Module Structure of the Space of Holomorphic Polydifferentials

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Outline

Setting

k algebraically closed field (usually assume $\mathrm{char}(k) = p$)

X smooth projective curve over k

G finite group acting on X

Ω_X sheaf of relative differentials of X over k

For integer $m \geq 1$, $\Omega_X^{\otimes m} = \underbrace{\Omega_X \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \Omega_X}_{m \text{ times}}$

General Problem

Definition

Define the *space of holomorphic m-polydifferentials of* X *over* k to be the global sections of the sheaf $\Omega_X^{\otimes m}$.

Remarks:

- Zeroth cohomology gives global sections, denote by $H^0(X, \Omega_X^{\otimes m})$
- $\Omega_X^{\otimes m}$ is G-equivariant $\implies H^0(X, \Omega_X^{\otimes m})$ is a representation of G
- $\dim_k H^0(X, \Omega_X^{\otimes m}) = \begin{cases} g(X) & \text{if } m = 1 \\ (2m-1)(g(X)-1) & \text{otherwise} \end{cases}$
- $\Omega_X^{\otimes m} \cong \mathcal{O}_X(mK_X)$, where K_X is a canonical divisor on X
- If $m = 1$, refer to $H^0(X, \Omega_X)$ as the space of holomorphic differentials

General Problem

Question (Hecke, 1928): How does $H^0(X, \Omega_X^{\otimes m})$ decompose into a direct sum of indecomposable representations of G?

Solved if $\mathrm{char}(k) = 0$ (Chevalley and Weil, 1934)

Assume that $\mathrm{char}(k) = p$

Can vary:

- Value of m
- Ramification of the cover $X \to X/G$
- Type of group G

Previous Work

Tamagawa (1951), unramified cover $X \to X/G$, G cyclic

Nakajima (1976), tamely ramified cover $X \to X/G$

Bleher, Chinburg, and Kontogeorgis (preprint, 2017), $m = 1$, G has cyclic Sylow p-subgroups

Karanikolopoulos (2012), $m > 1$, G cyclic p-group

Result

Theorem

Let k be a perfect field of prime characteristic p and let G be finite group acting on a curve X over k. Assume that G has cyclic Sylow p-subgroups. For $m > 1$, the module structure of $H^0(X, \Omega_X^{\otimes m})$ is determined by the inertia groups of closed points $x \in X$ and their fundamental characters.

Assume k is algebraically closed

Conlon induction theorem $\Longrightarrow$ assume that $G = P \rtimes C$, P cyclic p-group, C cyclic group with $p \nmid |C|$

Representation Theory

$G = P \rtimes C$, k field of characteristic p

$|P| = p^n$, $|C| = c$

Representation theory of G over k is well known

Simple kG-modules are the simple kC-modules

There are $c \cdot p^n$ isomorphism classes of indecomposable representations of G, all uniserial

Determined by socle and dimension

Technique

Galois cover of curves $X \to X/G$

$$X \xrightarrow{\text{Wild}} Y \xrightarrow{\text{Tame}} X/G$$

$Y = X/Q$, $Q = \langle \sigma \rangle$, subgroup of P generated by Sylow p-subgroups of inertia groups

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Define $M^{(j)} =$ Kernel of action of $(\sigma - 1)^j$ on M

Understand

$$(H^0(X, \Omega_X^{\otimes m}))^{(j+1)}/(H^0(X, \Omega_X^{\otimes m}))^{(j)}$$

as $k[G/Q]$-modules

Wild Cover

$\pi : X \to Y$

Get effective divisor D_j on Y so that

$$\pi_*\Omega_X^{\otimes m,(j+1)}/\pi_*\Omega_X^{\otimes m,(j)} \cong \mathcal{O}_Y(D_j) \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes m}$$

Recall Riemann-Hurwitz formula

$$\pi_*\Omega_X = \pi_*\mathcal{D}_{X/Y}^{-1} \otimes_{\mathcal{O}_Y} \Omega_Y$$

Compare

$$(H^0(X, \Omega_X^{\otimes m}))^{(j+1)}/(H^0(X, \Omega_X^{\otimes m}))^{(j)}$$

and

$$H^0(Y, \pi_*\Omega_X^{\otimes m,(j+1)}/\pi_*\Omega_X^{\otimes m,(j)})$$

Quotients

Get injective map

$$(H^0(X,\Omega_X^{\otimes m}))^{(j+1)}/(H^0(X,\Omega_X^{\otimes m}))^{(j)} \hookrightarrow H^0(Y,\pi_*\Omega_X^{\otimes m,(j+1)}/\pi_*\Omega_X^{\otimes m,(j)})$$

Riemann-Roch Theorem $\Longrightarrow$ dimensions agree

$$(H^0(X,\Omega_X^{\otimes m}))^{(j+1)}/(H^0(X,\Omega_X^{\otimes m}))^{(j)} \cong H^0(Y,\pi_*\Omega_X^{\otimes m,(j+1)}/\pi_*\Omega_X^{\otimes m,(j)})$$

Understand

$$H^0(Y,\mathcal{O}_Y(D_j)\otimes_{\mathcal{O}_Y}\Omega_Y^{\otimes m})$$

as a $k[G/Q]$-module

Tame Cover

$Y \to X/G$ tamely ramified cover with Galois group G/Q

$\mathcal{O}_Y(D_j) \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes m} \cong \mathcal{O}_Y(D_j + mK_Y)$

Riemann-Roch Theorem $\Longrightarrow H^1(Y, \mathcal{O}_Y(D_j) \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes m}) = 0$

Nakajima (1986) $\Longrightarrow H^0(Y, \mathcal{O}_Y(D_j) \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes m})$ projective $k[G/Q]$-module, gives formula for Brauer character

Building $H^0(X, \Omega_X^{\otimes m})$

Know $k[G/Q]$-module structure of

$$(H^0(X, \Omega_X^{\otimes m}))^{(j+1)}/(H^0(X, \Omega_X^{\otimes m}))^{(j)}$$

All indecomposable kG-module are uniserial $\Longrightarrow$ get kG-module decomposition of $H^0(X, \Omega_X^{\otimes m})$

Modular Curves

$\ell \neq p$ prime, $X(\ell)$ modular curve of level ℓ, k algebraically closed, $\mathrm{char}(k) = p$

Get smooth projective model X of $X(\ell)$ over k

$G = PSL(2, \mathbb{F}_\ell)$ acts on X

$H^0(X, \Omega_X^{\otimes m})$ gives space of weight $2m$ holomorphic cusp forms

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For $p = 3$, proof of theorem gives method for determining the decomposition of $H^0(X, \Omega_X^{\otimes m})$ as a direct sum of indecomposable kG-modules

Uses Green correspondence, known structure of G, and known ramification of $X \to X/G$

Modular Curves, $p = 3$

The decomposition of $H^0(X, \Omega_X^{\otimes m})$ depends on $m \mod 6$

If $m \equiv 2 \mod 3$, then $H^0(X, \Omega_X^{\otimes m})$ is projective

Verifies result of Köck (2004) for weakly ramified covers

Modular Curves, $p = 3$, $m \equiv 0 \mod 3$

Write $\ell + 1 = 3^n \cdot 2 \cdot m''$
Simple kG-modules are T_0, $\widetilde{T}_t$ $(0 \le t \le (m''-1)/2)$, γ_1, γ_2, η^G

$$H^0(X, \Omega_X^{\otimes m}) = \left(\frac{m-a}{6} + c_m\right) P(T_0) \oplus \bigoplus_{t=0}^{(m''-1)/2} \frac{(2m-1)\ell + 5 - 14m}{12} P(\widetilde{T}_t) \oplus \langle \gamma_1, \beta\rangle P(\gamma_1) \oplus \langle \gamma_2, \beta\rangle P(\gamma_2)$$

$$\oplus \bigoplus_{\eta} \langle \eta^G, \beta\rangle P(\eta^G) \oplus i_m Str(n) \oplus (1 - i_m) U_{0,3^n-1} \oplus \bigoplus_{t=1}^{(m''-1)/2} U_{t,2\cdot 3^n-1}$$

where

$$m \equiv a \mod 6, \qquad \delta_m = \begin{cases} 1 & \text{if } m \equiv 0 \mod 6 \\ -1 & \text{if } m \equiv 3 \mod 6 \end{cases},$$

$$c_m = \begin{cases} -1 & \text{if } m \equiv 0 \mod 6 \\ 0 & \text{if } m \equiv 3 \mod 6 \end{cases}, \qquad i_m = \begin{cases} 1 & \text{if } m \equiv 0 \mod 6 \\ 0 & \text{if } m \equiv 3 \mod 6 \end{cases},$$

$$\langle \gamma_1, \beta\rangle = \begin{cases} \frac{(2m-1)\ell - 19 + \delta_m 12 - 10m}{24} & \text{if } \ell \equiv 1 \mod 8 \\ \frac{(2m-1)\ell - 19 - 10m}{24} & \text{if } \ell \equiv 5 \mod 8 \end{cases}$$

$$\langle \gamma_2, \beta\rangle = \begin{cases} \frac{(2m-1)\ell + 17 - 10m}{24} & \text{if } \ell \equiv 1 \mod 8 \\ \frac{(2m-1)\ell + 17 - \delta_m 12 - 10m}{24} & \text{if } \ell \equiv 5 \mod 8 \end{cases}$$

$$\langle \eta^G, \beta\rangle = \begin{cases} \frac{(2m-1)\ell - 1 - \delta_m 6 - 10m}{12} & \text{if } \eta(s) = -1 \\ \frac{(2m-1)\ell - 1 + \delta_m 6 - 10m}{12} & \text{if } \eta(s) = 1 \end{cases}$$

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