Math 26, Second Midterm Exam Review
April, 2006

On the exam, please show your work, and try to organize the work as coherently as possible. (It is very difficult to assign partial credit if the work is chaotic.)

1. Some formulas/methods you need to know.

Here are some formulas that you know, and that will *not* be given to you

(1) The arc length of a curve $y = f(x)$ from $x = a$ to $x = b$ is

$$\int_a^b \sqrt{1 + f'(x)^2}\, dx = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx$$

(2) Area of a surface of revolution. A surface obtained by rotating a curve $y = f(x)$ from $x = a$ to $x = b$ about the x–axis is

$$\int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx = \int_a^b 2\pi y\sqrt{1 + (\frac{dy}{dx})^2}\, dx,$$

assuming $f(x) \geq 0$. For a curve rotated around the y–axis, you have to reverse the roles of x and y.

(3) The pressure exerted by a fluid at depth h is

$$P = \delta h,$$

where δ is the weight density of the fluid. The hydrostatic force on a little bit of surface at depth h is

$$\Delta F = P\,\Delta A = \delta h\,\Delta A.$$

The force on a vertically oriented surface extending from depth a to depth b is

$$\delta \int_a^b x\, w(x)\, dx,$$

where $w(x)$ is the width of the surface at depth x.

(4) Consider a plate of uniform density in the x–y plane. The center of mass of the plate has coordinates $(\bar{x}, \bar{y})$. The x–coordinate of the CM is computed as follows: First, the x–moment of mass is

$$x\text{–moment} = \int_a^b x\, h(x)\, dx,$$

where the plate lies between $x = a$ and $x = b$, and $h(x)$ is the height of the plate at x. In particular, if the plate lies between the x–axis and a curve $y = f(x)$ from $x = a$ to $x = b$, then

$$x\text{–moment} = \int_a^b x\, f(x)\, dx,$$

Then,

$$\bar{x} = \frac{x\text{–moment}}{\text{area}}.$$

The formula given above corresponds to slicing the plane vertically. Sometimes you get a better integral to evaluate if you slice horizontally intead. Let's say the plate lies between curves $x = a(y)$ and $x = b(y)$ from $y = c$ to $y = d$. Then

$$x\text{–moment} = \int_c^d \frac{1}{2}(a(y) + b(y))(b(y) - a(y))\, dy.$$

This formula is explained as follows: Slice the plate horizontally. A slice between y and $y + \Delta y$ is approximately a rectangle reaching from $x = a(y)$ to $x = b(y)$ with height Δy. The area of the slice is $\Delta A = (b(y) - a(y))\Delta y$. The CM of the slice is at $x = (1/2)(b(y) + a(y))$. The slice contributes to the x–moment as if its entire mass were at the CM, so its contribution is

$$(1/2)(b(y) + a(y))\Delta A = (1/2)(b(y) + a(y))(b(y) - a(y))\Delta y.$$

For example, if the plate lies between the y–axis ($x = 0$) and the curve $x = b(y)$, then

$$x\text{–moment} = \int_c^d \frac{1}{2} b(y)^2\, dy.$$

For the y–moment and $\bar{y}$, you have to reverse the roles of x and y.

(5) A probability density function $f(x)$ must satisfy $f(x) \geq 0$ and

$$\int_{-\infty}^{\infty} f(x) dx.$$

The probability assigned to an interval $[a, b]$ is then

$$\int_a^b f(x) dx.$$

The average or mean of the probability distribution is

$$\int_{-\infty}^{\infty} x f(x) dx.$$

(6) The general solution for a separable differential equation

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

is

$$\int h(y)\,dy = \int g(x)\,dx + c.$$

In particular the differential equation for exponential growth or decay

$$\frac{dy}{dx} = ky$$

is of this sort, and yields the solution $y = Ke^{kt}$.

2. Some formulas I will give you for the exam.

(1) All the formulas that I gave you for exam 1.

(2) The logistic DE is

$$\frac{dP}{dt} = kP(1 - \frac{P}{K}).$$

This is separable, with solution:

$$P(t) = \frac{K}{1 + Ae^{-kt}},$$

where $A = \frac{K-P_0}{P_0}$ and P_0 is the initial population.

(3) A linear differential equation is one of the form:

$$\frac{dy}{dx} + P(x)y = Q(x).$$

The idea for finding a solution is to multiply both sides by an *integrating factor* $I(x)$ chose so that the left side become a perfect derivative:

$$I(x)(\frac{dy}{dx} + P(x)y) = \frac{d}{dx}(I(x)y).$$

This give a DE for $I(x)$, namely

$$I'(x) = I(x)P(x).$$

The general solution to this is

$$I(x) = A\exp(\int P(x)dx).$$

Once you know $I(x)$, the solution to the original DE is

$$I(x)y = \int I(x)Q(x)dx + C.$$

3. Some review questions

(1) What is a direction field? What does it have to do with a differential equation? How is a solution to a DE related to the direction field?
(2) What is an initial value problem? Do differential equations usually have unique solutions? Do initial value problems usually have unique solutions?

4. Some practice exercises

For practice exercises, please refer to the chapter reviews and to examples in the textbook.