

Basic Integration

This chapter contains the fundamental theory of integration. We begin with some problems to motivate the main idea: approximation by a sum of slices. The chapter confronts this squarely, and Chapter 13 concentrates on the basic rules of calculus that you use after you have found the integrand.

Definite integrals have important uses in geometry and physics. Both the geometric and the physical integral formulas are derived in the following way: First, find a formula for the quantity (volume, area, length, distance, etc.) in the case of either a constant or a linear function. Next, approximate the nonlinear quantity by a sum of “slices” using the constant or linear formula for each slice. The primary difficulty is usually in expressing the variable sizes of the approximating pieces. In the geometric problems, this step is analytical geometry - finding the formula that goes with the picture you want. The Fundamental Theorem of Integral Calculus 12.13 gives us a simple way to compute the limit of the sum approximations exactly once we have the integral formula.

In order for sum approximations to tend to an integral we need to write them in the form

$$f[a] \Delta x + f[a + \Delta x] \Delta x + f[a + 2\Delta x] \Delta x + \cdots + f[b - \Delta x] \Delta x$$

where Δx is the “thickness of the slice” and $f[x]$ is the variable “base of the slice,” so that $f[x] \Delta x$ is the variable amount of the slice at x . This symbolic expression is an important part of the way the formulas are expressed in integration; without the symbolic expression the more or less obvious approximations could not be computed exactly in a common way. The computer has a sum command that computes this expression

$$\sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [f[x] \Delta x]$$

Computer summation is studied in detail in Section 12.4 below. The integral is the limit of this sum as Δx tends to zero,

$$\int_a^b f[x] dx = \lim_{\Delta x \rightarrow 0} \sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [f[x] \Delta x]$$

The slicing approximation is the forest that you must strive to see through a tangle of technical trees. The first problem is in finding the symbolic sum.

12.1 Geometric Slice Approximations

The volume of a right circular cone with height h and base of radius r is

$$V = \frac{\pi}{3}r^2h$$

We want **you** to derive this formula by approximating a cone with a sum of cylindrical “disks.” Once you understand the step from a constant radius to a varying radius, you will be able to find integral formulas for general solids of revolution. This important generalization illustrates the power of integration theory.

The volume of a right cylinder is

$$V = Ah$$

the area of the base times the height. If the base is a circle of radius r , as shown in Figure 12.1:1, the formula becomes

$$V = \pi r^2 h$$

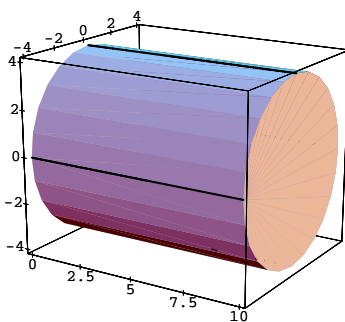
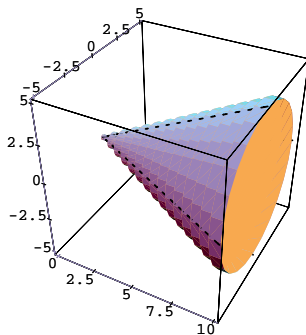


Figure 12.1:1: $V = \pi r^2 h$

Our next task is to let the radius vary with x , $r = R[x]$, and not be the same all along a cylinder.

12.1.1 The Volume of a Cone

Now we think of a cone as a figure with circular cross-sections that vary linearly, starting with zero radius and increasing to radius r when the distance from the tip is h . Let the variable x run down the axis of the cone with $x = 0$ at the tip as shown in Figure 12.1:2. The expression for the radius of the cross-section can be expressed as a function of x .

Figure 12.1:2: A cone along the x -axis

Example 12.1 *The Varying Radius of the Cross-Section of a Cone.*

We want a formula $R[x] = ?$ for the radius of the cross-section at x for a cone with tip at $x = 0$ and base radius r at height $x = h$. Since the side of the cone is a straight line, we want a function such that

$$\begin{aligned} R[x] &= m x + b, && \text{a linear function} \\ R[0] &= 0, && \text{zero cross-section at } x = 0, \text{ so } b = 0 \\ R[h] &= r, && \text{or } m x = r, \text{ when } x = h, \text{ so } m = \frac{r}{h} \end{aligned}$$

So, the radius of the cross-section at x of a cone of height h and base radius r is $R[x] = \frac{r}{h}x$.

Example 12.2 *The Volume of a Particular Cone*

Our approximation of the volume can be obtained by slicing the cone in steps of Δx , using the formula for the volume of a cylinder for each slice, $V_{\text{slice}} = \pi R^2[x] \Delta x$. For example, suppose we have a cone of height 10 and radius 5 at the base (in the same length units). In this case the formula for the radius x units below the apex is

$$R[x] = \frac{1}{2}x$$

If we let $\Delta x = 2$, or we slice the cone 5 times, we obtain 5 disks with radii at $x = 1, x = 3, \dots, x = 9$. The disks are generated by the horizontal segments shown in Figure 12.1:3. These segments generate the rough approximating pile of disks shown in Figure 12.1:4, and have volumes given as follows:

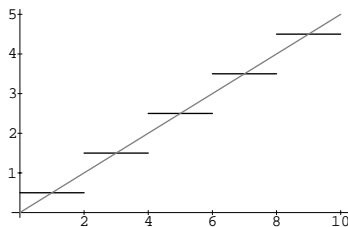


Figure 12.1:3: Generating lines for the cone approximation

1. (a) x from 0 to 2, $R(1) = \frac{1}{2}$, thickness of cylindrical disk = $\Delta x = 2$,
 $V_{slice1} = \pi \left[\frac{1}{2}\right]^2 2$
- (b) x from 2 to 4, $R(3) = \frac{3}{2}$, thickness of cylindrical disk = $\Delta x = 2$,
 $V_{slice2} = \pi \left[\frac{3}{2}\right]^2 2$
- (c) x from 4 to 6, $R(5) = \frac{5}{2}$, thickness of cylindrical disk = $\Delta x = 2$,
 $V_{slice3} = \pi \left[\frac{5}{2}\right]^2 2$
- (d) x from 6 to 8, $R(7) = \frac{7}{2}$, thickness of cylindrical disk = $\Delta x = 2$,
 $V_{slice4} = \pi \left[\frac{7}{2}\right]^2 2$
- (e) x from 8 to 10, $R(9) = \frac{9}{2}$, thickness of cylindrical disk = $\Delta x = 2$,
 $V_{slice5} = \pi \left[\frac{9}{2}\right]^2 2$

The total volume approximation is

$$\frac{\pi}{2}(1^2 + 3^2 + \dots + 9^2) = \frac{\pi 165}{2} \approx 259.$$

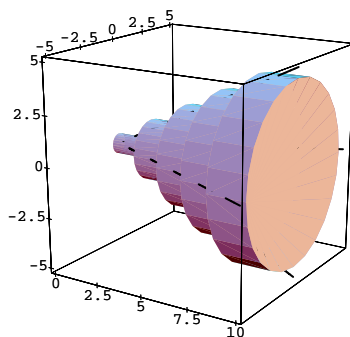


Figure 12.1:4: A cone approximated by 5 “disks”

We need to find the formula that expresses the sum of the volumes of the slices in terms of a variable for the thickness of the slice, Δx . The previous examples suggest an interesting special-purpose algebraic approach, but that is likely to work only for this specific cone. We can express the approximation in terms of the formula for the radius $R[x]$ of the cross-section at x and a thickness Δx as

$$\begin{aligned} & \pi[(R[\Delta x/2])^2 \Delta x + (R[3\Delta x/2])^2 \Delta x + (R[5\Delta x/2])^2 \Delta x + \cdots + (R[10 - \Delta x/2])^2 \Delta x] \\ &= \pi \sum_{\substack{x=\Delta x/2 \\ \text{step } \Delta x}}^{10-\Delta x/2} [(R[x])^2 \Delta x] \end{aligned}$$

12.1.2 A Parabolic Nose Cone

The simple linear formula for $R[x]$ in the case of the cone is easily generalized to more complicated shapes. A parabolic rocket “nose cone” may be described by the volume swept out as we rotate the curve $y = \sqrt{x}$ for $0 \leq x \leq 10$ about the x -axis shown in Figure 12.1:5.

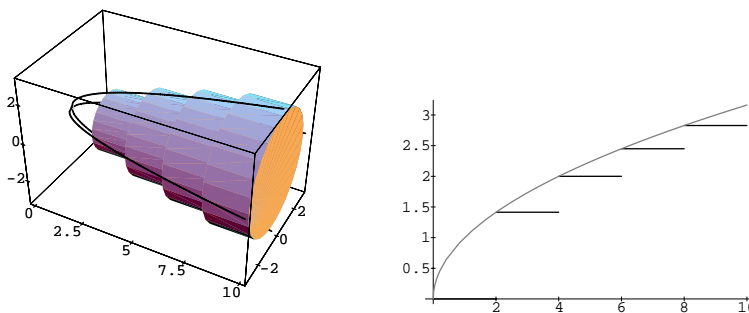


Figure 12.1:5: A parabolic nose cone

The formula for the radius of a cross-section is $R[x] = \sqrt{x}$, and the volume of a slice becomes $\pi R^2[x]h = \pi(\sqrt{x})^2 \Delta x = \pi x \Delta x$, making the approximating sum with the radius at the left side of the slices

$$\begin{aligned} & \pi(0 \Delta x + \Delta x \Delta x + 2\Delta x \Delta x + \cdots + (10 - \Delta x) \Delta x) \\ &= \pi \sum_{\substack{x=0 \\ \text{step } \Delta x}}^{10-\Delta x} [x \Delta x] \end{aligned}$$

12.1.3 A Spear Tip

A slender spear tip may be described by the volume swept out by the region between $y = \frac{x^2}{50}$ and the x -axis as the region is revolved about the x -axis. In this case, the radius of a cross-section

at x is $R[x] = \frac{x^2}{50}$, the volume of one slice is $\pi R^2[x]h = \pi \left[\frac{x^2}{50}\right]^2 \Delta x$, and the left radius sum by approximating disks is

$$\begin{aligned} & \pi \left(0 \Delta x + \frac{\Delta x^4}{2500} \Delta x + \frac{(2\Delta x)^4}{2500} \Delta x + \cdots + \frac{(10 - \Delta x)^4}{2500} \Delta x \right) \\ &= \frac{\pi}{2500} \sum_{\substack{x=0 \\ \text{step } \Delta x}}^{10-\Delta x} [x^4 \Delta x] \end{aligned}$$

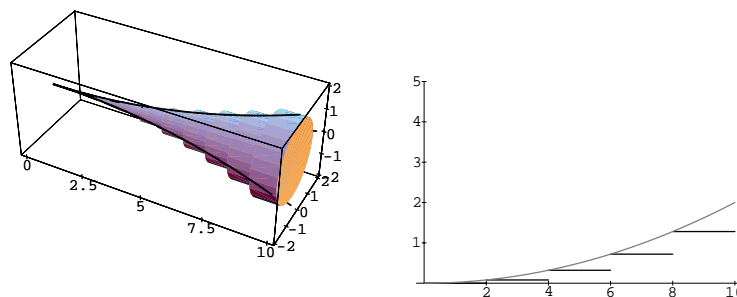


Figure 12.1:6: A cusped spear point and slice generators

12.1.4 The Area Between Two Curves

We need two formulas to build an approximation to the complicated areas between curves. First, the distance between points with real coordinates y_1 and y_2 , is $|y_2 - y_1|$. The signed quantity $y_2 - y_1$ gives the directed distance from y_1 to y_2 where a minus sign means moving opposite the direction from zero to one. (The absolute value will cause us technical difficulties in calculus.) Second, the area of a rectangle is the height times the width.

Example 12.3 *Area Between Sinusoids*

Slice the region between the sine and cosine curves from $-\pi$ to π into strips of width Δx as shown in Figure 12.1:7.

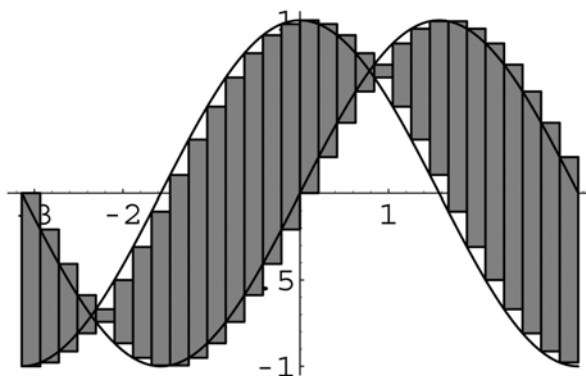


Figure 12.1:7: The area between the sine and cosine

The height of the slice with left point at x is given by $|\sin[x] - \cos[x]| = |\cos[x] - \sin[x]|$. The area of the slice is $|\cos[x] - \sin[x]| \Delta x$. The sum of all the approximating slices is

$$\begin{aligned}
 & (|\cos[-\pi] - \sin[-\pi]| \Delta x + \\
 & |\cos[-\pi + \Delta x] - \sin[-\pi + \Delta x]| \Delta x + \\
 & |\cos[-\pi + 2\Delta x] - \sin[-\pi + 2\Delta x]| \Delta x + \\
 & \quad \dots + |\cos[\pi - \Delta x] - \sin[\pi - \Delta x]| \Delta x) = \\
 & \sum_{\substack{x=-\pi \\ \text{step } \Delta x}}^{\pi-\Delta x} [|\cos[x] - \sin[x]| \Delta x]
 \end{aligned}$$

12.1.5 More Slices

In Chapter 14, we study geometric slicing approximations in greater detail. The first approximation in that chapter, the length of the sine curve, shows why extra care must sometimes be taken to find a correct approximation.

Exercise Set 12.1

1. Show that the volume approximation for the cone of height 10 and base radius 5 when cut into 10 slices with thickness 1 and radii at the midpoints, $\frac{1}{2}, \frac{3}{2}, \dots, \frac{19}{2}$ is

$$\frac{\pi}{4^2}(1^2 + 3^2 + 5^2 + \dots + 19^2) = \frac{\pi 1330}{4^2} \approx 261.$$

Begin by writing the volume approximation as a sum of the volumes of approximating disks as in the above list for 5 slices. See the program *Sums*.

2. Verify that your 10-slice approximation sum is a special case of this formula

$$\pi \sum_{\substack{x=a \\ \text{step } \Delta x}}^b [(R[x])^2 \Delta x, \{x, \Delta x/2, 10 - \Delta x/2, \Delta x\}]$$

in the case where $\Delta x = 1$ by writing out the steps of the sum command.

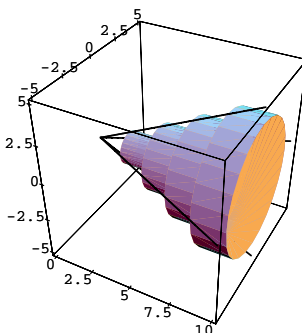


Figure 12.1:8: A cone approximated by 5 “disks”-radii at left

3.

- (a) Use the program *ConeVol* to draw a cone sliced into 20 slices.
 (b) Modify the program *ConeVol* to draw a sphere sliced into 20 slices.

If we measured our radii at the left end of the slice points instead of the midpoint, the approximation would be less accurate, because the slices fit completely inside the cone but the formula would be simpler,

$$\begin{aligned} & \pi[(R[0])^2 \Delta x + (R[1\Delta x])^2 \Delta x + (R[2\Delta x])^2 \Delta x + \dots + (R[10 - \Delta x])^2 \Delta x] \\ & = \pi \sum_{\substack{x=a \\ \text{step } \Delta x}}^b [(R[x])^2 \Delta x, \{x, 0, 10 - \Delta x, \Delta x\}] \end{aligned}$$

This has the form $f[a] \Delta x + f[a + \Delta x] \Delta x + f[a + 2\Delta x] \Delta x + \cdots + f[b - \Delta x] \Delta x$ where $f[x] = \pi R^2[x]$, $a = 0$ and $b = 10$.

4. The approximation for the area between the sine and cosine curves shown on Figure 12.1:7 is

$$\sum_{\substack{x=-\pi \\ \text{step } \Delta x}}^{\pi-\Delta x} [|\text{Cos}[x] - \text{Sin}[x]| \Delta x]$$

What function $f[x]$ and numbers a and b makes this sum have the form

$$\sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [f[x] \Delta x]$$

12.2 Distance When Speed Varies

If we drive 50 mph for an hour and a half, we compute the distance traveled by $D = R \cdot T$, “distance equals the rate times the time,” $D = 50 \cdot \frac{3}{2} = 75$ miles. If we vary our speed, we cannot use this formula. We can compute the distance traveled as an integral. This section shows how.

Suppose we get on the highway and accelerate. We speed up cautiously and drive 25 mph for 1 minute, 26 mph for 1 minute, 27 mph for 1 minute, \cdots , 49 mph for 1 minute and 50 mph for the remainder of the hour and a half as shown on Figure 12.2:9. How far do we go?

The distance traveled at 25 mph must be computed in the correct units,

$$D_1 = 25 \cdot \frac{1}{60} \approx 0.4167 \quad (\text{mph} \times \text{hours} = \text{miles})$$

The distance traveled at 26 mph is

$$D_2 = 26 \cdot \frac{1}{60} \approx 0.4333 \quad (\text{miles})$$

The distance traveled at 27 mph is

$$D_3 = 27 \cdot \frac{1}{60} \approx 0.45 \quad (\text{miles})$$

Each minute’s distance can be computed for 25 minutes, giving a sum of

$$\frac{25}{60} + \frac{26}{60} + \cdots + \frac{49}{60} \approx 15.42$$

The last part of the trip is $\frac{50 \cdot (90-25)}{60} \approx 54.17$ for a total of 69.59 miles.

There are several questions:

1. (a) What have we done symbolically?
- (b) How can we interpret the computation geometrically?
- (c) Why should we interpret the distance computation geometrically?
- (d) How can we extend this to continuously varying speed?

Symbolically, we have speed as a function of time in minutes, $S[1] = 25$, $S[2] = 26$, $S[3] = 27, \dots$ and

$$\text{distance traveled during acceleration} = \sum_{\substack{x=1 \\ \text{step } 1}}^{25} [S[m] \cdot \frac{1}{60}]$$

whereas we have

$$\text{distance traveled at 50 mph} = 50 \cdot \frac{65}{60}$$

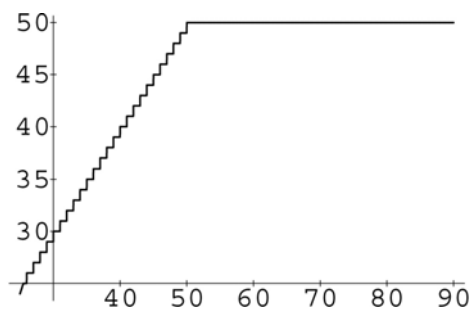


Figure 12.2:9: Speed vs. time in minutes

We may also think of this as a sum with a rate function defined at each minute, $S[m] = 50$, for all the minutes from 26 to 90. However, it is better to also use the correct units, $t =$ elapsed time in hours, and give the speed by the function

$$\begin{aligned} R[t] &= 25 && \text{for } 0 \leq t < \frac{1}{60} \\ R[t] &= 26 && \text{for } \frac{1}{60} \leq t < \frac{2}{60} \\ R[t] &= 27 && \text{for } \frac{2}{60} \leq t < \frac{3}{60} \\ R[t] &= 28 && \text{for } \frac{3}{60} \leq t < \frac{4}{60} \\ &\vdots && \\ R[t] &= 50 && \text{for } \frac{3}{60} \leq t \leq \frac{90}{60} \end{aligned}$$

In this case, we can combine the two pieces into one sum

$$\begin{aligned} \text{distance traveled} &= \sum_{\substack{t=0 \\ \text{step } \frac{1}{60}}}^{\frac{3}{2} - \frac{1}{60}} [R[t] \cdot \frac{1}{60}] \\ &= \sum_{\substack{x=a \\ \text{step } \Delta t}}^{b-\Delta t} [R[t] \cdot \Delta t] \end{aligned}$$

with $\Delta t = \frac{1}{60}$, $a = 0$ and $b = \frac{3}{2}$.

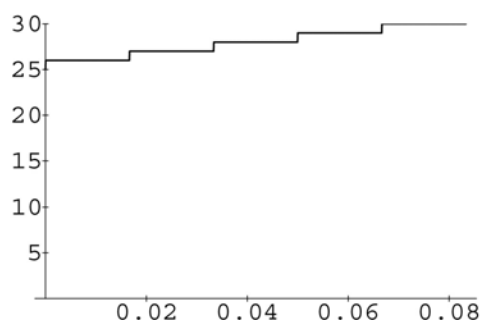


Figure 12.2:10: Speed vs. time in hours for five minutes

The products $25 \cdot \frac{1}{60}$, $26 \cdot \frac{1}{60}$, $27 \cdot \frac{1}{60}$, etc., can be associated with rectangles of height 25, 26, 27, etc., and width $\frac{1}{60}$ hours or 1 minute. Distance is not an area; but, in this case, the distance traveled at 25 mph is a product and that product is also an area of the rectangle under the segment on the graph of speed vs. time. This means that we can represent the distance traveled as the total area between the speed curve and the x -axis - provided our units are hours on the x -axis and miles per hour on the y -axis.

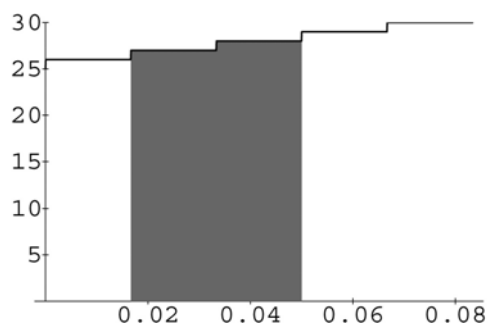


Figure 12.2:11: Distance represented as area

The area representation helps us see what happens if we accelerate continuously, instead of going exactly 25 mph for exactly 1 minute, 26 mph for another minute, and so forth. Suppose we accelerate linearly from 25 mph to 50 mph in 25 minutes, so that our speed is given by

$$R[t] = 25 + 60t \quad \text{for } 0 \leq t < \frac{25}{60}$$

and

$$R[t] = 50 \quad \text{for } \frac{25}{60} \leq t < \frac{90}{60}$$

We could calculate the distance traveled in each second using the speed at the beginning of the second,

$$25 \cdot \frac{1}{3600} = \frac{1}{144} \approx 0.006944$$

$$\left(25 + 60 \frac{2}{3600}\right) \cdot \frac{1}{3600} = \frac{1501}{216000} \approx 0.006949$$

$$\left(26 + 60 \frac{3}{3600}\right) \cdot \frac{1}{3600} = \frac{751}{108000} \approx 0.006954$$

$$\left(25 + 60 \frac{4}{3600}\right) \cdot \frac{1}{3600} = \frac{167}{24000} \approx 0.006958$$

This computation is no fun to do by hand, but the computer gives

$$\sum_{\substack{x=0 \\ \text{step } 1/3600}}^{1.5} [R[t] \cdot \Delta t] = \frac{20099}{288} \approx 69.788$$

when $\Delta t = 1/3600$.

The computer computation is a waste in this case, however, because we can associate the distance traveled with the area under the speed curve in Figure 12.2:12 and use the formula for a trapezoid for $0 \leq t < 25/60$ and for a rectangle for $25/60 \leq t \leq 90/60$,

$$\begin{aligned} \text{area} &= \frac{1}{2}(25 + 50) \cdot \frac{25}{60} + 50 \cdot \frac{65}{60} \\ &= \frac{1675}{24} \\ &\approx 69.7917 \end{aligned}$$

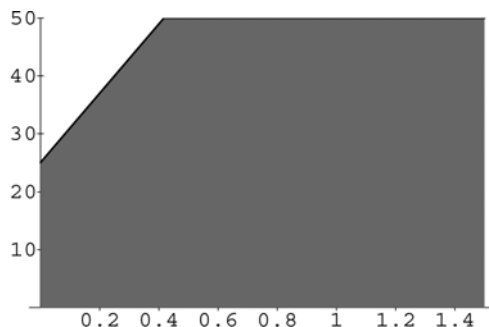


Figure 12.2:12: Linear acceleration and distance

If speed varies by a more complicated rule than linearly in pieces, we can still associate the area under the speed curve with the distance traveled, but now we have no formula to find the area. We will have to find that area and distance by definite integration. For a short interval of time, where speed does not change very much, the distance is approximately the product $R[t] \cdot \Delta t$.

$$\text{distance traveled from time } t \text{ to time } t + \Delta t = R[t] \cdot \Delta t + \varepsilon \cdot \Delta t$$

The approximation is actually close, even compared to Δt , which makes total distance an integral. We will see exactly what the approximation is and verify that it holds as long as $R[t]$ is a continuous function. This should be plausible, because we could also approximate by

$$\text{speed traveled from time } t \text{ to time } t + \Delta t \approx R[t']$$

for any t' between t and $t + \Delta t$. Continuity of $R[t]$ means that $R[t] \approx R[t']$, so $R[t] \Delta t = R[t'] \Delta t + \varepsilon \cdot \Delta t$, with $\varepsilon \approx 0$.

Exercise Set 12.2

1. *The Shortcut to Grandmother's*

You're late and rush to get to Thanksgiving dinner at Grandmother's house in 2 hours. Your speed is recorded on Figure 12.2:13. How far away does Grandmother live? (Note: 10 minutes = 0.16667 hours.) How much was the ticket?

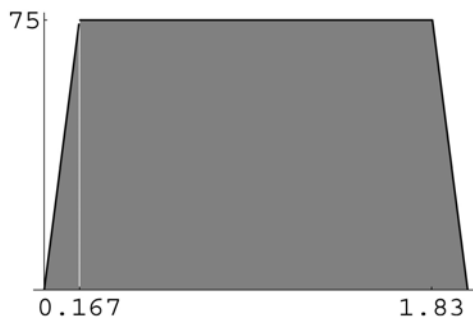


Figure 12.2:13: The speed on a trip to grandmother's

2. *The Return from Grandmother's*

A graph of your trip odometer is shown on Figure 12.2:14 on the return from Grandmother's to school. Graph your speed and show the distance as an area.

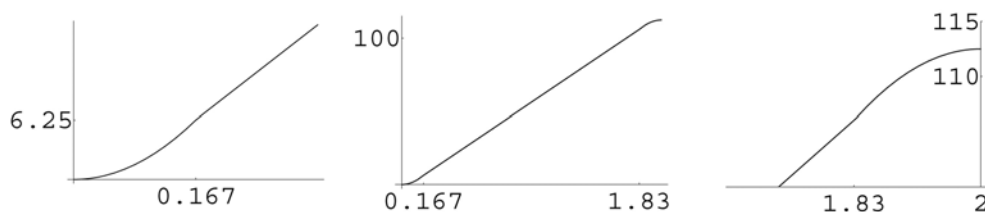


Figure 12.2:14: The speed on the trip back from grandmother's

12.3 The Definite Integral

This section records the definition of the integral that was motivated by ideas like the problem of computing distance from variable speeds and by the geometric area, length, and volume "slicing" problems.

Definition 12.1 *The Definite Integral in One Variable*

Let $f[x]$ be a continuous function defined on the interval $[a, b]$. The definite integral of $f[x]$ over

$[a, b]$ is given by the following limit:

$$\int_a^b f[x] dx = \lim_{\Delta x \rightarrow 0} \sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [f[x] \Delta x]$$

$$\approx \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\Delta x} [f[x] \delta x], \quad \text{when } \delta x, \approx 0$$

We need to use the algebraic properties of the summation function to deduce results about the integral. In either the limiting case $\Delta x \rightarrow 0$ or when we extend the $\sum_{\text{step } \Delta x}^{b-\Delta x}$ function to tiny increments, $\delta x \approx 0$, algebra of general sums extends to integrals. The most basic question is, “What is the role of continuity of $f[x]$ in the definition of $\int_a^b f[x] dx$?” The answer is, “Continuity guarantees us that the limit exists or, equivalently, that we get nearly the same value for each tiny δx .” Extension of the integral to certain discontinuous functions is possible, but involves extra mathematical complications. Four steps in an example of this limit are shown in Figure 12.3:15.

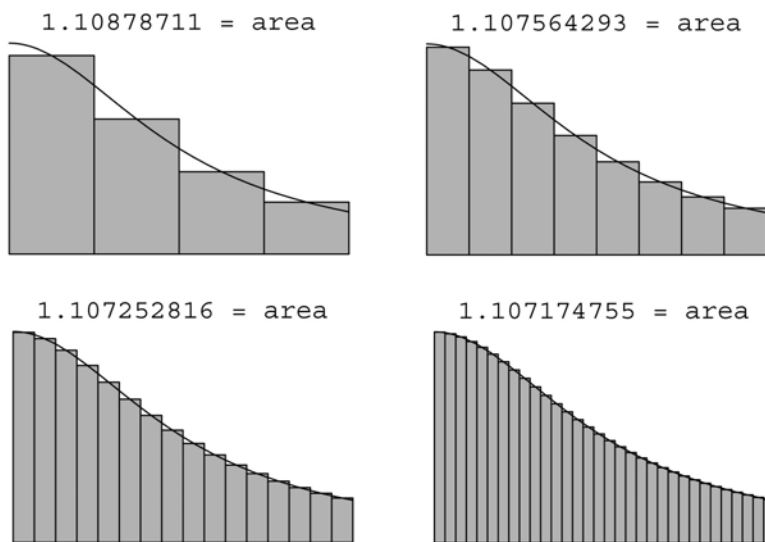


Figure 12.3:15: Approximations to the Area $\text{ArcTan}[2] \approx 1.10715$

One of the most important algebraic properties of summation, the Telescoping Sum Theorem 12.1, plus the differential approximation (or microscope equation) for a smooth function will give us half of the Fundamental Theorem of Integral Calculus 12.15. This theorem tells us how to find exact (symbolic) integrals without summing or taking a limit. We build lots of “techniques of integration” around that theorem because it gives us these exact answers. Summation is still important, however, because the “sums of little slices” idea is where nearly all applications of integration come from.

Exercise Set 12.3

1. Run the computer program **IntegrAprox**. Compare the convergence of the slice approximations with the exact area of a circle described there.

12.4 Computer Summation

Suppose that $F[x]$ is a function of x . To form the sum

$$F[a] + F[a + \Delta x] + F[a + 2\Delta x] + \cdots + F[b]$$

we may use a single computer command

$$\sum_{\substack{x=a \\ \text{step } \Delta x}}^b F[x]$$

This section studies this operator.

For example, if $F[x] = 1$, $a = 0$, $b = 1$ and $\Delta x = 1$

$$\sum_{\substack{x=a \\ \text{step } \Delta x}}^b 1 = F[0] + F[1] = 1 + 1 = 2$$

$$\sum_{\substack{x=a \\ \text{step } \Delta x}}^b 1$$

If we change to $\Delta x = \frac{1}{2}$, then

$$\sum_{\substack{x=a \\ \text{step } \Delta x}}^b 1 = F[0] + F\left[\frac{1}{2}\right] + F[1] = 3$$

The way $\sum_{\substack{x=a \\ \text{step } \Delta x}}^b F[x]$ works on the computer is as follows: We start with $x = a$ and start with $\sum_{\substack{x=a \\ \text{step } \Delta x}}^b = F[x] = F[a]$. Next, we add Δx to x and ask whether $x > b$. If not, we add $F[x] = F[a + \Delta x]$ to $\sum_{\substack{x=a \\ \text{step } \Delta x}}^b$, increment x again, check $x > b$, add $F[x]$ to $\sum_{\substack{x=a \\ \text{step } \Delta x}}^b$, and continue until $x > b$.

Suppose we have $F[x] = x$, $a = 0$, $b = 1$ and $\Delta x = \frac{1}{2}$, then

$$\sum_{\substack{x=a \\ \text{step } \Delta x}}^b F[x] = F[0] + F\left[\frac{1}{2}\right] + F[1] = 0 + \frac{1}{2} + 1 = \frac{3}{2}$$

Keeping the other values, but changing to $\Delta x = \frac{1}{4}$ gives

$$\begin{aligned} \sum_{\substack{x=a \\ \text{step } \Delta x}}^b x &= F[0] + F\left[\frac{1}{4}\right] + F\left[\frac{2}{4}\right] + F\left[\frac{3}{4}\right] + F[1] \\ &= 0 + \frac{1}{4} + \frac{2}{4} + \frac{3}{4} + 1 = \frac{10}{4} \end{aligned}$$

Values of Δx that do not exactly divide $b - a$ are also allowed, so when $\Delta x = .15$,

$$\sum_{\substack{x=a \\ \text{step } \Delta x}}^b x = 0 + .15 + .30 + .45 + .60 + .75 + .90 = 3.15$$

In integration, we will take Δx smaller and smaller and still reach a limiting value in our sums by having a factor Δx in the summand. The next exercise shows you how this works.

Exercise Set 12.4

- Write out all the terms of the sums $\sum_{\substack{x=a \\ \text{step } \Delta x}}^b F[x]$ defined by the following functions and increments, always using $a = 0$ and $b = 1$
 - $F[x] = x$, $\Delta x = 1/5$
 - $F[x] = x\Delta x$, $\Delta x = 1/2$
 - $F[x] = x\Delta x$, $\Delta x = 1/3$
 - $F[x] = x\Delta x$, $\Delta x = 1/4$
 - $F[x] = x\Delta x$, $\Delta x = 1/5$
- Use the computer program **Sums** to compare the summed values of $\sum_{\substack{x=a \\ \text{step } \Delta x}}^b x$ and $\sum_{\substack{x=a \\ \text{step } \Delta x}}^b x\Delta x$ for $\Delta x = \frac{1}{2}$, $\Delta x = \frac{1}{3}$, $\Delta x = \frac{1}{4}$, $\Delta x = \frac{1}{5}$.

12.5 The Algebra of Summation

This section writes simple properties of summation in terms of the summation operator $\sum_{\text{step } \Delta x}^b_{x=a}$. These properties allow us to develop an algebra of integration in the following section.

Our first result is an algebraic method of computing exact sums.

Theorem 12.1 *The Telescoping Sum Theorem*

If $F[x]$ is defined for $a \leq x \leq b$, then the sum of differences below equals the last value minus the first,

$$\sum_{\substack{x=a \\ \text{step } \Delta x}}^b F[x + \Delta x] - F[x] = F[b'] - F[a]$$

where b' is the last value of x of the form $a + n\Delta x$, which is less than or equal to b . If Δx divides $b - a$ exactly, $b' = b$.

PROOF:

$$\begin{aligned} \sum_{\substack{x=a \\ \text{step } \Delta x}}^b F[x + \Delta x] - F[x] = & \\ & (F[a + \Delta x] - F[a]) + (F[a + 2\Delta x] - F[a + \Delta x]) \\ & + (F[a + 3\Delta x] - F[a + 2\Delta x]) + (F[a + 4\Delta x] - F[a + 3\Delta x]) \\ & + \cdots + (F[b'] - F[b' - \Delta x]) \end{aligned}$$

The part $F[a + \Delta x]$ from the first term cancels the part $-F[a + \Delta x]$ from the second term. The part $F[a + 2\Delta x]$ from the second term cancels the part $-F[a + 2\Delta x]$ from the third term, and so on. Each term has a positive and part that cancels the corresponding negative part from the next term. The negative part from the first term and the positive part from the last term are never canceled, so the sum “telescopes” to $F[b'] - F[a]$.

Notice that we sum to $b - \Delta x$ (in steps of Δx), so that the last term is $F[b]$ when Δx divides $b - a$ (rather than $(F[b + \Delta x] - F[b])$)

Example 12.4 *Finding the Difference*

The difficulty in using the Telescoping Sum Theorem is in finding an expression of the form $F[x + \Delta x] - F[x]$ for the summand. Consider the sum

$$\sum_{\substack{x=a \\ \text{step } \Delta x}}^b \left[\frac{\Delta x}{x(x + \Delta x)} \right]$$

Assuming that neither $x = 0$ nor $x + \Delta x = 0$, we can write

$$\frac{\Delta x}{x(x + \Delta x)} = \frac{1}{x} - \frac{1}{x + \Delta x}$$

because putting the right-hand side on a common denominator gives,

$$\begin{aligned} \frac{1}{x} - \frac{1}{x + \Delta x} &= \frac{x + \Delta x}{x(x + \Delta x)} - \frac{x}{x(x + \Delta x)} \\ &= \frac{x + \Delta x - x}{x(x + \Delta x)} = \frac{\Delta x}{x(x + \Delta x)} \end{aligned}$$

This makes computation of the sum easy,

$$\begin{aligned} \sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} \left[\frac{\Delta x}{x(x + \Delta x)} \right] &= \sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} \left[\frac{1}{x} - \frac{1}{x + \Delta x} \right] \\ &= \sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} \left[- \left(\frac{1}{x + \Delta x} - \frac{1}{x} \right) \right] \\ &= F[b'] - F[a] = - \left(\frac{1}{b'} - \frac{1}{a} \right) \\ &= \frac{1}{a} - \frac{1}{b'} \end{aligned}$$

Theorem 12.2 *Superposition of Summation*

Let α and β be constants, and let $F[x]$ and $G[x]$ be functions defined on $[a, b]$. Then

$$\sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [\alpha F[x] + \beta G[x]] = \alpha \sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [F[x]] + \beta \sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [G[x]]$$

PROOF:

$$\begin{aligned} & [\alpha F[a] + \beta G[a]] + [\alpha F[a + \Delta x] + \beta G[a + \Delta x]] \\ & + [\alpha F[a + 2\Delta x] + \beta G[a + 2\Delta x]] + \cdots + [\alpha F[b'] + \beta G[b']] \\ & = \\ & \alpha[F[a] + F[a + \Delta x] + F[a + 2\Delta x] + \cdots + F[b']] \\ & + \beta[G[a] + G[a + \Delta x] + G[a + 2\Delta x] + \cdots + G[b']] \end{aligned}$$

Example 12.5 $\sum_{\substack{x=a \\ \text{step } \Delta x}}^b [x\Delta x] = \frac{b^2 - a^2}{2} - \Delta x \frac{b-a}{2}$

Given that

$$\sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [\Delta x] = \sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [(x + \Delta x) - x] = b - a$$

$$\sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [[x + \Delta x]^2 - x^2] = b^2 - a^2$$

$$\text{and } [x + \Delta x]^2 - x^2 = 2x\Delta x + \Delta x^2$$

we can use superposition to find

$$\sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [x\Delta x] = ?$$

We have

$$\begin{aligned}
 b^2 - a^2 &= \sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [(x + \Delta x)^2 - x^2] \\
 &= \sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [2x \Delta x + \Delta x^2] \\
 &= 2 \sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [x \Delta x] + \Delta x \sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [\Delta x] \\
 &= 2 \sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [x \Delta x] + \Delta x (b - a)
 \end{aligned}$$

Solving for the unknown sum,

$$2 \sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [x \Delta x] = (b^2 - a^2) - \Delta x (b - a)$$

Dividing by 2, we have our formula:

$$\sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [x \Delta x] = \frac{1}{2} ((b^2 - a^2) - \Delta x (b - a))$$

Notice that if Δx is small, this formula says,

$$\sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [x \Delta x] \approx \frac{1}{2}(b^2 - a^2)$$

Theorem 12.3 *The Triangle Inequality*

For $F[x]$ defined on $[a, b]$,

$$\left| \sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [F[x]] \right| \leq \sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [|F[x]|]$$

PROOF:

Terms of opposite sign inside the sum $|F[a] + F[a + \Delta x] + \cdots + F[b]|$ could cancel and make that sum smaller than the sum with all positive terms, $|F[a]| + |F[a + \Delta x]| + \cdots + |F[b]|$.

Theorem 12.4 *Additivity of Summation*

Suppose $F[x]$ is defined on $[a, c]$, Δx divides $b - a$ and $a < b < c$. Then

$$\sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [F[x]] + \sum_{\substack{x=b \\ \text{step } \Delta x}}^{c-\Delta x} [F[x]] = \sum_{\substack{x=a \\ \text{step } \Delta x}}^{c-\Delta x} [F[x]]$$

PROOF:

$$\begin{aligned} & [F[a] + F[a + \Delta x] + \cdots + F[b - \Delta x]] + [F[b] + F[b + \Delta x] + \cdots + F[c']] \\ &= [F[a] + F[a + \Delta x] + \cdots + F[b - \Delta x] + F[b] + F[b + \Delta x] + \cdots + F[c']] \end{aligned}$$

Theorem 12.5 *Monotony of Summation*

If $F[x] \leq G[x]$, then

$$\sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [F[x]] \leq \sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [G[x]]$$

PROOF:

This is obvious because each term in the left sum is smaller than the corresponding term in the right sum.

Theorem 12.6 *Orientation of Summation*

Suppose $f[x]$ is defined on $[a, b]$ with $a < b$ and Δx divides $b - a$, then

$$\sum_{\substack{x=b \\ \text{step } -\Delta x}}^a [f[x](-\Delta x)] = - \sum_{\substack{x=a \\ \text{step } \Delta x}}^b [f[x]\Delta x]$$

PROOF: To go from b to a , we must take negative steps.

Exercise Set 12.51. *What's the Difference?*

Use the *Telescoping Sum Theorem* to give exact answers to the following sums in the case where Δx divides $b - a$.

$$(a) \sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [\Delta x]$$

$$(b) \sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [2x\Delta x + \Delta x^2]$$

$$(c) \sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [3x^2\Delta x + 3x\Delta x^2 + \Delta x^3]$$

The question really is, "What $F[x + \Delta x] - F[x]$ equals the expressions that you are asked to sum?"

(a) Use telescoping sums, superposition, and known sums to find

$$\sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [x^2\Delta x] = ?$$

Notice that

$$\sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [\Delta x] = (b - a)$$

$$\sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [x\Delta x] = \frac{1}{2} ((b^2 - a^2) - \Delta x (b - a))$$

$$\text{and } (x + \Delta x)^3 - x^3 = 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3$$

(b) Use telescoping sums, superposition and known sums to find

$$\sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [x^3 \Delta x] = ?$$

2. Give an example of a sum with the cancellation described in the proof of the triangle inequality.
3. Write out the three sums in the proof of Additivity for $a = 0$, $b = 1$, $c = 2$, $\Delta x = \frac{1}{2}$, and $F[x] = x\Delta x$.
4. Write out the two sums in the proof of Monotony for $a = 0$, $b = 1$, $\Delta x = \frac{1}{2}$, $F[x] = x^2\Delta x$, and $G[x] = x\Delta x$.
5. Write out the two sums in the proof of Orientation for $a = 0$, $b = 1$, $\Delta x = \frac{1}{2}$, and $F[x] = x\Delta x$.

12.6 The Algebra of Integration

For a fixed choice of the continuous real function $f[x]$, we know that

$$\int_a^b f[x] dx \approx \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} f[x] \delta x$$

when $\delta x \approx 0$. The sum properties of the previous section hold for all Δx , so, in particular, they hold when $\Delta x = \delta x \approx 0$ is small. This shows that integrals have the following properties.

Proofs of the properties of integrals are given in the Mathematical Background.

Theorem 12.7 *Superposition*

$$\int_a^b [\alpha f[x] + \beta g[x]] dx = \alpha \int_a^b f[x] dx + \beta \int_a^b g[x] dx$$

Theorem 12.8 *Triangle Inequality*

$$\left| \int_a^b f[x] dx \right| \leq \int_a^b |f[x]| dx, (a < b)$$

Theorem 12.9 *Monotony*

$$\int_a^b f[x] dx \leq \int_a^b g[x] dx, \quad \text{if } f[x] \leq g[x]$$

Theorem 12.10 *Additivity*

$$\int_a^b f[x] dx + \int_b^c f[x] dx = \int_a^c f[x] dx$$

Theorem 12.11 *Orientation*

$$\int_b^a f[x] dx = - \int_a^b f[x] dx$$

Note: The “dx” in the reverse-oriented integral can be thought of as negative to move us from b back to a .

Example 12.6 *Integral Computation the Hard Way*

We have seen that

$$\sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [x\Delta x] = \frac{1}{2} ((b^2 - a^2) - \Delta x(b - a))$$

so, when $\delta x \approx 0$,

$$\begin{aligned} \int_a^b x dx &\approx \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [x\delta x] \\ &= \frac{1}{2} ((b^2 - a^2) - \delta x(b - a)) \\ &\approx \frac{1}{2} (b^2 - a^2) \end{aligned}$$

and the two fixed quantities must be equal,

$$\int_a^b x dx = \frac{1}{2} (b^2 - a^2)$$

Theorem 12.12 *Estimation of Sums*

$$\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [|F[x, \delta x]| \delta x] \leq |F[x_{Max}, \delta x]| \times [b - a]$$

In particular,

$$\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [\varepsilon[x, \delta x] \delta x] \approx 0$$

if $\varepsilon \approx 0$ for all x in $[a, b]$ when $\delta x \approx 0$.

PROOF:

This is the last technical result we need from algebra. We use the finite maximum function. Define a function like the computer's maximum

$$\begin{aligned} & \text{Max}[F[x, \Delta x], \{x = a, b - \Delta x, \Delta x\}] = \\ & \text{maximum}[F[a, \Delta x], F[a + \Delta x, \Delta x], F[a + 2\Delta x, \Delta x], \dots, F[b', \Delta x]] \end{aligned}$$

We know that

$$\text{Max}[F[x, \Delta x], \{a, b - \Delta x, \Delta x\}] = F[x_M, \Delta x]$$

for some real x_M of the form $x_M = a + n\Delta x$, $a \leq x_M \leq b - \Delta x$. Apply the Max function to a small $\delta x \approx 0$,

$$\text{Max}[F[x, \delta x], \{a, b - \delta x, \delta x\}] = F[x_M, \delta x]$$

for some x_M of the form $x_M = a + n\delta x$, $a \leq x_M \leq b$.

The formula we need is the formal statement that we can estimate by making all the terms of a sum larger,

$$\begin{aligned} & \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [|F[x, \delta x]| \delta x] \\ & \leq \text{Max}[|F[x, \delta x]|, \{a, b - \delta x, \delta x\}] \times \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [\delta x] \\ & = |F[x_M, \delta x]| \times \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [\delta x] = |F[x_M, \delta x]| \times (b - a) \end{aligned}$$

Exercise Set 12.6

1. (a) Compute $\int_a^b x^2 dx$ by extending the exact formula you computed earlier for

$$\sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [x^2 \Delta x] \quad \text{to } \delta x \approx 0$$

- (b) Compute $\int_a^b x^3 dx$ by extending the exact formula for

$$\sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [x^3 \Delta x] \quad \text{to } \delta x \approx 0$$

- (c) Compute $\int_a^b x^n dx$ the hard way for integer n by showing

$$(x + \delta x)^n - x^n = n x^{n-1} \cdot \delta x + \varepsilon \cdot \delta x$$

with $\varepsilon = \varepsilon[x, \delta x] \approx 0$. Note that

$$\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [(x + \delta x)^n - x^n] \approx b^n - a^n$$

2. Estimation for $\int_a^b \frac{1}{x^2} dx$

We showed above that

$$\sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} \left[\frac{\Delta x}{x(x + \Delta x)} \right] = \frac{1}{a} - \frac{1}{b}$$

provided that neither $x = 0$ nor $x + \Delta x = 0$ for any term of the sum. Compute the integral

$$\int_a^b \frac{1}{x^2} dx = \frac{1}{a} - \frac{1}{b}$$

by estimating the difference between the unknown sum

$$\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} \left[\frac{\delta x}{x^2} \right]$$

and the sum above,

$$\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} \left[\frac{\delta x}{x(x + \delta x)} \right] = \frac{1}{a} - \frac{1}{b}$$

HINT: Calculate the difference $\left[\frac{1}{x^2} - \frac{1}{x(x+\delta x)}\right] \delta x$ to estimate

$$\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} \left[\frac{\delta x}{x^2}\right] - \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} \left[\frac{\delta x}{x(x+\delta x)}\right]$$

Is your computation valid if $a < 0 < b$?

3. Estimation for $\int_a^b \frac{1}{x^3} dx$

Show that

$$\int_a^b \frac{1}{x^3} dx = \frac{1}{2a^2} - \frac{1}{2b^2}$$

as follows: First, by putting the difference on a common denominator, show that

$$\frac{1}{x^3} = \frac{x}{(x^2 - \delta x^2)^2} + \varepsilon$$

with $\varepsilon \approx 0$ if $\delta x \approx 0$.

$$\varepsilon = \frac{1}{x^3} - \frac{x}{(x^2 - \delta x^2)^2}$$

Second, show that

$$\frac{4x \delta x}{(x^2 - \delta x^2)^2} = \frac{1}{(x - \delta x)^2} - \frac{1}{(x + \delta x)^2}$$

by putting the right-hand side on a common denominator. Third, show that

$$\frac{4x \delta x}{(x^2 - \delta x^2)^2} = \left(\frac{1}{(x - \delta x)^2} + \frac{1}{x^2}\right) - \left(\frac{1}{(x + \delta x)^2} + \frac{1}{x^2}\right)$$

and that

$$\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} \left[\frac{4x \delta x}{(x^2 - \delta x^2)^2}\right] = \left(\frac{1}{a^2} + \frac{1}{(a - \delta x)^2}\right) - \left(\frac{1}{b^2} + \frac{1}{(b - \delta x)^2}\right)$$

4. Estimation for $\int_a^b \frac{1}{x^3} dx$

You can sum $\frac{1}{(x-\delta x)^3} - \frac{1}{(x+\delta x)^3}$. What integral does that tell you?

5. Estimation for $\int_a^b \frac{1}{x^{n+1}} dx$

Show that

$$\int_a^b \frac{1}{x^{n+1}} dx = \frac{1}{n} \left(\frac{1}{a^n} - \frac{1}{b^n}\right)$$

as follows. First, show that

$$\frac{1}{x^n} - \frac{1}{(x + \delta x)^n} = \frac{n x^{n-1} \delta x}{x^n (x + \delta x)^n} + \varepsilon_1 \cdot \delta x$$

with $\varepsilon_1 \approx 0$ when $\delta x \approx 0$. Second,

$$\frac{x^{n-1}}{x^n(x + \delta x)^n} = \frac{1}{x^{n+1}} + \varepsilon_2$$

with $\varepsilon_2 \approx 0$ when $\delta x \approx 0$. Can we have $a < 0 < b$?

6. Estimation for $\int_a^b \frac{1}{2\sqrt{x}} dx$

You can sum $\sqrt{x + \delta x} - \sqrt{x}$ in general for $a \leq x \leq b$. Multiply this expression by $\sqrt{x + \delta x} + \sqrt{x}$

$$\sqrt{x + \delta x} - \sqrt{x} = \frac{(\sqrt{x + \delta x} - \sqrt{x})(\sqrt{x + \delta x} + \sqrt{x})}{\sqrt{x + \delta x} + \sqrt{x}}$$

to find

$$\int_a^b \frac{1}{2\sqrt{x}} dx$$

7. Estimation for $\int_a^b \sqrt{x} dx$

Sum $\sqrt{(x + \delta x)^3} - \sqrt{x^3}$ to find

$$\int_a^b \sqrt{x} dx$$

12.7 Fundamental Theorem, Part 1

This section gives the theory to find integrals by antiderivatives without taking limits or approximating with tiny increments. (Chapter 13 gives techniques for using the theory.) The theory results from an estimate and a telescoping sum. The general estimate is even simpler than the specific telescoping sum exercises of the previous section.

Recall the differential approximation for a smooth function from Chapter 5 at this time, because we are about to use it.

Theorem 12.13 First Half of the Fundamental Theorem

Given an integrand $f[x] dx$, suppose that we can find a function $F[x]$ such that $dF[x] = f[x] dx$ (or $\frac{dF}{dx} = f[x]$) for all $a \leq x \leq b$, then

$$\int_a^b f[x] dx = F[b] - F[a]$$

We also sometimes write

$$\int_a^b f[x] dx = \int_a^b dF[x] = F[x]|_a^b = F[b] - F[a]$$

The notation $F[x]|_a^b$ simply means $F[b] - F[a]$.

PROOF:

The differential approximation (microscope equation, see Chapter 5) for $F[x]$ is

$$F[x + \delta x] - F[x] = f[x]\delta x + \varepsilon \cdot \delta x$$

for all x satisfying $a \leq x \leq b$, where $\varepsilon \approx 0$ when $\delta x \approx 0$. (This forces $f[x]$ to be a continuous function for the ordinary derivative defined in Chapter 5. See Section 5.5.)

The telescoping sum and superposition properties say

$$\begin{aligned} F[b] - F[a] &= \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [F[x + \delta x] - F[x]] = \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [f[x]\delta x + \varepsilon \cdot \delta x] \\ &= \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [f[x]\delta x] + \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [\varepsilon \cdot \delta x] \\ &\approx \int_a^b f[x] dx + \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [\varepsilon \cdot \delta x] \end{aligned}$$

We conclude the proof by using Theorem 12.12, which shows

$$\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [\varepsilon \cdot \delta x] \approx 0$$

We have shown that the fixed quantities satisfy

$$F[b] - F[a] \approx \int_a^b f[x] dx$$

which forces them to be equal and proves the theorem.

Strange though it sounds, we have shown two things with the proof of the first half of the Fundamental Theorem: 1) the integral exists! and 2) its value is $F[b] - F[a]$. (Instructor Note: The usual continuity hypothesis is hidden in our ability to find a function $F[x]$ satisfying the uniform microscope equation derivable with $f[x] = F'[x]$ for all x in $[a, b]$.)

Example 12.7 *Computation of $\int_a^b x^2 dx$, the Easy Way*

Compare the following use of the Fundamental Theorem with your direct computation from the last section. We have $f[x] = x^2$; so, if we take $F[x] = x^3/3$, then $dF[x] = x^2 dx$ and

$$\int_a^b x^2 dx = \frac{1}{3}x^3 \Big|_a^b = \frac{1}{3}(b^3 - a^3)$$

Example 12.8 *Integral of Cosine*

$$\int_a^b \cos [\theta] d\theta = \sin [\theta] \Big|_a^b = \sin [b] - \sin [a]$$

It would be quite difficult to compute this integral without the help of the Fundamental Theorem.

Exercise Set 12.71. *Fundamental Drill*

Show that $dF[x] = f[x] dx$ for the given pairs of functions $F[x]$ and $f[x]$. Use this to compute the integrals.

(a) $F[x] = \frac{1}{4}x^4$, $f[x] = x^3$, $\int_0^2 x^3 dx =$

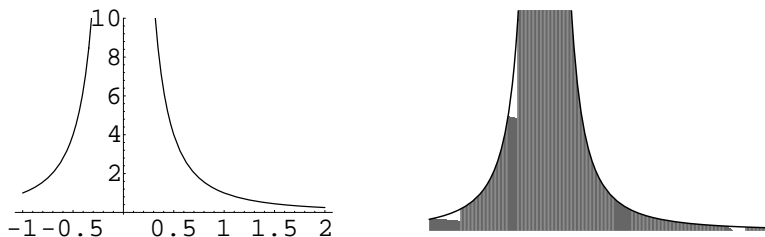
(b) $F[x] = \sqrt{x}$, $f[x] = \frac{1}{2\sqrt{x}}$, $\int_4^9 \frac{1}{\sqrt{x}} dx =$

(c) $F[x] = \frac{2}{3}\sqrt{x^3}$, $f[x] = \sqrt{x}$, $\int_4^9 \sqrt{x} dx =$

(d) $F[x] = -\frac{1}{x^n}$, $f[x] = \frac{n}{x^{n+1}}$, $\int_2^3 \frac{1}{x^3} dx =$

2. Can you use the formula $F[x] = -\frac{1}{x^n}$ implies $dF[x] = \frac{1}{x^{n+1}} dx$ to find $\int_2^3 \frac{1}{x} dx$? What function $F[x]$ has $dF[x] = \frac{1}{x} dx$?

If $f[x]$ is not continuous everywhere on $[a, b]$, the Fundamental Theorem does not apply and we shall see that using $F[b] - F[a]$ anyway will lead to errors, even if $\frac{dF}{dx} = f[x]$ at all but one point of $[a, b]$

Figure 12.7:16: Area under the curve $y = 1/x^2$

3. Is the following computation correct?

$$\begin{aligned} \int_{-1}^2 \frac{1}{x^2} dx &= \int_{-1}^2 x^{-2} dx \\ &= -x^{-1} \Big|_{-1}^2 = \frac{1}{x} \Big|_{-1}^2 \\ &= -1 - \frac{1}{2} = -\frac{3}{2} \end{aligned}$$

Note that if $F[x] = -x^{-1}$, then $dF[x] = x^{-2} dx$, but how could the “area” under a positive curve be a negative number? Draw a picture and compute the areas $\int_{.0001}^1 \frac{1}{x^2} dx$, $\int_{-1}^{-.0001} \frac{1}{x^2} dx$ and $\int_{+1}^2 \frac{1}{x^2} dx$. What do you think the area $\int_{-.0001}^{.0001} \frac{1}{x^2} dx$ should be?

Incorrect programs can produce incorrect results.

4. Garbage In, Garbage Out

What is wrong with the following “computer computation” of $\int_0^2 \frac{1}{\text{Cos}^2[x]} dx$?

In[1]

```
f := 1/(Cos[x]) ^ 2;      Integrate[f, {x, 0, 2}] < Enter >
```

Out[1]

```
 $\frac{\text{Sin}[2]}{\text{Cos}[2]} \approx -2.18504$ 
```

Can the integral of a positive function be negative? Is the function $F[x] = \text{Sin}[x]/\text{Cos}[x]$ an antiderivative for the function $f[x] = 1/[\text{Cos}[x]]^2$ over the whole interval $0 \leq x \leq 2$? That is, does $dF[x] = f[x] dx$ for all these values of x ? Use the computer to plot the integrand function over the interval $0 \leq x \leq 2$.

12.8 Fundamental Theorem, Part 2

The second part of the Fundamental Theorem of Integral Calculus says that the derivative of an integral of a continuous function is the integrand,

$$\frac{d}{dX} \int_a^X f[x] dx = f[X]$$

The function $A[X] = \int_a^X f[x] dx$ can be thought of as the “accumulated area” under the curve $y = f[x]$ from a to X shown in Figure 12.8:17. The “accumulation function” can also be thought of as the reading of your odometer at time X for a given speed function $f[x]$

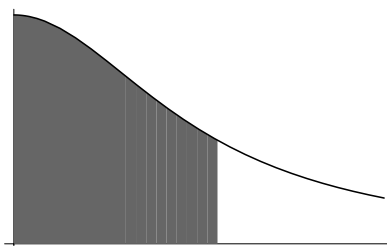


Figure 12.8:17: $A[X] = \int_a^X f[x] dx$

When we cannot find an antiderivative $F[x]$ for a given $f[x]$, we sometimes still want to work directly with the definition of the integral. A number of important functions are given by integrals of simple functions. The logarithm and arctangent are elementary examples. Some other functions like the probability function of the “bell-shaped curve” of probability and statistics do not have elementary formulas but do have integral formulas. The second half of the Fundamental Theorem justifies the integral formulas, which are useful as approximations. The **NumIntAprx** computer program shows us efficient ways to estimate the limit of sums directly.

Continuity of $f[x]$ is needed to show that the limit actually “converges.” With discontinuous integrands, it is possible to make the sums oscillate as $\Delta x \rightarrow 0$. In these cases, numerical integration by computer will likely give the wrong answer.

We need naked existence of the limit for the second half of the Fundamental Theorem. The first half of the Fundamental Theorem is used often, whereas the second half is seldom used. This is why we have postponed a general proof of existence (without antiderivatives). (Our proof of the First Half of the Fundamental Theorem gave existence and a formula at the same time.)

Theorem 12.14 *Existence of the Definite Integral*

Let $f[x]$ be a continuous function on the interval $[a, b]$. Then there is a real number I such that

$$\sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [f[x] \Delta x] \rightarrow I \text{ as } \Delta x \rightarrow 0$$

or, equivalently,

$$\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [f[x] \delta x] \approx I \text{ for any } \delta x \approx 0$$

PROOF:

First, by the Extreme Value Theorem for Continuous Functions, $f[x]$ has a min, m , and a Max, M , on the interval $[a, b]$. Monotony of summation tells us

$$m \times [b - a] \leq \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [f[x] \delta x] \leq M \times [b - a]$$

so that $\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [f[x] \delta x]$ is a finite number and thus near some fixed value $I[\delta x] \approx \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [f[x] \delta x]$.

What we need to show is that if we choose a different increment, for example $\delta u \approx 0$, then $I[\delta x] = I[\delta u]$ or

$$\sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [f[x] \delta x] \approx \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta u} [f[u] \delta u]$$

Draw your own picture of two different rectangular approximations to clarify this idea. If we superimpose the different partitions and consider two overlapping rectangles, the areas only differ by a small amount on a scale of the increments because $f[x] \approx f[u]$ by continuity for $x \approx u$. The proof that we do get the same number is taken up in detail in the book, Foundations of Infinitesimal Calculus at <http://www.math.uiowa.edu/~stroyan>.

Theorem 12.15 *Second Half of the Fundamental Theorem*

Suppose that $f[x]$ is a continuous function on an interval containing a and we define a new function by “accumulation,”

$$A[X] = \int_a^X f[x] dx$$

Then $A[X]$ is smooth and $\frac{dA}{dX}[X] = f[X]$; in other words,

$$\frac{d}{dX} \int_a^X f[x] dx = f[X]$$

PROOF:

We show that $A[X]$ satisfies the differential approximation $A[X + \delta X] - A[X] = f[X] \delta X + \varepsilon \cdot \delta X$ with $\varepsilon \approx 0$ when $\delta X \approx 0$. This proves that $f[X]$ is the derivative of $A[X]$.

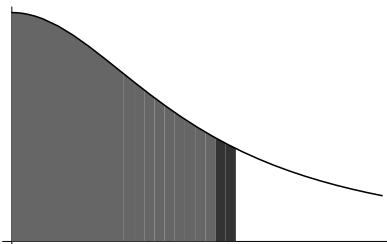


Figure 12.8:18: $\int_a^{X+\delta X} f[x] dx$

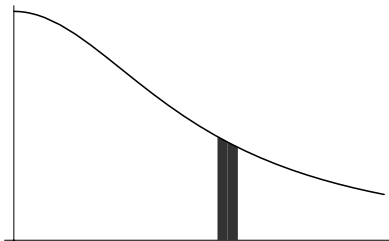
By definition of A and the additivity property of integrals, we have

$$\begin{aligned} A[X + \delta X] &= \int_a^{X+\delta X} f[x] dx = \int_a^X f[x] dx + \int_X^{X+\delta X} f[x] dx \\ &= A[X] + \int_X^{X+\delta X} f[x] dx \end{aligned}$$

So, we need to show that

$$\int_X^{X+\delta X} f[x] dx = f[X] \delta X + \varepsilon \cdot \delta X$$

with $\varepsilon \approx 0$, when $\delta X \approx 0$.

Figure 12.8:19: $\int_X^{X+\delta X} f[x] dx$

This is when we use the continuity hypothesis about $f[x]$. The Extreme Value Theorem for Continuous Functions 11.3 says that $f[x]$ has a max and a min on the interval $[X, X + \delta X]$, $m = f[X_m] \leq f[x] \leq M = f[X_M]$ for all $X \leq x \leq X + \delta X$. Monotony of the integral gives us the estimates

$$m\delta X = \int_X^{X+\delta X} m dx \leq \int_X^{X+\delta X} f[x] dx \leq \int_X^{X+\delta X} M dx = M \cdot \delta X$$

Since both X_m and X_M lie in the interval $[X, X + \delta X]$, we know that $X_m \approx X_M$ when $\delta X \approx 0$. Continuity of $f[x]$ means that $f[X_m] \approx f[X_M] \approx f[X]$ in this case, so

$$\int_X^{X+\delta X} f[x] dx = f[X] \delta X + \varepsilon \cdot \delta X$$

using upper and lower estimates of the integral based on the max and the min of the function over the small subinterval.

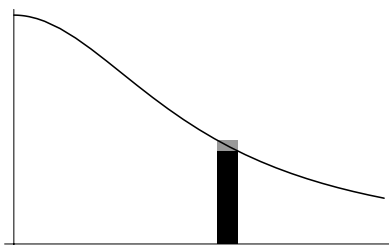


Figure 12.8:20: Upper and lower estimates

This proves the theorem because we have verified the microscope equation from Chapter 5,

$$A[X + \delta X] = A[X] + A'[X] \delta X + \varepsilon \cdot \delta X$$

with $A'[X] = f[X]$.

Exercise Set 12.8

1. Use the Second Half of the Fundamental Theorem to explain how YOU could compute $\text{Log}[x]$ and $\text{ArcTan}[x]$ directly using a numerical integration program. We know that

$$\frac{d\text{Log}[x]}{dx} = \frac{1}{x} \quad \text{and} \quad \frac{d}{dX} \int_1^X \frac{1}{x} dx = \frac{1}{X}$$

and

$$\frac{d\text{ArcTan}[x]}{dx} = \frac{1}{1+x^2} \quad \text{and} \quad \frac{d}{dX} \int_0^X \frac{1}{1+x^2} dx = \frac{1}{1+X^2}$$

Use the Numerical Integration program *NumIntAprx* to compute the various numerical integral approximations for these functions, and compare your results with the computer's built-in algorithms for these functions. For example, how do the following compare:

$$\text{Log}[7.38905] \approx \text{NIntegrate}[1/x, \{x, 1, 7.38905\}]$$

and

$$4 \times \text{ArcTan}[1] \approx 4 \text{NIntegrate}[1/(1+x^2), \{x, 0, 1\}]$$

Explain in terms of the exact symbolic computations why we have chosen these particular numbers (7.38905 in log and the 4 in arctangent). See the basic left- right- and midpoint- approximations in the program *IntegrAprx*.

Sometimes it is not possible to find an antiderivative in terms of elementary functions, but even in these cases, the integral defines a function and the numerical approximations give a direct way to compute these functions.

2. *Non-elementary Integrals*

Use the computer's numerical integral numerical integration command to approximate the following integrals:

$$\text{a) } \int_0^1 e^{-x^2} dx = \quad \text{b) } \int_0^1 \frac{\text{Sin}[x]}{x} dx = \quad \text{c) } \int_0^1 \text{Sin}[x^2] dx =$$

Also use the computer's symbolic integration to find expressions for these integrals. These expressions are not combinations of elementary functions. See the symbolic integration program *SymbolicIntegr* for details.