The focusing energy-critical Hartree equation

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Received 13 April 2008; revised 28 May 2008
Available online 13 June 2008

Abstract

We consider the focusing energy-critical nonlinear Hartree equation \( iu_t + \Delta u = -(|x|^{-4} * |u|^{2})u \). We proved that if a maximal-lifespan solution \( u : I \times \mathbb{R}^d \to \mathbb{C} \) satisfies \( \sup_{t \in I} \|\nabla u(t)\|_2 < \|\nabla W\|_2 \), where \( W \) is the static solution of the equation, then the maximal-lifespan \( I = \mathbb{R} \), moreover, the solution scatters in both time directions. For spherically symmetric initial data, similar result has been obtained in [C. Miao, G. Xu, L. Zhao, Global wellposedness, scattering and blowup for the energy-critical, focusing Hartree equation in the radial case, Colloq. Math., in press]. The argument is an adaptation of the recent work of R. Killip and M. Visan [R. Killip, M. Visan, The focusing energy-critical nonlinear Schrödinger equation in dimensions five and higher, preprint] on energy-critical nonlinear Schrödinger equations.

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MSC: 35Q55

1. Introduction

We consider the initial value problem of focusing energy-critical nonlinear Hartree equation:

\[
\begin{align*}
  iu_t + \Delta u &= F(u), \\
  u(0) &= u_0 \in \dot{H}^1_x(\mathbb{R}^d),
\end{align*}
\]

(1.1)

where \( u(t, x) \) is a complex-valued function on \( \mathbb{R} \times \mathbb{R}^d, d \geqslant 5 \), and \( F(u) = -(| \cdot |^{-4} * |u|^{2})u \).
The name “energy-critical” refers to the fact that the natural scaling
\[ u(t, x) \rightarrow u_{\lambda}(t, x) = \lambda^{-\frac{d-2}{2}} u \left( \frac{t}{\lambda^2}, \frac{x}{\lambda} \right) \]  
(1.2)
leaves both the equation and the energy invariant. Here the energy is defined as the following
\[ E(u(t)) = \frac{1}{2} \left\| \nabla u(t) \right\|_2^2 - \frac{1}{4} \int \int \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^4} \, dx \, dy. \]  
(1.3)

In this paper, we consider the global wellposedness and scattering for the initial value problem (1.1). Before stating the main results, we introduce some background materials. We begin by making the notion of solution more precise.

**Definition 1.1 (Solution).** A function \( u : I \times \mathbb{R}^d \rightarrow \mathbb{C} \) on a nonempty time interval \( t_0 \in I \) is a strong solution to (1.1) if it lies in \( C_t H^1_x (\mathbb{R}^d) \cap L^2_{t,x} (K \times \mathbb{R}^d) \) for any compact \( K \subset I \) and obeys the Duhamel’s formula:
\[ u(t) = e^{i(t-t_0)\Delta} u(t_0) - i \int_{t_0}^{t} e^{i(t-s)\Delta} F(u(s)) \, ds. \]  
(1.4)

That the solution lies in the space \( L^2_{t,x} \) locally in time is natural since by Strichartz estimate, the linear flow always lies in this space. Also, the finiteness of the norm on maximal-lifespan implies the solution is global and scatters in both time directions. In view of this, we define
\[ S_I(u) = \int_I \int_{\mathbb{R}^d} \left| u(t, x) \right|^{\frac{2(d+2)}{d-2}} \, dx \, dt \]
as the scattering size of \( u \). Closely associated with the notion of scattering is the notion of blowup:

**Definition 1.2.** We call a maximal-lifespan solution \( u : I \times \mathbb{R}^d \rightarrow \mathbb{C} \) blows up forward in time if there exists \( t_0 \in I \) such that \( S_{[t_0, \sup I]}(u) = \infty \); similarly, \( u(t, x) \) blows up backward in time if \( S_{[\inf I, t_0]}(u) = \infty \).

The local theory of (1.1) was worked out by Cazenave and Weissler [2], we record their results as follows.

**Theorem 1.3 (Local theory).** (See [2].) Given \( u_0 \in \dot{H}^1_x (\mathbb{R}^d) \) and \( t_0 \in \mathbb{R} \), there exists a unique maximal-lifespan solution \( u : I \times \mathbb{R}^d \rightarrow \mathbb{C} \) to (1.1) with initial data \( u(t_0) = u_0 \). The solution also has the following properties:
- (Local existence) \( I \) is an open neighborhood of \( t_0 \).
- (Energy conservation) The energy of \( u \) is conserved, that is \( E(u(t)) = E(u_0) \).
- (Blowup criterion) If \( \sup(I) \) is finite, then \( u \) blows up forward in time; if \( \inf(I) \) is finite, then \( u \) blows up backward in time.
• (Scattering) If \( \sup(I) = \infty \) and \( u \) does not blow up forward in time, then \( u \) scatters forward in time, that is, there exists a unique \( u_+ \in \dot{H}^1_\infty(\mathbb{R}^d) \) such that
\[
\lim_{t \to \infty} \| u(t) - e^{it\Delta} u_+ \|_{\dot{H}^1_\infty} = 0.
\]

• (Small data scattering) If \( \| \nabla u_0 \|_2 \) is sufficiently small, then \( u \) exists globally and scatters in both directions.

In the defocusing case, i.e., \( F(u) = (|x|^{-4} \ast |u|^2)u \), it was verified in [12] that all finite energy solutions exist globally and scatter. In the focusing case, things are more subtle.

According to the local theory Theorem 1.3, solutions with small kinetic energy exist globally and scatter. Solutions with big kinetic energy may blow up. An explicit example is the following. Let \( W \) be the unique positive solution of nonlinear elliptic equation
\[
\Delta W + (|x|^{-4} \ast |W|^2)W = 0 \tag{1.5}
\]
(the existence and uniqueness of the solution are established in [14,15]). Then \( W \in \dot{H}^1_\infty(\mathbb{R}^d) \) and \( W \) is a global solution of (1.1) but blows up in both time directions. It was believed that the kinetic energy of \( W \) is the minimal threshold for the solution blowing up. In this paper, we prove the following

**Theorem 1.4.** Let \( u : I \times \mathbb{R}^d \to \mathbb{C} \) be the maximal-lifespan solution of (1.1) satisfying
\[
E_* := \sup_{t \in I} \| \nabla u(t) \|_2 < \| \nabla W \|_2.
\]

Then \( I = \mathbb{R} \) and
\[
\int_{\mathbb{R} \times \mathbb{R}^d} |u(t,x)|^{2(d+2)/(d-2)} \, dx \, dt \leq C(E_*)
\]

As a consequence of this theorem and the coercive property of \( W \) (see Lemma 2.4), we have

**Corollary 1.5.** Let \( u_0 \in \dot{H}^1_\infty(\mathbb{R}^d) \) be such that \( \| \nabla u_0 \|_2 < \| \nabla W \|_2 \) and \( E(u) < E(W) \), then the corresponding solution to (1.1) is global, moreover,
\[
\int \int_{\mathbb{R} \times \mathbb{R}^d} |u(t,x)|^{2(d+2)/(d-2)} \, dx \, dt < \infty.
\]

**Remark 1.6.** We note that when the initial data \( u_0 \) is spherically symmetric, Corollary 1.5 was proved in [13]. We also note that the proof of Theorem 1.4 also applies to the defocusing nonlinear Hartree equation, thus give a simplification of the argument in [12].

The results in Corollary 1.5 is sharp. More precisely, we have
**Proposition 1.7** (Blowup). (See [11,13].) Let $u_0 \in \dot{H}^1_x(\mathbb{R}^d)$ be such that $E(u_0) < E(W)$ and $\|\nabla u_0\|_2 \geq \|\nabla W\|_2$. Assume also that $xu_0 \in L^2_x(\mathbb{R}^d)$ or $u_0 \in H^1_x(\mathbb{R}^d)$ is spherically symmetric. Then the corresponding solution blows up in finite time.

In the setting of energy-critical nonlinear Schrödinger equation, i.e., $F(u) = \pm |u|^{4/d - 2}u$, where $+$, $-$ corresponds to the defocusing case and focusing case, the problem was extensively studied. In the defocusing case, the global wellposedness and scattering was established by Bourgain [1], Grillakis [4] and Tao [18] for spherically symmetric initial data, and by Colliander, Keel, Staffilani, Takaoka and Tao [3] Ryckman and Visan [16] and Visan [21,22] for arbitrary initial data. In the focusing case, this problem was first investigated by Kenig and Merle [8]. They proved in dimension $d = 3, 4, 5$, all the spherically symmetric solutions with kinetic energy smaller than that of the ground state exist globally and scatter. Here the ground state is

$$W_{nls}(x) = \left(1 + \frac{|x|^2}{d(d-2)}\right)^{-\frac{d-2}{2}} \in \dot{H}^1_x(\mathbb{R}^d)$$

and solves the static nonlinear Schrödinger equation

$$\Delta W_{nls} + |W_{nls}|^{\frac{4}{d-2}} W_{nls} = 0.$$ 

It is a very challenging problem to remove the radial assumption. Recently in [11], Killip and Visan achieved this in dimension $d \geq 5$. In this paper, we will adapt the argument in [11] to prove the main result Theorem 1.4. In the following, we describe the ideas of the proof of Theorem 1.4.

1.1. Outline of the proof of Theorem 1.4

The basic strategy is: suppose for a contradiction that Theorem 1.4 does not hold, then we will see that the failure of Theorem 1.4 is caused by a special class of solutions; on the other hand, these solutions have so many good properties that they do not exist. Thus we get a contradiction. The most important properties of the solutions are that they are well localized in both physical and frequency spaces. We refer this as

**Definition 1.8** (Almost periodicity modulo symmetries). Let $d \geq 5$, a solution $u : I \times \mathbb{R}^d \to \mathbb{C}$ is called almost periodic modulo symmetries if there exist functions $N : I \to \mathbb{R}^+$, $x : I \to \mathbb{R}^d$ and $C : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $t \in I$ and $\eta > 0$,

$$\int_{|x-x(t)| \leq C(\eta)N(t)} |\nabla u(t,x)|^2 dx \leq \eta$$

and

$$\int_{|\xi| \geq C(\eta)N(t)} |\xi|^2 |\hat{u}(t,\xi)|^2 d\xi \leq \eta.$$ 

We refer to $N$ as the frequency scale function and $x$ the spatial center function and to $C$ as the compactness modulus function.
Remark 1.9. By Ascoli–Arzela Theorem, \( u \) is almost periodic modulo symmetry if and only if the set

\[
\left\{ N(t) \frac{d-2}{d} u \left( t, x(t) + \frac{x}{N(t)} \right), \quad t \in I \right\}
\]

falls in a compact set in \( \dot{H}_x^{1/2} (\mathbb{R}^d) \). The following are consequences of this statement. If \( u \) is almost periodic modulo symmetry, then

\[
\int_{|x-x(t)| \geq \frac{C(n)}{N(t)}} |u(t,x)|^{\frac{2d}{d-2}} \, dx \leq \eta. \tag{1.6}
\]

And there exists \( c(\eta) > 0 \) such that

\[
\int_{|x-x(t)| \leq \frac{C(n)}{N(t)}} \left| \nabla u(t,x) \right|^2 + \int_{|\xi| \leq c(\eta) N(t)} |\hat{u}(t,\xi)|^2 \, d\xi \leq \eta. \tag{1.7}
\]

We are ready to state

Theorem 1.10 (Reduction to almost periodic solutions). Let \( d \geq 5 \) such that Theorem 1.4 fails. Then there exists minimal kinetic blowup solution \( u \) in the sense that:

1. \( u \) is a maximal-lifespan solution on \( I \times \mathbb{R}^d \to \mathbb{C} \) to (1.1) and blows up in both time directions;
2. \( u \) is almost periodic modulo symmetries;
3. \( u \) has the minimal kinetic energy among all blowup solutions. More precisely, let \( v : J \times \mathbb{R}^d \to \mathbb{C} \) be a maximal-lifespan solution which blows up at least in one time direction, then

\[
\sup_{t \in I} \| \nabla u(t) \|_2 \leq \sup_{t \in J} \| \nabla v(t) \|_2.
\]

One refers [13] for the proof of a similar result. The only difference is that instead of proving the existence of minimal kinetic energy blowup solution, the authors proved the existence of minimal energy blowup solution. However, as the kinetic energy is not conserved in time, that needs nontrivial modification of the argument. The modification can be made in the same spirit with [11] by Killip and Visan on energy-critical nonlinear Schrödinger equations or with [9] by Kenig and Merle on \( \dot{H}_x^{1/2} \)-critical problem. In view of this, we will not repeat the argument here.

Remark 1.11. We should notice that most of the properties of minimal kinetic energy solution are the consequence of the minimality. However, the existence of a minimal solution was first proved in the pioneering work by Keraani [10] for mass-critical NLS. Kenig and Merle [8] adapted the argument to the energy-critical NLS, and first applied this to study the wellposedness problem.

So far, we do not have any control on \( N(t) \). However, the following theorem shows that no matter how small the set of minimal kinetic energy solution is, we will inevitably encounter at least one of the following three enemies. Thus the proof of Theorem 1.4 is reduced to showing the nonexistence of the three enemies.
Theorem 1.12 (Three enemies). Suppose \( d \geq 5 \) is such that Theorem 1.4 fails. Then there must exist minimal kinetic energy blowup solution \( u \) with maximal-lifespan \( I \) and frequency scale function \( N(t) \) satisfying one of the three restrictions:
1. (Finite time blowup) \( \inf I < \infty \), or \( \sup I < \infty \).
2. (Soliton-like solution) \( I = \mathbb{R} \) and \( N(t) = 1 \) for all \( t \in \mathbb{R} \).
3. (Low-to-high cascade) \( I = \mathbb{R} \) and \( \inf_{t \in \mathbb{R}} N(t) \geq 1 \), and \( \limsup_{t \to \infty} N(t) = \infty \).

We refer [11] for the proof of this theorem.

To kill the three enemies, we will use virial argument, this requires the solution lies in \( L^2_x(\mathbb{R}^d) \) or even the space with negative regularity \( L^\infty_t H^{-\epsilon}_x \). Finite time blowup scenario is precluded in Section 3 where the boundedness of the \( L^2_x \)-norm comes from the finiteness of the blowup time. To kill the last two enemies, we need to downgrade the regularity of the solution to some negative regularity space, this will be done in Section 4. Finally, we use virial argument and the low regularities to kill the last two enemies in Section 5. As we will see, the argument follow closely after that of [11] on nonlinear Schrödinger equation.

2. Notation and useful lemmas

2.1. Some notation

We use \( X \lesssim Y \) or \( Y \gtrsim X \) whenever \( X \leq CY \) for some constant \( C > 0 \). We use \( O(Y) \) to denote any quantity \( X \) such that \( |X| \lesssim Y \). We use the notation \( X \sim Y \) whenever \( X \lesssim Y \lesssim X \). The fact that these constants depend upon the dimension \( d \) will be suppressed.

We use the ‘Japanese bracket’ convention \( \langle x \rangle := (1 + |x|^2)^{1/2} \).

We write \( L^q_t L^r_x \) to denote the Banach space with norm
\[
\|u\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^d)} := \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^d} |u(t,x)|^r \, dx \right)^{q/r} \, dt \right)^{1/q},
\]
with the usual modifications when \( q \) or \( r \) are equal to infinity, or when the domain \( \mathbb{R} \times \mathbb{R}^d \) is replaced by a smaller region of spacetime such as \( I \times \mathbb{R}^d \). When \( q = r \) we abbreviate \( L^q_t L^q_x \) as \( L^q_t L^q_x \).

2.2. Basic harmonic analysis

Let \( \varphi(\xi) \) be a radial bump function supported in the ball \( \{ \xi \in \mathbb{R}^d : |\xi| \leq \frac{11}{10} \} \) and equal to 1 on the ball \( \{ \xi \in \mathbb{R}^d : |\xi| \leq 1 \} \). For each number \( N > 0 \), we define the Fourier multipliers
\[
\widehat{P_{\leq N} f}(\xi) := \varphi(\xi/N) \hat{f}(\xi),
\]
\[
\widehat{P_{>N} f}(\xi) := (1 - \varphi(\xi/N)) \hat{f}(\xi),
\]
\[
\widehat{P_N f}(\xi) := \psi(\xi/N) \hat{f}(\xi) := (\varphi(\xi/N) - \varphi(2\xi/N)) \hat{f}(\xi)
\]
and similarly $P_{<N}$ and $P_{\geq N}$. We also define

$$P_{M<\leq N} := P_{\leq N} - P_{\leq M} = \sum_{M<\leq N} P_{N'},$$

whenever $M < N$. We will usually use these multipliers when $M$ and $N$ are dyadic numbers (that is, of the form $2^n$ for some integer $n$); in particular, all summations over $N$ or $M$ are understood to be over dyadic numbers. Nevertheless, it will occasionally be convenient to allow $M$ and $N$ to not be a power of 2. Note that $P_N$ is not truly a projection; to get around this, we will occasionally need to use fattened Littlewood–Paley operators:

$$\tilde{P}_N := P_{N/2} + P_N + P_{2N}. \quad (2.1)$$

These obey $P_N \tilde{P}_N = \tilde{P}_N P_N = P_N$.

As with all Fourier multipliers, the Littlewood–Paley operators commute with the propagator $e^{it\Delta}$, as well as with differential operators such as $i\partial_t + \Delta$. We will use basic properties of these operators many times, including

**Lemma 2.1 (Bernstein estimates).** For $1 \leq p \leq q \leq \infty$,

$$\| |\nabla|^s P_N f \|_{L^p_x(\mathbb{R}^d)} \sim N^{s+1} \| P_N f \|_{L^p_x(\mathbb{R}^d)},$$

$$\| P_{\leq N} f \|_{L^2_x(\mathbb{R}^d)} \lesssim N^{d/p - d/q} \| P_{\leq N} f \|_{L^p_x(\mathbb{R}^d)},$$

$$\| P_N f \|_{L^2_x(\mathbb{R}^d)} \lesssim N^{d/p - d/q} \| P_N f \|_{L^p_x(\mathbb{R}^d)}.$$  

**2.3. Strichartz estimates**

From the explicit formula

$$e^{it\Delta} f(x) = \frac{1}{(4\pi it)^{d/2}} \int_{\mathbb{R}^d} e^{i|x-y|^2/4t} f(y) dy,$$

we deduce the standard dispersive inequality

$$\| e^{it\Delta} f \|_{L^\infty_x(\mathbb{R}^d)} \lesssim \frac{1}{|t|^{d/2}} \| f \|_{L^1_x(\mathbb{R}^d)} \quad (2.2)$$

for all $t \neq 0$. Interpolating between this and the conservation of mass, gives

$$\| e^{it\Delta} f \|_{L^p_x(\mathbb{R}^d)} \lesssim |t|^{\frac{d}{p} - \frac{d}{2}} \| f \|_{L^{p'}_x(\mathbb{R}^d)} \quad (2.3)$$

for all $t \neq 0$ and $2 \leq p \leq \infty$. Here $p'$ is the dual of $p$, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

We also record the following standard Strichartz estimates:
Lemma 2.2 (Strichartz). Let $I$ be an interval, let $t_0 \in I$, and let $u_0 \in L^2_x(\mathbb{R}^d)$ and $f \in L^{2(d+2)/(d+4)}_{t,x}(I \times \mathbb{R}^d)$, with $d \geq 3$. Then, the function $u$ defined by

$$u(t) := e^{i(t-t_0)\Delta}u_0 - i \int_{t_0}^{t} e^{i(t-t')\Delta} f(t') \, dt'$$

obeys the estimate

$$\|u\|_{C^0_t L^2_x} + \|u\|_{L^2_t L^{\frac{2d}{d+4}}_x} \lesssim \|u_0\|_{L^2_x} + \|f\|_{L^{2(d+2)/(d+4)}_{t,x}},$$

where all spacetime norms are over $I \times \mathbb{R}^d$.

Proof. See, for example, [5, 17]. For the endpoint see [7].

As we will see in Section 3, the low regularity is obtained through a recursive argument where one needs to apply the following Gronwall type inequality. The proof of the inequality is originally in [11].

Lemma 2.3 (Gronwall). Given $\gamma > 0$, $0 < \eta < \frac{1}{2}(1 - 2^{-\gamma})$ and $\{b_k\} \in l^\infty(\mathbb{Z}^+)$, let $x_k \in l^\infty(\mathbb{Z}^+)$ be a non-negative sequence obeying

$$x_k \leq b_k + \eta \sum_{l=0}^{\infty} 2^{-\gamma(k-l)}x_l, \quad \forall k \geq 0.$$

Then

$$x_k \lesssim \sum_{l=0}^{k} r^{k-l} b_l, \quad \forall k \geq 0,$$

where $r = r(\eta) \in (2^{-\gamma}, 1)$, moreover $r \to 2^{-\gamma}$ as $\eta \to 0$.

2.4. Properties for $W$

We will also need the coercive property of the ground state $W$. The following properties are established in [13], which is based on earlier work of Kenig and Merle [8].

Lemma 2.4 (Coercivity I). Assume $E(u_0) \leq (1 - \delta_0)E(W)$ for some $\delta_0 > 0$. Let $u(t)$ be the corresponding solution on $I$ with $u(t_0) = u_0$, $t_0 \in I$. Then there exist two positive constants $\delta_1 > 0$, $\delta_2 > 0$ (depending on $\delta_0$) such that:

(a) If $\|\nabla u_0\|_2 \leq \|\nabla W\|_2$, then

$$\|\nabla u(t)\|_2^2 \leq (1 - \delta_1)\|\nabla W\|_2^2 \quad \text{for all} \quad t \in I.$$
(b) If $\|\nabla u_0\|_2 \geq \|\nabla W\|_2$, then
\[
\|\nabla u(t)\|_2^2 \geq (1 + \delta_2)\|\nabla W\|_2^2 \quad \text{for all } t \in I.
\]

Moreover,
\[
\int_{\mathbb{R}^d} |\nabla u(t, x)|^2 \, dx - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^4} \, dx \, dy \leq -c(\delta_2, d).
\]

We also have

Lemma 2.5 (Coercivity II). Let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a solution to (1.1). Suppose $\sup_{t \in I}\|\nabla u(t)\|_2 \leq (1 - \delta)\|\nabla W\|_2$ for some $\delta > 0$, then for all $t \in I$,
\[
E(u(t)) \sim \|\nabla u(t)\|_2^2 \sim \|\nabla u_0\|_2^2
\]
and
\[
\int_{\mathbb{R}^d} |\nabla u(t, x)|^2 \, dx - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^4} \, dx \, dy \geq \|\nabla u_0\|_2^2.
\]

3. Finite time blowup

In this section, we preclude the existence of finite time blowup solution in the sense of Theorem 1.12. The argument is an adaptation of the that in [8] in nonlinear Schrödinger case. We have

Theorem 3.1 (No finite time blowup). Let $d \geq 5$, then there are no maximal-lifespan solutions $u : I \times \mathbb{R}^d \to \mathbb{C}$ that are almost periodic modulo symmetries, obey
\[
S_I(u) = \infty, \quad \sup_{t \in I}\|\nabla u(t)\|_2^2 < \|\nabla W\|_2^2
\]
and
\[
\sup I < \infty, \quad \text{or} \quad |\inf I| < \infty.
\]

Proof. Suppose for contradiction there exists such a solution $u$. Without loss of generality, we assume that $\sup I < \infty$. As a consequence, we have
\[
\liminf_{t \to \sup I} N(t) = \infty
\]
(see for example [11,20]). We now show that the blowup of $N(t)$ implies
\[
\limsup_{t \to \sup I} \int_{|x| \leq R} |u(t, x)|^2 \, dx = 0 \quad \text{for all } R > 0.
\]
Indeed, let $0 < \eta < 1$ and $t \in I$. By Hölder inequality, Sobolev embedding and (3.1), we have

$$
\int_{|x| \leq R} |u(t, x)|^2 \, dx \leq \int_{|x - x(t)| \leq \eta R} |u(t, x)|^2 \, dx + \int_{|x| \leq R, |x - x(t)| > \eta R} |u(t, x)|^2 \, dx
$$

$$
\lesssim \eta^2 R^2 \|u(t)\|_{L^2_x}^2 + R^2 \left( \int_{|x - x(t)| > \eta R} |u(t, x)| \frac{2d}{d - 2} \, dx \right)^{\frac{d-2}{d}}
$$

$$
\lesssim \eta^2 R^2 \|\nabla W\|_2^2 + R^2 \left( \int_{|x - x(t)| > \eta R} |u(t, x)| \frac{2d}{d - 2} \, dx \right)^{\frac{d-2}{d}}.
$$

From (1.6) and (3.2), we see that

$$
\lim_{t \to \sup I} \sup_{|x - x(t)| > \eta R} \int |u(t, x)| \frac{2d}{d - 2} \, dx = 0.
$$

This proves (3.3). Next we show that $u$ is identically zero, thus contradicts (3.1). We define

$$
M_R(t) = \int_{\mathbb{R}^d} \phi \left( \frac{|x|}{R} \right) |u(t, x)|^2 \, dx,
$$

where $\phi$ is smooth cutoff function such that $\phi(r) = 1$ as $r \leq 1$ and $\phi(r) = 0$ as $r \geq 2$. By (3.3), we have

$$
\lim_{t \to \sup I} M_R(t) = 0 \quad \text{for all } R > 0. \quad (3.4)
$$

On the other hand, using Hardy’s inequality, we have

$$
|\partial_t M_R(t)| \lesssim \|\nabla u(t)\|_2 \left\| \frac{|u(t)|}{|x|} \right\|_2 \lesssim \|\nabla u(t)\|_2^2 \lesssim \|\nabla W\|_2^2.
$$

By Fundamental Theorem of Calculus,

$$
M_R(t_1) = M_R(t_2) + \int_{t_2}^{t_1} \partial_t M_R(t) \, dt \lesssim M_R(t_2) + |t_2 - t_1| \|\nabla W\|_2^2
$$

for all $t_1, t_2 \in I$ and $R > 0$. Letting $t_2 \to \sup I$ and using (3.4), we get

$$
M_R(t_1) \lesssim |I - t_1| \|\nabla W\|_2^2.
$$

Letting $R \to \infty$ and using the conservation of mass, we obtain $u_0 \in L^2_x$. Now letting $t_1 \to \sup I$ we conclude $u_0 = 0$ which contradicts (3.1). \qed
4. Negative regularity

In this section we prove that

**Theorem 4.1 (Negative regularity).** Let \( d \geq 5 \) and \( u \) be a global solution to (1.1) which is almost periodic modulo symmetries. Suppose also

\[
\sup_{t \in \mathbb{R}} \| \nabla u(t) \|_2 < \infty
\]

and

\[
N(t) \geq 1,
\]

(4.1)

then \( u \in L^\infty_t \dot{H}^{-\epsilon}_x \) for some \( \epsilon > 0 \). In particular, \( u \in L^\infty_t L^2_x \).

We apply the idea of [11] to prove this. In view of (1.7) and (4.1), there exists \( N_0 = N_0(\eta) \) such that

\[
\| P_{\leq N_0} u(t) \|_{\dot{H}^{1}_x} \leq \eta.
\]

(4.2)

We define

\[
A(N) = N^{-\frac{1}{2}} \sup_{t \in \mathbb{R}} \| u_{N}(t) \|_{L_{x}^{\frac{2d}{d-3}}}.
\]

By Bernstein inequality, we know

\[
A(N) \leq \| u \|_{L^\infty_t \dot{H}^{1}_x} < \infty.
\]

Our first step is to establish the following

**Lemma 4.2 (Recursive formula for \( A(N) \)).** For all \( N \leq 8N_0 \), we have

\[
A(N) \lesssim \left( \frac{N}{N_0} \right)^{\frac{1}{2}} + \eta^{2} \sum_{N_1 \leq \frac{N}{8}} \left( \frac{N_1}{N} \right)^{\frac{1}{2}} A(N_1) + \eta^{2} \sum_{\frac{N}{8} < N_1 \leq N_0} \left( \frac{N}{N_1} \right)^{\frac{1}{2}} A(N_1).
\]

(4.3)

**Proof.** Fix \( N \leq 8N_0 \), by time translation symmetry, it suffices to prove that

\[
N^{-\frac{1}{2}} \| u_{N}(0) \|_2 \lesssim \left( \frac{N}{N_0} \right)^{\frac{1}{2}} + \eta^{2} \sum_{N_1 \leq \frac{N}{8}} \left( \frac{N_1}{N} \right)^{\frac{1}{2}} A(N_1) + \eta^{2} \sum_{\frac{N}{8} < N_1 \leq N_0} \left( \frac{N}{N_1} \right)^{\frac{1}{2}} A(N_1).
\]

(4.4)

To this end, we write Duhamel formula forward in time and use triangle, Bernstein inequality to estimate
\[
\|u_N(0)\|_{L^{\infty}_t L^2_x}^{2d+3} \leq \left\| \int_0^{N^{-2}} e^{-is\Delta} P_N F(u(s))\ ds \right\|_{L^{\infty}_s L^2_x}^{2d} + \left\| \int_{N^{-2}}^{\infty} e^{-is\Delta} P_N F(u(s))\ ds \right\|_{L^{\infty}_s L^2_x}^{2d}
\]
\[
\leq N^{-\frac{1}{2}} \|P_N F(u)\|_{L^{\infty}_s L^2_x} + \int_{N^{-2}}^{\infty} s^{-\frac{3}{2}} \|P_N F(u(s))\|_{L^{\infty}_s L^2_x}^{2d} \ ds
\]
\[
\leq N \|P_N F(u)\|_{L^{\infty}_t L^2_x}^{2d+3}.
\]
Comparing this with the desired estimate (4.4), we need to establish
\[
\|P_N F(u)\|_{L^{\infty}_t L^2_x}^{2d+3} \lesssim N^{-1} \|u_N\|_{L^{\infty}_t L^2_x}^{2d+3} + \eta^2 \sum_{N_1 < \frac{N}{8}} N_1^{-1} \|u_N\|_{L^{\infty}_t L^2_x}^{2d+3}.
\]
(4.5)
Now we split
\[
u = u_{\leq \frac{N}{8}} + u_{\frac{N}{8} < \cdot \leq N_0} + u_{> N_0},
\]
and further split the nonlinearity \(F(u)\). Depending on where the different frequency appears, we classify \(F(u)\) as three types, each type corresponds to the contribution of the left side of (4.5). In the following, we will use \(|\nabla|^{4-d}\) to express \(|\cdot|^{4-d}\) as they are equivalent up to a constant.

**Type 1.** There is one \(P_{> N_0} u\). It can appear inside or outside of the convolution, so we need to consider two subcases:

(1a) \(|\nabla|^{4-d}(u_{> N_0} u)\);
(2a) \(|\nabla|^{4-d}(|u|^2) u_{> N_0}\).

**Type 2.** All three \(u\) are in frequency \(\leq N_0\). Moreover, there are at least one very low frequency \(P_{\leq \frac{N}{8}}\). Again, we have two subcases:

(2a) \(|\nabla|^{4-d}(u_{\leq N_0} u_{\leq \frac{N}{8}}) u_{\leq N_0}\);
(2b) \(|\nabla|^{4-d}(|u|^2) u_{\leq \frac{N}{8}}\).

**Type 3.** All three \(u\) are in medium frequency \(\frac{N}{8} < \cdot \leq N_0\),
\[
|\nabla|^{4-d}(|u_{\frac{N}{8} < \cdot \leq N_0}|^2) u_{\frac{N}{8} < \cdot \leq N_0}.
\]
We first consider the contribution from the first type.
Using Hölder and Bernstein inequality and discarding the projection \(P_N\), we estimate the contribution from type (1a) as follows
\[ \| \nabla |^{4-d} (u > N_0 u) \|_{L^2_x} \lesssim \| \nabla |^{4-d} (u > N_0 u) \|_{L^2_x} \| u \|_{L^2_x}^{2d} \]
\[ \lesssim \| u > N_0 u \|_{L^2_x} \| u \|_{L^2_x}^{2d} \]
\[ \lesssim \| u > N_0 u \|_{L^2_x} \| u \|_{L^2_x}^{2d} \]
\[ \lesssim N_0^{-\frac{1}{2}} \| \nabla u \|_{L^2_x}^2 \lesssim N_0^{-\frac{1}{2}}. \]

Type (1b) can be estimated similarly:

\[ \| \nabla |^{4-d} (|u|^2) u > N_0 \|_{L^2_x} \lesssim \| \nabla |^{4-d} (|u|^2) \|_{L^2_x} \| u > N_0 \|_{L^2_x}^{2d} \]
\[ \lesssim \| |u|^2 \|_{L^2_x} \| \nabla |^{\frac{1}{2}} u > N_0 \|_2 \lesssim N_0^{-\frac{1}{2}}. \]

This completes the estimate of the first type, which gives the contribution of the first term on the right side of (4.5).

Now we consider the estimate of type (2a). We have

\[ \| P_N (|\nabla|^{4-d} (u \leq N_0 u \leq N) u \leq N) \|_{L^2_x} \]
\[ \lesssim \sum_{N_1 \leq N} \| P_{\frac{N}{4}} (|\nabla|^{4-d} (u \leq N_0 u N_1) u \leq N) \|_{L^2_x} \]
\[ + \sum_{N_1 \leq N} \| |\nabla|^{4-d} (u \leq N_0 u N_1) P_{\frac{N}{4}} u \leq N_0 \|_{L^2_x} \]
\[ \lesssim \sum_{N_1 \leq N} \| P_{\frac{N}{4}} (|\nabla|^{4-d} (u \leq N_0 u N_1) u \leq N) \|_{L^2_x} \| u \leq N_0 \|_{L^2_x}^{2d} \]
\[ + \sum_{N_1 \leq N} \| |\nabla|^{4-d} (u \leq N_0 u N_1) \|_{L^2_x} \| P_{\frac{N}{4}} u \leq N_0 \|_2 \]
\[ \lesssim \eta \sum_{N_1 \leq N} N^{-1} \| |\nabla|^{3-d} (u \leq N_0 u N_1) \|_{L^2_x} \]
\[ + \eta \sum_{N_1 \leq N} N^{-1} \| u \leq N_0 u N_1 \|_{L^2_x} \]
\[ \lesssim \eta^2 \sum_{N_1 \leq N} N^{-1} \| u \leq N_0 u N_1 \|_{L^2_x} \]

We now estimate type (2b). We have
\[
\|P_N(|\nabla|^{4-d}(u_{\leq N_0}^2)u_{\leq N_0}^2)\|_{L_x^{2d/3}} \lesssim \sum_{N_1 \leq \frac{N}{3}} \|P_{\geq N_1}|\nabla|^{4-d}(u_{\leq N_0}^2)u_{N_1}\|_{L_x^{2d/3}}
\]
\[
\lesssim \sum_{N_1 \leq \frac{N}{3}} \|u_{N_1}\|_{L_x^{2d/3}} \|P_{\geq N_1}|\nabla|^{4-d}(u_{\leq N_0}^2)\|_{L_x^{2d/3}}
\]
\[
\lesssim \sum_{N_1 \leq \frac{N}{3}} N^{-1} \|u_{N_1}\|_{L_x^{2d/3}} \|u_{\leq N_0}\|^2_{L_x^{2d/3}}
\]
\[
\lesssim \eta^2 \sum_{N_1 \leq \frac{N}{3}} N^{-1} \|u_{N_1}\|_{L_x^{2d/3}}.
\]

We see that type 2 contributes to the second term on RHS of (4.4).

To estimate the third type, we discard \(P_N\) and split \(u\) again into dyadic pieces, we have

\[
\|\nabla|^{4-d}(u_{\frac{N}{3}}^N u_{\leq N_0}^2)u_{\leq N_0}^2\|_{L_x^{2d/3}} \lesssim \sum_{N_0 \geq N_1 \geq N_2 \geq \frac{N}{3}} \sum_{N_0 \geq N_1 \geq N_2 \geq \frac{N}{3}} \|\nabla|^{4-d}(u_{N_1}u_{N_2})u_{N_3}\|_{L_x^{2d/3}}.
\]

In the case when \(N_3\) is the smallest, we have

\[
\sum_{N_1 \geq N_2 \geq N_3} \|\nabla|^{4-d}(u_{N_1}u_{N_2})u_{N_3}\|_{L_x^{2d/3}} \lesssim \sum_{N_1 \geq N_2 \geq N_3} \|u_{N_3}\|_{L_x^{2d/3}} \|\nabla|^{4-d}(u_{N_1}u_{N_2})\|_{L_x^{2d/3}}
\]
\[
\lesssim \sum_{N_1 \geq N_2 \geq N_3} \|u_{N_3}\|_{L_x^{2d/3}} \|u_{N_1}u_{N_2}\|_{L_x^{2d/3}}
\]
\[
\lesssim \sum_{N_1 \geq N_2 \geq N_3} \|u_{N_3}\|_{L_x^{2d/3}} \|u_{N_1}\|_{L_x^{2d/3}} \|u_{N_2}\|_{L_x^{2d/3}}
\]
\[
\lesssim \eta^2 \sum_{N_1 \geq N_2 \geq N_3} N_{1}^{-\frac{1}{2}} N_{2}^{-\frac{1}{2}} \|u_{N_3}\|_{L_x^{2d/3}}
\]
\[
\lesssim \eta^2 \sum_{N_1 \geq N_2 \geq N_3} N_{3}^{-1} \|u_{N_3}\|_{L_x^{2d/3}}.
\]

In the case when \(N_2\) is the smallest among the three, we plug in the estimate

\[
\|\nabla|^{4-d}(u_{N_1}u_{N_2})u_{N_3}\|_{L_x^{2d/3}} \leq \|\nabla|^{4-d}(u_{N_1}u_{N_2})\|_{L_x^{2d/3}} \|u_3\|_{L_x^{2d/3}} \lesssim \eta^2 N_1^{-\frac{1}{2}} N_3^{-\frac{1}{2}} \|u_{N_2}\|_{L_x^{2d/3}}.
\]
and run the same argument to get
\[
\sum_{N_1, N_3 > N_2} \| \nabla |^{4-d} (u_{N_1} u_{N_2}) u_{N_3} \|_{L^{2d/(d+3)}} \lesssim \eta^2 \sum_{N_2 < N_3 \leq N_0} N_2^{-1} \| u_{N_2} \|_{L^{2d/(d+3)}}.
\]
This completes the estimate of type 3 which contributes the third term on the RHS of (4.4). Therefore, (4.4) is established, so is Lemma 4.2. □

Combining this lemma with the Gronwall’s inequality Lemma 2.3, it follows that
\[
A(N) \lesssim N^{1/2}
\]
for all \(N \leq 8N_0\). From the definition of \(A(N)\), this implies
\[
\| u_N \|_{L^\infty_t L^p_x} \lesssim N^{1/2}.
\] (4.6)

As the implication of this estimate, we have

**Lemma 4.3 (\(L^p\) estimate).** For any \(\frac{4d}{2d+5} < p \leq \frac{2d}{d-2}\),
\[
\| u \|_{L^\infty_t L^p_x} \lesssim 1.
\]

Moreover, let \(r\) be such that \(\frac{2d}{d+5} < r < \frac{2d}{d+4}\), we have
\[
\| \nabla F(u) \|_{L^\infty_t L^r_x} \lesssim 1.
\]

**Proof.** We first estimate the \(L^p\)-norm of \(u\). Splitting \(u\) into dyadic pieces and using (4.6), interpolation, Bernstein, we have
\[
\| u \|_{L^\infty_t L^p_x} \lesssim \sum_{N \leq 8N_0} \| u_N \|_{L^\infty_t L^p_x} + \sum_{N > 8N_0} \| u_N \|_{L^\infty_t L^p_x}
\]
\[
\lesssim \sum_{N \leq 8N_0} \| u_N \|_{L^\infty_t L^r_x}^{1-\theta} \| u_N \|_{L^\infty_t L^r_x}^{\theta} + \sum_{N > 8N_0} \| u_N \|_{L^\infty_t L^r_x}^{\frac{2d}{d+5}}
\]
\[
\lesssim \sum_{N \leq 8N_0} N^{2\theta-1} + \sum_{N > 8N_0} N^d(\frac{1}{2} - \frac{1}{p})^{-1} \| u_N \|_{L^\infty_t \dot{H}^{\frac{1}{2}}} \lesssim 1.
\]

Here \(\theta = \frac{d(p-2)}{3p}\) satisfies \(2\theta - 1 > 0\) which is guaranteed under the restriction on \(p\). The estimate of \(F(u)\) will follow from Hölder inequality,
\[
\| \nabla F(u) \|_{L^\infty_t L^r_x} \lesssim \| \nabla |^{4-d} (|u|^2) \|_{L^\infty_t L^{\frac{2d}{d-r(d-2)}}} \| u \|_{L^\infty_t L^{\frac{2d}{d}}}
\]
\[
+ \| \nabla |^{4-d} (|u|^2) \|_{L^\infty_t L^{\frac{2d}{d-2}}} \| \nabla u \|_{L^\infty_t L^{\frac{2d}{d-2}}}
\]
\[
\lesssim \| \nabla u \|_{L^\infty_t L^r_x} \| u \|^2_{L^\infty_t L^{\frac{4d}{d-2(r-d-2)}}}.
\]
For the chosen \( r \), we verify that the exponent \( \frac{4rd}{2d+r-8r} \in (\frac{4d}{2d-3}, \frac{2d}{d-2}) \), thus is bounded. This ends the proof of the lemma. □

As in [11], we will use Lemma 4.3 and double Duhamel trick to downgrade the regularity of the solution. One refers [19] for the first appearance of the double Duhamel trick. We have the following

**Proposition 4.4 (Some negative regularity).** Assume \(|\nabla|^s F(u) \in L^\infty_t L^r_x\) for \( \frac{2d}{d+5} < r < \frac{2d}{d+4} \) and \( 0 \leq s \leq 1 \), then there exists \( s_0 := s_0(r, d) > 0 \) such that \( u \in L^\infty_t H^{s-s_0+}_x \).

**Proof.** First we notice that it is enough to prove

\[
\| |\nabla|^s u_N \|_{L^\infty_t L^2_x} \lesssim N^{s_0} \tag{4.7}
\]

for all \( N \lesssim 1 \) and \( s_0 := \frac{d}{r} - \frac{d}{2} - 2 \). Indeed, using Bernstein, we have

\[
\| |\nabla|^{s-s_0+} u \|_{L^\infty_t L^2_x} \leq \sum_{N \lesssim 1} \| |\nabla|^{s-s_0+} u_N \|_{L^\infty_t L^2_x} + \sum_{N \gtrsim 1} \| |\nabla|^{s-s_0+} u_N \|_{L^\infty_t L^2_x} \\
\leq \sum_{N \lesssim 1} N^{-s_0+} \| |\nabla|^s u_N \|_{L^\infty_t L^2_x} + \sum_{N \gtrsim 1} N^{-1+s-s_0+} \| \nabla u_N \|_{L^\infty_t L^2_x} \lesssim 1.
\]

We now prove (4.7). By time translation invariant, it suffices to prove

\[
\| |\nabla|^s u_N(0) \|_2 \lesssim N^{s_0}. \tag{4.8}
\]

We use Duhamel formula forward and backward in time to write

\[
\| |\nabla|^s u_N(0) \|_2^2 = \langle |\nabla|^s u_N(0), |\nabla|^s u_N(0) \rangle \leq \lim_{T \to \infty} \lim_{T' \to -\infty} \left( i \int_0^T e^{-it\Delta} |\nabla|^s F(u(t)) \, dt, -i \int_{T'}^0 e^{-i\tau\Delta} P_N |\nabla|^s F(u(\tau)) \, d\tau \right) \]

\[
\leq \int_{0}^{-\infty} \int_0^\infty \| P_N |\nabla|^s F(u(t)), e^{i(t-\tau)\Delta} P_N |\nabla|^s F(u(\tau)) \| \, dt \, d\tau.
\]

The inner product can be bounded by using Hölder inequality and Lemma 4.3:

\[
\langle P_N |\nabla|^s F(u(t)), e^{i(t-\tau)\Delta} P_N |\nabla|^s F(u(\tau)) \rangle \lesssim \| P_N |\nabla|^s F(u) \|_{L^r_t} \| e^{i(t-\tau)\Delta} P_N |\nabla|^s F(u) \|_{L^{r'}_t} \]

\[
\lesssim |t - \tau|^{\frac{d}{r} - \frac{d}{r'}} \| |\nabla|^s F(u) \|_{L^\infty_t L^2_x}^2,
\]

or by using Bernstein,
\[
\left\{ P_N |\nabla|^s F(u(t)), e^{i(t-\tau)\Delta} P_N |\nabla|^s F(u(\tau)) \right\} \\
\lesssim \left\| P_N |\nabla|^s F(u) \right\|_{L^2_x} \left\| e^{i(t-\tau)\Delta} P_N |\nabla|^s F(u) \right\|_{L^2_x} \\
\lesssim N^{2(d(1-\frac{1}{\tau})-1)} \left\| |\nabla|^s F(u) \right\|_{L^\infty_t L^\infty_x}^2.
\]

Using the two estimates, we obtain

\[
\left\| |\nabla|^s u_N(0) \right\|_{L^2_x}^2 \lesssim \int_0^\infty \int_{-\infty}^0 \min\{|t-\tau|^{-\frac{1}{2}}, N^2\} \frac{d}{dt} \frac{d}{d\tau} dt d\tau \\
\lesssim N^{2s} \left\| |\nabla|^s F(u) \right\|_{L^\infty_t L^\infty_x}^2 \leq N^{2s_0}.
\]

In the last inequality we used the fact that \( \frac{d}{r} - \frac{d}{2} > 2 \) since \( r < \frac{2d}{d+4} \). This concludes the proof of the proposition.

Theorem 4.1 will follow from iterating this proposition many times.

**Proof of Theorem 4.1.** In view of Lemma 4.3, we conclude that \( u \in L^\infty_t H^{1-s_0+} \) for some \( s_0 = s_0(r,d) \). This implies that \( |\nabla|^{1-s_0+} F(u) \in L^\infty_t L^r_x \). Using Proposition 4.4, we obtain \( u \in H^{-\varepsilon}_x \) for some \( \varepsilon > 0 \). This proves Theorem 4.1.

5. The low-to-high cascade

In this section, we will use Theorem 4.1 to preclude the low-to-high cascade in the sense of Theorem 1.12. More precisely, we will prove

**Theorem 5.1 (No low-to-high cascade).** There are no global solutions to (1.1) that are low-to-high cascade in the sense of Theorem 1.12.

**Proof.** Suppose \( u \) is a global solution that is low-to-high cascade, we aim to derive a contradiction. From Theorem 4.1, we know that \( u \in L^\infty_t H^s_x \), for all \( s \in (-\varepsilon, 1] \). In particular \( \|u(t)\|_2 \) is finite and conserved in time. As \( S(u) = \infty \), \( \|u(t)\|_2 > 0 \). On the other hand, by compactness (Remark 1.9), for \( \eta > 0 \), there exists \( c(\eta) > 0 \) such that

\[
\| P_{\leq c(\eta)N(t)} \nabla u(t) \|_2 < \eta.
\]

Hence we have

\[
0 < \text{const} = \|u(t)\|_2 \leq \| P_{\leq c(\eta)N(t)} u(t) \|_2 + \| P_{> c(\eta)N(t)} u(t) \|_2 \\
\lesssim \|u\|_{L^\infty_t H^{s_0}_x} \|\nabla P_{\leq c(\eta)N(t)} u(t)\|_{L^\infty_t L^2_x} + \frac{1}{c(\eta)N(t)} \|\nabla u(t)\|_2 \\
\lesssim \eta^{\frac{1}{r+2}} + \frac{1}{c(\eta)N(t)}.
\]

\[
0 < \text{const} = \|u(t)\|_2 \leq \| P_{\leq c(\eta)N(t)} u(t) \|_2 + \| P_{> c(\eta)N(t)} u(t) \|_2 \\
\lesssim \|u\|_{L^\infty_t H^{s_0}_x} \|\nabla P_{\leq c(\eta)N(t)} u(t)\|_{L^\infty_t L^2_x} + \frac{1}{c(\eta)N(t)} \|\nabla u(t)\|_2 \\
\lesssim \eta^{\frac{1}{r+2}} + \frac{1}{c(\eta)N(t)}.
\]

Suppose along a sequence $t_n \to \infty$, $N(t_n) \to \infty$, the last line of the above formula can be made arbitrarily small, which is clearly a contradiction. This completes the proof of this theorem. 

6. The soliton

In this section, we use virial argument to preclude the existence of soliton-like solution described in Theorem 1.12. First we note that for nonlinear Hartree equation, the momentum

$$P(u) = 2 \text{Im} \int \bar{u}(t, x) \nabla u(t, x) \, dx$$

(6.1)

is conserved in time. Moreover, for minimal kinetic energy solution $u$, the momentum is 0. This result is included in the following proposition. The original proof was given in [6].

**Proposition 6.1 (Zero momentum).** If $u$ is minimal kinetic energy blowup solution obeying $u \in L_t^\infty H_x^1$, then

$$P(u) = 0.$$ 

**Proof.** Applying Galilean transform to $u$, we get another solution of the equation:

$$\tilde{u}(t, x) = e^{ix\xi_0} e^{-it|\xi_0|^2} u(t, x - 2\xi_0 t),$$

which has the same scattering size with $u$, therefore,

$$S(\tilde{u}) = S(u) = \infty.$$ 

(6.2)

However, on the other hand, direct computation gives us

$$\|\nabla \tilde{u}(t)\|^2 = \|\nabla u(t)\|^2 + |\xi_0|^2 M(u) + \xi_0 P(u)$$

$$= \|\nabla u(t)\|^2 - \left[4M(u)\right]^{-1} P(u)^2,$$

by choosing $\xi_0 = -\frac{P(u)}{2M(u)}$. So, if $P(u) \neq 0$, then $\sup \|\nabla \tilde{u}(t)\|^2 < \sup \|\nabla u(t)\|^2$, in view of (6.2) and the fact that $u$ is minimal kinetic energy blow up solution, we get a contradiction. 

Next we prove

**Lemma 6.2 (Compactness in $L^2$).** $\forall \eta > 0$, there exists $C(\eta)$ such that

$$\sup_{t \in \mathbb{R}} \int_{|x - x(t)| > C(\eta)} |u(t, x)|^2 \, dx \leq \eta.$$ 

(6.3)
Proof. By space translation invariance, we can assume \( x(t) = 0 \). Let \( N \) be a dyadic number and \( R \) a large number to be specified later, we decompose \( u = u_{\leq N} + u_{> N} \), for the low frequency, we have

\[
\|u_{\leq N}\|_{L^2(|x| > R)} \leq \|u_{\leq N}\|_2 \leq N^\varepsilon \|\nabla\|^{-\varepsilon} u\|_{L^\infty_t L^2_x} \lesssim N^\varepsilon \leq \eta_{1/2}
\]

by taking \( N \) sufficiently small. For the chosen \( N \), we estimate the high frequency:

\[
\|u_{> N}\|_{L^2(|x| > R)} = \|\chi_{> R} u_{> N}\|_2 \\
\leq \|\chi_{> R} \Delta^{-1} \nabla P_{> N} \chi_{\leq \frac{R}{100}} \nabla u_{> N}\|_2 + \|\chi_{> R} \Delta^{-1} \nabla P_{> N} \chi_{> \frac{R}{100}} \nabla u_{> N}\|_2 \\
\leq N^{-1} (RN)^{-m} \|\nabla u_{> N}\|_2 + N^{-1} \|\chi_{> \frac{R}{100}} \nabla u_{> N}\|_2.
\]

Note in the last inequality, we have used the fact that: for any \( m > 0 \)

\[
\|\chi_{> R} \Delta^{-1} \nabla P_{> N} \chi_{\leq \frac{R}{100}} \nabla u_{> N}\|_2 \leq N^{-1} (RN)^m,
\]

which is a simple application of Schur’s test. Now by taking \( R \) sufficiently large and using the compactness of the solution, we conclude that

\[
\|u_{> N}\|_{L^2(|x| > R)} \leq \eta_{1/2}.
\]

Summing the estimates for low and high frequencies, we obtain (6.3). The lemma is proved. \( \square \)

Following the same argument in [6,11], we have the control over \( x(t) \). For the sake of the completeness, we record the proof.

Lemma 6.3 (Control over \( x(t) \)). Fix \( d \geq 5 \), and let \( u \) be the soliton in the sense of Theorem 1.12. Then

\[
x(t) = o(t) \quad \text{as} \quad t \to \infty.
\]

Proof. If by contradiction there exists \( \delta > 0 \) and a sequence \( t_n \to \infty \) such that

\[
|x(t_n)| > \delta t_n \quad \text{for all} \quad n \geq 1.
\]

By spatial translation invariance, we may assume that \( x(0) = 0 \).

Let \( \eta > 0 \) be a small constant to be chosen later. By compactness and Lemma 6.2, we have

\[
\sup_{t \in \mathbb{R}} \int_{|x-x(t)| > C(\eta)} (|\nabla u(t)|^2 + |u(t,x)|^2) \, dx \leq \eta. \tag{6.4}
\]

Define

\[
T_n := \inf_{t \in [0, t_n]} \{ t : x(t) = |x(t_n)| \} \leq t_n, \quad R_n := C(\eta) + \sup_{t \in [0, T_n]} |x(t)|. \tag{6.5}
\]
Let \( \phi \) be a smooth cutoff function such that \( \phi(r) = 1 \) as \( r \leq 1 \) and \( \phi(r) = 0 \) as \( r \geq 2 \), we define
\[
X_R(t) = \int x\phi\left(\frac{|x|}{R}\right)|u(t, x)|^2 \, dx.
\]
Using Lemma 4.1 and (6.4), we have
\[
|X_R(0)| \leq \int_{|x| \leq C(\eta)} x\phi\left(\frac{|x|}{R}\right)|u(0, x)|^2 \, dx + \int_{|x| \geq C(\eta)} x\phi\left(\frac{|x|}{R}\right)|u(0, x)|^2 \, dx \\
\leq C(\eta)M(u) + 2\eta R.
\]
On the other hand, by the triangle inequality combined with (6.4) and (6.5) we have
\[
|X_{R_n}(T_n)| \geq |x(T_n)|M(u) - |x(T_n)|\left|\int \left[1 - \phi\left(\frac{|x|}{R_n}\right)\right]u(T_n, x)|^2 \, dx\right| \\
- \left|\int_{|x-x(T_n)| \leq C(\eta)} \left[x - x(T_n)\right]\phi\left(\frac{|x|}{R_n}\right)|u(T_n, x)|^2 \, dx\right| \\
- \left|\int_{|x-x(T_n)| \geq C(\eta)} \left[x - x(T_n)\right]\phi\left(\frac{|x|}{R_n}\right)|u(T_n, x)|^2 \, dx\right| \\
\geq |x(T_n)||M(u) - \eta - C(\eta)M(u)| - 3C(\eta)M(u).
\]
Thus, taking \( \eta > 0 \) sufficiently small we have
\[
|X_{R_n}(T_n) - X_{R_n}(0)| \geq |x(T_n)| - C(\eta).
\]
A simple computation establishes
\[
\partial_t X_R(t) = 2 \text{Im} \int x\phi\left(\frac{|x|}{R}\right)\nabla u(t, x)\bar{u}(t, x) \, dx \\
+ 2 \text{Im} \int \frac{x}{|x|} \phi'\left(\frac{|x|}{R}\right)x \cdot \nabla u(t, x)\bar{u}(t, x) \, dx.
\]
By Proposition 6.1, \( P(u) = 0 \); together with Cauchy–Schwarz and (6.4) this yields
\[
|\partial_t X_{R_n}(t)| \leq 2 \text{Im} \int \left[1 - \phi\left(\frac{|x|}{R_n}\right)\right]\nabla u(t, x)\bar{u}(t, x) \, dx \\
+ 2 \text{Im} \int \frac{x}{|x|} \phi'\left(\frac{|x|}{R_n}\right)x \cdot \nabla u(t, x)\bar{u}(t, x) \, dx \\
\leq 6\eta
\]
for all \( t \in [0, T_n] \).
By the Fundamental Theorem of Calculus,

$$|x(T_n)| - C(\eta) \lesssim \eta T_n.$$ 

Recalling that $|x(T_n)| = |x(t_n)| > \eta t_n \geq \delta T_n$ and letting $n \to \infty$ we derive a contradiction. \(\square\)

We finally preclude the existence of soliton-like solution by using the finite mass property Theorem 4.1 and the virial argument. Again, we will follow the idea in [11]. Note in the radial case, the argument can also be found in [8]. We will prove that

**Theorem 6.4 (No soliton).** Let $d \geq 5$. A minimal kinetic energy blowup solution cannot be a soliton in the sense of Theorem 1.12.

**Proof.** Suppose for a contradiction that there exists such a solution $u$. Let $\eta > 0$ be specified later. Then we have

$$\sup_{t \in \mathbb{R}} \int_{|x-x(t)| > C(\eta)} \left( |\nabla u(t, x)|^2 + \left| u(t, x) \right|^2 \right)^{\frac{2d}{d-2}} dx \leq \eta. \quad (6.6)$$

Moreover, by Lemma 6.3, $|x(t)| = o(t)$ as $t \to \infty$. Thus, there exists $T_0 = T_0(\eta) \in \mathbb{R}$ such that

$$|x(t)| \leq \eta t \quad \text{for all } t \geq T_0. \quad (6.7)$$

Now let $\phi(r)$ be a smooth cutoff function such that $\phi(r) = 1$ as $r \leq 1$ and $\phi(r) = 0$ as $r \geq 2$, and define

$$V_R(t) = \int a(x) |u(t, x)|^2 dx,$$

where $R > 0$ and

$$a(x) = R^2 \phi \left( \frac{|x|^2}{R^2} \right).$$

Differentiating $V_R$ with respect to time, we find

$$\partial_t V_R(t) = 2 \text{Im} \int \phi' \left( \frac{|x|^2}{R^2} \right) \bar{u}(t, x) x \cdot \nabla u(t, x) dx.$$ 

From Theorem 4.1, we see that

$$|\partial_t V_R(t)| \lesssim R \|\nabla u(t)\|_2 \|u(t)\|_2 \lesssim R$$

for all $t$ and $R > 0$.

For a solution $u$ satisfying

$$iu_t + \Delta u = \mathcal{N},$$
further computation establishes that

$$
\partial_t^2 V_R(t) = \int_{\mathbb{R}^d} (\nabla \Delta) a(x) |u(t,x)|^2 \, dx + 4 \Re \int_{\mathbb{R}^d} a_{jk}(x) \bar{u}_j(t,x) u_k(t,x) \, dx \\
+ 2 \int_{\mathbb{R}^d} \{N, u\}_p(t,x) \nabla a(x) \, dx,
$$

(6.8)

where $\{f, g\}$ is the momentum bracket and is defined as $\Re(f \nabla \bar{g} - g \nabla \bar{f})$.

Now, plugging $N = -(|x|^{-4} \ast |u|^2) u$, we have

Lemma 6.5. We have that

$$
\partial_t^2 V_R(t) = 8 \int_{\mathbb{R}^d} |\nabla u(t,x)|^2 \, dx - 8 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(t,x)|^2 |u(t,y)|^2}{|x-y|^4} \, dx \, dy
$$

(6.9)

$$
+ O\left( \int_{|x| \geq R} |\nabla u(t,x)|^2 \, dx \right)
$$

(6.10)

$$
+ O\left( \frac{1}{R^2} \int_{R \leq |x| \leq 2R} |u(t,x)|^2 \, dx \right)
$$

(6.11)

$$
+ 16 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[ 1 - \phi'\left( \frac{|x|^2}{R^2} \right) \right] x(x-y)|u(t,y)|^2 |u(t,x)|^2 \frac{|x-y|^6}{|x-y|^6} \, dx \, dy.
$$

(6.12)

Proof. To estimate the first term in (6.8), we use the fact that

$$
|\Delta \Delta a(x)| \lesssim \frac{1}{R^2} \quad \text{and} \quad \text{supp} \ \Delta \Delta a(x) \in \left\{ x ; \ R \leq |x| \leq 2R \right\}.
$$

This contributes to term (6.11) in virial equality. The second term in (6.8) can be calculated as follows

$$
a_{jk}(x) u_j \bar{u}_k = (\nabla \bar{u})^T \nabla \nabla a \nabla u
$$

$$
= (\nabla u)^T \left( 2\phi'\left( \frac{|x|^2}{R^2} \right) \right) \nabla u + (\nabla \bar{u})^T \left( \frac{4}{R^2} \phi''\left( \frac{|x|^2}{R^2} \right) \right) x^T \nabla u
$$

$$
= 2\phi'\left( \frac{|x|^2}{R^2} \right) |\nabla u|^2 + \frac{4}{R^2} \phi''\left( \frac{|x|^2}{R^2} \right) |x \cdot \nabla u|^2.
$$
Therefore
\[
\int_{\mathbb{R}^d} a_{jk}(x) u_j(t,x) \tilde{u}_k(t,x) \, dx = 2 \int_{\mathbb{R}^d} |\nabla u(t,x)|^2 \, dx + O\left( \int_{|x| \geq R} |\nabla u(t,x)|^2 \, dx \right).
\]

Finally we compute the last term in (6.8). Note that by direct calculation
\[
\{\mathcal{N}, u\}_p(t,x) = -4 \int_{\mathbb{R}^d} \frac{(x-y)|u(t,y)||u(t,x)|^2}{|x-y|^6} \, dy
\]
we have
\[
2 \int_{\mathbb{R}^d} \{\mathcal{N}, u\}_p(t,x) \nabla a(x) \, dx = 4 \int_{\mathbb{R}^d} x \phi'\left( \frac{|x|^2}{R^2} \right) \{\mathcal{N}, u\}_p(t,x) \, dx
\]
\[
= -16 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{x(x-y)|u(t,y)||u(t,x)|^2}{|x-y|^6} \, dx \, dy
\]
\[
= -16 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{x(x-y)|u(t,y)||u(t,x)|^2}{|x-y|^6} \, dx \, dy \\
+ 16 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[ 1 - \phi'\left( \frac{|x|^2}{R^2} \right) \right] \frac{x(x-y)|u(t,y)||u(t,x)|^2}{|x-y|^6} \, dx \, dy
\]
\[
= -8 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(t,y)|^2|u(t,x)|^2}{|x-y|^4} \, dx \, dy \\
+ 16 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[ 1 - \phi'\left( \frac{|x|^2}{R^2} \right) \right] \frac{x(x-y)|u(t,y)||u(t,x)|^2}{|x-y|^6} \, dx \, dy.
\]

In the last step, we have used the symmetrization. This ends the proof of the lemma. \[\Box\]

We continue the proof of the theorem. From the property of \(W\) in Lemma 2.5, we know that
\[
\int_{\mathbb{R}^d} |\nabla u(t,x)|^2 \, dx - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(t,x)|^2|u(t,y)|^2}{|x-y|^4} \, dx \, dy \gtrsim \|\nabla u_0\|_2^2,
\]
now we show this implies
\[
\partial_{tt} V_R(t) \gtrsim \|\nabla u_0\|_2^2,
\] (6.13)
if we take $R$ large enough, say $R = C(\eta) + \sup_{T_0 \leq t \leq T_1} |x(t)|$. Indeed, as a consequence of compactness and finite mass property Theorem 4.1, we have

$$\tag{6.11} (6.10) \leq \eta.$$  

We now turn our attention to (6.12), which by symmetrization, can be written as

$$8 \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|x\psi(|x|) - y\psi(|y|)|(x - y)|u(t, y)|^2|u(t, x)|^2}{|x - y|^6} \, dx \, dy,$$  

where $\psi(|x|) = 1 - \frac{1}{|x|^2}$. In the region where $|x| \sim |y|$, we have

$$|x| \sim |y| \gtrsim R,$$

since otherwise (6.14) vanish. Moreover, note that

$$x\psi(|x|) - y\psi(|y|) \lesssim |x - y|,$$

we use Hölder inequality to control the contribution to (6.14) from this regime by

$$\int \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x - y|^4} X_{|y| \gtrsim R} |u(t, y)|^2 X_{|x| \gtrsim R} |u(t, x)|^2 \, dx \, dy$$

$$\lesssim \|\nabla|^4-d (X_{|y| \gtrsim R} |u(t)|^2)\|_{L^2_{\xi}} \|X_{|x| \gtrsim R} |u(t)|^2\|_{L^{d-2}_{\xi}}$$

$$\lesssim \|X_{|y| \gtrsim R} u\|_{L^{\frac{4d}{d-2}}_{\xi}} \lesssim \eta.$$  

In the region where $|x| \ll |y|$, we use the fact that

$$|x| \ll |y| \sim |x - y| \quad \text{and} \quad |y| \gtrsim R$$

to estimate the contribution from this regime by

$$\int \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x - y|^4} X_{|y| \gtrsim R} |u(t, y)|^2 |u(t, x)|^2 \, dx \, dy \lesssim \eta.$$  

The last line follows from the same computation as the first case. Finally, since the remaining region $|y| \ll |x|$ can be estimated in the same way, we conclude that

$$\tag{6.12} \lesssim \eta.$$  

Taking $\eta$ sufficiently small, we verify (6.13).

Finally, we use (6.13) to derive a contradiction. Using the Fundamental Theorem of Calculus on the interval $[T_0, T_1]$, we obtain

$$ \tag{6.15} (T_1 - T_0) \|\nabla u_0\|_2^2 \lesssim R \lesssim C(\eta) + \sup_{T_0 \leq t \leq T_1} |x(t)|.$$
for all $T_1 \geq T_0$. Invoking (6.7) and taking $\eta$ sufficiently small, we derive a contradiction unless $u_0 = 0$, which is not consistent with the fact that $S_R = \infty$. □

Acknowledgments

C. Miao is supported under NSF grant No. 10725102. D. Li and X. Zhang are supported under NSF grant DMS-0635607. X. Zhang is also supported under NSF grant No. 10601060, Project 973 and the Knowledge Innovation Program of CAS. The authors would also like to thank the anonymous referee for the careful reading of the manuscript.

References