Mathematics 121 Final Review
May, 2006

In this review, references to the text are to the *latest posted versions.* This does not mean that you have to print out the latest versions. See the review for the midterm for material on modules and linear algebra.

Definitions and theorem statements you should know:

1. Conditions for a field extension: finite, normal, Galois.
2. Algebraic element in a field extension, algebraic field extension. Transcendental element in a field extension.
3. Separable polynomial, separable element in a field extension, separable field extension.

Theorems you should be able to prove:

1. Multiplicativity of dimensions of field extensions.
2. "Algebraic over algebraic is algebraic." (Prop. 8.1.1)
3. A field extension is finite if and only if it is algebraic and finitely generated.
4. The composite of algebraic extensions is algebraic (Exercises 8.1.1-8.1.3).
5. Existence of an extension field in which a given polynomial has a root. Existence and uniqueness of splitting fields.
6. $\dim_K(K(\alpha))$ = degree of minimal polynomial for α in $K[x]$.
7. The Galois group of a polynomial acts faithfully on the set of roots of the polynomial in a splitting field. The action is transitive on the roots of each irreducible factor of the polynomial.
8. If L is the splitting field of a separable polynomial $f(x) \in K[x]$, then $\text{Fix}(\text{Aut}_K(L)) = K$ (Theorem 8.4.12).
9. If $K \subseteq L$ is a finite field extension and $\text{Fix}(\text{Aut}_K(L)) = K$, then $K \subseteq L$ is normal and separable, and is the splitting field of a separable polynomial in $K[x]$. (Converse to previous result.)
10. "Artin's Lemma".
11. Use Artin's Lemma to show: if $K \subseteq L$ is any finite field extension, then $|\text{Aut}_K(L)| \leq \dim_K(L)$ ("Proposition A")
12. I will not ask you to prove the following (Proposition B): If L is a field, $G \leq \text{Aut}(L)$ is a finite subgroup, and $F = \text{Fix}(G)$, then $\dim_F(L) \leq |G|$.
13. Use Propositions A and B to show: If $K \subseteq L$ is a Galois extension, then $\dim_K(L) = |\text{Aut}_K(L)|$.
14. Use Propositions A and B to show: If L is a field, $G \leq \text{Aut}(L)$ is a finite subgroup, and $F = \text{Fix}(G)$, then $F \subseteq L$ is a finite Galois extension, $G = \text{Aut}_F(L)$, and $\dim_F(L) = |G|$.
15. State the fundamental theorem of Galois theory. Prove it, quoting previous results as needed.
16. The Galois group of the general polynomial of degree n is S_n.

Be able to compute:

1. The Galois group of a cubic polynomial over $\mathbb{Q}$ (assuming I compute the discriminant for you).
2. Inverse of an element in $K[x]/(p(x))$, where $p(x)$ is an irreducible polynomial of low degree.