BEST APPROXIMATION

Given a function $f(x)$ that is continuous on a given interval $[a, b]$, consider approximating it by some polynomial $p(x)$. To measure the error in $p(x)$ as an approximation, introduce

$$E(p) = \max_{a \le x \le b} |f(x) - p(x)|$$

This is called the *maximum error* or *uniform error* of approximation of $f(x)$ by $p(x)$ on $[a, b]$.

With an eye towards efficiency, we want to find the 'best' possible approximation of a given degree n. With this in mind, introduce the following:

$$\begin{aligned}\rho_n(f) &= \min_{\deg(p) \le n} E(p) \\ &= \min_{\deg(p) \le n} \left[\max_{a \le x \le b} |f(x) - p(x)| \right]\end{aligned}$$

The number $\rho_n(f)$ will be the smallest possible uniform error, or *minimax error*, when approximating $f(x)$ by polynomials of degree at most n. If there is a polynomial giving this smallest error, we denote it by $m_n(x)$; thus $E(m_n) = \rho_n(f)$.

Example. Let $f(x) = e^x$ on $[-1, 1]$. In the following table, we give the values of $E(t_n)$, $t_n(x)$ the Taylor polynomial of degree n for e^x about $x = 0$, and $E(m_n)$.

	Maximum Error in:	
n	$t_n(x)$	$m_n(x)$
1	7.18E − 1	2.79E − 1
2	2.18E − 1	4.50E − 2
3	5.16E − 2	5.53E − 3
4	9.95E − 3	5.47E − 4
5	1.62E − 3	4.52E − 5
6	2.26E − 4	3.21E − 6
7	2.79E − 5	2.00E − 7
8	3.06E − 6	1.11E − 8
9	3.01E − 7	5.52E − 10

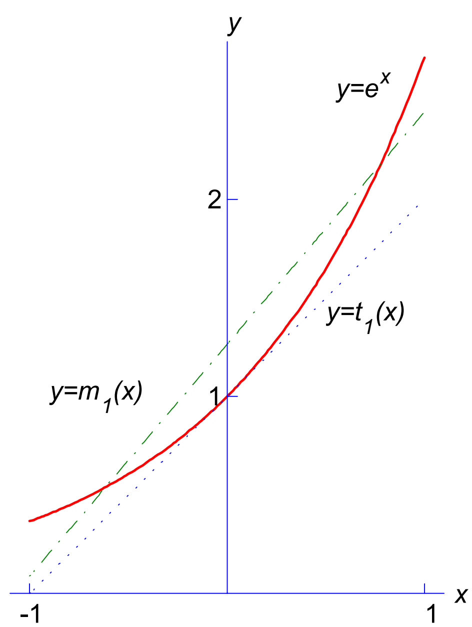
Consider graphically how we can improve on the Taylor polynomial

$$t_1(x) = 1 + x$$

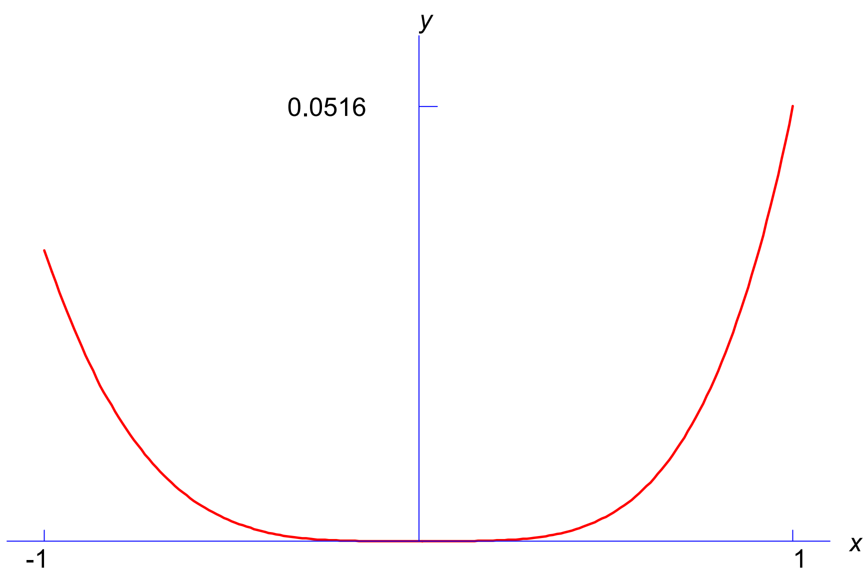
as a uniform approximation to e^x on the interval $[-1, 1]$.

The linear minimax approximation is

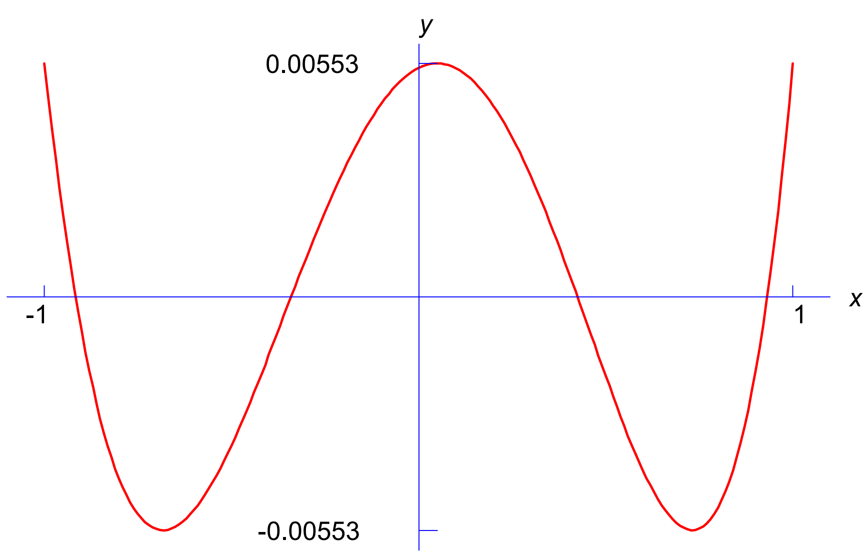
$$m_1(x) = 1.2643 + 1.1752x$$



Linear Taylor and minimax approximations to e^x



Error in cubic Taylor approximation to e^x



Error in cubic minimax approximation to e^x

Accuracy of the minimax approximation.

$$\rho_n(f) \le \frac{[(b-a)/2]^{n+1}}{(n+1)!2^n} \max_{a \le x \le b} \left| f^{(n+1)}(x) \right|$$

This error bound does not always become smaller with increasing n, but it will give a fairly accurate bound for many common functions $f(x)$.

Example. Let $f(x) = e^x$ for $-1 \le x \le 1$. Then

$$\rho_n(e^x) \le \frac{e}{(n+1)!2^n} \qquad (*)$$

n	*Bound* (*)	$\rho_n(f)$
1	6.80E − 1	2.79E − 1
2	1.13E − 1	4.50E − 2
3	1.42E − 2	5.53E − 3
4	1.42E − 3	5.47E − 4
5	1.18E − 4	4.52E − 5
6	8.43E − 6	3.21E − 6
7	5.27E − 7	2.00E − 7