

Chapter 3

Linearity vs. Local Linearity

This chapter helps you understand the main approximation of differential calculus: Small changes in smooth functions are approximately linear. Calculus lets us use this approximation to replace nonlinear problems by linear ones. We view this first graphically by magnification, then numerically, and finally symbolically.

It is much easier to answer the question: “For which x is $f[x]$ equal to $g[x]$?” if $f[x]$ and $g[x]$ are of the form $f[x] = 1 + 0.07x$ and $g[x] = 0.3 + 0.27x$ than if $f[x] = \text{ArcTan}[x]$ and $g[x] = \text{Log}[x]$. Graphically, this is a question of finding a point of intersection (cf. Figures 3.1 and 3.2).

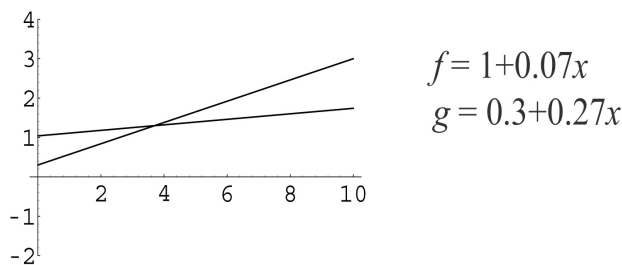


Figure 3.1: A common point on two linear graphs

Symbolically, the linear problem is simple, and the nonlinear one is not. Calculus lets us approximate smooth functions linearly on a small scale. (If you look very closely at the linear and nonlinear graphs near the point of intersection, they look the same.)

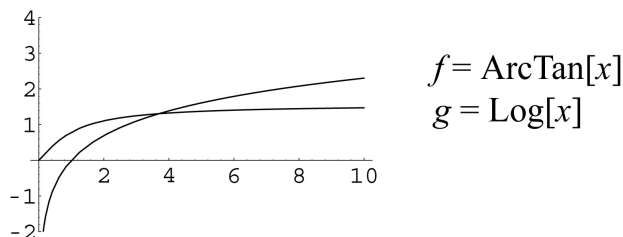


Figure 3.2: A common point on two nonlinear graphs

A linear function has the form $dy = m \cdot dx$ in local (dx, dy) -coordinates. The parameter m is the slope

of the line that graphs the function. This linear function is increasing if $m > 0$, decreasing if $m < 0$, and horizontal (or constant) if $m = 0$.

Calculus tells when a nonlinear function $y = f[x]$ is increasing by computing an approximating linear function, the differential, $dy = m \cdot dx$. The slope m of the differential is the derivative, $m = f'[x]$. It is easy to tell if the linear function is increasing; we simply ask if the slope m is positive. A nonlinear function also increases where its linear approximation increases and the approximation is valid. The “uniform derivatives” in this book make the approximation valid at least in a small interval.

The “local approximation” of differential calculus means that a microscopic view of a smooth function appears to be linear as in Figure 3.3. In other words, calculus can be used to compute what we would see in a powerful microscope focused on the graph of a nonlinear function. If we “see” an increasing linear function, then the nonlinear function is increasing in the range of the microscope.

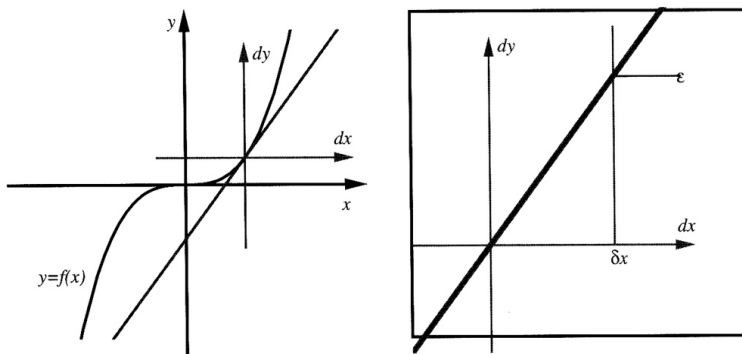


Figure 3.3: A highly magnified smooth graph

“Approximation” can mean many things. We take up a perfectly natural kind of approximation in the CO₂ Project in the accompanying separate book on Scientific Projects. Long-range approximation turns out not to be the kind that is so useful in calculus, but it is worth thinking about a little just for comparison. The next section gives you a clever case for comparison. “Local” microscopic approximation is probably not the first kind of approximation you would think to use, but it is the kind of approximation that makes calculus so successful.

3.1 Linear Approximation of Ox-bows

This section shows you one dramatic example of a nonlocal linear “approximation.” The local linearity of calculus is only good for small steps; we hope Mark Twain’s wit will help you remember this.

The chapter *Cut-offs and Stephen* of Mark Twain’s *Life on the Mississippi* contains the following excerpt about ox-bows on the lower Mississippi. In a flood, the river can jump its banks and cut off one of its meandering loops, thereby shortening the river and creating an ox-bow lake.

In the space of one hundred and seventy-six years the Lower Mississippi has shortened itself two hundred and forty-two miles. That is an average of a trifle over a mile and a third per year. Therefore, any calm person, who is not blind or idiotic, can see that in the Old Oolitic Silurian Period, just a million years ago next November, the Lower Mississippi River was upwards of one million three hundred thousand miles long, and stuck out over the Gulf of Mexico like a fishing-rod. And by the same token any person can see that

seven hundred and forty-two years from now the Lower Mississippi will be only a mile and three-quarters long, Cairo and New Orleans will have joined their streets together, and be plodding comfortably along under a single mayor and mutual board of aldermen. There is something fascinating about science. One gets such wholesome returns of conjecture out of such a trifling investment of fact.

Exercise Set 3.1

1. *Linear Ox-bows*

How wholesome are Twain's returns? Express the length of the river (in miles) as a function of time in years using the implicit mathematical assumption of Twain's statement. What is this assumption?

We are not given the length of the Lower Mississippi at the time of Twain's statement, but we can find it from the information. What is it? What is the moral of this whole exercise in terms of long-range prediction?

3.1.1 Chapter Plan

Here is the plan for the rest of the chapter. First, in Section 3.2, we use the computer to observe that the graphs of “typical” functions look “smooth.” Specifically, when we “zoom in” at high magnification to any point on the graph $y = f[x]$, we see a straight line. We will identify this line as $dy = f'[x]dx$ in the local coordinates described in Chapter 1.

Of course, there are exceptional “non-typical” functions, and we look briefly at them at the end of the next section.

Section 3.4 is our first real effort to measure the deviation from straightness. A smooth graph becomes indistinguishable from the straight differential line because the gap between the curve and the tangent decreases as we increase the magnification. This section measures the gap in some specific cases and concludes with a general summary problem.

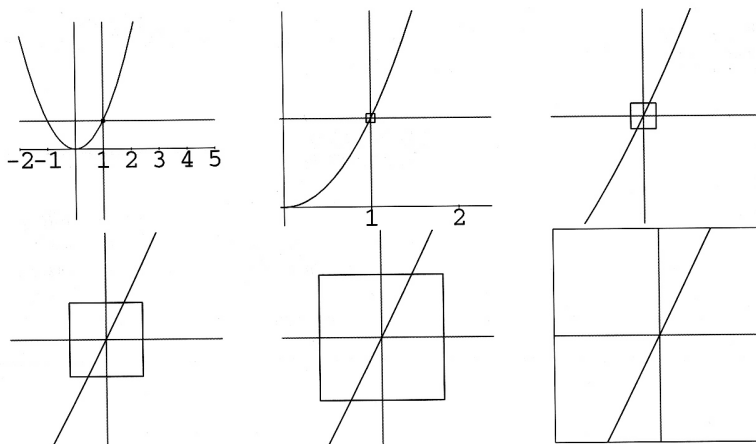
Section 3.5 uses the general formula first to verify your results from the previous special cases and then to extend the idea. Chapter 5 takes up the symbolic approach to the gap computations in general, whereas Chapter 4 shows how the gap formula looks in an approximate solution of a differential equation.

Chapter 6 gives the rules that allow us to find the derivative function $f'[x]$ by a “calculus.” These rules are actually theorems that guarantee that the gap in the microscope is small at high enough magnification. If you want to take that on faith, you could skip Sections 3.4, 3.5, and Chapter 5, where we compute the gap graphically, numerically, and symbolically.

3.2 Graphical Increments

This section uses computer “microscopes” to give one of the most important interpretations of differentiability or “local linearity.”

At high enough magnification, the graphs $y = x^2$, $y = x^3$, $y = x^{27}$, $y = 1/x$, and $y = \sqrt{x}$ appear straight (except at $x = 0$ on $1/x$ and $x \leq 0$ on \sqrt{x} .) The computer programs **Microscope** and **Zoom** will let you experiment with these and other functions yourself. The sequence of graphs in Figure 3.4 shows a “zoom” at $x = 1$ on the graph of $y = x^2$.

Figure 3.4: Successive magnifications of $y = x^2$ at $x = 1$

The computer programs **SecantGapZ** and **Zoom** help give you the geometric idea of the Increment Principle. These programs contain “animations” - a computer generated “movie” of graphs being magnified.

Exercise Set 3.2

1. Computer Zooming

- Run the computer program *Zoom* with its built-in example function, $y = f[x]$. Read the instructions in the program on making the computer run an animation. You should see a “movie” of an expanding graph. Each “frame” of the movie is expanded 1.5 times. Once the magnification is high enough, the graph appears to be linear.
- Focus the microscope at the point $x = 0$ instead of the built-in $x = 1$ and reenter the two computation cells.
- Use the computer program *Zoom* and redefine the function $f[x]$ to make animations of microscopic views of some of the functions:

- | | | |
|-----------------------------------|------------------------------|----------------------------|
| i) $f[x] = x^2$ | ii) $f[x] = x^3$ | iii) $f[x] = \frac{1}{x}$ |
| iv) $f[x] = \sqrt{x}$ | v) $f[x] = \text{ArcTan}[x]$ | vi) $f[x] = \text{Log}[x]$ |
| vii) $f[x] = \text{Exp}[x] = e^x$ | viii) $f[x] = \text{Cos}[x]$ | ix) $f[x] = \text{Sin}[x]$ |

Try focusing the microscope over the points $x = 1.5$, $x = 0$, and $x = -1$ when all these are possible. (Say why they if they cannot be done.)

2. Measuring the Derivative

Views of a function $y = f[x]$ are shown in the three figures below magnified by 50 and focused on the graph above three different x 's. Measure the derivative at each point by finding the slope of the approximating straight line.

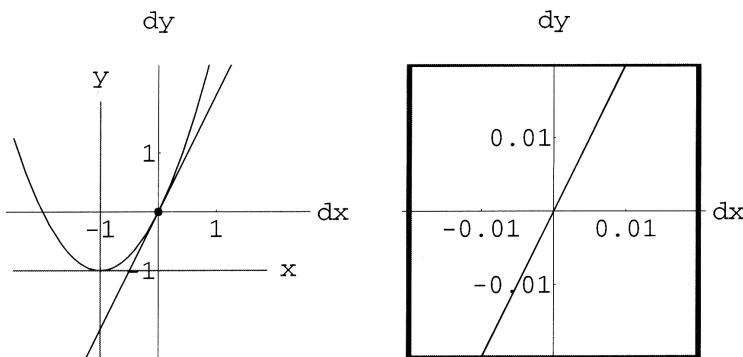


Figure 3.5: $y = f[x]$ magnified at $x = 1$, $f'[1] = ?$

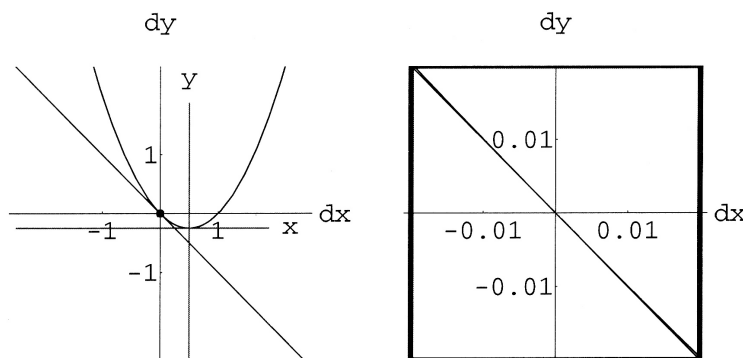


Figure 3.6: $y = f[x]$ magnified at $x = 3/2$, $f'[3/2] = ?$

Functions with Kinks and Jumps

Not all functions are smooth or locally linear. This means that when we magnify some functions, their graphs do not become more and more like straight lines. Which functions do, and which do not? Symbolic rules of calculus will answer this question easily, and it turns out that “most” functions are smooth so that microscopic views of their graphs do appear linear. Two exercises below experiment with some exceptional cases.

3 Jump Functions

Run the computer program *Zoom on the functions*

$$j_1[x] = \frac{\sqrt{x^2 + 2x + 1}}{x + 1} \quad \text{and} \quad j_2[x] = \frac{\sqrt{x^2 - x^4/2}}{x}$$

Focus the microscope at the points $x = +1$, $x = 0$, and $x = -1$. Where are the functions smooth and locally linear? Where is the “trouble spot”? What happens to the functions at $x =$ the trouble spot? Why?

4 Kink Functions

Run the computer program *Zoom on the functions*

$$k_1[x] = \sqrt{x^2 + 2x + 1} \quad \text{and} \quad k_2[x] = \sqrt{x^2 - x^4/2}$$

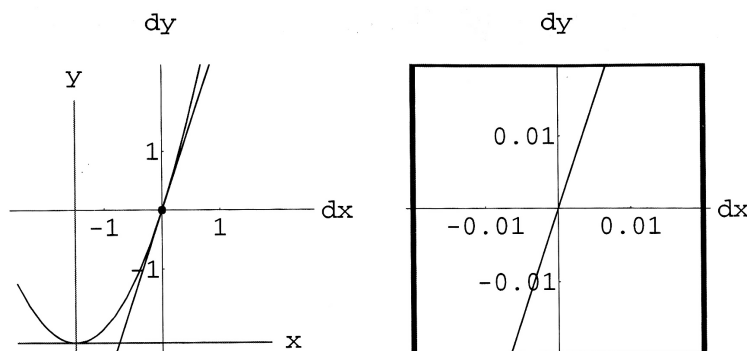


Figure 3.7: $y = f[x]$ magnified at $x = 3/2$, $f'[3/2] = ?$

Focus the microscope at the points $x = +1$, $x = 0$, and $x = -1$. Where are the functions smooth and locally linear? Where is the “trouble spot”? What happens to the functions at $x =$ the trouble spot? Why?

3.3 Continuous, But Not Smooth

The main ideas of differential calculus are based on approximating small changes in the output of a function when a small change is made in its input. We need some notation to indicate a small change. If x_1 and x_2 are nearly equal, we will write

$$x_1 \approx x_2$$

For now, this will just be an intuitive notion; we will not say exactly how close they have to be in order to write $x_1 \approx x_2$.

Definition 3.1 Informal Definition of Continuity

We say that a function $f[x]$ is continuous when small changes in x only produce small changes in the value of the function,

$$x_1 \approx x_2 \Rightarrow f[x_1] \approx f[x_2]$$

This is an intuitive formulation of the expression

$$\lim_{x_1 \rightarrow x_2} f[x_1] = f[x_2]$$

and it is important for you to have an idea of what this means before we try to formalize it technically (in Chapter 5.) The definition is a kind of approximation: $f[x_2]$ is approximately $f[x_1]$ when x_2 is approximately x_1 , but it is NOT an approximation by a linear function. Local linear approximation graphically says the graph looks like a straight line at a small scale. (Naturally, a linear function is continuous because the function changes by a multiple of the change in x . We formulate linear approximation symbolically in the Section 3.5.)

Problem 3.1 CONTINUITY WITHOUT LOCAL LINEARITY

Explain in terms of the definition above why the functions $j_i[x]$ in Exercise Set 3.2:3 are not continuous (at certain points) but the functions $k_i[x]$ in Exercise 3.2:4 are continuous (at least where they are defined).

The jump functions $j_i[x]$ are discontinuous at certain points, and the functions $k_i[x]$ are continuous but not locally linear (or smooth or differentiable) at certain points. The function (shown in Figure 3.8)

$$W[x] = \cos[x] + \frac{\cos[3x]}{2} + \frac{\cos[3^2x]}{2^2} + \frac{\cos[3^3x]}{2^3} + \dots + \frac{\cos[3^kx]}{2^k} + \dots$$

is continuous but is not locally linear at *any* point. It has a kink at every point on the graph! Weierstrass discovered this function, and its graph looks like lots of different size “Ws” all hooked together. It is an old example of a “fractal.” We want you to graph it and look at it on several scales. When we study series later in the course, we will return to Weierstrass’ function. (See Problem 3.2.)

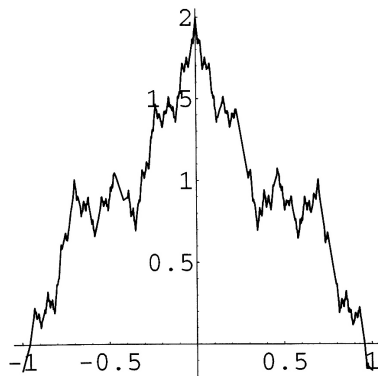


Figure 3.8: Weierstrass’ nowhere differentiable function

Problem 3.2 WEIERSTRASS WILD WIGGLES

Graph Weierstrass’ function using the program **Weierstrass**. Try several scales, “delta”. The program keeps the width of the x-axis and the width of the y-axis on your graph the same. This prevents distortion of slopes.

3.4 Algebra of Microscopes

The goal of this section is to numerically and symbolically calculate the error of deviation from straightness in microscopic views of graphs.

As a warm up to magnifying graphs, think about this question: *If you magnify a segment by one million and it appears to be 1 cm long, how long is it?* We need a general formula to answer this kind of question

so that we can predict when *sufficient* magnification will make a graph appear linear. We begin with some numerics to help present the idea.

3.4.1 Three Specific Magnifications

We begin by magnifying the graph of $y = x^3$ at the point $(x, y) = (3/4, 27/64) \approx (0.75, 0.421875)$ and comparing it with the line $dy = \frac{27}{16} \cdot dx$ in local (dx, dy) -coordinates centered at $(x, y) = (3/4, 27/64)$. We look at changes in x of $\Delta x = 1/2, 1/4, 1/8$ and magnify each graph to look at the difference between the curve and the line. We will make measurements in centimeters and adjust scales according to the amount of magnification.

Example 3.1 Magnification of $\Delta x = 1/2$ at $x = 3/4$

The graph in Figure 3.9 at the left has 1 cm for each unit. The graph at the right is magnified by 2 so 1 cm equals $1/2$ unit. Points are indicated at $(x, y) = (3/4, 27/64)$ or $(dx, dy) = (0, 0)$, and above $dx = \Delta x = 1/2$.

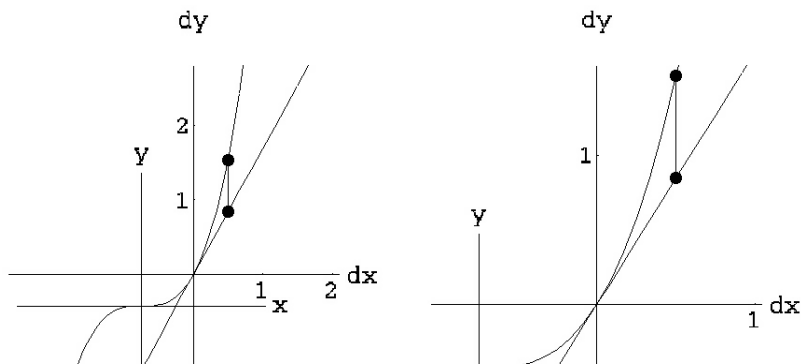


Figure 3.9: $y = x^3$ and $dy = \frac{27}{16} \cdot dx$ at $(\frac{3}{4}, \frac{27}{64})$, $\Delta x = \frac{1}{2}$

Example 3.2 Magnification of $\Delta x = 1/4$ at $x = 3/4$

The graph in Figure 3.10 at the left has 1 cm for each unit. The graph at the right is magnified by 4 so 1 cm equals $1/4$ unit. Points are indicated at $(x, y) = (3/4, 27/64)$ or $(dx, dy) = (0, 0)$, and above $dx = \Delta x = 1/4$.

Example 3.3 Magnification of $\Delta x = 1/8$ at $x = 3/4$

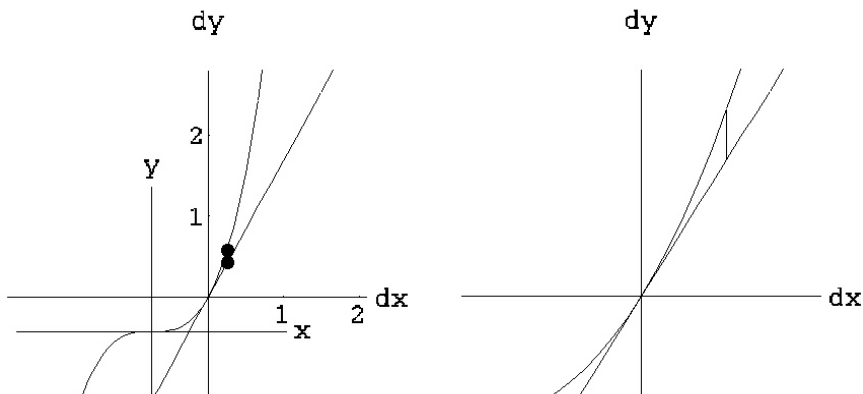


Figure 3.10: $y = x^3$ and $dy = \frac{27}{16} \cdot dx$ at $(\frac{3}{4}, \frac{27}{64})$, $\Delta x = \frac{1}{4}$

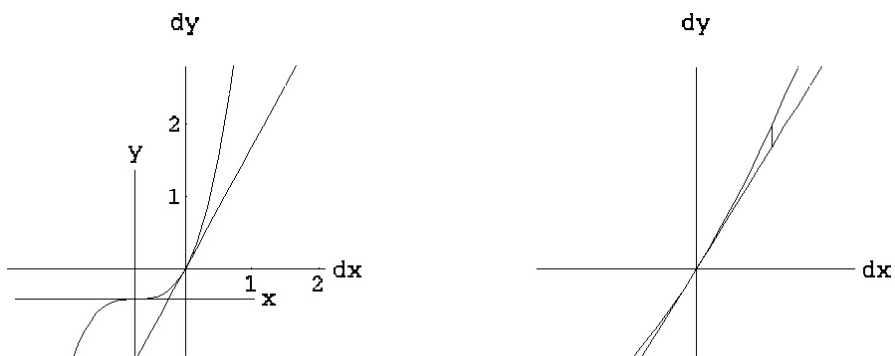


Figure 3.11: $y = x^3$ and $dy = \frac{27}{16} \cdot dx$ at $(\frac{3}{4}, \frac{27}{64})$, $\Delta x = \frac{1}{8}$

The graph Figure 3.11 at the left has 1 cm for each unit. The graph at the right is magnified by 8 so 1 cm equals 1/8 unit. Points are indicated at $(x, y) = (3/4, 27/64)$ or $(dx, dy) = (0, 0)$, and above $dx = \Delta x = 1/8$.

Example 3.4 *A Specific Gap of $y = x^3$ at a specific magnification*

We are interested in the gap between the curve and the straight line in the last microscopic view (where the change in x was $dx = 1/8$.) We will call the amount we measure (in cm) in the microscopic view “epsilon,” ϵ , greek “E” (for error). In this example, we want to know

- (1) How big is ϵ (in cm) in the microscopic view?
- (2) How big is the gap in original unmagnified coordinates?

Remember that if we magnify by one million and see 1 cm on the microscopic image, we actually have an error of 10^{-6} cm, one one millionth of the apparent error.

The actual distance from the dx -axis, where $y = f[x] = f[3/4] = 27/64$, to the point on the curve above $dx = \Delta x = 1/8$ is given by the difference,

$$f[x + \Delta x] - f[x] = \left(\frac{7}{8}\right)^3 - \left(\frac{3}{4}\right)^3 = \frac{127}{512} \approx 0.248047$$

because the input to $f[\cdot]$ is $3/4 + 1/8 = 7/8$ at the new point. The magnified segment of the curve is shown at the left in Figure 3.12. The actual height of the vertical segment from the dx -axis to the curve is $f[x + \Delta x] - f[x]$. The measured length after magnification is $(f[x + \Delta x] - f[x])/\Delta x$. The magnified linear approximation is shown at the right.

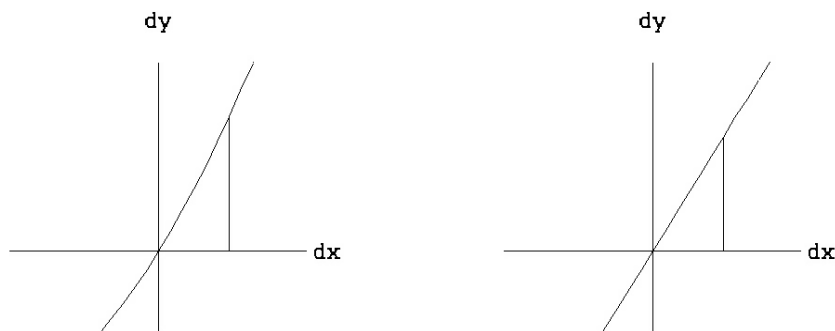


Figure 3.12: $y = x^3$ and $dy = \frac{27}{16} \cdot dx$ at $(\frac{3}{4}, \frac{27}{64})$, $\Delta x = \frac{1}{8}$

The distance from the dx -axis to the point on the line $dy = m dx$ above the point where $dx = 1/8$ is

$$dy = m dx = \frac{27}{16} \cdot \frac{1}{8} = \frac{27}{128} \approx 0.210937$$

The magnified vertical segment at the right measures $8 \cdot \frac{27}{16} \cdot \frac{1}{8} = \frac{27}{16} = m$ because we magnify by $1/\Delta x$.

Finally, we want to compare the difference between these vertical distances. Figure 3.13 shows the nonlinear and linear graphs at magnification 8 and shows a small segment connecting the linear graph to the nonlinear one above $x + \Delta x$. The actual length of the vertical segment connecting the (dx, dy) -point on the tangent with the (x, y) -point on the curve is the difference between these two values

$$\begin{aligned} f[x + \Delta x] - f[x] - m \cdot \Delta x &= f\left[\frac{3}{4} + \frac{1}{8}\right] - f\left[\frac{3}{4}\right] - \frac{27}{16} \cdot \frac{1}{8} \\ &= \left(\frac{7}{8}\right)^3 - \frac{27}{64} - \frac{27}{128} = \frac{19}{512} \approx 0.037109 \end{aligned}$$

The error magnified by 8 (as shown) measures

$$\varepsilon = \frac{19}{512} \cdot 8 = \frac{19}{64} \approx 0.296875 \text{ (cm)}$$

The moral of this whole chapter is that the error ε gets smaller and smaller as the x -increment, Δx , gets smaller, even when we view this gap at magnification $1/\Delta x$. This section only has some numerical examples to get us started at measuring this gap.

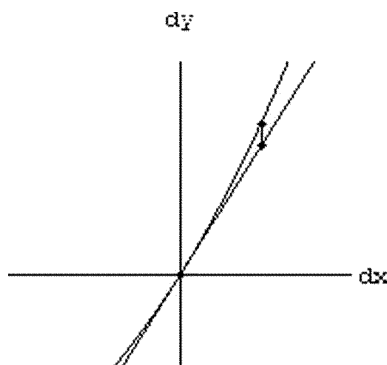


Figure 3.13: The magnified gap between $y = f[x]$ and $dy = m dx$

Exercise Set 3.3

In the following exercises, draw the sketches accurately on good graph paper. Be careful about the method you use to calculate your results so you are connecting numerics, symbolics, and graphics. You can use the computers if you wish, but you will still need to do algebra. If you magnify by $1/\Delta x$, so that a segment of length Δx appears to be unit size, and you observe another segment of apparent length ε , the actual length of the unmagnified segment is $\varepsilon \cdot \Delta x$. The first exercise tries to help you understand this formula.

1. Let $\varepsilon = 0.33$ and $\Delta x = 0.01$. Draw a line with a scale of 1 unit = 1 cm on the left half of your paper. Put dots at $x = 0$, $\Delta x = 0.01$, and $\varepsilon \cdot \Delta x = 0.033$. (What is the difficulty when you draw in centimeter units?) On the right half of your paper, draw a line starting at 0 magnified by 100. How far in centimeters on the magnified picture is the dot for Δx and $\varepsilon \cdot \Delta x$?

2. Your First Magnification

- (a) Sketch the graph of $y = x^3$ for $-1.5 \leq x \leq 1.5$ using equal x and y scales with 1 cm = 1 unit.
- (b) We want to focus our microscope over the x -value $x = 2/3$. What is the corresponding y -value?
- (c) Draw the local (dx, dy) -axes with its origin at the (x, y) -point, $(x, y) = (2/3, 8/27)$ on the same graph. The (dx, dy) origin lies at the center of your microscope. We will think of x as fixed and vary dx and dy .
- (d) The straight line in local coordinates $dy = (4/3)dx$ is tangent to $y = x^3$ at $x = 2/3$. Sketch this as a dotted line on the same graph.
- (e) Magnify a portion of your graph by 10 and draw the microscopic view with the same scale you used for your original (x, y) -plot. This way $(1/10)$ will appear unit size, that is, measure 1 cm on your graph paper.

Your result from Exercise 2 should look something like the Figure 3.14. Compare your work to the computer animation in the program **SecantGapZ**.

3. Measuring the Specific Gap

Carefully draw the curve $y = x^3$ between the values $x = \frac{2}{3}$ and $x + \Delta x = \frac{2}{3} + \frac{1}{10}$ on a scale where $1/10$ of a unit is 1 cm, that is, magnified by 10 if 1 cm is the original unit. The curve $y = x^3$ lies above your tangent line $dy = \frac{4}{3} dx$, where the local dx - dy axes lie at the center of the “microscope” magnifying the figure.

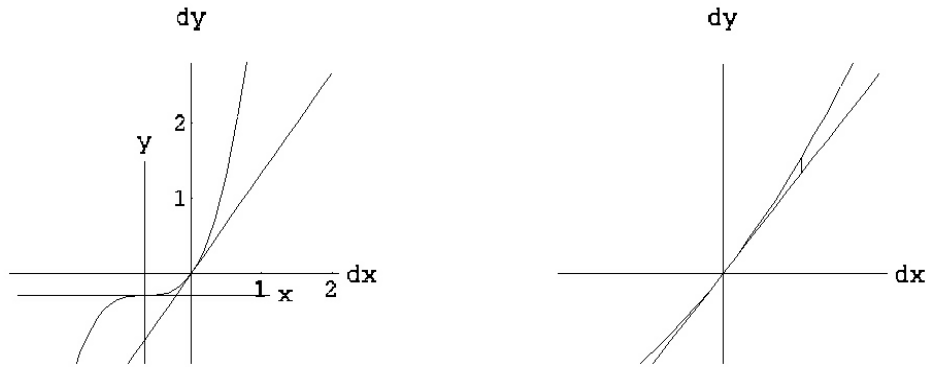


Figure 3.14: $y = x^3$ and $dy = \frac{4}{3} \cdot dx$ at $(\frac{2}{3}, \frac{8}{27})$, $\Delta x = \frac{1}{10}$

- What are the (x, y) -coordinates of the point where $(dx, dy) = (0, 0)$?
- The x -coordinate of the point $dx = 1/10 = 0.1$ is $0.7666\dots$, because $dx = 0$ corresponds to $x = 2/3 = 0.6666\dots$. What is the y -coordinate on $y = x^3$ when $x = 0.7666\dots$? What is the change in y from the dx -axis to the point on the curve above $x = 2/3 + 1/10$?
- What is the change in dy from $dy = 0$ to the point on $dy = (4/3)dx$ above $dx = \Delta x = 1/10$?
- What we see in the microscope as ε on our centimeter scale is the magnified difference between the answer to Part 2 and the answer to Part 3, so its actual size is given by applying the formula from Exercise 1 to this difference. How much is it?

4. A Gap in Function Notation

Let $f[x] = x^3$ in the following computations, and verify that the function procedures produce the answers to the measurement of ε as in the examples and exercises above, especially Exercise 3.4.1. Basic units are measured in centimeter, whereas magnified units appear 10 times larger.

- When $x = 2/3$ and the (dx, dy) -coordinates are centered on $y = f[x]$ above this value of x , then the (x, y) -value of $(dx, dy) = (0, 0)$ is $(x, f[x]) = (2/3, f[2/3]) = (2/3, ??)$.
- On the sketch of the portion of $y = f[x]$ between $x = \frac{2}{3}$ and $x + \Delta x = \frac{2}{3} + \frac{1}{10}$ with a scale of $1/10$ unit = 1 cm, the length of the vertical segment from the dx -axis to the (x, y) -point $(x + \Delta x, f[x + \Delta x])$ is 10 times the unit value of

$$f[x + \Delta x] - f[x] = \left(\frac{2}{3} + \frac{1}{10}\right)^3 - \left(\frac{2}{3}\right)^3 = ?$$

or simply,

$$\frac{f[x + \Delta x] - f[x]}{\Delta x}$$

- If we sketch the line $dy = m dx$, with $m = 4/3$, the length of the vertical segment (in original units) between the dx -axis and the point above $dx = \Delta x = 1/10$ is

$$dy = m \Delta x$$

If we magnify by $1/\Delta x$ so that Δx appears unit size or specifically, $1/10$ unit equals 1 cm, then this distance measures

$$m = 4/3$$

(d) The magnified gap ε measures

$$\varepsilon = \frac{f[x + \Delta x] - f[x]}{\Delta x} - m$$

(e) The actual unmagnified size of the gap between the curve and its tangent approximation above $x + \Delta x$ is

$$f[x + \Delta x] - f[x] = m \Delta x + \varepsilon \cdot \Delta x$$

3.4.2 The General Gap

Now we help you find the formula that expresses the quantities we see in a microscopic view of an unknown function $y = f[x]$ when magnified by an arbitrary $(1/\Delta x)$.

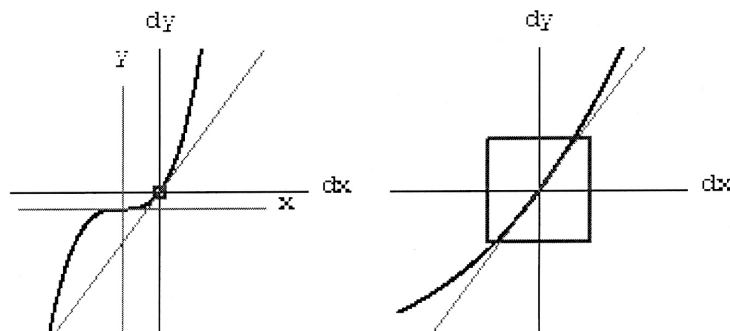


Figure 3.15: Slight magnification of $y = f[x]$

Figure 3.15 is a sketch of a general function $y = f[x]$ with 1 unit = 1 cm. A pair of local coordinate (dx, dy) -axes is centered on the graph over a fixed point x . A line in local coordinates $dy = m dx$ is shown in grey.

On the right, an image of these graphs is shown magnified by $1/\Delta x$, so that the small number Δx appears unit size. Since we magnify by an amount that makes Δx appear 1 cm in size, if we measure a distance ε in the microscopic image, the actual size is really $\varepsilon \cdot \Delta x$. (Check this formula intuitively when $\Delta x = 1/1,000,000$. We magnify by one million and see a gap of 0.3, for example, but it is really only a gap of $0.3/1000000$.)

Problem 3.3 SYMBOLIC MAGNIFICATION FOR AN UNKNOWN $Y=F[X]$

Explain the following statements about Figure 3.15:

1. The vertical distance from the dx -axis up to the curve is

$$f[x + \Delta x] - f[x]$$

but the magnified view of this vertical segment measures

$$\frac{f[x + \Delta x] - f[x]}{\Delta x}$$

2. The vertical distance from the dx -axis up to the line $dy = m dx$ above the point $dx = \Delta x$ is

$$m \Delta x$$

but the magnified view of this vertical segment measures

$$m$$

3. The magnified gap ε measures

$$\varepsilon = \frac{f[x + \Delta x] - f[x]}{\Delta x} - m$$

but the actual change in the function as x moves to $x + \Delta x$ is

$$f[x + \Delta x] - f[x] = m \Delta x + \varepsilon \cdot \Delta x$$

3.5 Symbolic Increments

The following formula for the change in a general function

$$f[x + \Delta x] - f[x] = f'[x] \cdot \Delta x + \varepsilon \cdot \Delta x$$

gives the gap ε one would measure at magnification $1/\Delta x$ between a straight line of slope $f'[x]$ and the curve as we move from x to $x + \Delta x$.

The condition for local linearity of a graph $y = f[x]$ is that the magnified error gap between the curve and line is small, $\varepsilon \approx 0$, when the magnification is large. In other words, if the local change in x , $dx = \delta x \approx 0$, then the MAGNIFIED change along the curve is $\varepsilon \approx 0$ -close to the change along the line. (The lowercase [small] Greek delta, δ , indicates intuitively that the difference in x is a very small amount.)

When the gap is small for $\delta x \approx 0$, the slope of the local linear approximation, $f'[x]$, is called the “derivative.” This is what we saw in the examples and exercises of the last section:

Summary of The Graphical Approximations or " $f[x]$ is the Derivative of $f[x]$ "

The condition of (uniform) “tangency” is expressed by the microscopic error formula

$$f[x + \delta x] - f[x] = f'[x] \cdot \delta x + \varepsilon \cdot \delta x$$

provided the magnified error is small, $\varepsilon \approx 0$, whenever the change in x is small, $\delta x \approx 0$.

The approximation $\varepsilon \approx 0$ means that a microscopic view of a tiny piece of the graph $y = f[x]$ looks the same as the linear graph $dy = m \cdot dx$ on the scale of δx . This looks like

When we say $f'[x]$ is the derivative of $f[x]$, we mean that this local approximation is valid, $\varepsilon \approx 0$ when $\delta x \approx 0$, or $\varepsilon \rightarrow 0$ as $\Delta x \rightarrow 0$.

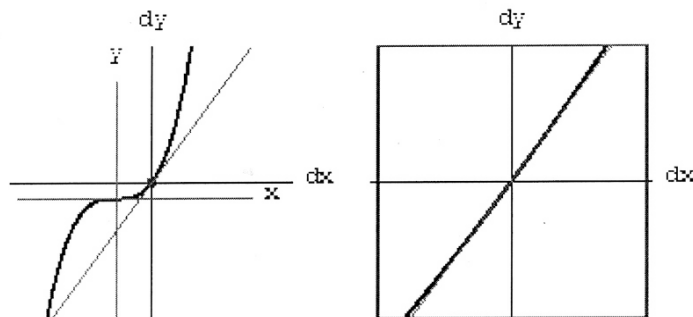


Figure 3.16: A Symbolic Microscope

Chapter 5 is devoted to symbolic computations of the gap ε and symbolic ways to show that it becomes small when δx is small. This is easy to verify in the case of $y = x^3$.

We expand

$$\begin{aligned}
 f[x + \delta x] - f[x] &= (x + \delta x)^3 - x^3 \\
 &= x^3 + 3x^2 \delta x + 3x \delta x^2 + \delta x^3 - x^3 \\
 &= 3x^2 \delta x + 3x \delta x^2 + \delta x^3 \\
 &= 3x^2 \delta x + (\delta x \cdot (3x + \delta x)) \cdot \delta x \\
 f[x + \delta x] - f[x] &= f'[x] \cdot \delta x + \varepsilon \cdot \delta x
 \end{aligned}$$

with $f'[x] = 3x^2$ and $\varepsilon = \delta x \cdot (3x + \delta x)$. Because $\varepsilon = \delta x \cdot (3x + \delta x)$ contains the number δx as a factor, ε is small when $\delta x \approx 0$ is small (as long as x is bounded. See Chapter 5 for details.)

Exercise Set 3.4

1. *Check Your General Formula with $y = x^3$*
When $f[x] = x^3$, the microscopic gap is $\varepsilon = \delta x \cdot (3x + \delta x)$.
 - 1. Use this formula for ε to check the errors you measured in Exercise 3.4.1 and Problem 3.3.
 - 2. We are interested in the size of the gap ε at $x + 1/100$. How much is this gap when in a microscope of power 100 focused at the point $x = 2/3$? How much is it really? Use your general formula to show that $\varepsilon = 0.0201$ and $\varepsilon \cdot \delta x = 0.000201$ (see Figure 3.17).
- *Your General Formula with $y = x^4$*
Let $f[x] = x^4$ and show that the microscope equation

$$f[x + \delta x] - f[x] = f'[x] \cdot \delta x + \varepsilon \cdot \delta x$$

becomes

$$(x + \delta x)^4 - x^4 = 4x^3 \cdot \delta x + \varepsilon \cdot \delta x$$

with $\varepsilon = \delta x \cdot (6x^2 + 4x\delta x + \delta x^2)$.

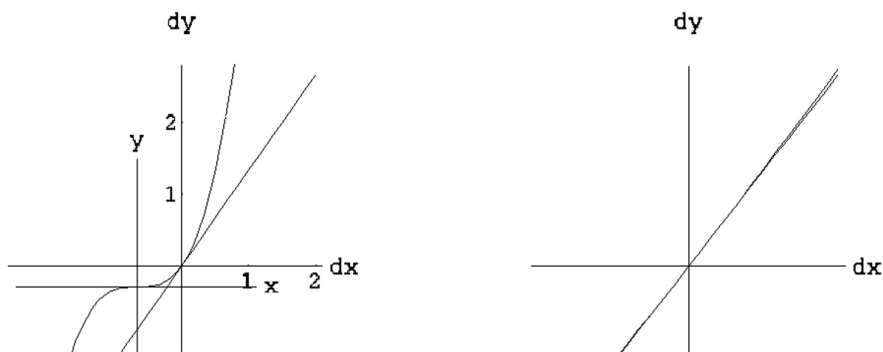


Figure 3.17: $y = x^3$ and $dy = \frac{4}{3} \cdot dx$ at $(\frac{2}{3}, \frac{8}{27})$, $\Delta x = \frac{1}{100}$

- There is only one possible line that can ‘fit’ a graph $y = f[x]$ when magnified. To see this, let $y = f[x] = \frac{1}{2}x$ near $x = 0$, but suppose the approximation we try is $dy = dx$, or $m = 1$, so the microscope equation is

$$f[x + \Delta x] - f[x] = \Delta x + \varepsilon \cdot \Delta x$$

(see Figure 3.18).

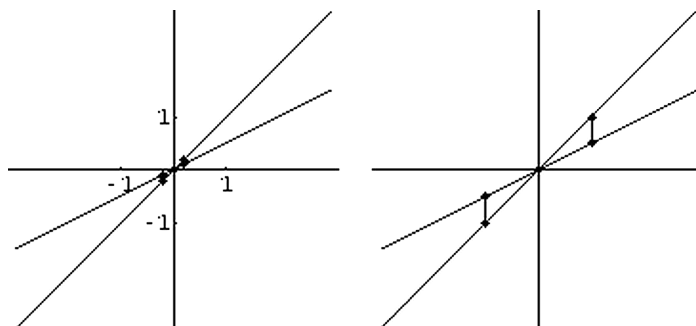


Figure 3.18: $y = x/2$ and $dy = dx$ near $(x, y) = (0, 0)$, magnified $\times 5$

Show that no matter what magnification, we always have $\varepsilon = -1/2$ above the point that appears to be one unit to away from the intersection. (Do this by writing $f[x]$ and $f'[x]$ explicitly in the microscope equation and solving for ε .)

Problem 3.4

The graph of the function

$$y = f[x] = x + \frac{3}{2}\sqrt{x^2 - 2x + 1}$$

is actually two half lines meeting at the point $(1, 1)$. To the right, the line has slope $\frac{5}{2}$, and to the left it has slope $-\frac{1}{2}$. There is no tangent line at this point because the ‘gap’ does not go to zero. (Rules of differentiation from Chapter 6 applied to this formula give a formula that is not defined at $x = 1$.) We

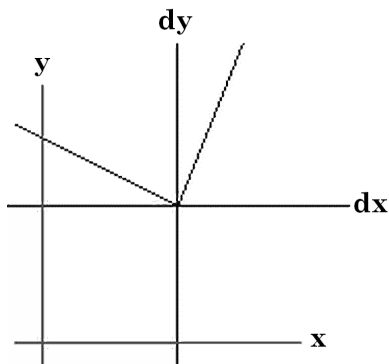


Figure 3.19: $y = x + \frac{3}{2}\sqrt{x^2 - 2x + 1}$ near $(x, y) = (1, 1)$

want to magnify this graph at $(1, 1)$ anyway and compare it with the “best” local linear approximation we can make. Since the slopes average out to slope 1, compute the gap between this function and the line $dy = dx$. Begin with several numerical cases. Let $x = 1$ and calculate ε in the equation

$$f[x + \Delta x] - f[x] = \Delta x + \varepsilon \cdot \Delta x$$

for $\Delta x = \pm 1, \pm 1/10, \pm 1/100, \pm 1/1000$. (ANS: $\varepsilon = \pm 3/2$) Graph the function and the line $dy = dx$ on the same axes for the scales in the previous part of the problem and show that the gap remains the same size at all these scales. Magnifying the graph never makes $y = f[x]$ appear linear and never makes the gap between $y = f[x]$ and $dy = dx$ get smaller.

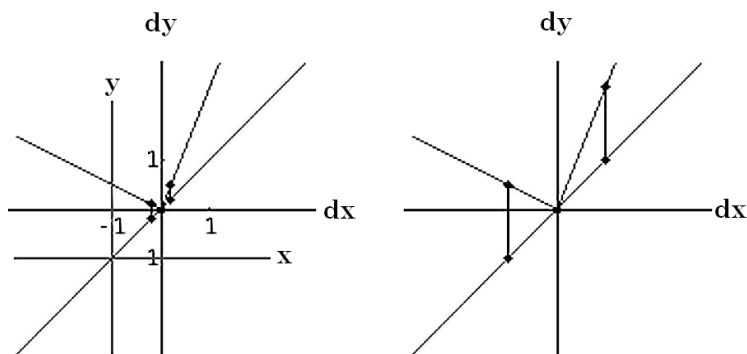


Figure 3.20: $y = x + \frac{3}{2}\sqrt{x^2 - 2x + 1}$ near $(x, y) = (1, 1)$, magnified $\times 5$

3.6 Projects

Projects for this chapter are:

3.6.1 CO2 Data

The Scientific Project on CO2 Data fits a linear function to data and compares it with long range prediction.

3.6.2 A Project on Functional Linearity

Linearity in function notation has a peculiar appearance. This is not difficult, just different. This project shows you the linear case of the main formula underlying differential calculus.

3.6.3 A Project on Functional Identities

The Project on Functional Identities includes further study of the additive identity

$$f[x + y] = f[x] + f[y]$$

This is related to a famous problem posed by Cauchy and solved by Hamel. If $f[0] = 0$ and $f[x]$ satisfies the identity $f[x + y] = f[x] + g[y]$,

$$f[x + \Delta x] - f[x] = g[\Delta x]$$

$$f[0 + \Delta x] - f[0] = g[\Delta x]$$

$$f[\Delta x] = g[\Delta x]$$

so that

$$f[x + \Delta x] - f[x] = f[\Delta x]$$

$$f[x + \Delta x] = f[x] + f[\Delta x]$$

Changing variables, we get, $f[x + y] = f[x] + f[y]$. In particular, if $f[x] = mx$, then $f[x]$ satisfies $f[x + y] = f[x] + f[y]$. Cauchy's question was: If an unknown function satisfies $f[x + y] = f[x] + f[y]$, does the unknown function have to be $f[x] = mx$ for some constant m ? You answer this question in the exercises for the important case where $f[x]$ is smooth.

Notice that the identity

$$f[x + \Delta x] - f[x] = m \cdot \Delta x$$

is the microscope approximation

$$f[x + \Delta x] - f[x] = m \cdot \Delta x + \varepsilon \cdot \Delta x$$

with zero error, $\varepsilon = 0$.