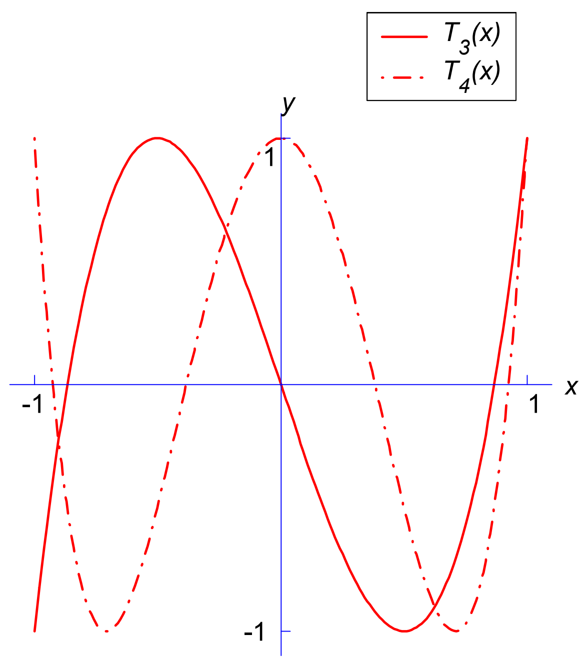
$n = 1$

$$T_1(x) = \cos(\theta) = x$$

$n = 2$

$$T_2(x) = \cos(2\theta) = 2\cos^2(\theta) - 1 = 2x^2 - 1$$


y
1
-1
x
1
-1
$T_0(x)$
$T_1(x)$
$T_2(x)$


$T_3(x)$
$T_4(x)$
y
1
-1
x
1
-1

The triple recursion relation. Recall the trigonometric addition formulas,

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

Let $n \geq 1$, and apply these identities to get

$$\begin{aligned} T_{n+1}(x) &= \cos[(n+1)\theta] = \cos(n\theta + \theta) \\ &= \cos(n\theta)\cos(\theta) - \sin(n\theta)\sin(\theta) \\ T_{n-1}(x) &= \cos[(n-1)\theta] = \cos(n\theta - \theta) \\ &= \cos(n\theta)\cos(\theta) + \sin(n\theta)\sin(\theta) \end{aligned}$$

Add these two equations, and then use (1) and (3) to obtain

$$T_{n+1}(x) + T_{n-1} = 2\cos(n\theta)\cos(\theta) = 2xT_n(x)$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad n \geq 1 \tag{4}$$

This is called the *triple recursion relation* for the Chebyshev polynomials. It is often used in evaluating them, rather than using the explicit formula (1).

Example. Recall

$$T_0(x) = 1, \qquad T_1(x) = x$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad n \geq 1$$

Let $n = 2$. Then

$$\begin{aligned} T_3(x) &= 2xT_2(x) - T_1(x) \\ &= 2x(2x^2 - 1) - x \\ &= 4x^3 - 3x \end{aligned}$$

Let $n = 3$. Then

$$\begin{aligned} T_4(x) &= 2xT_3(x) - T_2(x) \\ &= 2x(4x^3 - 3x) - (2x^2 - 1) \\ &= 8x^4 - 8x^2 + 1 \end{aligned}$$

The minimum size property. Note that

$$|T_n(x)| \leq 1, \qquad -1 \leq x \leq 1 \tag{5}$$

for all $n \geq 0$. Also, note that

$$T_n(x) = 2^{n-1}x^n + \text{lower degree terms}, \qquad n \geq 1 \tag{6}$$

This can be proven using the triple recursion relation and mathematical induction.

Introduce a modified version of $T_n(x)$,

$$\widetilde{T}_n(x) = \frac{1}{2^{n-1}}T_n(x) = x^n + \text{lower degree terms} \tag{7}$$

From (5) and (6),

$$\left|\widetilde{T}_n(x)\right| \leq \frac{1}{2^{n-1}}, \qquad -1 \leq x \leq 1, \qquad n \geq 1 \tag{8}$$

Example.

$$\widetilde{T}_4(x) = \frac{1}{8}\left(8x^4 - 8x^2 + 1\right) = x^4 - x^2 + \frac{1}{8}$$

A polynomial whose highest degree term has a coefficient of 1 is called a *monic polynomial*. Formula (8) says the monic polynomial $\widetilde{T}_n(x)$ has size $1/2^{n-1}$ on $-1 \leq x \leq 1$, and this becomes smaller as the degree n increases. In comparison,

$$\max_{-1 \leq x \leq 1} |x^n| = 1$$

Thus x^n is a monic polynomial whose size does not change with increasing n.

Theorem. Let $n \geq 1$ be an integer, and consider all possible monic polynomials of degree n. Then the degree n monic polynomial with the smallest maximum on $[-1, 1]$ is the modified Chebyshev polynomial $\widetilde{T}_n(x)$, and its maximum value on $[-1, 1]$ is $1/2^{n-1}$.

This result is used in devising applications of Chebyshev polynomials. We apply it to obtain an improved interpolation scheme.