Chapter 4
Differential Equations and Derivatives

This chapter uses the "microscope" approximation to view derivatives as time rates of change and describe some practical changes.

The microscope approximation of the last chapter can be written

$$f[x + \Delta x] = f[x] + f'[x] \cdot \Delta x + \varepsilon \cdot \Delta x$$

If our independent variable is t rather than x and our function is $s[t]$ rather than $f[x]$, the equation becomes

$$s[t + \Delta t] = s[t] + s'[t] \cdot \Delta t + \varepsilon \cdot \Delta t$$

This is related to the differential equations in Chapter 2.

In Chapter 2 we approximated the susceptible and infectious fractions in a population by recursively computing

$$\begin{aligned} s[t + \Delta t] &= s[t] + s'[t]\Delta t \\ i[t + \Delta t] &= i[t] + i'[t]\Delta t \end{aligned}$$

where we computed the "prime" terms using the formulas:

$$s'[t] = -a\, s[t]\, i[t] \qquad \text{and} \qquad i'[t] = (a\, s[t] - b) i[t]$$

so

$$\begin{aligned} s[t + \Delta t] &= s[t] - (a\, s[t]\, i[t])\Delta t \\ i[t + \Delta t] &= i[t] + (a\, s[t]\, i[t] - b\, i[t])\Delta t \end{aligned}$$

The recursively computed functions $s[t]$ and $i[t]$ actually depended on the step size Δt, but we observed in practical terms that the approximate solutions converged with only a modestly small Δt. In any case, the exact solutions satisfy

$$\begin{aligned} s[t + \Delta t] &= s[t] + s'[t] \cdot \Delta t + \varepsilon_s \cdot \Delta t \\ i[t + \Delta t] &= i[t] + i'[t] \cdot \Delta t + \varepsilon_i \cdot \Delta t \end{aligned}$$

The recursive approximation discards the errors $\varepsilon_s \cdot \Delta t$ and $\varepsilon_i \cdot \Delta t$ at each successive step.

In this chapter, we formulate differential equations such as

$$\frac{dC}{dt} = -k\ C$$

and use the microscope approximation

$$C[t + \Delta t] = C[t] + C'[t] \cdot \Delta t + \varepsilon \cdot \Delta t$$

and the formula $C'[t] = -k\ C[t]$, so

$$C[t + \Delta t] = C[t] - k\ C[t] \cdot \Delta t + \varepsilon \cdot \Delta t$$

We approximate the exact solution $C[t]$ by discarding the error $\varepsilon \cdot \delta t$. The approximate solution is computed recursively: $C_a[0] = c_0$, and for later times,

$$C_a[t + \delta t] = C_a[t] - k\ C_a[t]\ \delta t$$

This satisfies $C_a[t] \approx C[t]$ when $\delta t \approx 0$ is small.

4.1 The Cool Canary

This section studies change and another kind of linearity - change that is linear in the dependent variable.

It was a cool January night in Iowa, about 0° F outside but cozy and around 75° F in our snug farm house. Jonnie was still mad at sister Sue for getting a better report card, so at 8:00 pm he put her covered canary cage outside. Ten minutes later the canary chirped as he always does at 60°F.

The canary will die when the temperature in the cage reaches freezing (32° F). How long do we have to rescue him?

This scenario is a little silly, but we are getting at some serious mathematics. The urgent need of the poor canary should hold your attention. We let C equal the temperature in degrees Fahrenheit and t equal the time in minutes, measured with $t = 0$ at the time Jonnie put the cage outside. Our data are $(t, C) = (0, 75)$ and $(t, C) = (10, 60)$. A linear model of the data looks like Figure 4.1, but is a linear model correct?

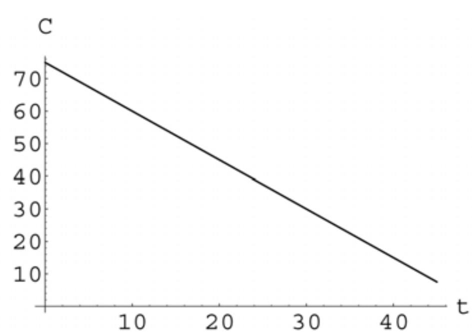

Figure 4.1: Linear cooling?

Will the temperature of the cage drop below zero? Of course not. Is the limiting value of the temperature zero? In other words, will the little canary's body temperature be zero when Susie finds him in the morning? Could the temperature decline linearly to zero and then abruptly stop declining and stay at zero?

A linear graph of temperature vs. time means that the rate of cooling remains the same for all temperatures. In other words, when the cage is much warmer than the outside air, it only cools as fast as when it has cooled down to almost zero. Does this seem plausible? Of course not. The graphs in Figure 4.1 and Figure 4.2 show a linear and a nonlinear fit to our same two data points and plot temperature for 50 minutes. Different nonlinear graphs are shown in Figure 4.3 for 600 minutes. Notice the "tapering off" of the nonlinear graph as the temperature gets near zero.

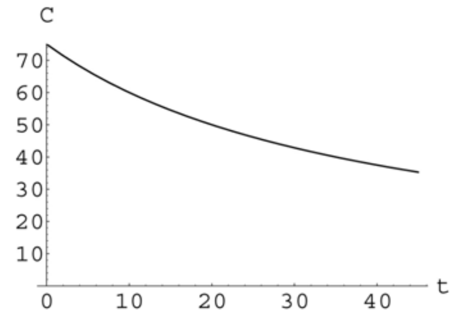

Figure 4.2: Realistic cooling?

There are many explicit formulas with the property that the rate of decrease itself decreases as the dependent variable tends toward zero. One student suggested a variation on $y = \frac{1}{x}$. If we sketch the graph of $y = \frac{1}{x}$ except make the variables C and t, we notice that there is a "singularity" at $t = 0$. We need to move the graph of $C = \frac{k}{t}$ to the left in order to make it at least a plausible model of the chilly canary. (Certainly the temperature does not tend to infinity as t tends to zero!) Why not also try a family of curves like $C = \frac{k}{(t+a)^2}$ or $C = \frac{k}{\sqrt{t+a}}$? This approach is explored in Exercise 4.1:3.

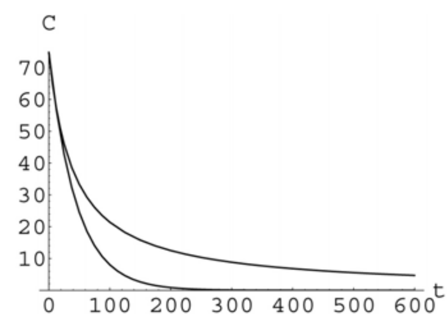

Figure 4.3: Two nonlinear cooling models

Guessing explicit formulas for temperature vs. time is a hard way to build a model of the cooling canary. It is better to go right to the underlying question of how the temperature *changes*. We want a mathematical relationship that says, "The rate of cooling decreases to zero as C decreases to zero." The simplest relationship is a linear equation between C and the rate of change of C with respect to t, C'. How do you "say" that C' is a linear function of C? This is Problem 4.1.

You should see that the ideas of the poor freezing canary's plight are simple - at least qualitatively. The rate of cooling decreases as the temperature decreases, but it is hard to express the simple ideas in terms of variables. This is because we do not have the proper mathematical *language*. A language for change needs symbols describing the change in other quantities.

Exercise Set 4.1

1. A Practical Two-Point Form
   What is the equation of the line that passes through the points $(0, 75)$ *and* $(10, 60)$*? Use your equation to predict the temperature of the canary at 6:00 am, 10 hours after Jonnie does his vicious deed.*

2. Tapering Off is a ?? in The Rate of Cooling
   What happens to the rate of cooling on the nonlinear graphs above as the temperature decreases toward zero? Does the rate of cooling decrease or increase as the temperature decreases toward zero? We expect that when C *is big, the rate of cooling is big, and that when* C *is near zero, the rate of cooling is small. Why do we expect this? Which graphs in the Figures above have this feature?*

3. Mindless Fits
   Show that all of the functions below satisfy $C[0] = 75$ *and* $C[10] = 60$.

   (a) $C = \frac{a}{t+k}, \quad a = 3000, \quad k = 40$

   (b) $C = \frac{a}{\sqrt{t+k}}, \quad a = 100\sqrt{10}, \quad k = 160/9$

   (c) $C = \frac{a}{(t+k)^2}, \quad a = 10000(27 + 12\sqrt{5}) \approx 538328, \quad k = 40 + 20\sqrt{5} \approx 84.7214$

   (d) $C = 75\ e^{-k\,t}, \quad k = \text{Log}[5/4]/10 \approx 0.0223144$

   Use the computer to graph each of these functions for 10 hours.

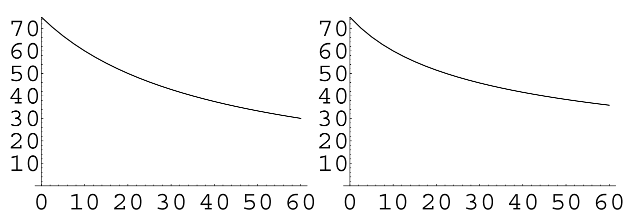

Figure 4.4: $C = \frac{a}{t+k}$ $\qquad C = \frac{a}{\sqrt{t+k}}$

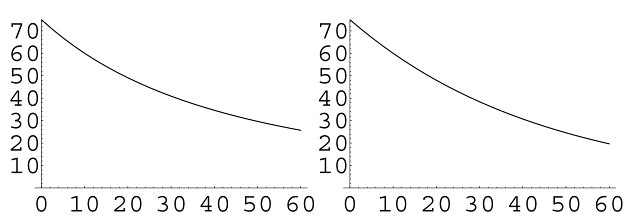

Figure 4.5: $C = \frac{a}{(t+k)^2}$ $\qquad C = \frac{a}{e^{kt}}$

Problem 4.1 LINEAR COOLING

Let C' stand for the rate of cooling of the canary cage of temperature C at time t. What is the relationship between rate of cooling, C', and the temperature, C? In other words, what does "rate of cooling" mean? For example, which units would you use to measure C'? What feature of the graph of C vs. t represents C'? What is the physical meaning of C'? Write an equation that says, "The rate of cooling is proportional to the temperature."

4.2 Instantaneous Rates of Change

Time rates of change are one important way to view derivatives. This section contains two examples. The reason that $-\frac{dC}{dt}$ is the instantaneous rate of decrease in temperature can be seen from the increment approximation.

Here is why $\frac{dC}{dt}$ is the rate of change of temperature - instantaneously. The change in temperature for a small change ($\delta t \approx 0$) in time, from time t to $t + \delta t$, is

$$C[t + \delta t] - C[t]$$

(in units of degrees). The rate at which it changes during this time interval of duration δt is the ratio

$$\frac{C[t + \delta t] - C[t]}{\delta t} = \frac{\delta C}{\delta t}$$

(in units of degrees per minute). The increment approximation for $C[t]$ is

$$C[t + \delta t] - C[t] = \frac{dC}{dt} \cdot \delta t + \varepsilon \cdot \delta t$$

with $\varepsilon \approx 0$, for $\delta t \approx 0$. Dividing results in the expression

$$\frac{C[t + \delta t] - C[t]}{\delta t} \approx \frac{dC}{dt}$$

In an "instant" δt becomes zero, the left hand side of the previous equation no longer makes sense mathematically but "tends" to the instantaneous rate of change as ε becomes zero. This means that

$$\text{the rate of change of } C \text{ in an instant of } t = \frac{dC}{dt}$$

Exactly what an instant is may not be clear, but the description of cooling makes it clear that it is useful.

4.2.1 The Dead Canary

The derivative $\frac{dC}{dt}$ of a function $C[t]$ is the instantaneous rate at which C changes as t changes. In the first section, we told you about Jonnie putting sister Susie's canary cage out into an Iowa winter night. The initial temperature was 75° F and 10 minutes later the temperature was 60° F. The rate of cooling would be fast when the temperature is high and gradually become slower as the temperature gets closer to zero. When the temperature reaches zero, the cage wouldn't cool any more. The simplest relationship with this property is that the rate of cooling is proportional to the temperature, or

$$\text{the rate of cooling} = kC$$

The poor canary has long since died, but that makes the mathematical model simpler anyway (since he no longer generates even a little heat.) We want to finish the story, mathematically speaking, before we bury the canary.

The simplest way to measure the rate of cooling of the canary cage would be to measure the rate at which the temperature *decreases* as time increases. In other words,

$$\text{the rate of cooling} = -\frac{dC}{dt}$$

Because the rate of cooling of the canary cage is proportional to the temperature, we have

$$\begin{aligned}
\text{rate of drop in temperature} &\propto \text{temperature} \\
\text{rate of drop in temperature} &= k \cdot \text{temperature} \\
-\frac{dC}{dt} &= kC \\
\frac{dC}{dt} &= -kC
\end{aligned}$$

We also know that $C = 75$ when $t = 0$ and $C = 60$ when $t = 10$.

Exercise Set 4.2

1. Euler Solution of the Canary Problem
   Use the program EulerApprox to compute the temperature of Susie's canary. Run your program with various values of k until $C = 60$ when $t = 10$. (Hint: Start with $k = 0.03$ and 0.01.) Once you determine the value of k that makes $C[10] = 60$, change the final time to 10 hours $= 600$ and plot the whole canary's plight, so that you find the temperature at 6:00 am in particular.

2. Graphical Euler's Method
   You only need to know where C starts ($C[0] = 75$) and your equation for change $\frac{dC}{dt} = -kC$ to sketch its curve. (This is not very accurate but gives the right shape.)

   (a) *Sketch the curve $C = C[t]$ when $k = 1/50$, so $\frac{dC}{dt} = -\frac{C}{50}$. When $C = 75$, this makes the slope of the curve $-3/2$. Move a small distance along the line of slope $-3/2$ from the point $(0, 75)$. If $C = 60$, how much is the slope of the curve? If $C = 5$, how much is the slope of the curve? If $C = 0$, what is the slope of the curve?*

   (b) *Sketch the curve $C = C[t]$ on the same axes when $k = 0.01$, so $\frac{dC}{dt} = -C/100$.*

(c) *Explain the idea in the computer's recursive "Euler" computation of the temperature. Write a few sentences covering the main ideas without giving a lot of technical details but rather only giving a description and a few sample computations. In other words, convince us that you understand the computer program and its connection with sketching the curve $C = C[t]$ from a starting value and differential equation.*

The language of change in Problem 4.1 says, "the cage cools slower as temperature approaches zero." Specifically, the rate of cooling is a linear function of the temperature. This is called "Newton's Law of Cooling," and it really does work (for inanimate objects). But math is not magic, as the next problem shows.

4.2.2 The Fallen Tourist

You have a chance to visit Pisa, Italy, and are fascinated by the leaning tower. You decide to climb up and throw some things off to see if Galileo was right. Luckily, you have a camera and your glasses to help with the experiment. You drop your camera and glasses simultaneously and are amazed that they smash simultaneously on the sidewalk below. Unconvinced that a heavy object does not fall faster, you race down and retrieve a large piece and a small piece of the stuff that is left.

Dropping the pieces again produces the same simultaneous smash and you also notice, of course, that the objects speed up as they fall. Not only that, but both the heavy object and the light object speed up at the same rates. You know that a feather would be affected by air friction; but you theorize that in vacuum, there must be just one "law of gravity" to govern falling objects.

About then, the Italian police arrive and take you to a nice quiet room to let you think about the grander meaning of your experiments while an interpreter arrives from the American consulate.

Problem 4.2 GALILEO'S FIRST CONJECTURE

Since all objects speed up as they fall, and at the same rate neglecting air friction, you theorize that

the speed of a falling object is proportional to the distance it has fallen.

Let D be the distance the object has fallen from the top of the tower at time t (in the units of your choice.)

1. (a) *What is the speed in terms of the rate of change of D with respect to t?*
   (b) *Formulate your "law of gravity" as a differential equation about D.*
   (c) *What is the initial value of D in your experiment?*
   (d) *Use the computer or mathematics to decide if your "law of gravity" is correct.*

We will follow up on the Tourist's progress later in the course when we study Bugs Bunny's Law of Gravity in Problem 8.5, Wiley Coyote's Law of Gravity in Problem 21.8, and Galileo's Law in Exercise 10.2.

4.3 Projects

4.3.1 The Canary Resurrected

The Scientific Project on the canary compares actual cooling data with the model of this chapter.