On Phase Reduction and Time Period of Noisy Oscillators

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Abstract— **We study phase reduction for noisy oscillator models by deriving a reduced order stochastic differential equation describing the phase evolution using the first and second order Phase Response Curves (PRCs). We discuss direct methods and ordinary differential equations for computing these PRCs, and derive approximate first and second moments of the time period of the oscillator models in terms of functions of the PRCs. We illustrate the theoretical results on a noisy Hopf bifurcation normal form, on a noisy Van der Pol oscillator, and on a noisy bursting neuron model.**

I. INTRODUCTION

Oscillator models are fundamental in modeling systems with rhythmic behavior, including systems in ecology, neuroscience, and engineering [1–7]. Phase Response Curves (PRCs) provide fundamental information about how these oscillator models perform in a neighborhood of a stable limit cycle and facilitate a reduction of a high dimensional model to a one-dimensional phase model. Furthermore, when multiple oscillator models interact with each other, such one-dimensional reduced models enable the development of coupled oscillator models that use only the phase information and relative timing of their limit cycles.

PRC theory is typically developed for small deterministic perturbations around the stable limit cycle and in such cases it is sufficient to consider only the first order effects of the perturbation on the limit cycle. In this paper, we consider stochastic perturbations to the limit cycle and develop a stochastic phase reduced model.

The idea of phase reduction goes back at least to [8] and has been expanded and formalized in subsequent works, including [1, 9, 10]. The references [11–13] provide a good tutorial introduction to the topic.

Phase reduction for noisy oscillators has also received some attention. Moehlis [14] considers oscillators under small noise and studies mitigation techniques to reduce the effect of noise on the time periods of the oscillators. The small noise assumption implies that the second order terms in the Ito expansion can be discarded. Here, we relax this assumption and towards this end use the notion of second order phase response curves, which was introduced in [15]. Teramae et al. [16, 17] consider a setup very similar to that studied in the present paper. However, their computations

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rely on the Stratonovich interpretation of stochastic differential equations, which leads to different reduced order models than those derived below using the Ito interpretation.

Bonnin [18] considers a system similar to the one studied here and analyzes the amplitude deviation and phase dynamics. The effect of noise on oscillators has also been analyzed in [3, 4] and references therein. Compared with these works, we provide complementary techniques that illuminate phase reduction from a PRC perspective.

In this paper, we consider stochastic perturbations to the limit cycle and develop a stochastic phase reduced model. Towards this end, we use the notion of second order phase response curves. We show how the reduced system can be used to determine distributions and moments of the noisy oscillators' time periods. We also derive series expansion approximations to the first and second moments of the time periods of the oscillator models. We illustrate the techniques developed with three oscillator models, the noisy Hopf bifurcation normal form, the noisy Van der Pol oscillator and a noisy bursting neuron model.

The remainder of the paper is organized as follows. In Section II, we recall some background on PRCs and phase reduction. In Section III, we derive the phase reduced model for noisy oscillators. In Section IV we develop computational techniques to determine second order PRCs. We discuss computation of distributions and moments of the time period in Section V, and finally conclude in Section VI.

II. BACKGROUND

A. Phase Reduction for Oscillators

In this section we first review the phase reduction technique for oscillators. Consider the autonomous system

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^n, \quad n \geq 2, \tag{1}$$

with an asymptotically stable hyperbolic limit cycle $\boldsymbol{x}^\gamma(t)$ of period T and frequency $\omega = \frac{2\pi}{T}$. The *phase* of an oscillator, denoted by $\theta(t)$, is the time that has elapsed as its state moves around the limit cycle $\boldsymbol{x}^\gamma(t)$, starting from an arbitrary reference point $\bar{\theta}$ on the cycle, called relative phase. The phase, defined by $\theta(t) = \omega t + \bar{\theta} \pmod{2\pi}$, reduces the dynamics of (1) to the following scalar phase equation

$$\dot{\theta}(t) = \omega, \tag{2}$$

where $\dot{\theta}$ denotes $\frac{d\theta}{dt}$. Note that there is a one to one correspondence between phase θ and each point $\boldsymbol{x}$ on the limit cycle $\boldsymbol{x}^\gamma(t)$. This correspondence defines the following *phase map* [10, 11] on the basin of attraction of $\boldsymbol{x}^\gamma(t)$:

$$\Phi(\boldsymbol{x}(t)) := \theta(t) = \omega t + \bar{\theta}, \quad \boldsymbol{x} \in \boldsymbol{x}^\gamma, \tag{3}$$

with dynamics

$$\dot{\Phi}(\boldsymbol{x}(t)) = \nabla\Phi(\boldsymbol{x})\cdot\dot{\boldsymbol{x}} = \nabla\Phi(\boldsymbol{x})\cdot\boldsymbol{F}(\boldsymbol{x}) = \omega, \quad \boldsymbol{x}\in\boldsymbol{x}^\gamma, \quad (4)$$

where $\cdot$ denotes the inner product.

We now consider the effect of small perturbations to the dynamics of (1), which no longer leave $\boldsymbol{x}^\gamma(t)$ invariant. To this end, we first generalize the definition of the phase map to a neighborhood of $\boldsymbol{x}^\gamma(t)$. Since $\boldsymbol{x}^\gamma(t)$ is asymptotically stable, for any point $\boldsymbol{y}$ in the basin of attraction of $\boldsymbol{x}^\gamma(t)$, there exists an $\boldsymbol{x}\in\boldsymbol{x}^\gamma(t)$ such that as $t\to\infty$, $\|X(t,\boldsymbol{x}) - X(t,\boldsymbol{y})\|\to 0$, where $X(\cdot,\boldsymbol{x})$ is the unique solution of (1) with initial condition $\boldsymbol{x}$, and $\|\cdot\|$ is an arbitrary norm in $\mathbb{R}^n$. The set of all such points $\boldsymbol{y}$ is called the *isochron* of $\boldsymbol{x}$. For any $\boldsymbol{x}\in\boldsymbol{x}^\gamma$, all the points on the isochron of $\boldsymbol{x}$ have the same phase as $\boldsymbol{x}$, i.e., Φ can be extended to the basin of attraction of $\boldsymbol{x}^\gamma$ as follows:

$$\Phi(\boldsymbol{y}) := \Phi(\boldsymbol{x}) = \theta, \quad \forall\boldsymbol{y}\in\text{isochron of }\boldsymbol{x}.$$

Note that the isochron of $\boldsymbol{x}$ is a level set of $\Phi(\boldsymbol{x})$.

Now consider (1) with a small perturbation

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}) + \epsilon\boldsymbol{G}(\boldsymbol{x},\ldots), \quad \boldsymbol{x}\in\mathbb{R}^n, \quad 0 < \epsilon \ll 1. \quad (5)$$

Then, using (4), we have

$$\dot{\Phi}(\boldsymbol{x}(t)) = \nabla\Phi\cdot\dot{\boldsymbol{x}} = \nabla\Phi\cdot(\boldsymbol{F} + \epsilon\boldsymbol{G}) = \omega + \epsilon\,\nabla\Phi\cdot\boldsymbol{G}. \quad (6)$$

Therefore, by definition of the phase map, the dynamics of (5) can be reduced to the following phase equation

$$\dot{\theta} = \omega + \epsilon\,\nabla\Phi\cdot\boldsymbol{G}.$$

The gradient of the phase map, $\nabla\Phi$, which is called *the phase response curve* (PRC) and captures changes in the phase per unit perturbation for small perturbations, plays an important role in reducing the system (5). In what follows we review two important methods to compute PRCs.

B. Computation of Phase Response Curves

The PRC, denoted by $\boldsymbol{Z}(\theta)$, is defined by the gradient of the phase map $\nabla\Phi(\boldsymbol{x})$ at the point on the limit cycle associated with phase θ. The PRC can be computed using a *direct method* in which a perturbation is introduced at each point of the limit cycle and the resulting change in phase is recorded,

$$Z_i(\theta) = \frac{\partial\Phi}{\partial x_i}(\boldsymbol{x}) = \lim_{r\to 0}\frac{1}{r}\left(\Phi(\boldsymbol{x} + r\hat{\boldsymbol{i}}) - \Phi(\boldsymbol{x})\right), \quad \boldsymbol{x}\in\boldsymbol{x}^\gamma, \quad (7)$$

where $\hat{\boldsymbol{i}}$ is the i-th coordinate vector.

Alternatively, the *adjoint method* can be used that solves the following ODE in reverse time [6, 19]

$$\frac{d}{dt}\nabla\Phi(\boldsymbol{x}^\gamma(t)) = -D\boldsymbol{F}^\top(\boldsymbol{x}^\gamma(t))\nabla\Phi(\boldsymbol{x}^\gamma(t)), \quad (8)$$

with constraint

$$\nabla\Phi(\boldsymbol{x}^\gamma(0))\cdot\boldsymbol{F}(\boldsymbol{x}^\gamma(0)) = \omega, \quad (9)$$

where $D\boldsymbol{F}^\top$ denotes the transpose of the Jacobian of $\boldsymbol{F}$. Note that due to the negative sign in front of $-D\boldsymbol{F}^\top$, the stability of the adjoint equation (8) is the opposite of the stability of the limit cycle. Hence, the adjoint equation needs to be solved in reverse time.

III. PHASE REDUCTION FOR NOISY OSCILLATORS

We now focus on the effect of noise in the dynamics of equation (1) on its phase reduction. Consider (1) with additive white noise

$$d\boldsymbol{x} = \boldsymbol{F}(\boldsymbol{x})dt + \sigma B(\boldsymbol{x})d\boldsymbol{W}(t), \quad (10)$$

where $B(\boldsymbol{x})\in\mathbb{R}^{n\times n}$ is the input matrix, $\sigma\ll 1$ is a constant determining the variability of noise, and $d\boldsymbol{W}(t)$ is a standard n-dimensional Weiner process increment.

Before we discuss the phase reduction of noisy oscillators, we first introduce the second order PRC, denoted by $H(\theta)$ and defined by the Hessian of the phase map $\nabla^2\Phi(\boldsymbol{x})$ at the point on the limit cycle associated with phase θ.

*Proposition 1 (**Phase reduction of noisy oscillators**):* For the noisy oscillator (10) with an asymptotically stable limit cycle in the absence of noise, the dynamics of the phase in a neighborhood of the limit cycle is

$$d\theta = \left(\omega + \frac{\sigma^2}{2}\text{tr}\left(B(\boldsymbol{x}^\gamma(t))^\top H(\theta)B(\boldsymbol{x}^\gamma(t))\right)\right)dt + \sigma\boldsymbol{Z}(\theta)^\top B(\boldsymbol{x}^\gamma(t))d\boldsymbol{W}(t), \quad (11)$$

where $\text{tr}(\cdot)$ is the trace operator.

Proof: We apply the Ito formula [20, Theorem 4.16] to the phase map $\Phi(\boldsymbol{x}(t))$ to obtain

$$\begin{aligned}
d\Phi &= \nabla\Phi\cdot\boldsymbol{F}(\boldsymbol{x})dt + \sigma\nabla\Phi\cdot B(\boldsymbol{x})d\boldsymbol{W}(t) \\
&\quad + \frac{1}{2}\left(\boldsymbol{F}(\boldsymbol{x})dt + \sigma B(\boldsymbol{x})d\boldsymbol{W}(t)\right)^\top\nabla^2\Phi(\boldsymbol{x}) \\
&\qquad \times \left(\boldsymbol{F}(\boldsymbol{x})dt + \sigma B(\boldsymbol{x})d\boldsymbol{W}(t)\right) \\
&= \left(\omega + \frac{\sigma^2}{2}\text{trace}\left(B(\boldsymbol{x})^\top\nabla^2\Phi(\boldsymbol{x})B(\boldsymbol{x})\right)\right)dt \\
&\quad + \sigma\nabla\Phi(\boldsymbol{x})\cdot B(\boldsymbol{x})d\boldsymbol{W}(t),
\end{aligned}$$

which yields the desired result similarly to [20, Theorem 4.16]. ∎

Remark 1: If $B(\boldsymbol{x})$ is the identity, (11) reduces to

$$d\theta = \left(\omega + \frac{\sigma^2}{2}\sum_{i=1}^n H_{ii}(\theta)\right)dt + \sigma\boldsymbol{Z}(\theta)^\top d\boldsymbol{W}(t). \quad (12)$$

*Example 1: (**Phase reduction of noisy Hopf bifurcation normal form**.)* We consider the normal form of a supercritical Hopf bifurcation with additive noise:

$$\begin{aligned}
dx_1 &= (\mu x_1 - \omega x_2 - (x_1^2 + x_2^2)x_1)dt + \sigma dW_1(t), \\
dx_2 &= (\omega x_1 + \mu x_2 - (x_1^2 + x_2^2)x_2)dt + \sigma dW_2(t).
\end{aligned} \quad (13)$$

Recall that, without noise, the dynamics of (13) yields a circular limit cycle centered at the origin with radius $\sqrt{\mu}$ and frequency ω. Furthermore, given the phase θ on the limit cycle, the associated point $(x_1^\gamma, x_2^\gamma) = (\sqrt{\mu}\cos(\theta), \sqrt{\mu}\sin(\theta))$. Equivalently, $\theta = \Phi(x_1^\gamma, x_2^\gamma) = \tan^{-1}(x_2^\gamma/x_1^\gamma)$. It follows immediately that

$$\begin{aligned}
\boldsymbol{Z}(\theta) &= \frac{1}{\sqrt{\mu}}\begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix}, \\
H(\theta) &= \frac{1}{\mu}\begin{pmatrix} \sin(2\theta) & -\cos(2\theta) \\ -\cos(2\theta) & -\sin(2\theta) \end{pmatrix}.
\end{aligned} \quad (14)$$

Therefore, using equation (12), the phase reduction for the noisy Hopf bifurcation normal form (13) is

$$dθ = ωdt + \frac{σ}{\sqrt{μ}}dW(t). \tag{15}$$

$\square$

IV. COMPUTATION OF SECOND ORDER PHASE RESPONSE CURVES

Similar to the PRC, the second order PRC, denoted by $H(θ) = ∇^2Φ(\boldsymbol{x})$, can be computed using a *direct method* as follows.

$$H_{ij}(θ) = \frac{∂^2Φ}{∂x_i∂x_j}(\boldsymbol{x}) = \lim_{r→0} \frac{1}{r}\left(\frac{∂Φ}{∂x_j}(\boldsymbol{x} + r\hat{\boldsymbol{i}}) - \frac{∂Φ}{∂x_j}(\boldsymbol{x})\right).$$

Alternatively, the second order PRC solves the following ODE in reverse time

$$\dot{H}(θ) = -∇^2\mathbf{F}(\boldsymbol{Z}(θ) ⊗ \boldsymbol{I}) - D\boldsymbol{F}^⊤H(θ) - H(θ)D\boldsymbol{F}, \tag{16}$$

where all the arguments $(\boldsymbol{x}^γ(t))$ from $∇^2\boldsymbol{F}$ and $D\boldsymbol{F}$ are dropped; $∇^2\boldsymbol{F} = [∇^2\boldsymbol{F}_1 \cdots ∇^2\boldsymbol{F}_n]$ is an $n × n^2$ matrix and represents the Hessian matrix of the vector field $\boldsymbol{F}$, $⊗$ is the Kronecker product and $\boldsymbol{I}$ is the $n × n$ identity matrix. The initial condition is determined by the following constraint

$$\boldsymbol{F}^⊤(\boldsymbol{x}^γ(0))H(θ)\boldsymbol{F}(\boldsymbol{x}^γ(0)) = -\left(\boldsymbol{F}^⊤D\boldsymbol{F}^⊤\right)(\boldsymbol{x}^γ(0))\boldsymbol{Z}(θ). \tag{17}$$

*Proposition 2 (**Computing the second order PRC**):*
Consider system (1) with an asymptotically stable limit cycle $\boldsymbol{x}^γ(t)$ and its corresponding phase map $Φ$. Let $∇^2Φ$ be the Hessian matrix of the phase map $Φ$. Then $∇^2Φ$ solves (16) with constraint (17). $\square$

A proof of Proposition 2 can be found in [15, Section 2].

Remark 2: The following choices of initial conditions for $∇Φ$ and $∇^2Φ$ guarantee the constraints given in (9) and (17), respectively:

$$∇Φ(\boldsymbol{x}^γ(0)) = \frac{ω}{(\boldsymbol{F}^⊤\boldsymbol{F})(\boldsymbol{x}^γ(0))}\boldsymbol{F}(\boldsymbol{x}^γ(0)),$$

$$∇^2Φ(\boldsymbol{x}^γ(0)) = -\frac{ω}{2(\boldsymbol{F}^⊤\boldsymbol{F})(\boldsymbol{x}^γ(0))}(D\boldsymbol{F} + D\boldsymbol{F}^⊤)(\boldsymbol{x}^γ(0)).$$

Remark 3: In what follows, we vectorize [21] equation (16) and combine the corresponding equations of the PRC and the second order PRC. Let $H_v = \text{vec}(∇^2Φ)$ be the vectorization of $∇^2Φ$. Then

$$\frac{d\boldsymbol{Z}}{dt} = -D\boldsymbol{F}^⊤\boldsymbol{Z},$$

$$\begin{aligned}\frac{dH_v}{dt} &= -(\boldsymbol{I} ⊗ D\boldsymbol{F}^⊤ + D\boldsymbol{F}^⊤ ⊗ \boldsymbol{I})H_v \\ &\quad - (\boldsymbol{I} ⊗ ∇^2\boldsymbol{F})\text{vec}(\boldsymbol{Z} ⊗ \boldsymbol{I}), \\ &= -(\boldsymbol{I} ⊗ D\boldsymbol{F}^⊤ + D\boldsymbol{F}^⊤ ⊗ \boldsymbol{I})H_v \\ &\quad - (\boldsymbol{Z}^⊤ ⊗ \boldsymbol{I} ⊗ \boldsymbol{I})\text{vec}(∇^2\boldsymbol{F}),\end{aligned} \tag{18}$$

with constraints

$$\boldsymbol{F}^⊤(\boldsymbol{x}^γ(0))\boldsymbol{Z}(0) = ω,$$

$$(\boldsymbol{F}^⊤ ⊗ \boldsymbol{F}^⊤)(\boldsymbol{x}^γ(0))H_v(0) = \left(\boldsymbol{F}^⊤D\boldsymbol{F}^⊤\boldsymbol{Z}\right)(\boldsymbol{x}^γ(0)),$$

where the following vectorization equalities are used for arbitrary matrices A, B, and C:

$$\text{vec}(AB) = (\boldsymbol{I} ⊗ A)\text{vec}(B) = (B^⊤ ⊗ \boldsymbol{I})\text{vec}(A),$$
$$\text{vec}(ABC) = (C^⊤ ⊗ A)\text{vec}(B).$$

Here $\boldsymbol{I}$ is an identity matrix of the appropriate size.

Remark 4: Due to the negative sign in the right hand side of (16), or equivalently (18), its stability is the opposite of the stability of the limit cycle. Hence, the equation needs to be solved in reverse time.

*Example 2: (**Hopf bifurcation normal form.**)* For the Hopf bifurcation dynamics (13) with $σ = 0$, it can be verified that the matrix H derived in (14) satisfies (16).

*Example 3: (**Van der Pol oscillator.**)* We now consider the van der Pol oscillator with additive white noise

$$dx_1 = \left(x_1 - \frac{1}{3}x_1^3 - x_2\right)dt + σdW_1(t) \tag{19a}$$

$$dx_2 = x_1dt + σdW_2(t). \tag{19b}$$

Fig. 1 shows the PRC and the second order PRC for dynamics (19) with $σ = 0$. These 2-component PRCs are computed by numerically solving (18) with initial conditions discussed in Remark 2.
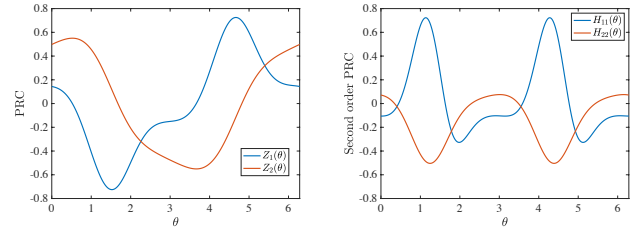

Fig. 1: PRC (left) and 2nd order PRC (right) for the Van der Pol oscillator.

*Example 4: (**Bursting neuron model.**)* This bursting neuron model, containing a system of 3 ODEs, was developed for an insect central pattern generator (CPG) [22, 23]. It describes the dynamics of trans-membrane cell voltages and slow and fast ionic gates, as follows:

$$\begin{aligned}C\dot{v} &= -\{I_{Ca} + I_K + I_{KS} + I_L\} + I_{ext}, \\ \dot{m} &= \frac{γ}{τ_m(v)}[m_∞(v) - m], \\ \dot{w} &= \frac{δ}{τ_w(v)}[w_∞(v) - w],\end{aligned} \tag{20}$$

where the ionic currents are of the following forms

$$\begin{aligned}I_{Ca}(v) &= \bar{g}_{Ca}n_∞(v)(v - E_{Ca}), \\ I_K(v, m) &= \bar{g}_K\, m\, (v - E_K), \\ I_{KS}(v, w) &= \bar{g}_{KS}w\, (v - E_{KS}), \\ I_L(v) &= \bar{g}_L(v - E_L).\end{aligned}$$

The steady state gating variables associated with ion channels

and their time scales take the forms

$$m_\infty(v) = \frac{1}{1 + e^{-2k_K(v - v_K)}},$$

$$w_\infty(v) = \frac{1}{1 + e^{-2k_{KS}(v - v_{KS})}},$$

$$n_\infty(v) = \frac{1}{1 + e^{-2k_{Ca}(v - v_{Ca})}},$$

and

$$\tau_m(v) = \text{sech}(0.5\, k_K(v - v_K)),$$

$$\tau_w(v) = \text{sech}(0.5\, k_K(v - v_{KS})).$$

CPGs are neural networks responsible for rhythmic behaviors such as breathing and walking. Therefore, the model's parameters, specified in Table I, are chosen such that equation (20) possesses an attracting hyperbolic limit cycle. As shown in Fig. 2, the limit cycle contains a burst of spikes, followed by a quiescent phase at low voltage with $m \approx 0$.

I_{ext}	$\bar{g}_{Ca}$	$\bar{g}_K$	$\bar{g}_{KS}$	$\bar{g}_L$	E_{Ca}	E_K	E_{KS}	E_L
35.65	4.4	9.0	0.5	2.0	120	-80	-80	-60

k_{Ca}	k_K	k_{KS}	v_{Ca}	v_K	v_{KS}	C	γ	δ
0.056	0.1	0.8	-1.2	2	-26	1.2	5.0	0.027

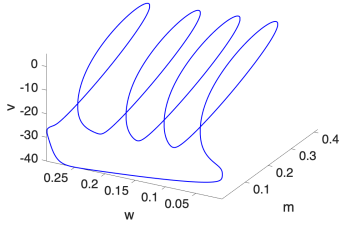TABLE I: The constant parameters in the bursting neuron model.

Fig. 2: A limit cycle of the bursting neuron model (20), in (v, m, w) space. The parameters are specified in Table I.

In [23, 24] a system of 6 weakly coupled bursting neuron models like (20) were employed to study locomotion patterns in insects. Using phase reduction techniques and the Averaging Theorem, the 24 bursting neuron equations were reduced to a system of 6 coupled phase equations in which the PRC played a crucial role (see (6)). To reduce a system of 24 *noisy* bursting neuron equations, one also needs to compute the second order PRC which we will do below. Note that we will postpone the derivation of the 6 coupled phase equations of noisy coupled bursting neuron models to future works.

In Fig. 3, the voltage and the first entries of the PRC and the second order PRC of the bursting neuron model are shown. We show only the first entries because perturbations to a bursting neuron such as equation (20) enter from other neurons via the external current I_{ext}, so only the first entries of the PRC $\boldsymbol{Z}$ and the second-order PRC H appear in the phase equations. These are computed by solving equations (8) and (16) in reverse time with initial conditions given in Remark 2.
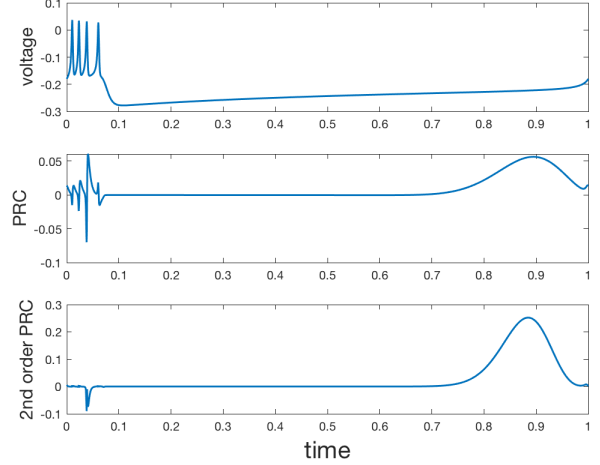

Fig. 3: The voltage and first entry of PRC and second order PRC of the bursting neuron model (20) are shown.

V. TIME-PERIOD OF NOISY OSCILLATORS

In this section, we focus on leveraging the phase reduction of a noisy oscillator towards computation of various statistical properties of the time period of the noisy limit cycle. Towards this end, we recall that one period of the noisy limit cycle can be interpreted as the first passage/hitting time of the phase variable θ evolving according to equation (11) with absorbing boundary at $\theta = 2\pi$ and initial condition at $\theta = 0$. Let the first passage time of the stochastic process $\theta(t)$ with respect to boundary 2π be defined by

$$\mathcal{T}(\theta_0) = \inf\{\tau > 0 \mid \theta(\tau) = 2\pi,\ \theta(0) = \theta_0\}. \quad (21)$$

Then, the time period $T = \mathcal{T}(0)$.

We now discuss computation of the distribution of the time period of a noisy oscillator. For the first passage time defined in (21) and initial condition $\theta(0) = \theta_0$, let $G(\theta_0, t)$ be defined by

$$G(\theta_0, t) = \mathbb{P}(\mathcal{T} \geq t | \theta(0) = \theta_0).$$

For simplicity of exposition, in the following we assume that $B(\boldsymbol{x}) = \text{diag}(\beta_1, \ldots, \beta_n)$. Let

$$\Pi(\theta_0) = \frac{1}{2} \sum_{i=1}^{n} \beta_i^2 H_{ii}(\theta_0),\ \text{and}\ \zeta(\theta_0) = \frac{1}{2} \sum_{i=1}^{n} \beta_i^2 Z_i(\theta_0)^2. \quad (22)$$

It is known [25, Section 5.5.1] that G is the solution to the following Fokker-Planck equation

$$\frac{\partial G}{\partial t} = (\omega + \sigma^2 \Pi(\theta_0)) \frac{\partial G}{\partial \theta_0} + \sigma^2 \zeta(\theta_0) \frac{\partial^2 G}{\partial \theta_0^2}, \quad (23)$$

with initial condition $G(\theta_0, 0) = 1$, for each $\theta_0 \in (-k\pi, 2\pi)$, and boundary conditions $G(2\pi, t) = G(-k\pi, t) = 0$, where $k \to \infty$. Note that two boundary conditions are required to solve the above Fokker-Planck equation, hence an absorbing boundary is assumed at $\theta = -\infty$.

While the distribution of the time period contains all its statistical information, solution of the partial differential

equation (23) can be tedious. Sometimes we are interested only in computation of certain moments of the time period. The moments of the passage/hitting time can be computed by solving an ordinary differential equation (ODE) instead of equation (23). Let $M_n = \mathbb{E}[\mathcal{T}^n]$ be the n-th moment of the time period. Then, it is known [25, Section 5.5.1] that M_n is the solution to the following ODE

$$(\omega + \sigma^2 \Pi(\theta_0)) \frac{dM_n}{d\theta_0} + \sigma^2 \zeta(\theta_0) \frac{d^2 M_n}{d\theta_0^2} = -n M_{n-1}(\theta_0), \quad (24)$$

with boundary conditions $M_n(2\pi) = 0$ and $M_n(-k\pi) = 0$, $k \to \infty$, and using the fact that $M_0 = 1$.

We now compute approximate first and second moments of the time period.

*Proposition 3 (**Approximate Moments of Time Period**):* For the noisy oscillator (10) and the associated reduced model (11) with $B(x) = \text{diag}(\beta_1, \ldots, \beta_n)$, the following statements hold

  i) The mean time period is

$$\mathbb{E}[T] = \frac{2\pi}{\omega} - \frac{\sigma^2}{\omega^2} \int_0^{2\pi} \Pi(\alpha) d\alpha + o(\sigma^2);$$

  ii) the second moment of the time period is

$$\mathbb{E}[T^2] = \frac{4\pi^2}{\omega^2} - \frac{2\sigma^2}{\omega^3} \int_0^{2\pi} (2\pi - \alpha) \Pi(\alpha) d\alpha$$
$$+ \frac{2\sigma^2}{\omega^3} \int_0^{2\pi} \zeta(\alpha) d\alpha - \frac{2\sigma^2}{\omega^3} \int_0^{2\pi} \int_\xi^{2\pi} \Pi(\alpha) d\alpha d\xi + o(\sigma^2).$$

The functions $\Pi(\alpha)$ and $\zeta(\alpha)$ are defined in equation (22). Proposition 3 is proved in the appendix.

*Example 5: (**Time period of the noisy Hopf bifurcation normal form**.*) Recall that the phase reduction of a noisy Hopf bifurcation normal form (13) is a diffusion equation with constant drift and diffusion terms (15). The first hitting time properties of this equation are well studied [25, 26], and its time-period satisfies a Wald distribution.

$$p(t) = \sqrt{\frac{2\pi\mu}{\sigma^2 t^3}} \exp\left(-\frac{\mu(2\pi - \omega t)^2}{2\sigma^2 t}\right).$$

The mean and variance of the time period are

$$\mathbb{E}[T] = \frac{2\pi}{\omega}, \qquad \text{Var}[T] = \frac{2\pi\sigma^2}{\omega^3 \mu}.$$

Remark 5: The first hitting time (21) of equation (15) with respect to a single fixed threshold has been successfully used to model interval timing behavior [27], which studies how humans perceive time-interval lengths. The connection with noisy oscillators provides a normative account to such models. Similar investigations with bursting neurons could produce more biophysically-grounded models. We intend to investigate this in future work.

*Example 6: (**Time period of noisy Van der Pol oscillator**.*) We now study the time-period of the noisy Van der Pol oscillator (19). In Fig. 4, we compare the above analytic approximations to the mean and second moment of the time period derived via Proposition 3 with Monte Carlo simulations performed on the dynamics of equation (19) and averaged over 5000 samples. It can be seen that the
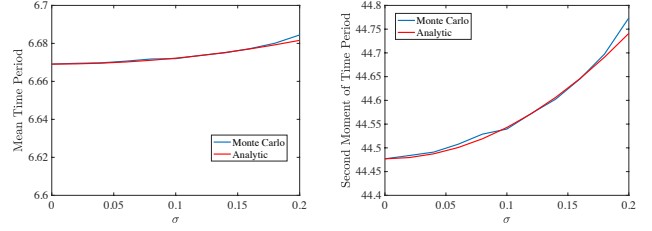

Fig. 4: Comparison of time period statistics obtained using 5000 Monte Carlo simulations of the noisy Van der Pol oscillator with analytic approximations derived in Proposition 3.

approximations in Proposition 3 are fairly accurate for the range of σ shown here.

VI. CONCLUSIONS

We studied phase reduction of noisy oscillator dynamics. We leveraged the first and second order PRCs to derive a scalar stochastic differential equation that describes phase evolution in oscillator models. We discussed the direct method and ODEs for computing the PRCs. We also discussed computation of the distribution and moments of noisy oscillators' time periods, deriving analytic approximations to the first and second moments of these time periods in terms of functions of their PRCs.

Future directions for investigation include design of control protocols to minimize the influence of noise on the oscillator time period statistics. Another interesting direction is to extend the proposed methodology to coupled oscillators and to study the influence of the second order PRC on their synchronization.

APPENDIX

PROOF OF PROPOSITION 3

We start by establishing the first statement. Let $M_1(\theta_0) = T_0(\theta_0) + \sigma T_1(\theta_0) + \sigma^2 T_2(\theta_0) + o(\sigma^2)$. Then, from equation (24), we have

$$(\omega + \sigma^2 \Pi(\theta_0)) \left(\frac{dT_0}{d\theta_0} + \sigma \frac{dT_1}{d\theta_0} + \sigma^2 \frac{dT_2}{d\theta_0} \right) +$$
$$\sigma^2 \zeta(\theta_0) \left(\frac{d^2 T_0}{d\theta_0^2} + \sigma \frac{d^2 T_1}{d\theta_0^2} + \sigma^2 \frac{d^2 T_2}{d\theta_0^2} \right) + o(\sigma^2) = -1.$$

Collecting σ^i terms, for $i \in \{0, 1, 2\}$:

$$\omega \frac{dT_0}{d\theta_0} = -1 \implies T_0 = -\frac{\theta_0}{\omega} + c_1$$
$$\omega \frac{dT_1}{d\theta_0} = 0 \implies \omega \frac{dT_1}{d\theta_0} = 0 \implies T_1 = c_2$$
$$\Pi(\theta_0) \frac{dT_0}{d\theta_0} + \omega \frac{dT_2}{d\theta_0} + \zeta(\theta_0) \frac{d^2 T_0}{d\theta_0^2} = 0$$
$$\implies \omega \frac{dT_2}{d\theta_0} = \frac{\Pi(\theta_0)}{\omega} \implies T_2 = \frac{1}{\omega^2} \int_0^{\theta_0} \Pi(\alpha) d\alpha + c_3.$$

Therefore,

$$M_1(\theta_0) = -\frac{\theta_0}{\omega} + \frac{\sigma^2}{\omega^2} \hat{\Pi}(\theta_0) + c_1 + \sigma c_2 + \sigma^2 c_3 + o(\sigma^2),$$

where $\hat{\Pi}(\theta_0) = \int_0^{\theta_0} \Pi(\alpha)d\alpha$.

Using the boundary condition $M_1(2\pi) = 0$, we get

$$c_1 = \frac{2\pi}{\omega}, \; c_2 = 0, \quad \text{and} \quad c_3 = -\frac{1}{\omega^2}\hat{\Pi}(2\pi).$$

Therefore,

$$M_1(\theta_0) = \frac{2\pi - \theta_0}{\omega} - \frac{\sigma^2}{\omega^2}(\hat{\Pi}(2\pi) - \hat{\Pi}(\theta_0)) + o(\sigma^2)$$
$$= \frac{2\pi - \theta_0}{\omega} - \frac{\sigma^2}{\omega^2}\int_{\theta_0}^{2\pi}\Pi(\alpha)d\alpha + o(\sigma^2).$$

Thus,

$$\mathbb{E}[T] = M_1(0) = \frac{2\pi}{\omega} - \frac{\sigma^2}{\omega^2}\int_0^{2\pi}\Pi(\alpha)d\alpha + o(\sigma^2).$$

We now establish the second statement. Let $M_2(\theta_0) = S_0(\theta_0) + \sigma S_1(\theta_0) + \sigma^2 S_2(\theta_0) + o(\sigma^2)$. Then,

$$(\omega + \sigma^2\Pi(\theta_0))\left(\frac{dS_0}{d\theta_0} + \sigma\frac{dS_1}{d\theta_0} + \sigma^2\frac{dS_2}{d\theta_0}\right) +$$
$$\sigma^2\zeta(\theta_0)\left(\frac{d^2S_0}{d\theta_0^2} + \sigma\frac{d^2S_1}{d\theta_0^2} + \sigma^2\frac{d^2S_2}{d\theta_0^2}\right)$$
$$= -2\left(\frac{2\pi - \theta_0}{\omega} - \frac{\sigma^2}{\omega^2}\int_{\theta_0}^{2\pi}\Pi(\alpha)d\alpha\right) + o(\sigma^2).$$

Following the same steps as in the proof of statement (i), we obtain

$$M_2(\theta_0) = \frac{\theta_0^2}{\omega^2} - \frac{4\pi\theta_0}{\omega^2} + \frac{2\sigma^2}{\omega^3}\hat{\Pi}_1(\theta_0)$$
$$- \frac{2\sigma^2}{\omega^3}\hat{\Pi}_2(\theta_0) + \frac{2\sigma^2}{\omega^3}\hat{\Pi}_3(\theta_0) + K_2(\sigma) + o(\sigma^2),$$

where $\hat{\Pi}_1(\theta_0) = \int_0^{\theta_0}(2\pi - \alpha)\Pi(\alpha)d\alpha$, $\hat{\Pi}_2(\theta_0) = \int_0^{\theta_0}\zeta(\alpha)d\alpha$, $\hat{\Pi}_3(\theta_0) = \int_0^{\theta_0}\int_\xi^{2\pi}\Pi(\alpha)d\alpha d\xi$ and $K_2(\sigma)$ is the integration constant.

Using the boundary condition $M_2(2\pi) = 0$, we get

$$K_2(\sigma) = \frac{4\pi^2}{\omega^2} - \frac{2\sigma^2}{\omega^3}\hat{\Pi}_1(2\pi) + \frac{2\sigma^2}{\omega^3}\hat{\Pi}_2(2\pi) - \frac{2\sigma^2}{\omega^3}\hat{\Pi}_3(2\pi).$$

Therefore,

$$M_2(\theta_0) = \frac{(2\pi - \theta_0)^2}{\omega^2} - \frac{2\sigma^2}{\omega^3}\int_{\theta_0}^{2\pi}(2\pi - \alpha)\Pi(\alpha)d\alpha$$
$$+ \frac{2\sigma^2}{\omega^3}\int_{\theta_0}^{2\pi}\zeta(\alpha)d\alpha - \frac{2\sigma^2}{\omega^3}\int_{\theta_0}^{2\pi}\int_\xi^{2\pi}\Pi(\alpha)d\alpha d\xi + o(\sigma^2).$$

Thus,

$$\mathbb{E}[T^2] = M_2(0) = \frac{4\pi^2}{\omega^2} - \frac{2\sigma^2}{\omega^3}\int_0^{2\pi}(2\pi - \alpha)\Pi(\alpha)d\alpha$$
$$+ \frac{2\sigma^2}{\omega^3}\int_0^{2\pi}\zeta(\alpha)d\alpha - \frac{2\sigma^2}{\omega^3}\int_0^{2\pi}\int_\xi^{2\pi}\Pi(\alpha)d\alpha d\xi + o(\sigma^2).$$

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