Chapter 14
Applications of Integration

This chapter explores deeper applications of integration, especially integral computation of geometric quantities.

The most important parts of integration are setting the integrals up and understanding the basic techniques of Chapter 13. Proficiency at basic techniques will allow you to use the computer to correctly perform complicated symbolic integration, but the computer cannot tell if the integral formula is a correct approximation. Chapter 12 began with this "slicing" approximation and this chapter returns to it in more detail. The problems at the end of the chapter ask *you* to derive some integral formulas yourself. The problems are hard because of the analytical geometry, but solving them will give you important insight.

14.1 The Length of a Curve

We begin by showing that some care must be taken in the more delicate geometric approximation problems in order to be sure that a sum of many small errors does not build up. The secret in all the integral approximations is to measure the error on a scale of the *increment* of the independent variable, Δx. (See Theorem 14.1.) It is *not* sufficient to have the error of each slice tend to zero.

The way you "slice" sometimes matters. The approximation to the length of a curve by sloping line segments that connect $(x, f[x])$ and $(x + \Delta x, f[x + \Delta x])$ is given in Example 14.1. This is a "good" approximation that converges to the actual length. "Slices" that just run horizontally out from the slice points do not approximate the length even though the smaller and smaller segments do get closer and closer to the curve.

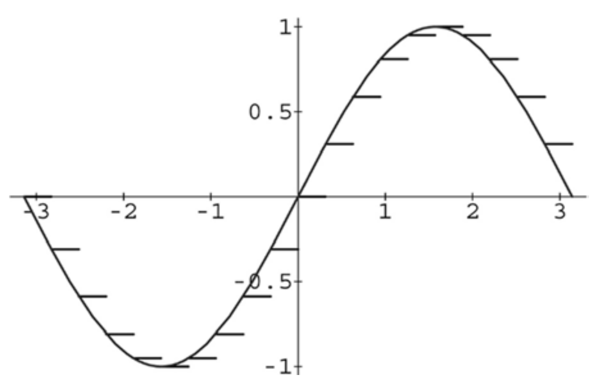

Figure 14.1:1: An incorrect "approximation" to the length

Horizontal slices are a good approximation to the area, but not the length. (See Figure 12.3.15.)

Example 14.1 *Length* $\approx \sum_{\substack{x=a \\ step\ \Delta x}}^{b-\Delta x} [\sqrt{(\Delta x)^2 + (f[x+\Delta x] - f[x])^2}]$

The Pythagorean Theorem shows that the length of a line segment from the point $(x, \text{Sin}[x])$ to the point $(x + \Delta x, \text{Sin}[x + \Delta x])$ is

$$\sqrt{(x + \Delta x - x)^2 + (\text{Sin}[x + \Delta x] - \text{Sin}[x])^2} = \sqrt{\Delta x^2 + (\text{Sin}[x + \Delta x] - \text{Sin}[x])^2}$$

The sum of the lengths of the segments in Figure 14.1:2 is

$$\sum_{\substack{x=-\pi \\ \text{step } \Delta x}}^{\pi-\Delta x} [\sqrt{\Delta x^2 + (\text{Sin}[x + \Delta x] - \text{Sin}[x])^2}]$$

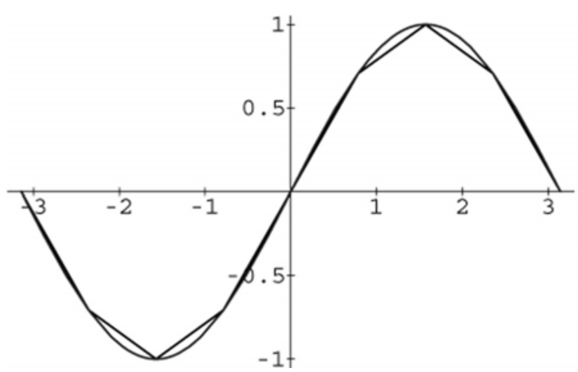

Figure 14.1:2: Approximate Length of the Sine Curve

More generally, if the curve is $y = f[x]$, the sum of the lengths of connecting segments is

$$\sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [\sqrt{\Delta x^2 + (f[x + \Delta x] - f[x])^2}]$$

This is a correct approximation but it is not in the form of an integral approximation

$$\sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [F[x]\Delta x]$$

However, when we replace Δx by a tiny increment δx, we can use the differential approximation of Definition 5.2 to write this sum in the form of a definite integral. First,

$$f[x + \delta x] - f[x] = f'[x] \cdot \delta x + \varepsilon \cdot \delta x$$

with $\varepsilon \approx 0$ for each x in $[a, b]$. This means

$$\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [\sqrt{(\delta x)^2 + (f[x+\delta x] - f[x])^2}] = \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [\sqrt{(\delta x)^2 + (f'[x]+\varepsilon)^2(\delta x)^2}]$$

$$= \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [\sqrt{1 + (f'[x]+\varepsilon)^2}\ \delta x]$$

where $\varepsilon \approx 0$. In Problem 1, you can use properties of summation to show that

$$\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [\sqrt{1 + (f'[x]+\varepsilon)^2}\ \delta x] \approx \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [\sqrt{1 + (f'[x])^2}\ \delta x]$$

We return to the arclength formula below in Example 14.4.

Exercise Set 14.1

1. Pythagoras meets Descartes
   Given two points (a, b) and (u, v) in the (x, y)-plane, show that the Pythagorean Theorem gives

   $$\text{the length of the line segment connecting the points} = \sqrt{(u-a)^2 + (v-b)^2}$$

   HINT: Make the segment the hypotenuse of a right triangle with horizontal and vertical legs. What are the lengths of these legs?

   Now apply your formula to a function graph.

2. Pythagoras and the Increment
   This exercise asks you to find a certain formula on a general function graph $y = f[x]$. If you wish, you can begin with a specific nonlinear function like $f[x] = x^2$, but the goal is an expression in terms of a general $f[x]$.

   (a) *Sketch a figure showing a graph $y = f[x]$ in the (x, y)-plane.*
   (b) *Put a dot on the graph of your function at one point $(x, y) = (x, f[x])$.*
   (c) *Put a second dot on your graph of $y = f[x]$ at a nearby point $(x + \Delta x, ??)$.*
   (d) *Express the y-value of the point on your graph above $x + \Delta x$ in terms of $f[\cdot]$.*
   (e) *Draw the straight line segment connecting the two dots on your graph. We want a formula for the length of this segment.*
   (f) *Apply the formula from Exercise 1 to show that the length of the segment connecting your two points is*

   $$\sqrt{(\Delta x)^2 + (f[x+\Delta x] - f[x])^2}$$

The next exercise has you show what can go wrong when an "approximation" is not accurate.

3. Horizontal Slices Do Not Approximate Length
This exercise has you find a sum expression for the attempt at approximating the length of a curve by horizontal slices. Then it has you explain why it is a bad approximation.

   (a) *Sketch your general curve* $y = f[x]$ *for* $a \le x \le b$.

   (b) *Draw a dot on your curve at a particular point* $(x, f[x])$.

   (c) *Draw a small horizontal segment of length* Δx *beginning at* $(x, f[x])$. *This should look similar to one of the ones shown on Figure 14.1:3 below. Your segment begins at the point* $(x, f[x])$ *and ends at a point a distance* Δx *to the right and at the same y-height, so its coordinates are* $(x + \Delta x, f[x])$.

   (d) *Sketch a sequence of these horizontal segments beginning at* $(a, f[a])$, *each having length* Δx *and continuing until one segment goes beyond* $x = b$. *(The specific curve* $y = \text{Sin}[x]$ *for* $-\pi \le x \le \pi$ *is shown in Figure 14.1:3.)*

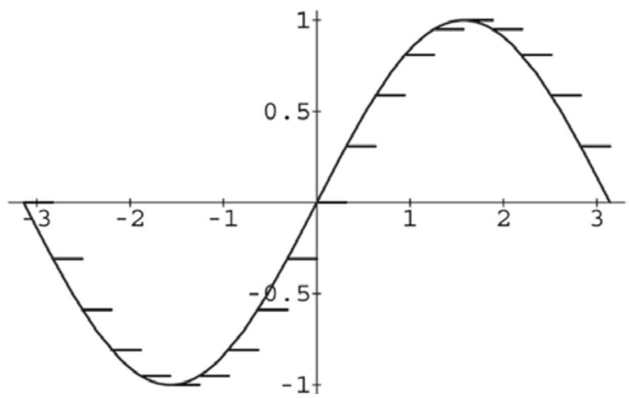

Figure 14.1:3: An incorrect "approximation" to the length

   (e) *The length of one segment is* Δx *because that was the instruction. The sum of the lengths of the segments is* Δx *times the number of segments, or, to be fancy,*

$$\begin{aligned}\sum_{\substack{x=a\\ \text{step } \Delta x}}^{b-\delta x} \Delta x &= \sum_{\substack{x=a\\ \text{step } \Delta x}}^{b-\delta x} [(x + \Delta x) - (x)] \\ &= \sum_{\substack{x=a\\ \text{step } \Delta x}}^{b-\delta x} [F[x + \Delta x] - F[x]]\end{aligned}$$

with $F[x] = x$. *Show that the sum of the lengths of the horizontal segments is approxi-*

mately $b-a$.

$$\sum_{\substack{x=a\\ \text{step } \Delta x}}^{b-\delta x} \Delta x \approx b-a$$

(The answer $b-a$ is exact if Δx divides the interval $[a,b]$ exactly. Think about how many terms there are or use Theorem 12.2.)

(f) *According to your correct result from the last part of this exercise, the sum of the lengths of the horizontal segments does not depend on the particular function $f[x]$. In particular, show that the sum is 2π, for Figure 14.1:3, the function $f[x]=0$ for $-\pi \le x \le \pi$, and the function $f[x]=x$ for $-\pi \le x \le \pi$.*

Why can't the sum of the horizontal slices be used to find the length of a graph?

Problem 14.1

Let $f[x]$ be a smooth function on an interval $[a,b]$. Show that

$$\sum_{\substack{x=a\\ step\ \delta x}}^{b-\delta x} [\sqrt{(\delta x)^2 + (f[x+\delta x]-f[x])^2}] \approx \sum_{\substack{x=a\\ step\ \delta x}}^{b-\delta x} [\sqrt{1+(f'[x])^2}]\delta x$$

so that length is given by the integral

$$\int_a^b \sqrt{1+(f'[x])^2}\; dx$$

Use the following hints.

First, use Definition 5.2 to show that

$$\sqrt{(\delta x)^2 + (f[x+\delta x]-f[x])^2} = \sqrt{1+(f'[x]+\varepsilon)^2}\delta x$$

with $\varepsilon \approx 0$ when $\delta x \approx 0$. Next, show that

$$\sqrt{1+(f'[x]+\varepsilon)^2} = \sqrt{1+(f'[x])^2} + \iota$$

with $\iota \approx 0$. Because

$$\begin{aligned}
&\sqrt{1+(f'[x]+\varepsilon)^2} - \sqrt{1+(f'[x])^2} = \\
&= \frac{\left(\sqrt{1+(f'[x]+\varepsilon)^2} - \sqrt{1+(f'[x])^2}\right)\left(\sqrt{1+(f'[x]+\varepsilon)^2} + \sqrt{1+(f'[x])^2}\right)}{\sqrt{1+(f'[x]+\varepsilon)^2} + \sqrt{1+(f'[x])^2}} \\
&= \frac{\varepsilon(2f'[x]+\varepsilon)}{\sqrt{1+(f'[x]+\varepsilon)^2} + \sqrt{1+(f'[x])^2}}
\end{aligned}$$

Finally, show that a sum with $\iota \approx 0$, for all x, satisfies

$$\sum_{\substack{x=a \\ step\ \delta x}}^{b-\delta x} [\iota \cdot \delta x] \approx 0$$

(See Theorem 12.12.)

14.2 Duhamel's Principle

Duhamel's Principle gives us an accuracy test for integral formulas.

In order to give a general result, we need to formulate the problem in terms of an "additive" quantity. Additivity expresses a simple property that many geometric functions such as accumulated area, length, volume, and surface area, have. This means that we can apply Duhamel's Principle to finding integral formulas of many geometric quantities. For example, the accumulated area used in the second half of the Fundamental Theorem of Integral Calculus is additive. Figure 14.2:4 shows the area accumulated from a to x:

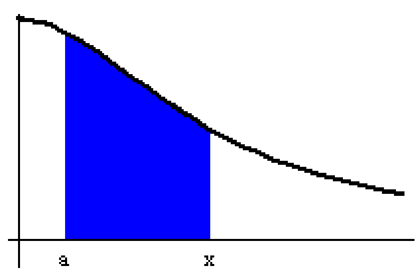

Figure 14.2:4: $A[a,x]$, the area from a to x

Additivity of $A[a,b]$ means that the area from a to b plus the area from b to c equals the whole area from a to c (where $a < b < c$) as shown in Figure 14.2:5:

Additivity is an important property of the integral. If we define a function of two variables, $I[u,v] = \int_u^v f[x]\ dx$ (for $f[x]$ continuous on $[a,b]$), then we have the additivity property

$$\int_u^v f[x]\ dx + \int_v^w f[x]\ dx = \int_u^w f[x]\ dx \qquad I[u,v] + I[v,w] = I[u,w]$$

for any $u < v < w$ in $[a,b]$.

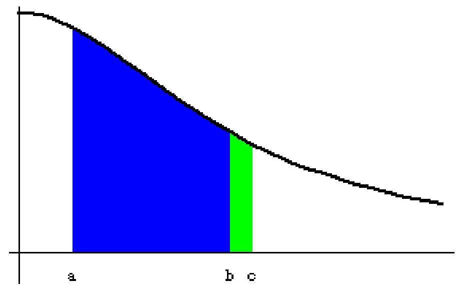

Figure 14.2:5: $A[a,c] = A[a,b] + A[b,c]$

Theorem 14.1 *Keisler's Infinite Sum Theorem or Duhamel's Principle*
Let $Q[u,v]$ be an additive quantity of a real variable, that is, satisfy

$$Q[u,v] + Q[v,w] = Q[u,w]$$

for $u < v < w$ in $[a,b]$. Suppose $f[x]$ is a continuous real function on $[a,b]$, such that for any tiny subinterval $[x, x+\delta x] \subseteq [a,b]$, with $\delta x \approx 0$,

$$Q[x, x+\delta x] = f[x]\ \ \delta x + \varepsilon \cdot \delta x$$

for a small error $\varepsilon \approx 0$. In other words, suppose $f[x]\,\delta x$ approximates a small "slice" of Q on a scale of δx. Then,

$$Q[a,b] = \int_a^b f[x]\,dx$$

First, notice that $Q[a,x] + Q[x,x+\delta x] = Q[a,x+\delta x]$ by additivity, so

$$Q[a,x+\delta x] - Q[a,x] = Q[x,x+\delta x]$$

Duhamel's approximation formula in the theorem above becomes

$$Q[a,x+\delta x] - Q[a,x] = f[x] \cdot \delta x + \varepsilon \cdot \delta x$$

So, if we let $F[x] = Q[a,x]$, Duhamel's approximation is the differential approximation 5.2 for $F[x]$. This shows that when we find an integral formula, we are "writing a differential equation for $Q[a,x]$" in terms of the slicing variable x.

PROOF:

By repeated use of additivity

$$\begin{aligned} Q\,[a,b] &= Q[a,a+\delta x] + Q[a+\delta x, a+2\delta x] + \ldots + Q[b-\delta x, b] \\ &= \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} \left[Q[x, x+\delta x]\right] \\ &= \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} \left[f\,[x]\,\delta x\right] + \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} \left[\varepsilon \cdot \delta x\right] \\ &\approx \int_a^b f\,[x]\,dx + \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} \left[\varepsilon \cdot \delta x\right] \end{aligned}$$

and

$$\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} \left[\varepsilon \cdot \delta x\right] \approx 0$$

by the triangle inequality estimate in Theorem 12.12. We have shown that the two fixed quantities satisfy $Q\,[a,b] \approx \int_a^b f\,[x]\,dx$, forcing them to be equal and proving the theorem.

Our first two examples of the use of Duhamel's Principle are simple and could be done directly by sandwiching the approximating sums between upper and lower estimates instead of using the general principle. These estimates do give an error of the form $\varepsilon \cdot \delta x$, too, however.

Example 14.2 *Distance Is an Integral*

We return to the distance example of Section 12.2 above. Suppose that $R[t]$ is any continuous rate (speed) function. By the Extreme Value Theorem 11.2, $R[t]$ has a max and a min over the interval $[t, t+\delta t]$. We denote these

$$R_m \le R[s] \le R_M \qquad \text{for} \quad t \le s \le t+\delta t$$

The distance traveled during this time interval must lie between the extremes

$$R_m \cdot \delta t \le \text{actual distance traveled} \le R_M \cdot \delta t$$

However, since $R[t]$ is continuous,

$$R_m \approx R[t] \approx R_M$$

so that

$$\text{actual distance traveled from time } t \text{ to time } t+\delta t = R[t]\cdot\delta t + \varepsilon\cdot\delta t$$

with $\varepsilon \approx 0$.

Finally, let $D[t_1, t_2]$ denote the distance traveled between the times t_1 and t_2. This is additive because if $t_1 < t_2 < t_3$, $D[t_1, t_3] = D[t_1, t_2] + D[t_2, t_3]$. Duhamel's Principle shows that

$$\text{total distance traveled} = \int_a^b R[t]\ dt$$

Example 14.3 *The General Disk Method*

When the graph of a positive continuous function, $y = f[x]$ for $a \le x \le b$, is revolved about the x-axis, the volume of the resulting solid is

$$V = \pi \int_a^b (f[x])^2\ dx$$

The general formula is explained in the Mathematical Background by an upper and lower estimate similar to the previous example.

Definite integrals *must* be written in the form $\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [F[x]\delta x]$ for some function $F[x]$; so although

$$\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [\sqrt{(\delta x)^2 + (f[x+\delta x] - f[x])^2}]$$

is a valid approximation for length, this approximation cannot be computed as an integral. The derivative formula or "microscope equation" allows us to express approximate length in the integral form using the "integrand" $F[x] = \sqrt{1 + (f'[x])^2}$. This means that the arclength can be expressed by

$$\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [\sqrt{1 + (f'[x])^2}\ \delta x] \approx \int_a^b \sqrt{1 + (f'[x])^2}\ dx = \text{length of the curve } y = f[x]$$

Example 14.4 *Length Formula* $L = \int_a^b \sqrt{1 + (f'[x])^2}\ dx$

Example 14.1, Exercise 2 and Problem 14.1 symbolically derive the arclength formula for a general explicit curve $y = f[x]$. A simpler way to summarize that work is to imagine measuring the length of a segment of the curve viewed inside a powerful microscope. We will show now that what we see in the microscope gives the same answer as the symbolic approach.

By Theorem 14.1, the formula for the length of the curve is

$$L = \int_a^b \sqrt{1 + (f'[x])^2}\ dx$$

provided that we can show that the length of a tiny segment between x and $x + \delta x$ satisfies

$$\text{arclength between } x \text{ and } x + \delta x = \sqrt{1 + (f'[x])^2}\ \delta x + \iota\ \delta x$$

for $\iota \approx 0$ when $\delta x \approx 0$.

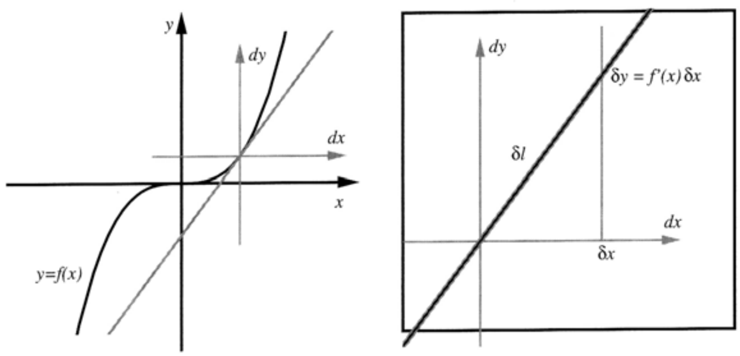

Figure 14.2:6: Microscopic view of length

We know that the line we see in a microscopic view of a smooth function has equation $dy = f'[x]\ dx$, so the height of the triangle with vertices $(x, f[x])$, $(x + \delta x, f[x + \delta x])$, and $(x + \delta x, f[x])$, is $f'[x]\ \delta x$, for a base of length δx. The Pythagorean Theorem says the square of the length is the sum of the squares of the sides,

$$\sqrt{(\delta x)^2 + (f'[x])^2 \delta x^2} = \sqrt{1 + (f'[x])^2}\ \delta x$$

This is the length of a segment of the curve except for errors that are tiny when viewed in the microscope of power $1/\delta x$. Errors that are small after magnification by $1/\delta x$ have actual magnitude $\varepsilon \cdot \delta x$ with $\varepsilon \approx 0$. Microscopic errors are exactly what Duhamel's Principle 14.1 allows, so the Theorem says that we can add the linear pieces we see in the microscopic views to obtain the full nonlinear quantity.

Example 14.5 *Parametric Length* $L = \int_a^b \sqrt{[dx[t]]^2 + [dy[t]]^2}$

The simple shortcut for finding the length integral with the view in a microscope also works for parametric curves. Sometimes parametric integral formulas are better behaved than explicit formulas, and sometimes curves do not even have single explicit formulas so that we must use parametric formulas.

What are parametric formulas for curves? Parametric curves are given by functions $x = f[t]$ and $y = g[t]$ where we plot (x, y) but not t. This is taken up in more detail in Chapter 16, but you should be familiar with the following example. A circle is given by the equations

$$x[\theta] = \text{Cos}[\theta]$$
$$y[\theta] = \text{Sin}[\theta]$$

These are parametric equations for a circle. We measure a distance θ along the unit circle starting at $(1, 0)$, and the point has coordinates $(x, y) = (\text{Cos}[\theta], \text{Sin}[\theta])$. See the section on trig functions in Chapter 28. The value of θ is not plotted (althought it appears as the length measured along the circle.) Parametric equations describing an ellipse are given in Problem 14.12.

We used this idea in Chapter 5 when we computed the increments of sine and cosine and proved the differentiation formulas for sine and cosine. A tiny increment of the circle looks as follows under a powerful microscope:

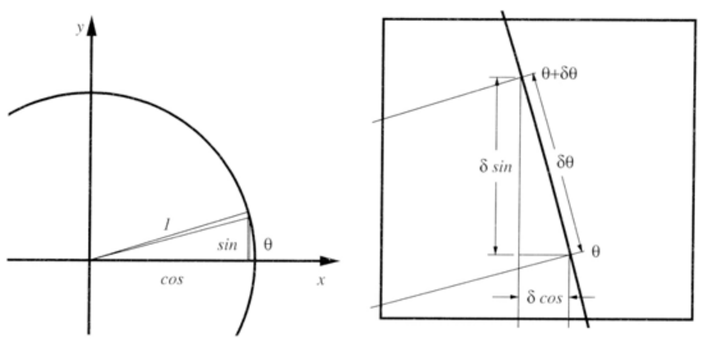

Figure 14.2:7: Increments of Sine and Cosine

The small triangle in the magnified view has

$$\text{Horizontal base} = -\delta\,\text{Cos}[\theta] = -(\text{Cos}[\theta + \delta\theta] - \text{Cos}[\theta])$$
$$\text{Vertical side} = \delta\,\text{Sin}[\theta] = \text{Sin}[\theta + \delta\theta] - \text{Sin}[\theta]$$
$$\text{Hypotenuse} = \delta\theta$$

The length of the "hypotenuse" is $\delta\theta$ because radian measure is defined to be the length measured along the unit circle. However, the side that looks like a hypotenuse is actually a magnified small piece of a circle. Intuitively, the error between the length of the circular arc and the approximating straight line is small compared to $\delta\theta$, so, by Theorem 14.1, the length of the circle is given by the integral

$$\int_0^{2\pi} d\theta$$

It is still worthwhile to see how this is related to the computation with increments.

Definition 5.2 and the formulas for the derivatives of sine and cosine give us

$$\delta x = \delta \operatorname{Cos}[\theta] = \operatorname{Cos}[\theta + \delta\theta] - \operatorname{Cos}[\theta] = -\operatorname{Sin}[\theta] \cdot \delta\theta + \iota_1 \cdot \delta\theta$$
$$\delta y = \delta \operatorname{Sin}[\theta] = \operatorname{Sin}[\theta + \delta\theta] - \operatorname{Sin}[\theta] = \operatorname{Cos}[\theta] \cdot \delta\theta + \iota_2 \cdot \delta\theta$$

with $\iota_j \approx 0$ whenever $\delta\theta \approx 0$, $j = 1, 2$.

The Pythagorean Theorem says that the length of the true straight hypotenuse is the square root of the sum of the squares of the lengths of the legs,

$$\begin{aligned}
\text{Small hypotenuse} &= \sqrt{[\delta x]^2 + [\delta y]^2} \\
&= \sqrt{(\operatorname{Cos}[\theta + \delta\theta] - \operatorname{Cos}[\theta])^2 + (\operatorname{Sin}[\theta + \delta\theta] - \operatorname{Sin}[\theta])^2} \\
&= \sqrt{(-\operatorname{Sin}[\theta]\delta\theta + \iota_1\delta\theta)^2 + (\operatorname{Cos}[\theta]\delta\theta + \iota_2\delta\theta)^2} \\
&= \sqrt{(-\operatorname{Sin}[\theta]\delta\theta + \iota_1)^2 + (\operatorname{Cos}[\theta]\delta\theta + \iota_2)^2}\ \delta\theta \\
&= \sqrt{(-\operatorname{Sin}[\theta])^2 + (\operatorname{Cos}[\theta])^2}\ \delta\theta + \iota_3\delta\theta \\
&= \delta\theta + \iota_3\delta\theta
\end{aligned}$$

since $(\operatorname{Cos}[\theta])^2 + (\operatorname{Sin}[\theta])^2 = 1$. By algebraic approximations, $\iota_3 \approx 0$ whenever $\delta\theta \approx 0$. This means that we may compute the length by the integral of $d\theta$ by Duhamel's Principle 14.1.

The general approximation idea for smooth functions $x[t]$, $y[t]$ is

$$\begin{aligned}
\text{Small hypotenuse} &= \sqrt{[\delta x]^2 + [\delta y]^2} \\
&= \sqrt{(x'[\theta]\delta\theta + \iota_1\delta\theta)^2 + (y'[\theta]\delta\theta + \iota_2\delta\theta)^2} \\
&= \sqrt{(x'[\theta] + \iota_1)^2 + (y'[\theta] + \iota_2)^2}\ \delta\theta \\
&= \sqrt{(x'[\theta])^2 + (y'[\theta])^2}\ \delta\theta + \iota_3\delta\theta
\end{aligned}$$

In the integral for the length of an ellipse in Problem 14.12, the expression $\sqrt{(x')^2 + (y')^2}$ is more complicated, and we cannot do the final step of replacing $\sqrt{(-\operatorname{Sin}[\theta])^2 + (\operatorname{Cos}[\theta])^2}$ by 1 as above.

Example 14.6 *A Simple Parametric Length*

Let us consider another example, the length of the parametric curve

$$x = t^2$$
$$y = t^3$$

for $0 \le t \le 1$. We have the increments

$$\begin{aligned}
\delta x &= x'[t]\delta t + \iota_1\delta t & \qquad \delta y &= y'[t]\delta t + \iota_2\delta t \\
&= 2t\delta t + \iota_1\delta t & &= 3t^2\ \delta t + \iota_2\delta t
\end{aligned}$$

so that the length of the hypotenuse of the triangle we would see in an tiny microscope is

$$\delta l = \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{(2t)^2 + (3t^2)^2}\ \delta t + \iota_3 \delta t$$

and Duhamel's Principle says,

$$L = \int_0^1 \sqrt{(2t)^2 + (3t^2)^2}\ dt = \frac{(13)^{3/2} - 8}{27} \approx 1.4971$$

Example 14.7 *Volume of a Half Bagel*

The upper half disk of radius 1 centered at $(2, 0)$ is revolved about the y-axis. The equation of the semicircular boundary of the half disk is

$$y = \sqrt{1 - (x - 2)^2}$$

The resulting figure looks like the top half of a sliced bagel as in Figure 14.2:9.

We approximate the accumulated volume between radius x and radius $x + \delta x$ by the cylindrical "shell" with inner radius x, outer radius $x + \delta x$, and height $ht\,[x] = \sqrt{1 - (x - 2)^2}$.

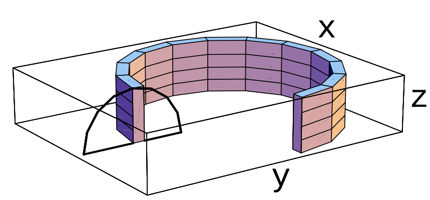

Figure 14.2:8: Half bagel shells

$$\begin{aligned} \text{Outer cylinder} - \text{inner cylinder} &= \\ \pi R^2 ht - \pi r^2 ht &= \pi((x + \delta x)^2 - x^2) ht \\ &= \pi(2x\delta x + \delta x^2) ht \\ &= \pi(2x\delta x + \delta x^2)\sqrt{1 - (x - 2)^2} \end{aligned}$$

The δx^2 term can be neglected in our integral formula by Duhamel's Principle because it produces a term of the form $\varepsilon \cdot \delta x$ with $\varepsilon \approx 0$.

$$\pi(2x\delta x + \delta x^2)\sqrt{1-(x-2)^2} =$$

$$\hat{u} = \left(2\pi\, x\sqrt{1-(x-2)^2}\right)\delta x + \left(\delta x \pi\sqrt{1-(x-2)^2}\right)\delta x$$
$$= \left(2\pi\, x\sqrt{1-(x-2)^2}\right)\delta x + \varepsilon \cdot \delta x$$

The flat top on our cylinder produces another error between the accumulated volume from x to $x+\delta x$. If we are on the rising side of the circle, the left height, $ht\,[x] = \sqrt{1-(x-2)^2}$ produces a shell that lies completely inside the bagel while the right height, $ht\,(x+\delta x) = \sqrt{1-([x+\delta x]-2)^2}$, produces a shell that includes the cylindrical slice of the bagel (with its curved top.) This means

$$\pi 2x\delta x\sqrt{1-(x-2)^2} + \varepsilon_1\delta x \le V\,[x, x+\delta x] \le \pi 2x\delta x\sqrt{1-([x+\delta x]-2)^2} + \varepsilon_2\delta x$$

On the falling part of the circle, the outside terms in this inequality are interchanged - the left height is above and the right height below. In either case,

$$V\,[x, x+\delta x] = \pi 2x\delta x\sqrt{1-([x+\delta x]-2)^2} + \varepsilon_3\delta x$$

with $\varepsilon_3 \approx 0$, so we have

$$V\,[a,b] = 2\pi\int_1^3 x\sqrt{1-(x-2)^2}dx$$

Example 14.8 *Computation of* $2\pi\int_1^3 x\sqrt{1-(x-2)^2}dx$

This integral can be computed with the computer or by hand with a trig substitution,

$$x-2 = \mathrm{Sin}\,[\theta] \qquad dx = \mathrm{Cos}\,[\theta]\,d\theta$$
$$x = 1 \Leftrightarrow \theta = -\pi/2 \qquad x = 3 \Leftrightarrow \theta = \pi/2$$

so the integral becomes

$$\begin{aligned}
V[a,b] &= 2\pi \int_{-\pi/2}^{\pi/2} (\mathrm{Sin}(\theta)+2)\sqrt{1-\mathrm{Sin}^2[\theta]}\,\mathrm{Cos}[\theta]\,d\theta \\
&= 2\pi \int_{-\pi/2}^{\pi/2} (\mathrm{Sin}(\theta)+2)\,\mathrm{Cos}^2(\theta)\,d\theta \\
&= 2\pi \int_{-\pi/2}^{\pi/2} \mathrm{Sin}(\theta)\,\mathrm{Cos}^2(\theta)\,d\theta + 2\pi \int_{-\pi/2}^{\pi/2} 2\mathrm{Cos}^2[\theta]\,d\theta \\
&= 2\pi \left(\frac{-1}{3}\mathrm{Cos}^3(\theta)\right)\Big|_{-\pi/2}^{\pi/2} + 2\pi \int_{-\pi/2}^{\pi/2} (1+\mathrm{Cos}(2\theta))\,d\theta \\
&= 0 + 2\pi[\theta + \frac{1}{2}\mathrm{Sin}(2\theta)]\Big|_{-\pi/2}^{\pi/2} \\
&= 2\pi^2
\end{aligned}$$

Example 14.9 *Trig Substitutions and Parametric Forms*

The change of variables in the Example 14.8 can be viewed as parametric equations for the surface of the bagel. A unit semicircle is given by the parametric equations

$$x = \mathrm{Sin}[\theta]$$
$$y = \mathrm{Cos}[\theta]$$

where x goes from -1 to $+1$ as θ goes from $-\pi/2$ to $+\pi/2$. You should verify that y takes only positive values in this range and that this pair traces out the top half circle of the unit radius centered at zero.

Adding 2 to the value of x moves the center of the semicircle to $(2,0)$, so it forms the outline semicircle of our half bagel,

$$x = 2 + \mathrm{Sin}[\theta]$$
$$y = \mathrm{Cos}[\theta]$$

The integral change of variables formula for dx can be thought of as expressing the thickness of the shells in terms of θ. It is also perfectly OK to think of the change of variables as a formal manipulation that makes the integral easier to compute.

Example 14.10 *The General Shell Method - Explicit Form*

If the region below the graph of a positive continuous function, $y = f[x]$ for $0 \le a \le x \le b$, is revolved about the y-axis, the volume of the solid obtained is

$$V = 2\pi \int_a^b x\, f[x]\; dx$$

This general result is given in "Foundations of Infinitesimal Calculus" at www.math.uiowa.edu/~stroyan/InfsmlCalculus/InfsmlCalc.htm.

Example 14.11 *The Icing on the Donut*

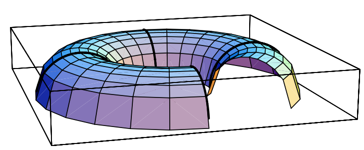

Figure 14.2:9: Area of donut icing

The surface of the top half of the torus is obtained by rotating the semicircle

$$y = \sqrt{1 - (x-2)^2}$$

about the y axis. We will approximate the area by considering what happens to a tiny segment of the semicircle as we rotate that segment about the y axis. The slope of the segment matters in calculating the area of one of these 'barrel hoops.'

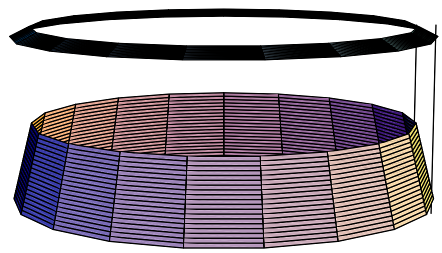

Figure 14.2:10: Hoops of different slant, but same radius and dx-thickness

The area is approximately the total length of the generating segment, δl, times the distance through which it travels. It is rotated about the y-axis, so it goes around a circle of radius x. The circumference of a circle is $C = 2\pi r = 2\pi\, x$, in this case. This makes the area of the hoop at x

$$\delta A = 2\pi\, x\ \delta l$$

We need an expression for δl in terms of the x variable so that we can form an integral.

The length of a tiny segment of the curve is

$$\delta l = \sqrt{\delta x^2 + \delta y^2} + \varepsilon_1 \cdot \delta x$$

with an error that is small compared to δx. Since we have $y = y[x]$ as an explicit function of x, we can write

$$\delta y = y'[x]\ \delta x + \varepsilon_2 \cdot \delta x$$

so that our length becomes $\delta l = \sqrt{\delta x^2 + (y'[x])^2 \delta x^2} + \varepsilon_3 \cdot \delta x = \sqrt{1 + (y'[x])^2}\ \delta x + \varepsilon_3 \cdot \delta x$ and our approximating area is

$$\delta A = 2\pi\ x\ \sqrt{1 + (y'[x])^2}\ \delta x + \varepsilon \cdot \delta x$$

Finally, by Duhamel's Principle,

$$\begin{aligned} A &= 2\pi \int_1^3 x\ \sqrt{1 + (y'[x])^2}\ dx \\ &= 2\pi \int_1^3 x\ \sqrt{1 + \frac{(x-2)^2}{1-(x-2)^2}}\ dx \\ &= 2\pi \int_1^3 \frac{x}{\sqrt{1-(x-2)^2}}\ dx \end{aligned}$$

This is a nasty integral, when $x = 1$ or $x = 3$. Of course, the computer might do the integration for us, but there is a more geometric way to see how to proceed.

Example 14.12 *A Second Approach to the Area of Revolution*

This time represent the generating semicircle of radius 1 centered at 2 on the x axis by parametric equations:

$$\begin{aligned} x[t] &= 2 + \mathrm{Sin}[t] \\ y[t] &= \mathrm{Cos}[t] \qquad \text{for } -\pi/2 \le t \le \pi/2 \end{aligned}$$

We go back to the length portion of the area of the approximating hoops. The length of a tiny segment of the curve is approximately

$$\delta l = \sqrt{\delta x^2 + \delta y^2} + \varepsilon_1 \cdot \delta t$$

but, in this case, the increments satisfy

$$\begin{aligned} \delta x[t] &= x[t + \delta t] - x[t] = x'[t]\ \delta t + \varepsilon_2\ \delta t \\ &= \mathrm{Cos}[t]\ \delta t + \varepsilon_2 \cdot \delta t \\ \delta y[t] &= y[t + \delta t] - y[t] = y'[t]\ \delta t + \varepsilon_3\ \delta t \\ &= -\mathrm{Sin}[t]\ \delta t + \varepsilon_3 \cdot \delta t \end{aligned}$$

so the length is $\delta l = \sqrt{\delta x^2 + \delta y^2} + \varepsilon \cdot \delta t = \sqrt{\mathrm{Cos}^2[t] + \mathrm{Sin}^2[t]}\ \delta t + \varepsilon \cdot \delta t = \delta t + \varepsilon \cdot \delta t$. We must express the increment of area in terms of t,

$$\begin{aligned} \delta A &= 2\pi\, x\ \delta l = 2\pi\, x\ \delta t + \varepsilon \cdot \delta t \\ &= 2\pi(2 + \mathrm{Sin}[t])\ \delta t + \varepsilon \cdot \delta t \\ &= 2\pi(2 + \mathrm{Sin}[t])\ \delta t + \varepsilon \cdot \delta t \end{aligned}$$

and the area is given by

$$\begin{aligned} A &= 2\pi \int_{-\pi/2}^{\pi/2} (2 + \mathrm{Sin}[t])\ dt \\ &= 2\pi \int_{-\pi/2}^{\pi/2} 2\ dt + 2\pi \int_{-\pi/2}^{\pi} \mathrm{Sin}[t]\ dt \\ &= 4\pi^2 + 0 \end{aligned}$$

since the integral of sine over a half period is zero.

Not only is this parametric integral for the area easy to compute, it also does not have the mathematical singularities of the cartesian form above when $x = 1$ and $x = 3$.

Example 14.13 *General Area of Revolution Formulas*

If the graph of a positive smooth function $y = f[x]$ for $a \le x \le b$ is revolved about the x-axis, the area of the surface obtained is

$$A = 2\pi \int_a^b f[x]\sqrt{1 + (f'[x])^2}\ dx$$

If a smooth parametric graph, $x[t] = f[t]$, $y[t] = g[t]$ for $a \le x \le b$, with $g[t] > 0$, is revolved about the x-axis the area of the surface so obtained is

$$A = 2\pi \int_a^b x[t]\sqrt{(x'[t])^2 + (y'[t])^2}\ dt$$

These results are explained in "Foundations of Infinitesimal Calculus" at www.math.uiowa.edu/~stroyan/InfsmlCalculus/InfsmlCalc.htm.

Exercise Set 14.2

1. *Define the accumulated energy function*

$$E[t_1, t_2]$$

   to be the amount of energy a household consumes from time t_1 to time t_2. Explain why $E(s,t)$ is additive. (This is just a matter of saying what the terms in the formula mean.)

2. *Explicit Length*
   Verify the parametric computation of Example 14.6 by using the explicit equation for arclength, $L = \int_a^b \sqrt{1 + [f'[x]]^2}\ dx$, on the curve $y = x^{3/2}$. This is the same as the parametric curve above, since $t = \sqrt{x} = x^{1/2}$, so $y = t^3 = (x^{1/2})^3 = x^{3/2}$.

14.3 A Project on Geometric Integrals

It is important for you to try "slicing" on your own. These problems are hard, but you need to do several to understand integration.

Each problem has the following steps

Procedure 14.1

1. Slice the figure and find an approximate formula for one slice of the form 'a function of the slice at x' times the 'thickness,' $f[x]\ \delta x$.
2. Find the limits of integration.
3. Compute using rules or numerical integration with or without the computer.

You have some freedom in your approximation as long as the error for one slice is small compared to the 'thickness' of the slice,

$$\text{amount of one slice } = f[x]\ \delta x + \varepsilon \cdot \delta x$$

General Instructions for the Problems:

Find integral formulas for the following quantities. Use Duhamel's Principle, Theorem 14.1, explicitly to verify the correctness of your formulas. If possible, compute your integral symbolically; otherwise use numerical integration. Use the computer if you wish.

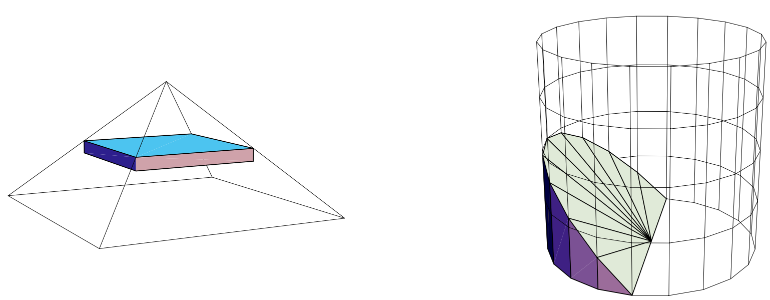

Figure 14.3:11: A sliced pyramid and a round wedge

Problem 14.2 Discs
Find the volume of a right circular cone of base radius r and height h.

Problem 14.3 SQUARE SLICES
Find the volume of a square pyramid with base area B and height h. Does it matter whether or not it is a right pyramid or slant pyramid? Does the base have to be square?

Problem 14.4 TRIANGLES
A wedge is cut from a (cylindrical) tree trunk of radius r by cutting the tree with two planes meeting on a diameter. One plane is perpendicular to the axis and the other makes an angle θ with the first. Find the volume of the wedge.

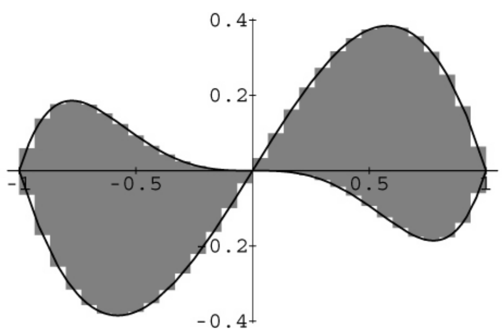

Figure 14.3:12: Area between $y = x^5 - x^3$ and $y = x - x^3$

Problem 14.5 AREA BETWEEN CURVES
Find the area of the bounded regions between the curves $y = x^5 - x^3$ and $z = x - x^3$. Notice that the curves define two regions, one with z on top and the other with y on top. (See the program ***AreaBetween****.)*

Problem 14.6
Find the area between the curves of Problem 6.14 and 6.15.

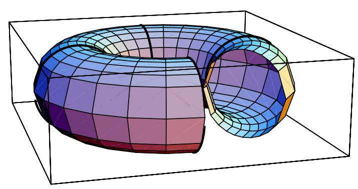

Figure 14.3:13: A torus

Problem 14.7 SLICE BY SHELLS
Find the volume of the solid torus (donut)

$$(x - R)^2 + y^2 \leq r^2 \qquad (0 < r < R)$$

Figure 14.3:14: Slice by washers or slice by shells

Problem 14.8 Slice by Washers or Shells
A (cylindrical) hole of radius r is bored through the center of a sphere of radius R. Find the volume of the remaining part of the sphere.

Figure 14.3:15: A Spherical cap and crossing cylinders

Problem 14.9 PART OF A SPHERE
Find the volume of the portion of a sphere that lies above a plane a distance c above the center of the sphere for $0 < c < r$, where r is the radius of the sphere.

Problem 14.10 INTERSECTING CYLINDERS
Two circular cylinders of equal radii r intersect through their centers at right angles. Find the volume of the common part. (HINT: The intersection can actually be sliced into square cross sections.)

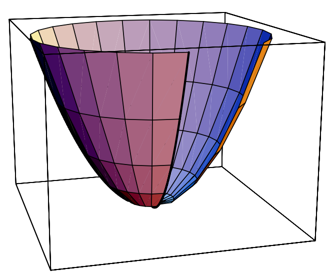

Figure 14.3:16: A parabolic antenna

Problem 14.11 SURFACE AREA
Find the surface area of the surface obtained by revolving $y = \frac{1}{2}x^2$ about the y-axis. (See the program ***AntennaArea****.)*

Problem 14.12 Arclength of an Ellipse

An ellipse is given parametrically by the pair of equations

$$x = 3\,\mathrm{Cos}\,[\theta]$$
$$y = 2\,\mathrm{Sin}\,[\theta]$$

with $-\pi < \theta < \pi$. Find an integral formula for the length of the curve by "looking in a powerful microscope," as we did in the computation of the length of a parametric curve above. In a microscope we will see a right triangle with the change in x on the horizontal leg, the change in y on the vertical leg and the length along the hypotenuse. The Pythagorean Theorem says that the corresponding increment of length is given by

$$\delta l = \sqrt{\delta x^2 + \delta y^2}$$

This time, the length will NOT be equal to the change in the angle, $\delta\theta$, because an ellipse is a circle that has been stretched different amounts in the x and y directions. The changes in the coordinates are function changes:

$$\delta x[\theta] = x(\theta + \delta\theta) - x[\theta]$$
$$\delta y[\theta] = y(\theta + \delta\theta) - y[\theta]$$

Use the increment approximation for these changes to express them approximately in terms of $\delta\theta$. Substitute the approximations into the Pythagorean expression above.

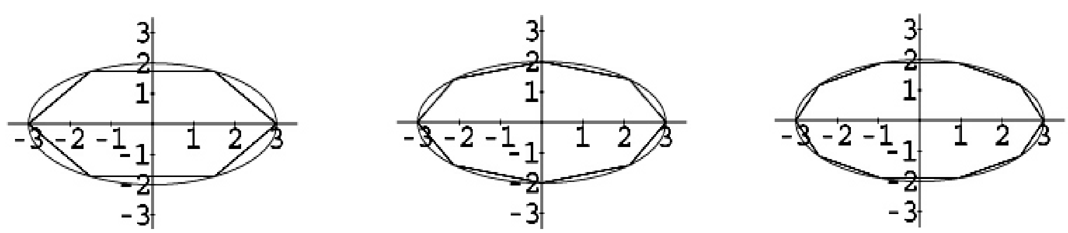

Figure 14.3:17: Equal θ partition of the ellipse

Test your formula on the circle (where you know the answer) as well as the "parametric" equations

$$x = \theta$$
$$y = 2\sqrt{1 - \left(\frac{\theta}{3}\right)^2}$$

(which can be compared to the explicit arclength formula $L = \int_a^b \sqrt{1 + [f'[x]]^2}\ dx$.)

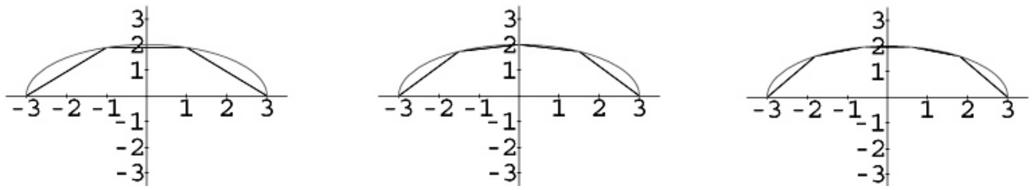

Figure 14.3:18: Equal x partition of the ellipse

The (correct) arclength formula for the ellipse cannot be computed by antidifferentiation so you must use numerical integration such as the computer NIntegrate[.]. Why is the parametric integral for the ellipse better behaved than the explicit formula?

14.4 Improper Integrals

"Improper" integrals, such as $\int_0^1 \frac{1}{\sqrt{x}}\,dx$, whose integrand tends to infinity and is discontinuous at $x = 0$ or $\int_1^\infty \frac{1}{x^2}\,dx$, which is integrated over an infinite interval, are studied in the book of projects and used in Chapter 18 on infinite series.