Mathematics 121 Final Exam – Fred Goodman
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Version A

Do all problems. Responses will be judged for accuracy, clarity and coherence. This exam has 5 questions and 2 pages.

1. Define the following:

(a) The ascending chain condition for ideals in a ring.
(b) A Euclidean domain.
(c) The dual basis to a basis of a finite dimensional vector space.
(d) The matrix of a linear transformation $T : V \longrightarrow V$ with respect to a (single) basis of the finite dimensional vector space V.
(e) A separable field extension $K \subseteq L$.
(f) A normal field extension $K \subseteq L$.

2. Consider the following conditions on a nonzero, nonunit element p in a **principal ideal domain** R:

- pR is a maximal ideal.
- pR is a prime ideal.
- p is prime.
- p is irreducible.

What implications hold among these conditions? Prove sufficiently many implications so that all valid implications are entailed. (If you show that A implies B and that B implies C, you don't have to tell me, or to prove, that A implies C.)

3. Consider the matrix

$$A = \begin{bmatrix} -1 & 0 & 0 & -3 & 3 \\ 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & -2 & 4 \end{bmatrix}$$

The Smith Normal Form of $x - A$ is

$$D(x) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 + x & 0 \\ 0 & 0 & 0 & 0 & (2 - 3x + x^2)^2 \end{bmatrix}$$

The last diagonal entry of $D(x)$ expands to

$$4 - 12x + 13x^2 - 6x^3 + x^4 = (-2 + x)^2(-1 + x)^2$$

(a) Determine the minimal and characteristic polynomials of A.
(b) Write down the Jordan Canonical Form and the Rational Canonical Form of A.

(c) Find a matrix S such that $S^{-1}AS$ is in rational canonical form. The following information is useful for this: One has

$$x - A = P(x)D(x)Q(x),$$

where $P(x)$ and $Q(x)$ are invertible 5–by–5 matrices with entries in $\mathbb{Q}[x]$. The matrix $P(x)^{-1}$ is

$$P(x)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & 0 & 0 & 0 \\ 0 & 0 & 2-x & -1+x & -1 \\ \frac{1}{3}(-1+x) & -1+x & -1+x & 1-x & 1 \\ \frac{2}{3} & -2+x & \frac{1}{3}(-2+x)(2+x) & -\frac{1}{3}(-2+x)(2+x) & \frac{1}{3}(1+x) \end{bmatrix}$$

The following matrices might also be useful to you:

$$A^2 = \begin{bmatrix} -5 & 0 & 0 & -6 & 9 \\ 1 & 4 & 0 & -6 & 3 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -6 & 0 & 0 & -4 & 10 \end{bmatrix}, \qquad A^3 = \begin{bmatrix} -13 & 0 & 0 & -9 & 21 \\ -3 & 8 & 0 & -19 & 15 \\ 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -14 & 0 & 0 & -6 & 22 \end{bmatrix}$$

$$A - 2 = \begin{bmatrix} -3 & 0 & 0 & -3 & 3 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -2 & 0 & 0 & -2 & 2 \end{bmatrix} \qquad (A-2)^2 = \begin{bmatrix} 3 & 0 & 0 & 6 & -3 \\ -3 & 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 4 & -2 \end{bmatrix}$$

$$(A - 1) = \begin{bmatrix} -2 & 0 & 0 & -3 & 3 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & -2 & 3 \end{bmatrix} \qquad (A-1)^2 = \begin{bmatrix} -2 & 0 & 0 & 0 & 3 \\ -1 & 1 & 0 & -4 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 3 \end{bmatrix}$$

4. Prove Artin's lemma: any (finite) set of field automorphisms of a field L is linearly independent (regarded as a subset of the L–vector space of L–valued functions on L).

5. State the fundamental theorem of Galois theory.