SMOOTHING ESTIMATES OF THE RADIAL
SCHRÖDINGER PROPAGATOR IN
DIMENSIONS $n \geq 2$

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Abstract The usual Kato smoothing estimate for the Schrödinger propagator in 1D takes the form
$$\|\partial_x \frac{1}{2} e^{it \Delta} u_0\|_{L^2_x L^2_t} \lesssim \|u_0\|_{L^2_x}.$$ In dimensions $n \geq 2$ the smoothing estimate involves certain localization to cubes in space. In this paper we focus on radial functions and obtain Kato-type sharp smoothing estimates which can be viewed as natural generalizations of the 1D Kato smoothing. These estimates are global in the sense that they do not need localization in space. We also present an interesting counterexample which shows that even though the time-global inhomogeneous Kato smoothing holds true, the corresponding time-local inhomogeneous smoothing estimate cannot hold in general.

Key words Schrödinger equation; smoothing estimate

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1 Introduction

In this paper we consider Kato type smoothing estimates for the Schrödinger propagator $e^{it \Delta}$ in dimensions $n \geq 2$. Kato smoothing estimates play an important role in local theory for semi-linear nonlinear Schrödinger equations. Typically one employs smoothing estimates to overcome the loss of derivatives caused by the semi-linear terms from the nonlinearity. The first such estimate dates back to [3], where Kato studied the Cauchy problem for the (generalized) Korteweg-de Vries equation

$$\begin{cases}
\partial_t u - \partial_{xxx} u + u \partial_x u = 0, & (x, t) \in \mathbb{R} \times \mathbb{R}, \\
u(x, 0) = u_0(x).
\end{cases}$$

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He proved a smoothing estimate for the linear propagator $e^{it\partial_x^2}$ which takes the form
\[
\int_{-T}^{T} \int_{-l}^{l} |(\partial_x u)(x,t)|^2 \, dx \, dt \leq C(T, l, \|u_0\|_{L^2}). \tag{1.1}
\]
Here and below we shall use the notation $C(A, B, \ldots)$ to denote constants which depend on quantities $A, B, \ldots$.

For the Schrödinger propagator $e^{it\Delta}$, the corresponding version of (1.1) was obtained in 
\cite{1}, \cite{8}, \cite{9}, and has the form
\[
\int_{-T}^{T} \int_{-l}^{l} \left| (1 - \partial_x^2)^{\frac{1}{2}} e^{it\partial_x^2} u_0 \right|^2 \, dx \, dt \leq C(T, l, \|u_0\|_{L^2}). \tag{1.2}
\]

The multi-dimensional analogue of (1.2) for the Schrödinger propagator $e^{it\Delta}$ was also obtained in 
\cite{1}, \cite{8}, \cite{9}. To state the results, denote $\{Q_\alpha\}_{\alpha \in \mathbb{Z}^n}$ as the family of non-overlapping cubes of size $l$ such that $\mathbb{R}^n = \bigcup_{\alpha \in \mathbb{Z}^n} Q_\alpha$. Then for $n \geq 2$,
\[
\sup_{\alpha \in \mathbb{Z}^n} \left( \int_{Q_\alpha} \int_{-\infty}^{\infty} \left| \nabla_x e^{it\partial_x^2} u_0 \right| \, dx \, dt \right)^{\frac{1}{2}} \leq C(n, l, \|u_0\|_{L^2}). \tag{1.3}
\]

The sharp versions of (1.2), (1.3) were proven by Kenig, Ponce and Vega \cite{4} and take the following form: for $n = 1$,
\[
\sup_x \left( \int_{-\infty}^{\infty} \left| \partial_x e^{it\partial_x^2} u_0 \right| \, dt \right)^{\frac{1}{2}} \leq C_1 \cdot \|u_0\|_{L^2}, \tag{1.4}
\]
and for $n \geq 2$,
\[
\sup_{\alpha \in \mathbb{Z}^n} \left( \int_{Q_\alpha} \int_{-\infty}^{\infty} \left| \nabla_x e^{it\partial_x^2} u_0 \right| \, dx \, dt \right)^{\frac{1}{2}} \leq C_2 \cdot l \cdot \|u_0\|_{L^2}. \tag{1.5}
\]
Here $C_1$ is an absolute constant and $C_2$ depends only on the dimension $n$. To study the initial value problem for semilinear nonlinear Schrödinger equations, one needs time-inhomogeneous version of (1.4)–(1.5). The following estimates were first proven by Kenig, Ponce and Vega \cite{5}:

for $n \geq 1$,
\[
\sup_x \left( \int_{-\infty}^{\infty} \left| \partial_x \int_{0}^{t} e^{i(t-s)\partial_x^2} F(\cdot, s) \right| \, ds \right)^{\frac{1}{2}} \leq C_3 \|F\|_{L^2_{t,x}(\mathbb{R} \times \mathbb{R}^n)}, \tag{1.6}
\]
and for $n \geq 2$,
\[
\sup_{\alpha \in \mathbb{Z}^n} \left( \int_{Q_\alpha} \int_{-\infty}^{\infty} \left| \nabla_x \int_{0}^{t} e^{i(t-s)\Delta} F(\cdot, s) \, ds \right| \, dx \, dt \right)^{\frac{1}{2}} \leq C_4 \cdot l \cdot \sum_{\alpha \in \mathbb{Z}^n} \left( \int_{Q_\alpha} \int_{-\infty}^{\infty} |F(x, t)|^2 \, dx \, dt \right)^{\frac{1}{2}}. \tag{1.7}
\]
Here $C_3$ is an absolute constant and $C_4$ depends only on the dimension $n$. Note that due the averaging effect in time, the gain of derivative in (1.6)–(1.7) is twice than that of (1.4)–(1.5).

Let $\chi \geq 0$ be a smooth cut-off function so that its Fourier transform $\hat{\chi} \in C_0^\infty(\mathbb{R}^n)$. A slightly variant of (1.5), often referred to as the Kato 1/2-smoothing estimate in $\mathbb{R}^n$, $n \geq 2$ takes the form
\[
\left\| \chi(x)(1 - \Delta)^{\frac{1}{4}} e^{it\partial_x^2} u_0 \right\|_{L^2_{t,x}(\mathbb{R} \times \mathbb{R}^n)} \leq C_5 \cdot \|u_0\|_{L^2_{x}(\mathbb{R}^n)}. \tag{1.8}
\]
Here the constant $C_0$ depends on $\chi$ and $n$. Roughly speaking, the meaning of (1.8) is that if $u_0 \in L^2$, then $|\nabla_x|^\frac{n}{2} e^{it\Delta} u_0 \in L^2_{\text{loc}}(\mathbb{R}^n \times \mathbb{R})$.

The Kato smoothing estimate is sharp. For example, one can consider a Gaussian wave packet with frequency localized to $\xi \approx \xi_0$. The average time that the wave packet spent in an $O(1)$ compact space region is roughly $O(1/|\xi_0|)$. Therefore the $L^2$ norm is $O(|\xi_0|^{-\frac{1}{2}})$ which precisely cancels the $|\xi_0|^{\frac{n}{2}}$ factor induced by the $1/2$-derivative.

The estimates (1.1)–(1.8) are by now very well-known and considered to be standard smoothing estimates. However one should notice some qualitative differences between the one-dimensional Kato smoothing (1.4) to higher dimensions under the radial assumption. Our assumption (1.11) implies (1.9) which naturally motivates us to consider whether there are genuine $L^\infty_t L^2_x$ type estimates in dimensions $n \geq 2$ which are in a sense more natural analogues of (1.4) and (1.6). The purpose of this paper is to answer this question at the expense of radial assumptions. Here are our main results.

**Theorem 1.1** (Kato smoothing for radial functions in dimension $n \geq 2$) Let the dimension $n \geq 2$. Then for any radial function $f$ on $\mathbb{R}^n$ we have

$$\left\| |x|^{-\frac{n-1}{2}} |\nabla| |x|^{-\frac{1}{2}} e^{it\Delta} f \right\|_{L^\infty_t L^2_x(\mathbb{R}^n \times \mathbb{R})} \lesssim \|f\|_{L^2_x L^2_t(\mathbb{R}^n \times \mathbb{R})}. \quad (1.9)$$

**Remark 1.2** One should view Theorem 1.1 as the natural generalization of the usual one-dimensional Kato smoothing (1.4) to higher dimensions under the radial assumption. Our intention here is to obtain global weighted estimates (for positive weights). The usual Kato smoothing in higher dimensions involves localization in space. To have an impression, compare (1.9) with (1.5).

From duality and a $T - T^*$ argument, (1.9) implies

**Theorem 1.3** (Kato smoothing, time-inhomogeneous global version) Let the dimension $n \geq 2$. Then for any radial function $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ we have

$$\left\| |x|^{-\frac{n}{2}} |\nabla| \int_{\mathbb{R}} e^{i(t-s)\Delta} |y|^{-\frac{n}{2}-1} f(s, y) ds \right\|_{L^\infty_t L^2_x(\mathbb{R}^n \times \mathbb{R})} \lesssim \|f\|_{L^2_x L^2_t(\mathbb{R}^n \times \mathbb{R})}. \quad (1.10)$$

Theorem 1.1 and Theorem 1.3 have applications in our recent work [6, 7]. On the other hand, one may wonder the time retarded estimate, where in (1.10) the integrating in time is from 0 to $t$, instead of the whole $\mathbb{R}$, still hold. Note that the retarded estimate is used in the nonlinear problem.

Surprisingly, we shall show that this is not the case. In other words, we shall prove in Section 3 that the following estimate

$$\left\| |x|^{-\frac{n}{2}} |\nabla| \int_0^t e^{i(t-s)\Delta} |y|^{-\frac{n}{2}-1} f(s, y) ds \right\|_{L^\infty_t L^2_x(\mathbb{R}^n \times \mathbb{R})} \lesssim \|f\|_{L^2_x L^2_t(\mathbb{R}^n \times \mathbb{R})} \quad (1.11)$$

fails in general. On the other hand, an $\epsilon$-version of (1.11) holds, namely

$$\left\| |x|^{-\frac{n}{2}} |\nabla| |1-\epsilon| \int_0^t e^{i(t-s)\Delta} |y|^{-\frac{n}{2}-1} f(s, y) ds \right\|_{L^\infty_t L^2_x(\mathbb{R}^n \times \mathbb{R})} \lesssim \|f\|_{L^2_x L^2_t(\mathbb{R}^n \times \mathbb{R})} \quad (1.12)$$

The proof of (1.12) uses a similar analysis used in the disproof of (1.11). We therefore leave it as an exercise to interested readers.

This paper is organized as follows. In Section 2 we give the proofs of Theorem 1.1 and Theorem 1.3. Section 3 is devoted to the disproof of (1.11).
2 Kato Smoothing Estimate for Radial Functions

In this section we give detailed proofs of Theorem 1.1 and Theorem 1.3. As we shall show, the proofs are not surprisingly low-tech since the original Kato smoothing proof is also very simple. We begin with

Proof of Theorem 1.1 By passing to radial coordinates, we can write
\[ |x|^{\frac{n-1}{2}} |\nabla|^{\frac{1}{2}} e^{it\Delta} f = |x|^{\frac{n-1}{2}} \int_0^{\infty} k^{\frac{1}{2}} e^{-ik^2} \hat{f}(k) k^{n-1} \left( \int_{|\omega|=1} e^{ik|x|\omega} d\sigma(\omega) \right) dk, \] (2.1)

where \( \hat{f} \) is the Fourier transform of the function \( f \) and \( d\sigma(\omega) \) is the surface measure on \( S^{n-1} \). Since the Fourier transform of a radial function is still radial, we can slightly abuse the notation \( \hat{f}(k) \) to denote the Fourier transform of \( f \). Now consider the function
\[ h(\rho) := \int_{|\omega|=1} e^{i\rho\omega} d\sigma(\omega). \]

It is clear that
\[ \rho^{\frac{n-2}{2}} h(\rho) = J_{\frac{n-2}{2}}(\rho), \]
where \( J_{\frac{n-2}{2}} \) is the usual Bessel function of order \( \frac{n-2}{2} \). Then by using the asymptotics for Bessel functions, it is not difficult to see that
\[ \sup_{\rho>0} \rho^{\frac{n-1}{2}} |h(\rho)| \leq C_1 < \infty, \] (2.2)

where \( C_1 \) is a constant depending only on the dimension \( n \). By Plancherel, one can show that for any one-dimensional function \( F \), we have
\[ \left\| \int_0^{\infty} e^{-ik^2} F(k) dk \right\|_{L^2_t}^2 = \frac{1}{2} \int_0^{\infty} |F(k)|^2 \frac{dk}{k}. \] (2.3)

Now by (2.1), (2.2), (2.3), we obtain
\[ \left\| |x|^{\frac{n-1}{2}} |\nabla|^{\frac{1}{2}} e^{it\Delta} f \right\|_{L^2_t}^2 = \frac{1}{2} \int_0^{\infty} |\hat{f}(k)|^2 \cdot |x|^{n-1} \cdot k^{2(n-1)} \cdot |h(k|x|)|^2 dk \leq \frac{1}{2} \int_0^{\infty} |\hat{f}(k)|^2 k^{n-1} dk \cdot \left( \sup_{\rho>0} \rho^{\frac{n-1}{2}} |h(\rho)| \right)^2 \leq C_1^2 \|f\|_{L^2}^2. \]

The theorem is proved.

We shall establish the inhomogeneous smoothing estimate similar to Theorem 1.1.

Proof of Theorem 1.3 By Theorem 1.1 and duality, we have
\[ \left\| |\nabla|^{\frac{1}{2}} \int_{\mathbb{R}} e^{-is\Delta} |y|^{\frac{n-1}{2}} f(s, y) ds \right\|_{L^2_t} \lesssim \|f\|_{L^2_{1,1}}. \] (2.4)

Now define the linear operator \( T \) as
\[ (Tf)(t, x) = |x|^{\frac{n-1}{2}} |\nabla| \int_{\mathbb{R}} e^{i(t-s)\Delta} |y|^{\frac{n-1}{2}} f(s, y) ds. \] (2.5)
Then (2.4) implies for any $f$,

$$\langle f, Tf \rangle_{x,t} \lesssim \|f\|_{L^1_t L^2_x}^2.$$  \hfill (2.6)

With (2.6) in hand, we now establish (1.11) by a polarization argument. Namely for any $h, g$ with $\|h\|_{L^1_t L^2_x} = \|g\|_{L^1_t L^2_x} = 1$, substituting $f = h + g$ and $f = h + ig$, we obtain

$$\langle h, Tg \rangle_{x,t} \lesssim 1.$$  \hfill (3.1)

Then for any $\tilde{h}, \tilde{g}$, we obtain

$$\langle \tilde{h}, T\tilde{g} \rangle_{x,t} \lesssim \|\tilde{h}\|_{L^1_t L^2_x} \cdot \|\tilde{g}\|_{L^1_t L^2_x}.$$  \hfill (3.1)

This immediately implies (1.11).

### 3 Time-local Inhomogeneous Flow Smoothing is not True: A Disproof

From Theorem 1.3, it is tempting to invoke the Christ-Kiselev Lemma [2] to obtain the retarded estimate

$$\left\| \left| x^{\frac{\lambda}{2}} |\nabla| \int_0^t e^{i(t-s)\Delta} |y|^{\frac{\lambda}{2}} f(s, y) ds \right| \right\|_{L^\infty_t L^2_x(\mathbb{R}^n \times \mathbb{R})} \lesssim \|f\|_{L^1_t L^2_x(\mathbb{R}^n \times \mathbb{R})}.$$  \hfill (3.1)

However, the Christ-Kiselev lemma fails to be applicable since we are considering the case with the endpoint exponent $2$. More surprisingly, we shall show that (3.1) is not true in general! To see this it is enough consider the 2D case (higher dimensional computations are similar). Since $f$ is radial, we write

$$\left| x^{\frac{\lambda}{2}} \nabla \right| \int_0^t e^{i(t-s)\Delta} |y|^{\frac{\lambda}{2}} f(s, y) ds$$

$$= \int |x|^{\frac{\lambda}{2}} \xi e^{i\xi \cdot x} \int_0^t e^{-i(t-s)|\xi|^2} |y|^{\frac{\lambda}{2}} f(s, |y|) e^{-i\xi \cdot y} dy ds d\xi$$

$$= \int_0^\infty \int_0^\infty \int_0^t |x|^{\frac{\lambda}{2}} \cdot k \cdot J_0(k|x|) \cdot e^{-i(t-s)k^2} \rho^{\frac{\lambda}{2}} f(s, \rho) J_0(k\rho) \cdot \rho k ds d\rho dk.$$

Now pass everything to radial coordinates and write $x$ as one-dimensional, we can reformulate (3.1) as the following equivalent one-dimensional problem:

$$\left\| \int_0^\infty \int_0^\infty \int_0^t x^{\frac{\lambda}{2}} \cdot k \cdot J_0(kx) \cdot e^{-i(t-s)k^2} \rho^{\frac{\lambda}{2}} f(s, \rho) J_0(k\rho) \cdot k ds d\rho dk \right\|_{L^\infty_t L^2_x(\mathbb{R} \times \mathbb{R})} \lesssim \|f\|_{L^1_t L^2_x(\mathbb{R} \times \mathbb{R})}.$$  \hfill (3.2)

Note here that we regard $f(\rho) \rho$ as $f(\rho)$ to absorb the volume factor since originally $f$ is a two-dimensional function. In (3.2) everything is one-dimensional. By convexity, to check whether (3.1) is true, it suffices to consider $f$ of the form $f = \delta(\rho - \rho')\eta(t)$, where $\eta \in L^2$ and $\rho'$ is arbitrary. So equivalently we need to check

$$\left\| \int_0^\infty \int_0^t x^{\frac{\lambda}{2}} \cdot k \cdot J_0(kx) \cdot e^{-i(t-s)k^2} \rho^{\frac{\lambda}{2}} \eta(s) J_0(k\rho) \cdot k ds d\rho dk \right\|_{L^\infty_t L^2_x(\mathbb{R} \times \mathbb{R})} \lesssim \|\eta\|_{L^2(\mathbb{R})}.$$  \hfill (3.3)
Next write the Fourier transform of $\eta(t)$ as $\hat{\eta}(\mu)$. We compute
\[
\int_0^t e^{-i(t-s)k^2} \eta(s) ds = \int_{-\infty}^t \int_0^\infty e^{-i(t-s)k^2} e^{i\mu \hat{\eta}(\mu)} ds d\mu \\
= \int_{-\infty}^\infty e^{i\mu - e^{-ik^2}} \hat{\eta}(\mu) d\mu \\
= - \int_{-\infty}^\infty \frac{e^{-ik^2}}{k^2 + \mu} \hat{\eta}(\mu) d\mu + \int_{-\infty}^\infty \frac{e^{i\mu}}{k^2 + \mu} \hat{\eta}(\mu) d\mu. \tag{3.4}
\]

We estimate separately the contributions of (a) and (b) to the LHS of (3.3). For (a), we have
\[
\left\| \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-ik^2} \hat{\eta}(\mu) \frac{d\mu}{k^2 + \mu} \cdot x^\frac{1}{2} k^2 J_0(kx) J_0(kp) \rho^2 dk \right\|_{L^2_t}^2 \\
= \left\| \int_{-\infty}^\infty e^{-ik^2} \left( \int_{-\infty}^\infty \hat{\eta}(\mu) \frac{d\mu}{k^2 + \mu} \right) \cdot (xp) \cdot x^\frac{1}{2} k^2 J_0(kx) J_0(kp) \right\|_{L^2_k}^2 \\
\lesssim \int \left\| \int_{-\infty}^\infty \hat{\eta}(\mu) \frac{d\mu}{k^2 + \mu} \right\|^2 dk \\
\lesssim \int_{0}^\infty \frac{k}{k^2 + \mu} d\mu \left\| \hat{\eta}(\mu) \cdot (xp) \cdot x^\frac{1}{2} k^2 J_0(kx) J_0(kp) \right\|_{L^2_{kx}} \\
\lesssim \left\| \frac{1}{\mu} \cdot \hat{\eta} \right\|_{L^2_{\mu_k}}^2 = \| H(\hat{\eta}) \|_{L^2_{\mu_k}}^2 \lesssim \| \eta \|_{L^2_{\mu_k}}^2.
\]

Therefore the contribution of (a) to the LHS of (3.3) is acceptable. Next consider (b), we compute
\[
\left\| \int_{-\infty}^\infty \int_{0}^\infty (xp) \frac{1}{2} k^2 J_0(kx) J_0(kp) \cdot \frac{e^{i\mu}}{k^2 + \mu} \hat{\eta}(\mu) d\mu dk \right\|_{L^2_t}^2 \\
= \left\| \int_{-\infty}^\infty \int_{0}^\infty (xp) \frac{1}{2} k^2 J_0(kx) J_0(kp) \cdot \frac{1}{k^2 + \mu} dk \right\|_{L^2_{kx}}^2 \left\| \hat{\eta}(\mu) \right\|^2_{L^2_{\mu_k}}.
\]

To finish the proof of (3.3), we would have to show that $G(x, \mu, \rho)$ is uniformly bounded. However one can check using the asymptotics of Bessel functions that this is not true!

**Remark 3.1** Although (3.1) is not true, we can show that an $\epsilon$-perturbation of (3.1) is true. Namely for $\epsilon > 0$, we have
\[
\left\| |x|^{\frac{n-1-\epsilon}{2}} |\nabla|^{1-\epsilon} \int_0^t e^{i(t-s)\Delta} |y|^{\frac{n-1-\epsilon}{2}} f(s, y) ds \right\|_{L^\infty_t L^2_{x,y}} \lesssim \| f \|_{L^1_t L^2_x}. \tag{3.5}
\]

We leave it as an exercise to interested readers.
References

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