

# Chapter 1

## Calculus with Computers

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*This book, with its software called “Mathematica NoteBooks” and “Maple Worksheets,” is a new approach to an old subject: calculus.*

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*Calculus* means “stone” or “pebble” in Latin. Roman numerals were so awkward that ancient Romans carried pebbles to compute their grocery bills and gamble and, as a result, “calculus” also came to be known as a method for computing things. The specific calculus of this book is “differential and integral calculus,” formerly called “infinitesimal calculus.” The fundamental things this calculus computes are rates of change of continuous variables through differentials and the inverse computation: accumulation of known rates of change through integrals. Derivatives and integrals are illustrated in Figures 1.1 and 1.2.

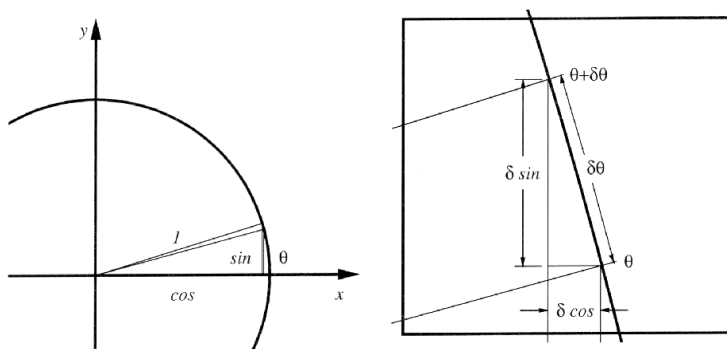


Figure 1.1: Derivatives of sine and cosine

Calculus is one of the great achievements of the human intellect. It has served as the language of change in the development of scientific thought for more than three centuries. During that time it became a coherent polished subject, growing hand in hand with physical science and technology. The contemporary importance of calculus is actually expanding into economics and the social sciences, as well as continuing to play a key role in its traditional areas of application. Chapter 2 previews its use in the study of epidemic diseases, where the models can show us such things as the differences in successful vaccination strategies for polio and measles. Similar models have helped make national health policy changes in screening for gonorrhea.

The model in Chapter 2 is a system of differential equations, so it may seem technically a little ahead of the story. The basic ideas are not difficult and require only high school math. There is a straightforward description of the daily rates of change in numbers of sick people that depends on how many people are

1.107174755 = area of rectangles

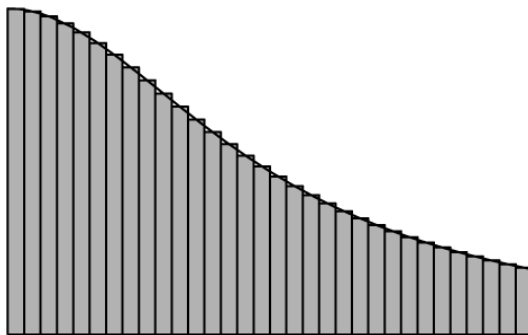


Figure 1.2: Arctangent as an integral

sick and how many are susceptible. The objective of Chapter 2 is to show that it is relatively easy to describe the spread of a disease such as measles by writing formulas that say how it changes. One new thing in our approach to calculus is that *you* will be expected to *use* calculus to describe applications that interest you. (We offer many choices, including mathematical ones, ranging from bungee jumping to drug absorption to price adjustment. See the accompanying book of projects.) This will show you the role of calculus as a language – you will “speak” it.

Computers make it possible for us to begin the course with a real-life problem such as the study of epidemics. By describing epidemic changes in computer notation, we will immediately get approximate answers to many questions about the spread of diseases. Throughout the course, we will use scientific computing, either *Mathematica* or *Maple*, that is both powerful and relatively easy to use. Computing will help us understand and explore mathematics and to expand the power and applicability of the mathematics we learn. You will leave this course with a good start on learning 21<sup>st</sup>-century scientific computing.

You will also learn all the old things from traditional calculus in this course. In particular, you will learn to compute derivatives and integrals. Good high school algebra preparation and effort during the semester are all you need. The rote computations that often dominate traditional calculus take practice but not very much if you are proficient at high school math. If you make this effort, you will be well prepared for your later science and math courses. *Mathematica* or *Maple* may even help you check some of your rote skills. You will be far better prepared than traditional calculus students in understanding the role of calculus in science and mathematics.

## 1.2 Previews of Coming Attractions

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*Besides basic computing, this chapter shows you some simple examples of the limit that defines the derivative:*

$$\lim_{\Delta x \rightarrow 0} \frac{f[x + \Delta x] - f[x]}{\Delta x} = f'[x]$$


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Examples of the relationship between a function  $f[x]$  and its derivative  $f'[x]$  are

if $y = f[x] = x^2$ ,	then $\frac{dy}{dx} = f'[x] = 2x^1$
if $y = f[x] = x^3$ ,	then $\frac{dy}{dx} = f'[x] = 3x^2$
if $y = f[x] = x^3 - x^2 - 5x$ ,	then $\frac{dy}{dx} = f'[x] = 3x^2 - 2x - 5$
if $y = f[x] = \text{Sin}[x]$ ,	then $\frac{dy}{dx} = f'[x] = \text{Cos}[x]$

The various derived functions  $f'[x]$  go with the original functions,  $f[x]$ . No doubt you can see a pattern in the first few results above, but do not worry if you have not seen this before. The rules for finding  $f'[x]$  from formulas for  $f[x]$  are the “little pebbles” of calculus. We return to a systematic development of rules of differentiation in Chapters 5 and 6.

Rules are important but are not the whole story. In this chapter, you will see how the computer can help with the “approximation” part of the rest of the story. Each of the expressions above means that the rate of change of the function  $f[x]$  is approximately  $f'[x]$  or that “small” changes in the function are approximately linear at rate  $f'[x]$ . (Linear functions change at a constant rate.) Where is the “approximation” in the foregoing examples? Where is the linearity? Use of rules alone skips the approximation and only gives the answer. The computer together with some basic high school math can help us see the approximation.

The idea of a limit is based on an approximation. The meaning of

$$\lim_{\Delta x \rightarrow 0} \frac{f[x + \Delta x] - f[x]}{\Delta x} = f'[x]$$

is that the expression  $\frac{f[x + \Delta x] - f[x]}{\Delta x}$  gets close to  $f'[x]$  as  $\Delta x$  gets small. Or,

$$\frac{f[x + \delta x] - f[x]}{\delta x} \approx f'[x], \quad \text{when } \delta x \approx 0$$

How close and how small are the technical details of the approximation that we treat later in the course.

In this chapter we will see two graphical approximations and animate them with the computer. We return to another view of the main approximation of differential calculus in Chapters 4 and 5. At that point, you will learn the rules and approximations more systematically.

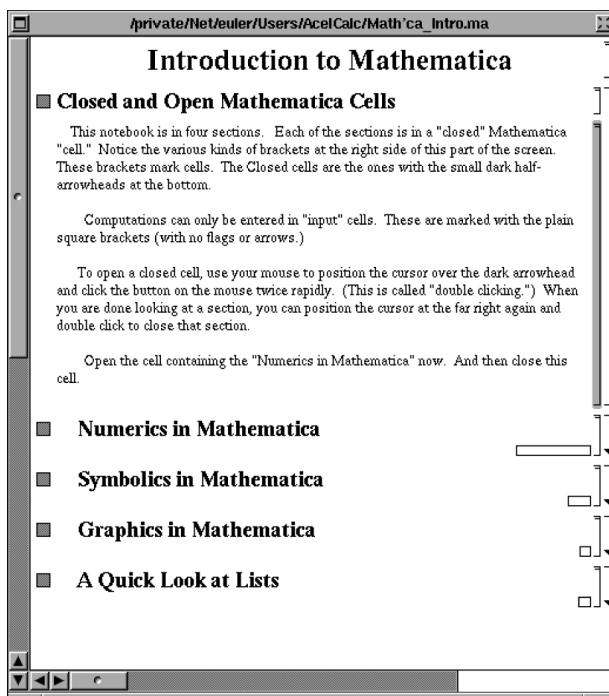
## 1.3 An Introduction to Computing

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*One goal of this chapter is to have you run your first Mathematica or Maple program. (They are similar, and you will use whichever one you or your school owns.)*

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Both *Mathematica* NoteBooks and *Maple* Worksheets include instructions to you, have space for instructions to the computer, and have places for you to type your own interpretations of results. These programs can do numeric, symbolic, and graphical computations in the same program.



A *Mathematica* NoteBook Window

The folder **aComputerIntro** uses features of the program to introduce you to *Mathematica* or *Maple*. When you open the programs from that folder, you will see a window that looks something like the one above, though the control buttons may be a little different in your version.

You may be using your own computer or using a network of various kinds of computers that belong to your school. Except for the location of the control buttons, *Mathematica* or *Maple* programs run the same way on all computers. The appropriate programs should be installed on your hard drive or your school's network.

There is a *Mathematica* version of the **aComputerIntro**, and there is a *Maple* version. They are different in detail but contain the same information. You need to run only one version of each program depending on your system.

If your machines are networked, you need to learn about the details from your instructor. These vary from place to place, even with the same kind of machine.

## Exercise Set 1.2

1. If your computers are networked, log in, change your password, and log out.

You need to learn to do "mouse" editing. Of course, it is a little frustrating sometimes to use the computer to get an introduction to the computer. Stick with it; once you learn a little "mouse editing," you will find that almost all the computer information you need is contained in the *Mathematica* or *Maple* programs. You could also ask a friend or lab monitor to help.

2. Log on to your account with your new password, run the programs in **aComputerIntro**. Read the instructions in the programs and do the exercises in it. When you are done, save it.

Your instructor will tell you how you should save your electronic homework. This depends on the kind of network and disk drives that are installed with your system.

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## 1.4 Linearity in Local Coordinates

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*Since we want to understand how calculus approximates nonlinear functions by linear ones, we will first reformulate linear functions in terms of their changes. This is describing a line by its “differential equation.”*

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We will look at this formulation symbolically (or algebraically), graphically (or geometrically), and numerically, since it is often helpful in mathematics to view a topic from all three of the computational points of view described in the programs of **aComputerIntro**.

The change in a variable is a difference, the new value minus the old value, so we abbreviate change by Greek capital delta,  $\Delta$  (for difference, small delta is written  $\delta$ ). Using these symbols, we have the definition

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### Definition 1.4.1 *y varies Linearly with x*

*Let  $y$  be a real valued function of a real variable  $x$ ,  $y = f[x]$ . We say the dependent variable,  $y$ , varies linearly with the independent variable,  $x$ , if the ratio of the change in  $y$  over a corresponding change in  $x$  always equals the same constant,  $m$ ,*

$$\frac{\Delta y}{\Delta x} = m$$

*for any pair of  $(x, y)$  points satisfying  $y = f[x]$ . For example, if  $y_1 = f[x_1]$ ,  $y_2 = f[x_2]$  and  $y_3 = f[x_3]$ , then the following ratios of differences are all equal,*

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2 - y_3}{x_2 - x_3} = \frac{y_3 - y_1}{x_3 - x_1} = m$$


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We use this definition to get equations for a line three ways: with “local coordinates” directly, and with ordinary coordinates to give the point-slope formula, and the two-point form of a line. Algebraic simplification of these forms leads to the familiar slope-intercept form of a line:

$$y = f[x] = m x + b$$

This algebraically simplified form is often not simplest when we are interested in the change of a nonlinear function near a certain point. Local coordinates put it in the form “ $y = m x$ ,” but since  $(x, y)$ -coordinates are already used, we introduce new variables  $(dx, dy)$  to do this.

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### Example 1.4.2 *Geometric Linearity*

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Two pairs of points on the graph of a linear function are shown in Figure 4.3 along with horizontal and vertical segments corresponding to the changes in  $x$  and  $y$ . Notice that the two triangles are similar.

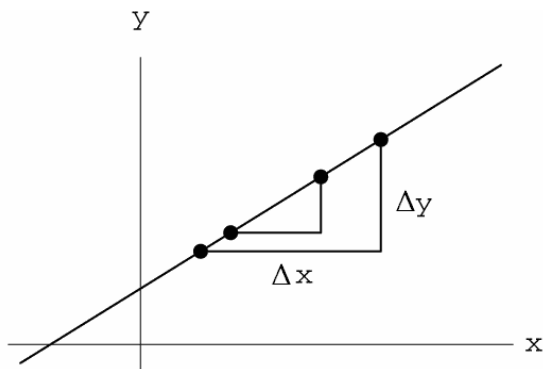


Figure 4.3: Similar triangles

Definition 1.4.1 is equivalent to the statement that any two triangles along the curve are similar. This is because similarity of the geometric figures means the ratios of corresponding sides are equal. Geometrically, the constant  $m$  in Definition 1.4.1 is the slope, so “a line has constant slope” is the equivalent geometric statement that  $y$  varies linearly with  $x$ .

**Example 1.4.3** *Numerical Linearity*

Data from an accurate experiment might look something like Table 1.1 (where the two place accuracy removes measurement errors).

Linear $(x, y)$ Data				
$x$	1.0	2.0	5.0	6.3
$y$	3.0	1.0	-5.0	-7.6

Table 1.1: Linear data

These data are linear because the ratios of all changes equal the same constant.

The change in  $x$  from the first point to the second is  $2.0 - 1.0 = 1.0$ , the change in  $y$  from first to second is  $1.0 - 3.0 = -2.0$  (taking the difference in the same order), and

$$\frac{\Delta y}{\Delta x} = \frac{1.0 - 3.0}{2.0 - 1.0} = -2.0$$

Similarly, the ratio of changes from second to third is

$$\frac{\Delta y}{\Delta x} = \frac{-5.0 - 1.0}{5.0 - 2.0} = \frac{-6.0}{3.0} = -2.0$$

The ratio of changes from third to fourth is

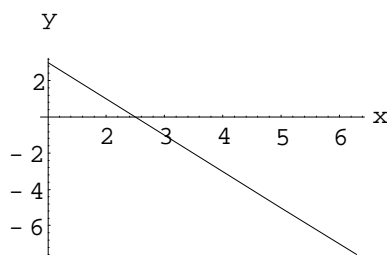
$$\frac{\Delta y}{\Delta x} = \frac{-7.6 - (-5.0)}{6.3 - 5.0} = \frac{-2.6}{1.3} = -2.0$$

The ratio of changes from second to fourth is

$$\frac{\Delta y}{\Delta x} = \frac{-7.6 - 1.0}{6.3 - 2.0} = \frac{-8.6}{4.3} = -2.0$$

The numerical point of Definition 1.4.1 is that any way we compute the ratio of changes, we get the same constant.

There is no reason to separate the numerical, graphical, and symbolic views of a topic; in particular, it is perfectly natural to graph the data in Table 1.1 as well as to find a formula that describes the graph and data. Both the graph and formula are more powerful representations in the sense that they condense information about all the points between the data points in convenient forms (although the data may be all that are scientifically known.)



Graph in Table 1.1

The formula derived from any two  $(x, y)$  points of Table 1.1 also provides a simpler way to demonstrate linearity. The remaining ones can just be checked by substitution into the formula.

Notice that the computer chose axes starting at the first point of the table, rather than at  $(0, 0)$ . In calculus, we often need a formula for the line with a given slope through a certain point. We will see this as the tangent line problem in the second half of this section. For this reason, we will develop special “local” coordinates for this problem. We start with a fresh example and leave a unified approach to the tabular data to one of your exercises.

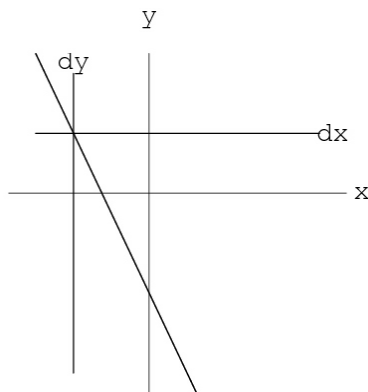
**Example 1.4.4** *Local Coordinates for the Point-Slope Form of a Line*

Suppose we know that a line goes through the point  $(-4, 3)$  and has slope  $-2$ . The easy way to sketch this line would be to start at the point  $(-4, 3)$  and move  $+1$  in the  $x$  direction, then  $-2$  in the  $y$  direction, put our ruler down, and draw. We want a symbolic version of this procedure of changing from a base point.

We define new coordinates (new  $x$ , new  $y$ ) =  $(dx, dy)$  called *local coordinates through  $(-4, 3)$*  (Figure 4.4). These have their origin at  $(-4, 3)$  and are parallel to the  $(x, y)$  coordinates. Notice that an amount of  $dx$  represents a *change* in  $x$  from the point  $-4$  and an amount of  $dy$  represents a change in  $y$  from  $3$ . [The quantities  $\Delta x$  and  $\Delta y$  of Definition 1.4.1 represent any differences, not necessarily based at  $(-4, 3)$ .] In local coordinates, Definition 1.4.1 says the equation of our line is

$$\frac{dy}{dx} = -2 \quad \text{or} \quad dy = -2 dx$$

**Example 1.4.5** *The  $(x, y)$  Point-Slope Formula from the Local Formula*

Figure 4.4: The line  $dy = -2 dx$ 

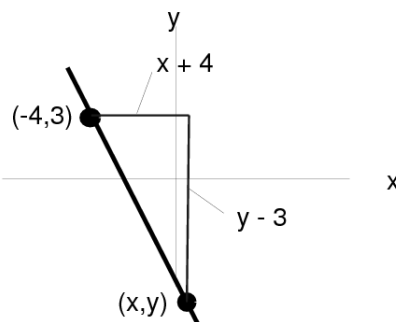
How can we change the local equation  $dy = -2 dx$  to our original  $(x, y)$ -coordinates? The  $(dx, dy)$  origin is the same as the  $(x, y)$  point  $(-4, 3)$ . Local coordinates measure changes in  $x$  and  $y$  from this point. This means we could have defined the local coordinates symbolically by

$$\begin{aligned} dx &= x - (-4) = x + 4 \\ dy &= y - 3 \end{aligned}$$

so the line can be written in either of the forms

$$\frac{dy}{dx} = -2 \quad \Leftrightarrow \quad \frac{y - 3}{x + 4} = -2$$

Here is another explanation. The change in  $y$  moving from 3 to the generic value  $y$  is the difference in  $y$ ,  $\Delta y = y - 3$ , and the change in  $x$  in moving from  $-4$  to the generic value  $x$  is the difference in  $x$ ,  $\Delta x = x - (-4) = x + 4$ , as shown in Figure ??.

Figure 4.5: Changes from  $(-4, 3)$  to  $(x, y)$ 

The ratio of any pair of corresponding differences equals the constant slope  $m$ ,  $\frac{\Delta y}{\Delta x} = m$ , and our particular changes are  $dx$  and  $dy$ , so the following are equivalent:

$$\frac{dy}{dx} = m \quad \Leftrightarrow \quad \frac{\Delta y}{\Delta x} = m \quad \Leftrightarrow \quad \frac{y - 3}{x + 4} = -2$$



We can simplify this equation algebraically, but it is often best left in this form when we are interested in the changes in the variable near  $(-4, 3)$ . Local coordinates build this feature in by moving the origin to this point. The algebraically simplified formula has the more compact form  $y = m x + b$ , but the difference form (above at the right) shows the point and the slope. Local coordinates are already simplified at their own origin.

In general, the local coordinates can be defined by

**Definition 1.4.6**  $(dx, dy)$  *Coordinates*

The local coordinates through the fixed  $(x, y)$  point  $(x_0, y_0)$  are given by

$$\begin{aligned} dx &= x - x_0 \\ dy &= y - y_0 \end{aligned}$$

Local coordinates are nothing mysterious, but they are very useful in calculus.

**Procedure 1.4.7** *The General  $(x, y)$  Point-Slope Formula*

Find the  $(x, y)$  equation of a line of slope  $m$  through the fixed point  $(x_0, y_0)$ .

1. Write the local equation:

$$\frac{dy}{dx} = m$$

2. Replace  $dy$  and  $dx$  by their definition in terms of  $x$  and  $y$ :

$$\frac{y - y_0}{x - x_0} = m$$

You should verify that this is the same as the point-slope formula if you memorized that in high school.

**Example 1.4.8** *Two-Point Formula from the Change Formula*

Here is another example of using change to describe a line. Suppose our line goes through the points  $(x, y) = (-2, 3)$  and  $(x, y) = (1, 4)$ .

Before we can use Procedure 1.4.7, we need to compute the slope. The two given points tell us the slope:

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \text{constant} \\ \frac{4 - 3}{1 - (-2)} &= \frac{4 - 3}{1 + 2} = \frac{1}{3} \end{aligned}$$

The differences needed to compute the slope are shown on Figure 4.6.

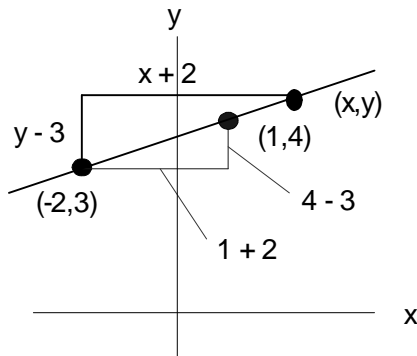


Figure 4.6: Two points determine the line  $y = \frac{1}{3}x + \frac{11}{3}$

Now step (1) of Procedure 1.4.7 says the equation in local coordinates at  $(x, y) = (-2, 3)$  is:

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \text{constant} \\ \frac{dy}{dx} &= \frac{1}{3}\end{aligned}$$

Step (2) replaces local coordinates at  $(x, y) = (-2, 3)$  by differences in the original coordinates:

$$\frac{dy}{dx} = \frac{y - 3}{x - (-2)} = \frac{y - 3}{x + 2} = \frac{1}{3}$$

The general differences  $y - 3$  and  $x + 2$  are shown on Figure 4.6.

We can also solve this problem using the change in moving from  $(1, 4)$  to the general  $(x, y)$  point:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{3} \\ \frac{y - 4}{x - 1} &= \frac{1}{3}\end{aligned}$$

In local coordinates the equation is  $\frac{dy}{dx} = \frac{1}{3}$ , but the point at which we localize affects the  $x - y$ - form. The equations we got from the different base points are equivalent:

$$\frac{y - 3}{x + 2} = \frac{1}{3} \quad \Leftrightarrow \quad \frac{y - 4}{x - 1} = \frac{1}{3}$$

The easiest way to see this is to simplify both algebraically, putting them in the form  $y = m x + b$ :

$$\begin{aligned}\frac{y - 3}{x + 2} &= \frac{1}{3} & \frac{y - 4}{x - 1} &= \frac{1}{3} \\ y - 3 &= \frac{1}{3}(x + 2) & y - 4 &= \frac{1}{3}(x - 1) \\ y - 3 &= \frac{1}{3}x + \frac{2}{3} & y - 4 &= \frac{1}{3}x - \frac{1}{3} \\ y &= \frac{1}{3}x + \frac{2}{3} + 3 & y &= \frac{1}{3}x - \frac{1}{3} + 4 \\ y &= \frac{1}{3}x + \frac{11}{3} & y &= \frac{1}{3}x + \frac{11}{3}\end{aligned}$$

In high school, you learned various symbolic forms of the equation of a line: the point-slope formula, the two-point formula, as well as slope-intercept formula. You do not need to remember the high school formulas as long as you can use Definition 1.4.1. In any case, start to think in terms of changes rather than only the fixed form  $y = m x + b$ .

### Example 1.4.9 *Linearity and Constant Rates*

When we view the independent variable as time, Definition 1.4.1 of linearity is equivalent to the statement that the dependent variable changes at a constant rate. Time rates of change are important in many applications of calculus, and we will see them often later in the course. Once we understand an idea in  $x$  and  $y$ , it is often advantageous to understand it in terms of other variables, such as time,  $t$ .

### Exercise Set 1.3

#### 1. Local Coordinates for a Line

- (a) Suppose a line goes through the point  $(5, -4)$  and has slope  $-3$ . Define local coordinates  $(dx, dy)$  with origin at the point  $(5, -4)$ , giving formulas for  $dx$  and  $dy$  in terms of  $x$  and  $y$ .  
 What is the  $(dx, dy)$  equation for this line?  
 What is the  $(x, y)$  equation of the line?  
 What is the  $(x, y)$  slope-intercept form  $(y = m x + b)$  of this line?
- (b) Suppose a line goes through the fixed point  $(x_0, y_0)$  and has slope  $m$ .  
 What are the local  $(dx, dy)$  coordinates through  $(x_0, y_0)$ ?  
 What is the  $(dx, dy)$  equation of the line?  
 What is the  $(x, y)$  equation of the line? What is the  $(x, y)$  slope-intercept form  $(y = m x + b)$  of this line ( $b = ?$ )?

#### 2. Point-Slope Form to Slope-Intercept Form

- (a) Algebraically simplify the expression  $\frac{y-3}{x+4} = -2$  showing that it is equivalent to the expression  $y = -2x - 5$  (when  $x \neq -4$ ).
- This line has slope  $-2$ . Use both formulas to show this.
  - This line contains the point  $(3, -4)$ . Use both formulas to show this.
  - This line crosses the  $y$ -axis at  $y = -5$ . Use both formulas to show this.
- (b) Simplify the following expressions and give the slope and  $y$ -intercept:
- $\frac{y-2}{x-3} = -4$
  - $\frac{y-2}{x+3} = 4$
  - $\frac{y+2}{x-3} = -4$
  - $\frac{y}{x-3} = 1$
  - $\frac{y-3}{x} = -2$
  - $\frac{y}{x} = 3$
- (c) Simplify the expression  $\frac{y-y_0}{x-x_0} = m$ , showing that it is equivalent to  $y = m x + b$  (when  $x \neq x_0$ ), where  $b = y_0 - m x_0$ . What is the slope of this line? Use both formulas to show that the line contains the point  $(x_0, y_0)$ .

### 3. Two Points Determine a Line

(a) Suppose a line goes through the points  $(3, -2)$  and  $(-4, 5)$ . Find its slope.

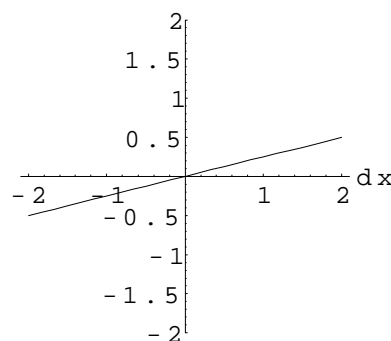
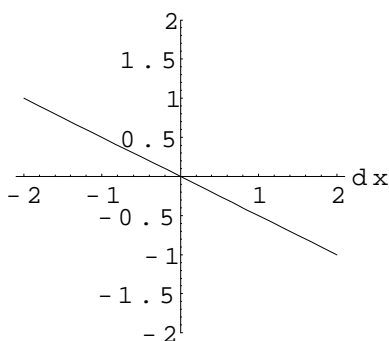
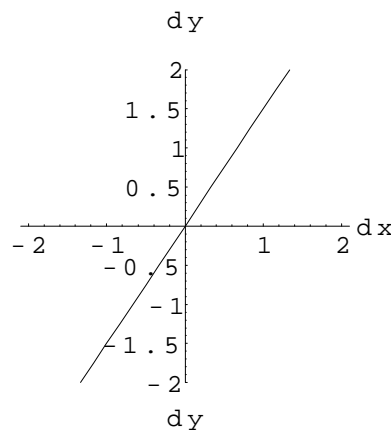
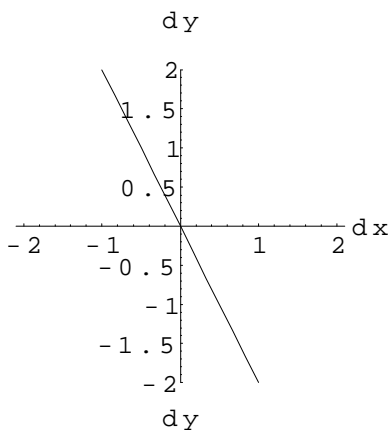
- i. Use the slope and the point  $(3, -2)$  to find its  $(x, y)$  equation.
- ii. Use the slope and the point  $(-4, 5)$  to find its  $(x, y)$  equation.
- iii. Compare the two equations by putting both in slope-intercept form.

(b) Find the slope-intercept equation of the line that passes through:

- i)  $(1, 2)$  and  $(3, 4)$     ii)  $(2, -1)$  and  $(-3, 4)$     iii)  $(-2, 3)$  and  $(5, -7)$

### 4. Graphics to Symbolics

Find an equation for each of the graphs shown below.



### 5. Numerics to Symbolics

- (a) Find an equation satisfied by all the points on Table 1.1.
- (b) (Optional) Solve this with the computer using the program *LinearIntro*.

### 6. The LinearIntro Program

Run the program *LinearIntro* in the Chapter 1 folder of the course software and answer the questions in it. Save the results in your computer account (or homework disk). *LinearIntro* shows you how to solve Exercise 4 with the computer. Comparing your paper-and-pencil work with your electronic work should help you begin the course computing.

7. *Newton's Law of Cooling* says that the rate of cooling of an object is proportional to the difference between the temperature of the object and the temperature of its surroundings. Let  $F$  denote the Fahrenheit temperature of the object and  $t$  denote the time in minutes beginning with  $t = 0$  at our first observation. (See Chapters 4 and 8.)

- Suppose an object is  $75^\circ$  when we first observe it and it cools to  $60^\circ$  ten minutes later. What is its (average) rate of cooling over this time period?
- Sketch the line segment passing through the points  $(t, F) = (0, 75)$  and  $(10, 60)$ .
- What is the slope of this segment?
- What is the equation of the line through these two points?

Newton's Law of Cooling does not give the temperature as a function of the time, but rather says,  $R = k(F - F_0)$ , where  $R$  is the rate of cooling,  $F_0$  is the temperature of the surroundings, and  $k$  is a constant. The line through the points above is not a good long-term prediction of temperature because its slope is constant and that slope is the rate referred to in Newton's Law. Calculus will give us a simple exponential formula.

8. *The downward velocity  $v$  of an object released in vacuum near the surface of the earth satisfies a linear equation,  $v = gt + v_0$ , where  $g$  is Galileo's universal gravitational constant,  $g = 9.8 \text{ m/sec}^2$  or  $g = 32 \text{ ft/sec}^2$ .* (See Chapter 10.)

- Suppose we first observe a falling object traveling  $13 \text{ ft/sec}$  downward. How fast will it be falling 2 seconds later?
- Suppose we first observe an object thrown  $13 \text{ ft/sec}$  upward. How fast will it be falling 2 seconds later?

## 1.5 Derivatives for Explicit Formulas

*This section begins with more data. The data are NOT linear because the ratios of all changes do NOT equal the same constant.*

NonLinear $(x, y)$ Data					
$x$	-1.000	0.000	1.000	1.500	2.000
$y$	-1.000	0.000	1.000	3.375	8.000

Table 1.2: Table Caption

Four differences from first to second, second to third, etc. are:

$$\frac{0.0 - (-1.0)}{0.0 - (-1.0)} = 1, \quad \frac{1.0 - 0.0}{1.0 - 0.0} = 1, \quad \frac{3.375 - 1.0}{1.5 - 1.0} = 4.75, \quad \frac{8.0 - 3.375}{2.0 - 1.5} = 9.25$$

A graph of the data is shown in Figure 5.7

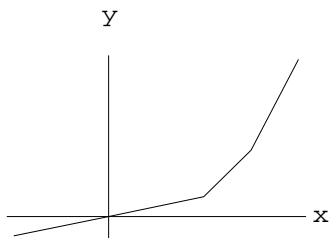


Figure 5.7: The data of Table 1.2 connected by lines

The slopes of segments connecting points on this curve do NOT remain the same.

These points lie on the curve  $y = f[x] = x^3$  as shown in Figure 5.8

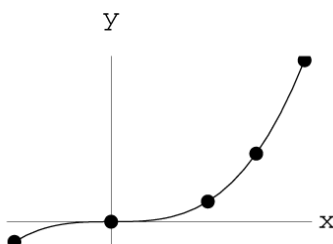


Figure 5.8: The data on  $y = f[x] = x^3$

The slope of the curve varies “continuously” as a point moves along it. The derivative measures this changing slope.

### Example 1.5.1 *The Symbolic Derivative of $y = x^3$*

One way to approach the derivative is to choose a fixed  $(x, y)$  point and look at the slope of the segment that differs by a small amount  $\Delta x$  in  $x$ . The corresponding change in  $y$  now must be computed with the function  $f[x] = x^3$  at the new  $x$  point  $x + \Delta x$ . This is

$$f[x + \Delta x] = (x + \Delta x)^3 = x^3 + 3x^2 \Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

The change in  $y$  is the new value minus the old value:

$$\begin{aligned} \Delta y &= f[x + \Delta x] - f[x] = (x + \Delta x)^3 - x^3 \\ &= x^3 + 3x^2 \Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3 \\ &= 3x^2 \Delta x + 3x(\Delta x)^2 + (\Delta x)^3 \\ &= 3x^2 \cdot \Delta x + (3x + \Delta x) \cdot (\Delta x)^2 \end{aligned}$$

The ratio of these changes is

$$\frac{\Delta y}{\Delta x} = 3x^2 + (3x + \Delta x) \cdot (\Delta x)$$

In particular, if  $x = 1.0$  and  $\Delta x = 1.0$ , we have

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= 3x^2 + (3x + \Delta x) \cdot (\Delta x) \\ &= 3 \cdot 1^2 + (3 \cdot 1 + 1) \cdot 1 \\ &= 7\end{aligned}$$

We could have seen this particular result more easily from

$$(f[x + \Delta x] - f[x]) / (\Delta x) = (2^3 - 1) / (1 - 0) = (8 - 1) / 1 = 7$$

but the symbolic expression is more powerful. It lets us see what happens as  $\Delta x$  gets small:

$$\begin{aligned}\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \rightarrow 0} 3x^2 + (3x + \Delta x) \cdot (\Delta x) \\ &= 3x^2 + \lim_{\Delta x \rightarrow 0} (3x + \Delta x) \cdot (\Delta x) \\ &= 3x^2 + 0\end{aligned}$$

The limit is  $3x^2$  because  $3x^2$  does not change as  $\Delta x$  gets smaller and smaller, whereas the remaining expression,

$$(3x + \Delta x) \cdot (\Delta x)$$

tends to zero. It gets small as  $\Delta x$  gets small since it is multiplied by the small number  $\Delta x$ . For example, suppose we want the error  $(3x + \Delta x) \cdot (\Delta x)$  to be less than  $1/1000$  and we know that  $|x| < 1,000,000$ . The term  $|3x + \Delta x| < 3,000,001$ , so if we take any  $|\Delta x| < 1/(4 \times 10^9)$ , the error is less than a thousandth.

To summarize the example so far, we have shown

$$\begin{aligned}\lim_{\Delta x \rightarrow 0} \frac{f[x + \Delta x] - f[x]}{\Delta x} &= f'[x] \\ \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} &= 3x^2\end{aligned}$$

What does this computation mean? Graphically, we have found that the slope of the segment from the point  $(x, f[x])$  to the point  $(x + \Delta x, f[x + \Delta x])$  gets closer and closer to  $3x^2$  as  $\Delta x$  gets smaller and smaller. In other words, the slope of the line through closer and closer points on the curve tends to  $3x^2$ . You can see this dynamically on the computer program **DerivLimit** in the course software. Figure 5.9 shows three successively closer points.

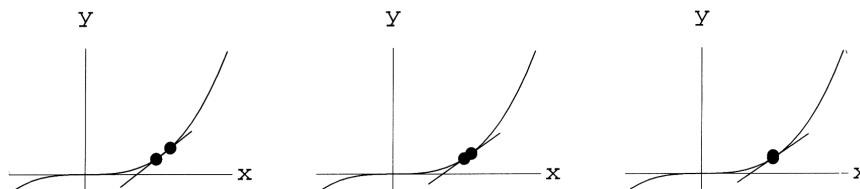
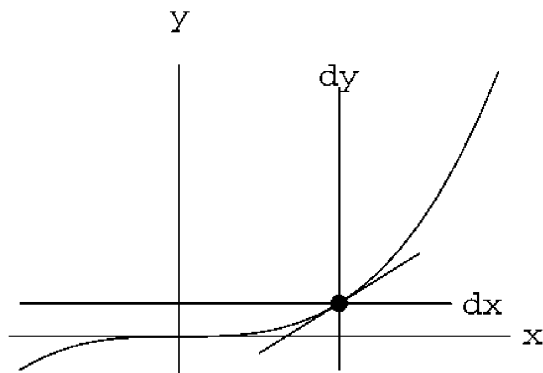


Figure 5.9: Successively closer points on  $y = x^3$

The successive lines tend toward the graph of the line  $dy = m dx$  in local coordinates through  $(x, f[x])$  where  $m = 3x^2$  and  $x$  is considered fixed for the  $(dx, dy)$ -plot. The equation

$$dy = 3x^2 dx$$

Figure 5.10:  $y = x^3$  and  $dy = 3x^2 dx$  on one graph

gives the local coordinates of the tangent line to the curve at the point shown in Figure 5.10:

We summarize the limit computation by the following statement:

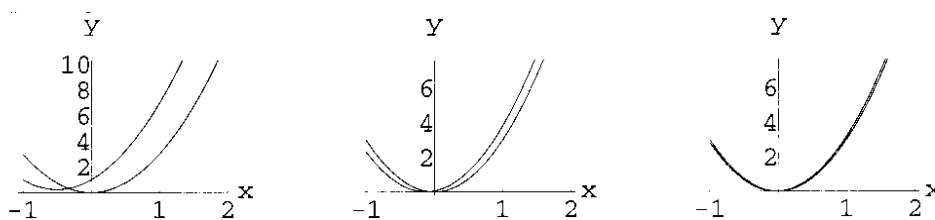
$$\text{If } y = f[x] = x^3, \text{ then } \frac{dy}{dx} = f'[x] = 3x^2$$

The new function  $f'[x] = 3x^2$  is called the derivative of  $f[x] = x^3$ , and we say that the slope of the curve at the single point  $x$  is  $f'[x]$  since its tangent line at this point has this slope.

We can also see the convergence of the limit

$$\lim_{\Delta x \rightarrow 0} \frac{f[x + \Delta x] - f[x]}{\Delta x} = f'[x]$$

by plotting both  $(f[x + \Delta x] - f[x])/\Delta x$  and the function  $f'[x]$  on the same  $(x, y)$  graph for smaller and smaller  $\Delta x$ . The program **DerivLimit** also shows the convergence of the approximate formulas to the limiting formula. It can even do this dynamically as an animation or computer generated movie.

Figure 5.11:  $\frac{f[x+\Delta x]-f[x]}{\Delta x}$  and  $f'[x]$  for  $\Delta x = 1/2, 1/8, 1/32$ 


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### Exercise Set 1.4

1. For the function  $f[x] = x^3$ , verify that when  $x = 1.0$  and  $\Delta x = 0.5$  the expression

$$\frac{\Delta y}{\Delta x} = \frac{f[x + \Delta x] - f[x]}{\Delta x} = 4.75$$



but that when  $x = 1.5$  and  $\Delta x = 0.5$  the expression

$$\frac{\Delta y}{\Delta x} = \frac{f[x + \Delta x] - f[x]}{\Delta x} = 9.25$$

2. Let  $f[x] = x^2$ . This exercise has you show step by step that  $f'[x] = 2x$ . Expand the expression  $f[x + \Delta x] = (x + \Delta x)^2$ . Use your expansion to show that  $f[x + \Delta x] - f[x] = 2x \cdot \Delta x + (\Delta x)^2$ . Show that the  $\lim_{\Delta x \rightarrow 0} \frac{f[x + \Delta x] - f[x]}{\Delta x} = 2x$ . Check your work with the Symbolics program in aComputerIntro.
3. Let  $f[x] = x^4$ . This exercise has you show step by step that  $f'[x] = 4x^3$ . Expand the expression  $f[x + \Delta x] = (x + \Delta x)^4$ . Use your expansion to show that  $f[x + \Delta x] - f[x] = 4x^3 \cdot \Delta x + (\Delta x)^2 \cdot (\text{stuff})$ . Show that the  $\lim_{\Delta x \rightarrow 0} \frac{f[x + \Delta x] - f[x]}{\Delta x} = 4x^3$ . Check your work with the Symbolics program in aComputerIntro.
4. Run the computer program *DerivLimit*, and verify convergence behind the rules

$$\text{if } y = f[x] = x^2, \quad \text{then } \frac{dy}{dx} = f'[x] = 2x^1$$

$$\text{if } y = f[x] = x^4, \quad \text{then } \frac{dy}{dx} = f'[x] = 4x^3$$

$$\text{if } y = f[x] = x^3 - x^2 - 5x, \quad \text{then } \frac{dy}{dx} = f'[x] = 3x^2 - 2x - 5$$

$$\text{if } y = f[x] = \text{Sin}[x], \quad \text{then } \frac{dy}{dx} = f'[x] = \text{Cos}[x]$$

## 1.6 Review of High School Math

Chapter 28 reviews some of the most important high school math topics needed in calculus. Take a brief look at it now, and return to it as needed throughout the course. Here are a few topics from the review chapter.

### 1.6.1 Function Notation and Substitution of Expressions in Functions

Chapter 28 has a careful explanation of function notation and substitution of expressions in functions such as the one used in our preview of derivatives:

$$\begin{aligned} \frac{f[x + \Delta x] - f[x]}{\Delta x} &= \frac{(x + \Delta x)^3 - x^3}{\Delta x} \\ &= \frac{(x^3 + 3x^2 \Delta x + 3x(\Delta x)^2 + (\Delta x)^3) - x^3}{\Delta x} \\ &= 3x^2 + 3x(\Delta x) + (\Delta x)^2 \end{aligned}$$

## 1.6.2 Types of Explicit Functions

The basic high school functions defined by explicit formulas are

<b>Linear Functions</b>	$y = mx + b$ or $\frac{\Delta y}{\Delta x} = m$
<b>Polynomials</b>	$y = a_0 + a_1x + \dots + a_nx^n, \quad a_j \text{ constants}$
<b>Trigonometric Functions</b>	$y = \text{Sin}[\theta], \quad y = \text{Cos}[\theta], \quad y = \text{Tan}[\theta]$
<b>Inverse Trig Functions</b>	$\theta = \text{ArcTan}[y]$
<b>Exponential Functions</b>	$y = a \cdot b^x, \quad a, b > 0 \text{ constants}$
<b>The Natural Logarithm</b>	$x = \text{Log}[y]$

*Note:* Natural log is sometimes denoted  $\ln(y)$  in high school texts, where  $\log(y)$  denotes the base 10 log.

## 1.6.3 Format for Homework

Chapter 28 includes a general approach to “word problems.” In homework problems involving applications

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### Procedure 1.6.1 *Homework Format*

- (a) *Explicitly list your variables with units and sketch a figure if appropriate.*
- (b) *Translate the information stated in the problem into formulas in your variables. (If this translation is difficult, you may not have chosen the best variables.) Often it is helpful to balance units on both sides of an equation.*
- (c) *Formulate the question in terms of your variables and solve.*
- (d) *Explicitly interpret your solution.*

There are practice word problem examples in Chapter 28.

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## 1.6.4 Review as You Go

We will review several important high school topics in the main text at the beginning of sections where they are needed. At those places, we will also reference the relevant parts of Chapter 28. We urge you to forge ahead with calculus and brush up on high school material as needed when you need it.

## 1.7 Free Advice

We know that the first few computer exercises take a long time and may be frustrating for people who are not used to “windows” and “mouse editing.” These initial frustrations will seem very simple in a week or so if you confront them now. **DO NOT GET BEHIND IN YOUR HOMEWORK.**

We encourage you to work with a friend on the computing in this course. Often two heads are better than one because one of you can concentrate on the machine details while the other thinks about the main mathematical task you are trying to accomplish. As long as you change roles occasionally, you will both learn faster.

There are several kinds of problems for you to work in this course.

### 1.7.1 Drill Exercises

Routine algorithms that just need practice and not much thinking are called “drill exercises.” Some of these are expanded on in the companion book, *Foundations of Infinitesimal Calculus*. Some instructors like to give drill tests as “barrier exams.” This means that you have several chances to pass a skill exam but that you must get 90% of the problems completely correct. You do not get a grade for the skill test but cannot pass the course unless you hurdle it. These are necessary skills, but the skills alone do not mean that you understand the material.

### 1.7.2 Exercises

Regular “exercises” may require a little more thought about the current material than drill exercises. Calculus is important because the basic skill algorithms have important meanings. The skills themselves are not the whole story, and exercises sometimes take a step in the direction of their meaning. Exercises are sometimes “electronic”; that is, they require you to use a *Mathematica* NoteBook or *Maple* Worksheet. Most instructors assign one large electronic homework assignment per week. These assignments show you how to do basic calculations on the computer. Once you have done these, you can use the computer to solve regular homework exercises and work projects.

### 1.7.3 Problems

“Problems” are larger exercises that are meant to help you organize your thinking. They go beyond exercises in that they usually have several parts and ask you to write summary explanations of the combined meaning of the parts. Your written explanations are very important. You will find that it is sometimes difficult to put your ideas into words. You “understand” but find it hard to explain the problem. Wrestling with this difficulty improves your understanding and helps you to combine the parts of calculus that you have learned into coherent understanding.

### 1.7.4 Projects

“Projects” are larger problems - serious applications of calculus, not fragmented parts of applications or text exercises contrived to come out in whole numbers. Scientific Projects comprise a paperback book accompanying this text. You should look through the table of contents of the Scientific Projects to get an idea of how widely YOU can apply calculus. There are projects on topics ranging from bungee jumping to extinction of whales.