Global Synchronization Analysis of Non-Diffusive Networks

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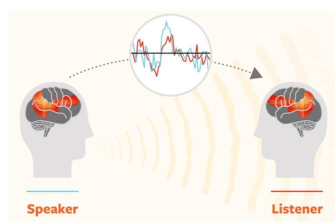
Motivation and Objectives

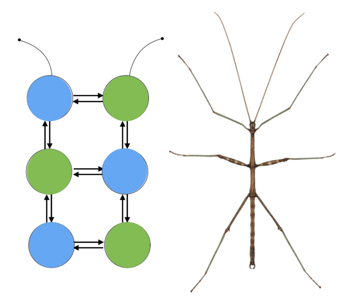
Synchronization of coupled dynamical systems is a widespread phenomenon in both biological and engineered networks, and understanding this behavior is crucial for controlling such systems. Our objective is to establish conditions that ensure the existence of complete synchronization solutions and determine their global stability in non diffusive coupled systems.

Synchronization in nature

Flashing fireflies



Brain to brain sync
(Hasson Lab, Princeton)



Animal locomotion

Network Model

Consider a connected network with N systems described by:

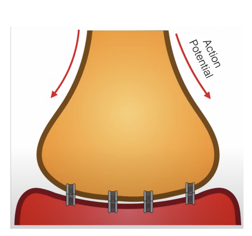
$$\dot{X}_i = F(X_i) + \sigma \sum_{j=1}^{N} \alpha_{ij} H(X_i, X_j) \qquad (\star)$$

- $X_i : [0,\infty) \to \mathbb{R}^n$: State of node i
- $F : \mathbb{R}^n \to \mathbb{R}^n$: Intrinsic dynamics of each node
- $\sigma \in \mathbb{R}_{\geq 0}$: Coupling strength
- $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$: Coupling function
- $[\alpha_{ij}]$: Connectivity matrix, $\alpha_{ij} > 0$ if j is connected to i, zero otherwise

Diffusive and Non-Diffusive Coupling

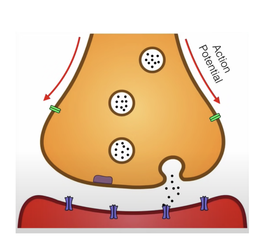
Diffusively Coupled

- $H(X_j, X_i) = 0$ when $X_j = X_i$
- Electrical gap junction: $V_j - V_i$



Non-Diffusively Coupled

- $H(X_j, X_i) \neq 0$ when $X_j = X_i$
- Chemical synaptic coupling: $S(V_j)(V_s - V_i)$



Diffusive coupling is well studied in the literature (e.g [1])

We are interested in non-diffusive coupling in this work.

Existence of a Synchronous Solution

A necessary condition for the existence of a synchronous solution is **input balance condition**, that is, $\exists\, k$ s.t. $\forall i,\ \sum_j \alpha_{ij} = k$

Algebraic Connectivity

The algebraic connectivity of a graph reflects how well connected the graph is. Let $\mathcal{L} = \text{diag}\{d_1, \dots, d_N\} - [a_{ij}]$ where $d_i = \sum_j a_{ij}$ be the Laplacian matrix with eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$

- For a connected undirected graph, the algebraic connectivity equals to $\lambda_2 \neq 0$
- For a digraph, the algebraic connectivity $\mathbf{a}(\mathcal{L})$ is defined as [4]:

$$\mathbf{a}(\mathcal{L}) = \min_{X \perp \mathbf{1}_N, X^\top X = 1} X^\top \mathcal{L} X.$$

Contraction Theory and Logarithmic Norm

- $\dot{x} = F(x)$ is *contractive*, if any two trajectories x & y converge to each other exponentially:
$$\|x(t) - y(t)\| \leq e^{\mu t}\|x(0) - y(0)\|, \quad \mu < 0$$
- We leverage Logarithmic norms to identify contractive systems.
- *Logarithmic norm* $\mu_{\mathcal{X}}[\cdot]$ induced by $\|\cdot\|_{\mathcal{X}}$ is defined as:
$$\mu_{\mathcal{X}}[A] = \lim_{h \to 0^+} \frac{1}{h}\left(\|I + hA\|_{\mathcal{X}\to\mathcal{X}} - 1\right)$$
- $\|x\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}}$ and $\mu_2[A] = \lambda_{\max}\left(\frac{1}{2}(A + A^\top)\right)$
- $\mu_{2,P}[A] = \mu_2[PAP^{-1}]$ is log norm induced by a $P-$ weighted L^2 norm

Main Result: Sufficient condition for Synchronization

Theorem: Consider $(\star)$ and assume

- Excitatory coupling: $D_2 H = \frac{\partial H}{\partial Y}(X,Y)$ is non-negative and diagonal
- $\exists P$ s.t. $PD_2H(X,X) + D_2H(X,X)P \succ 0$
- $c = \sup_X \mu_{2,\sqrt{P}}[DF(X) + \sigma k(D_1H + D_2H)(X,X) - \sigma \mathbf{a}(\mathcal{L}) D_2 H(X,X)]$

Then for any solution X, there exists a solution X_s such that

$$\|X(t) - \bar{X}(t)\|_{2,\sqrt{P}} \leq e^{ct}\|X(0) - \bar{X}(0)\|_{2,\sqrt{P}}$$

where $\bar{X} = (X_s, \dots, X_s)^\top$ is a synchronization solution.

Condition for global stability of $\bar{X}$: $c < 0$

Application 1: Networks of Hindmarsh-Rose (HR) Models

Consider a network of non-diffusively coupled HR oscillators [3]

$$\dot{v}_i = \alpha v_i^2 - v_i^3 - w_i - n_i + \sigma \sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{N}} \alpha_{\mathbf{ij}}(\mathbf{v_s} - \mathbf{v_i})\mathbf{\Gamma}(\mathbf{v_j})$$
$$\dot{w}_i = \beta v_i^2 - w_i$$
$$\dot{n}_i = \epsilon(\gamma v_i + \delta - n_i)$$

Corollary 1. Any HR networks completely synchronize if $\sigma > \dfrac{\bar{\mathcal{M}}}{k}$, where

$$\bar{\mathcal{M}} = \sup_v \frac{2\alpha v - 3v^2 + (\beta p v - \frac{1}{2p})^2}{\Gamma(v) - (v_s - v)\Gamma'(v)} \quad \& \quad \Gamma(v) = \frac{1}{1 + \exp(-\frac{2}{3}(v - \Theta_s))}$$

Numerical Simulations

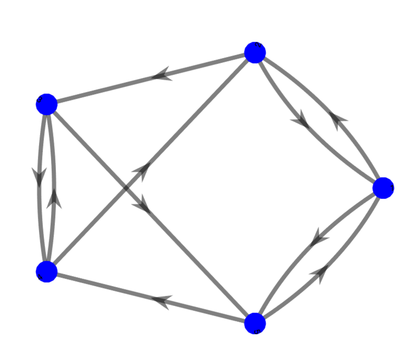


Figure 1. A digraph of five nodes with $k = 2$ and $\mathbf{a}_1(\mathcal{L}) = 1.382$

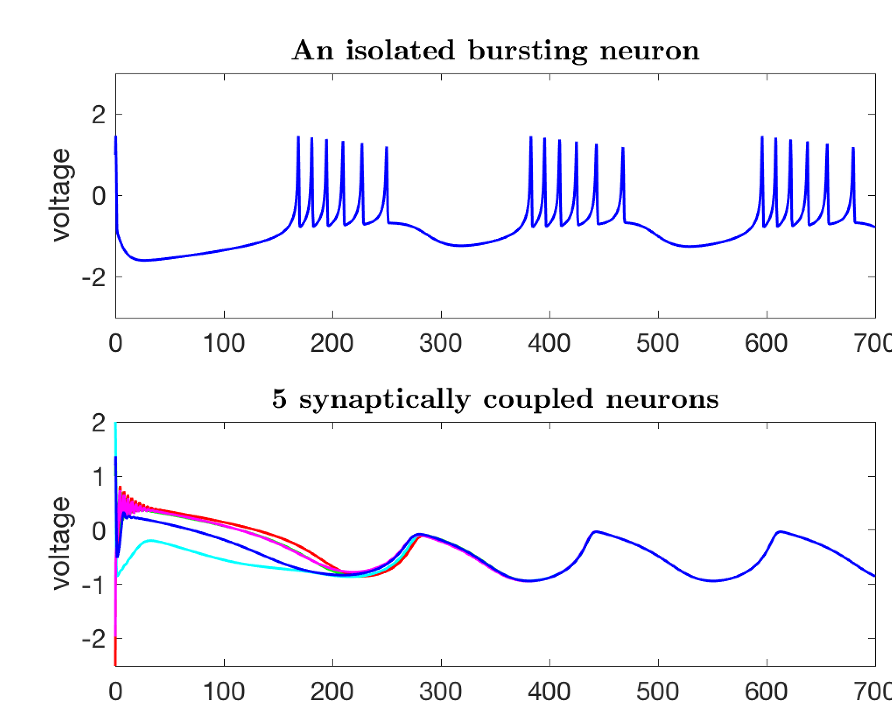


Figure 2. (Top) Isolated HR (Bottom) Five synchronized HRs with $\sigma > 0.704$ that are coupled via the digraph on the left. Parameters: $\alpha = 2.8,\ \beta = 4.4,\ \delta = 8,\ \gamma = 9,\ \epsilon = 1.6,\ v_s = 1, \Theta_s = -2.$

Application 2: Networks of FitzHugh-Nagumo (FN) Models

Consider a network of non-diffusively coupled FN oscillators

$$\dot{v}_i = v_i - v_i^3/3 - a - w_i + I + \sigma \sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{N}} \alpha_{\mathbf{ij}}(\mathbf{v_s} - \mathbf{v_i})\mathbf{\Gamma}(\mathbf{v_j})$$
$$\dot{w}_i = \epsilon(v_i - b w_i)$$

Corollary 2. Any FN networks completely synchronize if $\sigma > \dfrac{\bar{\mathcal{M}}}{k}$, where

$$\bar{\mathcal{M}} = \sup_v \frac{1 - v^2}{\Gamma(v) - (V_s - v)\Gamma'(v)} \quad \& \quad \Gamma(v) = \frac{1}{1 + \beta(1 + \exp(-0.1(v - \Theta_s)))}$$

Numerical Simulations

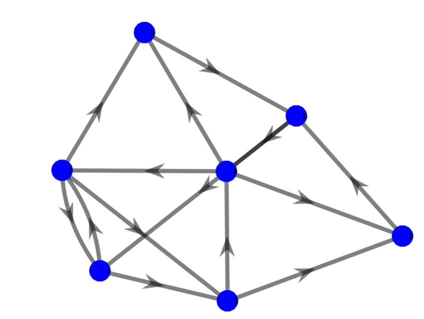


Figure 3. A digraph of five nodes with $k = 2$ and and $\mathbf{a}_1(\mathcal{L}) = 1.3465$

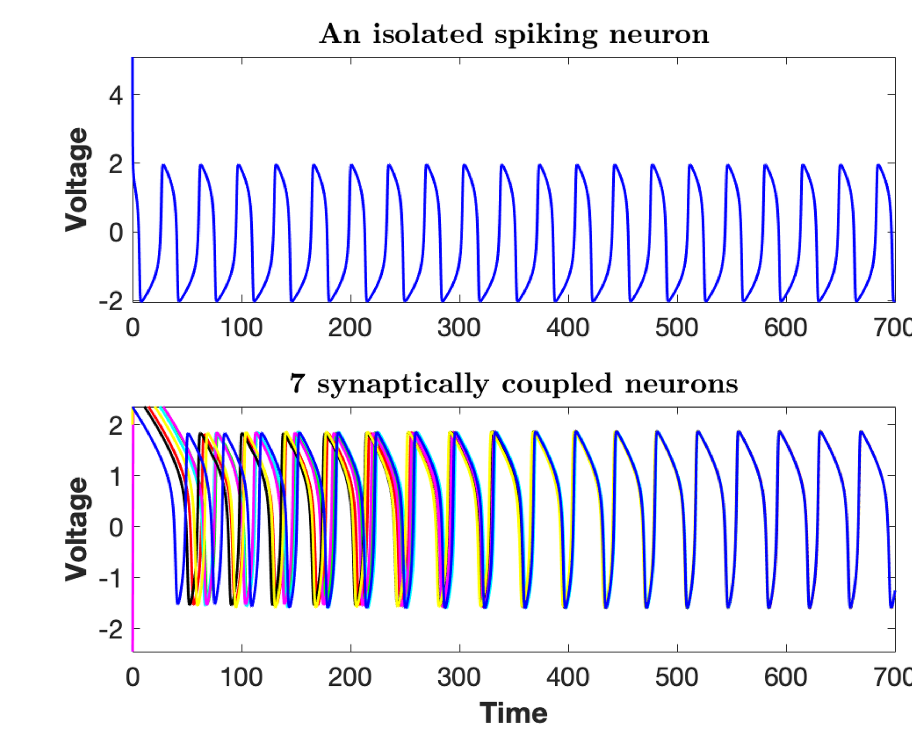


Figure 4. (Top) Single FN (Bottom) Seven synchronized FNs with $\sigma > 0.53$ that are coupled via the digraph on the left. Parametere: $a = 0.5,\ b = 0.1,\ \epsilon = 0.08,\ I = -2,\ v_s = 35,\ \Theta_s = -20,\ \beta = 0.5$

Conclusions

We have expanded the application of contraction theory to determine the global stability of synchronization patterns arising from non-diffusively coupled networks, with a specific focus on neuronal networks. Our analysis provides both a necessary condition for the existence of synchronization solutions and a sufficient condition for their global stability. We applied our theory to two different neuronal models, a network of bursting Hindmarsh-Rose and a network of spiking FitzHugh-Nagumo models. In both cases, the networks are formed by a synaptic (sigmoidal) coupling. We have shown that the bound for the coupling strength is inversely proportional to the digraphs' in-degree, which is consistent with results obtained in [2], proved for graphs with normal Laplacian matrices.

References

[1] Zahra Aminzare, Biswadip Dey, Elizabeth N. Davison, and Naomi Ehrich Leonard. Cluster synchronization of diffusively coupled nonlinear systems: A contraction-based approach. *Journal of Nonlinear Science*, 30(5):1–23, 2018.

[2] Igor Belykh and Martin Hasler. *Patterns of Synchrony in Neuronal Networks: The Role of Synaptic Inputs*, pages 1–28. Springer International Publishing, Cham, 2015.

[3] J. L. Hindmarsh and R. M. Rose. A model of neuronal bursting using three coupled first order differential equations. *Proc. R. Soc. B*, 221:87–102, 1984.

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