ROOTFINDING

We want to find the numbers x for which $f(x) = 0$, with f a given function. Here, we denote such *roots* or *zeroes* by the Greek letter α. Rootfinding problems occur in many contexts. Sometimes they are a direct formulation of some physical situtation; but more often, they are an intermediate step in solving a much larger problem.

An example with annuities Suppose you are paying into an account an amount of P_{in} per period of time, for N_{in} periods of time. The amount you are deposited is compounded at at interest rate of r per period of time. Then at the beginning of period $N_{in}+1$, you will withdraw an amount of P_{out} per time period, for N_{out} periods. In order that the amount you withdraw balance that which has been deposited including interest, what is the needed interest rate? The equation is

$$P_{in}\left[(1+r)^{N_{in}}-1\right] = P_{out}\left[1-(1+r)^{-N_{out}}\right]$$

We assume the interest rate r holds over all $N_{in}+N_{out}$ periods.

As a particular case, suppose you are paying in $P_{in} =$ \$1,000 each month for 40 years. Then you wish to withdraw $P_{out} =$ \$5,000 per month for 20 years. What interest rate do you need? If the interest rate is R per year, compounded monthly, then $r = R/12$. Also, $N_{in} = 40 \cdot 12 = 480$ and $N_{out} = 20 \cdot 12 = 240$. Thus we wish to solve

$$1000\left[\left(1+\frac{R}{12}\right)^{480} - 1\right] = 5000\left[1 - \left(1+\frac{R}{12}\right)^{-240}\right]$$

What is the needed yearly interest rate R? The answer is 2.92%. How did I obtain this answer?

This example also shows the power of compound interest.

THE BISECTION METHOD

Most methods for solving $f(x) = 0$ are *iterative methods*. We begin with the simplest of such methods, one which most people use at some time.

We assume we are given a function $f(x)$; and in addition, we assume we have an interval $[a, b]$ containing the root, on which the function is continuous. We also assume we are given an error tolerance $\varepsilon > 0$, and we want an approximate root $\widetilde{\alpha}$ in $[a, b]$ for which

$$|\alpha - \widetilde{\alpha}| \leq \varepsilon$$

We further assume the function $f(x)$ changes sign on $[a, b]$, with

$$f(a)\, f(b) < 0$$

Algorithm **Bisect**(f, a, b, ε). *Step 1*: Define

$$c = \frac{1}{2}(a + b)$$

Step 2: If $b - c \leq \varepsilon$, accept c as our root, and then stop.
Step 3: If $b - c > \varepsilon$, then check compare the sign of $f(c)$ to that of $f(a)$ and $f(b)$. If

$$\text{sign}(f(b)) \cdot \text{sign}(f(c)) \leq 0$$

then replace a with c; and otherwise, replace b with c. Return to Step 1.

Denote the initial interval by $[a_1, b_1]$, and denote each successive interval by $[a_j, b_j]$. Let c_j denote the center of $[a_j, b_j]$. Then

$$\left|\alpha - c_j\right| \leq b_j - c_j = c_j - a_j = \frac{1}{2}\left(b_j - a_j\right)$$

Since each interval decreases by half from the preceding one, we have by induction,

$$|\alpha - c_n| \leq \left(\frac{1}{2}\right)^n (b_1 - a_1)$$

EXAMPLE Find the largest root of

$$f(x) \equiv x^6 - x - 1 = 0$$

accurate to within $\epsilon = 0.001$. With a graph, it is easy to check that $1 < \alpha < 2$. We choose $a = 1$, $b = 2$; then $f(a) = -1, f(b) = 61$, and the requirement $f(a)\,f(b) < 0$ is satisfied. The results from *Bisect* are shown in the table. The entry n indicates the iteration number n.

n	a	b	c	$b - c$	$f(c)$
1	1.0000	2.0000	1.5000	0.5000	8.8906
2	1.0000	1.5000	1.2500	0.2500	1.5647
3	1.0000	1.2500	1.1250	0.1250	−0.0977
4	1.1250	1.2500	1.1875	0.0625	0.6167
5	1.1250	1.1875	1.1562	0.0312	0.2333
6	1.1250	1.1562	1.1406	0.0156	0.0616
7	1.1250	1.1406	1.1328	0.0078	−0.0196
8	1.1328	1.1406	1.1367	0.0039	0.0206
9	1.1328	1.1367	1.1348	0.0020	0.0004
10	1.1328	1.1348	1.1338	0.00098	−0.0096

Recall the original example with the function.

$$f(r) = P_{in}\left[(1+r)^{N_{in}} - 1\right] - P_{out}\left[1 - (1+r)^{-N_{out}}\right]$$

Checking, we see that $f(0) = 0$. Therefore, with a graph of $y = f(r)$ on $[0, 1]$, we see that $f(x) < 0$ if we choose x very small, say $x = .001$. Also $f(1) > 0$. Thus we choose $[a, b] = [.001, 1]$. Using $\varepsilon = .000001$ yields the answer

$$\widetilde{\alpha} = .02918243$$

with an error bound of

$$|\alpha - c_n| \leq 9.53 \times 10^{-7}$$

for $n = 20$ iterates. We could also have calculated this error bound from

$$\frac{1}{2^{20}}(1 - .001) = 9.53 \times 10^{-7}$$

Suppose we are given the initial interval $[a, b] = [1.6, 4.5]$ with $\varepsilon = .00005$. How large need n be in order to have

$$|\alpha - c_n| \le \varepsilon$$

Recall that

$$|\alpha - c_n| \le \left(\frac{1}{2}\right)^n (b - a)$$

Then ensure the error bound is true by requiring and solving

$$\left(\frac{1}{2}\right)^n (b - a) \le \varepsilon$$

$$\left(\frac{1}{2}\right)^n (4.5 - 1.6) \le .00005$$

Dividing and solving for n, we have

$$n \ge \log\left(\frac{2.9}{.00005}\right) = 15.82$$

Therefore, we need to take $n = 16$ iterates.

ADVANTAGES AND DISADVANTAGES

Advantages: 1. It always converges.
2. You have a guaranteed error bound, and it decreases with each successive iteration.
3. You have a guaranteed rate of convergence. The error bound decreases by $\frac{1}{2}$ with each iteration.

Disadvantages: 1. It is relatively slow when compared with other rootfinding methods we will study, especially when the function $f(x)$ has several continuous derivatives about the root α.
2. The algorithm has no check to see whether the ε is too small for the computer arithmetic being used. [This is easily fixed by reference to the *machine epsilon* of the computer arithmetic.]

We also assume the function $f(x)$ is continuous on the given interval $[a, b]$; but there is no way for the computer to confirm this .