Immersed surfaces and Seifert fibered surgery on Montesinos knots

Ying-Qing Wu

Abstract

We will use immersed surfaces to study Seifert fibered surgery on Montesinos knots, and show that if $\frac{1}{q_1-1} + \frac{1}{q_2-1} + \frac{1}{q_3-1} \leq 1$ then a Montesinos knot $K(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3})$ admits no atoroidal Seifert fibered surgery.

1 Introduction

Exceptional Dehn surgeries on arborescent knots have been studied extensively. They have been classified for arborescent knots of length at least 4 [Wu2], as well as for all 2-bridge knots [BW]. There is no reducible surgery on hyperbolic arborescent knots [Wu1], and toroidal surgeries on length 3 Montesinos knots have also been classified [Wu3]. Therefore atoroidal Seifert fibered surgeries on Montesinos knots of length 3 are the only ones on arborescent knots that have not been determined.

Atoroidal Seifert fibered surgery is much more difficult to deal with than other types of exceptional surgeries. For example, the minimum upper bounds for distances between two types of exceptional Dehn fillings on hyperbolic manifolds have all been determined when those are not atoroidal Seifert fibered, but no such bound is known when one of them is. See [GW] and the references there for works in that direction. The major difficulty to deal with atoroidal Seifert fibered surgery is that there is no embedded essential small surfaces (sphere, disk, annulus or torus) in such manifolds and therefore one cannot use those traditional combinatorial methods on intersection graphs for such surgery problems. Those with infinite fundamental group do contain immersed essential tori, but there had not been

[1]Mathematics subject classification: *Primary 57N10.*

[2]Keywords and phrases: Immersed surfaces, Dehn surgery, Seifert fibered manifolds, Montesinos knots

much success using them as tools in solving Dehn surgery problems. In this paper, however, we *will* use those immersed surfaces as a major tool. We will study the intersection of an immersed surface F with the tangle decomposition surfaces and the tangle spaces, and show that no such surface could exist when the tangles are not too simple.

A length 3 Montesinos knot $K(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3})$ is the cyclic union of three rational tangles $T(\frac{p_i}{q_i})$, where $q_i \geq 2$ and p_i, q_i are coprime. The integer part of the p_i/q_i can be shifted around, so we will always assume that $2|p_i| \leq q_i$ for $i = 1, 2$. Given a knot K in S^3, we use $K(r)$ to denote the manifold obtained by Dehn surgery on K along a slope r on $\partial N(K)$, where $N(K)$ is a tubular neighborhood of K. The following is our main theorem.

Theorem 1.1 *Suppose $K = K(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3})$ is a hyperbolic Montesinos knot of length 3. If $\frac{1}{q_1-1} + \frac{1}{q_2-1} + \frac{1}{q_3-1} \leq 1$ then K admits no atoroidal Seifert fibered surgery.*

Recently Ichihara and Jong [IJ2] showed that the only toroidal Seifert fibered surgery on Montesinos knots is the 0 surgery on the trefoil knot, hence the theorem is still true with the word "atoroidal" deleted. By [Wu1] there is no reducible surgery on hyperbolic Montesinos knots, so the following result follows immediately from Theorem 1.1 and the classification of toroidal surgeries on these knots [Wu3, Theorem 1.1].

Corollary 1.2 *Suppose $K = K(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3})$ is a hyperbolic Montesinos knot of length 3. If $\frac{1}{q_1-1} + \frac{1}{q_2-1} + \frac{1}{q_3-1} \leq 1$ and a Dehn surgery $K(r)$ is nonhyperbolic, then $|p_i| = 1$, r is the pretzel slope, and $K(r)$ is toroidal.*

We may assume $2 \leq q_1 \leq q_2 \leq q_3$. Corollary 1.2 and [BW, Wu2] provide a classification of exceptional Dehn surgeries on all arborescent knots except those $K(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3})$ with $q_1 = 2$, or $(q_1, q_2) = (3, 3)$, or $(q_1, q_2, q_3) = (3, 4, 5)$. Further restrictions on p_i and the surgery slopes for these cases will be given in Theorems 8.2.

Theorem 1.1 is known in the special case that $K(r)$ has finite fundamental group. The classification of finite surgeries on Montesinos knots has been completed by Ichihara and Jong [IJ1]. It used the results of Delman [De] on essential laminations, Mattman's result [Ma] on surgery on pretzel knots, and Ni's result [Ni] on Heegaard Floer homology and fibered knots. See also [Wa] and [FIKMS]. While our major goal is to prove Theorem 1.1 for infinite Seifert fibered surgeries, we will also provide an independent proof of it for the finite surgery case. We extend the thin position idea of Gabai [Ga] and use an immersed thin sphere to replace the immersed essential torus in the case that $K(r)$ has infinite fundamental group. This works fine in our setting and sometimes the proof is simpler than for infinite surgery case since the surface is now a sphere instead of a torus.

The paper is organized as follows. Section 2 is to set up some notations and conventions, and introduce some basic lemmas. Section 3 discusses immersed essential disks in tangle spaces. It will be shown that any such disk must intersect the axis of the tangle at some minimal number of points, and the disks are embedded and standard in certain cases. In Section 4 we define immersed surfaces that are in essential position with respect to an essential embedded surface and prove its existence in manifolds with finite fundamental groups. Section 5 defines elementary surfaces and shows that if the surface F coming from an immersed π_1-injective torus is elementary then the surgered manifold is either toroidal or the connected sum of two lens spaces. This is crucial to deal with the fact that some of the knots excluded in Theorems 1.1 and 8.2 admit toroidal surgeries and hence the surgered manifold does contain π_1-injective tori. In Section 6 we define intersection graphs and prove some basic properties of such graphs. Section 7 defines angled Euler numbers and show that it is additive. Section 8 completes the proof of the main theorems.

I would like to thank the referee for careful reading and some helpful comments.

2 Preliminaries

In this paper we will consider both embedded surfaces and non-embedded surfaces in 3-manifolds. We always assume that surfaces and curves intersect transversely. Unless otherwise stated, surfaces F in a 3-manifold M are assumed to have boundaries on the boundary of M, and a homotopy of F refers to a relative homotopy of the pair $(F, \partial F)$ in $(M, \partial M)$, i.e. a homotopy $f_t : F \to M$ such that $f_t(\partial F) \subset \partial M$ for all t. Similarly for arcs on surfaces.

An arc α on a surface F is trivial if it is rel $\partial\alpha$ homotopic to an arc on ∂F. If α is closed then it is trivial if and only if it is null homotopic on F. A (possibly non-embedded) disk D in a 3-manifold M is *nontrivial* if it is not rel ∂D homotopic to a disk on ∂M. If M is irreducible (in particular if M is the tangle space $E(t)$ below), then D in M is nontrivial if and only if ∂D is a nontrivial curve on ∂M. A curve or disk is *essential* if it is nontrivial. Two (possibly immersed) curves C_1, C_2 on a surface F *intersect minimally* if there are no subarc $a_i \subset C_i$ such that $\partial a_1 = \partial a_2$ and the loop $a_1 \cup a_2$

is null homotopic on F. When C_1, C_2 are embedded, this is equivalent to say that $C_1 \cup C_2$ contains no bigons on F, i.e. there is no arcs $a_i \subset C_i$ with $\partial a_1 = \partial a_2$ such that $a_1 \cup a_2$ bounds a disk on F with interior disjoint from $C_1 \cup C_2$.

In this paper a *tangle* is a triple $T = (B, t, m)$, where B is a fixed 3-ball, $t = t_1 \cup t_2$ is a pair of arcs properly embedded in B, and m is a simple loop on ∂B, cutting ∂B into two disks, called the left disk and the right disk, each containing two points of ∂t. The curve m is called the *axis* of the tangle. Two tangles (B, t, m) and (B', t', m') are *equivalent* if they are homeomorphic as a triple. They are *strongly equivalent* if $B = B'$ and the homeomorphism is the identity on ∂B.

Denote by $N(t)$ a regular neighborhood of t, and by $E(t) = B - \text{Int}N(t)$ the exterior of t, which will also be called the *tangle space* of (B, t, m). Denote by $A(t)$ the two annuli $A(t) = \partial N(t) \cap E(t) = A_1(t) \cup A_2(t)$ on $\partial E(t)$, by $P(t)$ the 4-punctured sphere $\partial B \cap E(t)$, and by $P_1(t) \cup P_2(t)$ the two twice punctured disks obtained by cutting $P(t)$ along m. If C is a properly embedded $n-1$ manifold in an n-manifold F, denote by $F|C$ the manifold obtained by cutting F along C.

Definition 2.1 A homotopy or isotopy h_x of $E(t)$ or $\partial E(t)$ is *P-preserving* if h_x maps each of the set $A(t), P_1(t), P_2(t), m$ to itself during the homotopy. Similarly, if C is a curve or surface in $E(t)$ then a P-preserving homotopy or isotopy h_x of C is such that $C \cap Y$ is mapped to Y for $Y = A(t), P_1(t), P_2(t), m$ and all $x \in [0, 1]$.

A tangle $T = (B, t, m)$ is a *p/q rational tangle*, denoted by $T(p/q)$, if B is isotopic to a pillowcase with the four points of ∂t as the cone points and m a vertical circle, and t is rel ∂t isotopic to a pair of arcs on ∂B of slope p/q. See [HT]. By definition $T(p/q)$ is equivalent to $T(p'/q')$ if $p/q \equiv p'/q'$ mod 1. The tangle is trivial if $q = 0$ or 1. We will always assume $q \geq 2$ and hence $T(p/q)$ is nontrivial. Denote by $E(p/q)$ the tangle space $E(t)$ if $(B, t, m) = T(p/q)$.

Definition 2.2 We use $\bar{p} = \bar{p}(p, q)$ to denote the mod q inverse of $-p$ with minimal absolute value, i.e., $\bar{p}$ satisfies $p\bar{p} \equiv -1$ mod q, and $2|\bar{p}| \leq q$. Similarly, $\bar{p}_i$ denotes $\bar{p}(p_i, q_i)$ throughout the paper.

By a deformation of B one can see that $T(p/q) = (B, t, m)$ can be isotoped so that t is rel ∂ isotopic to a pair of vertical arcs, and m is a curve of slope $\bar{p}/q$ on ∂B. See Figure 2.1, where (a) is the standard picture of a

$T(1/3)$, and (b) is $T(1/3)$ after the deformation. We have $\bar{p} = -1$ in this case, so m is a curve of slope $-1/3$ on ∂B. In general this can be proved by lifting the boundary map to the universal cover of ∂B, considered as a sphere with four cone points of order 2. It is represented by a matrix A with $\det(A) = 1$, which maps a line of slope p/q to $1/0$. One can show that A maps the line of slope $1/0$ to $\bar{p}/q$. We will use both points of view for $T(p/q)$ in below.

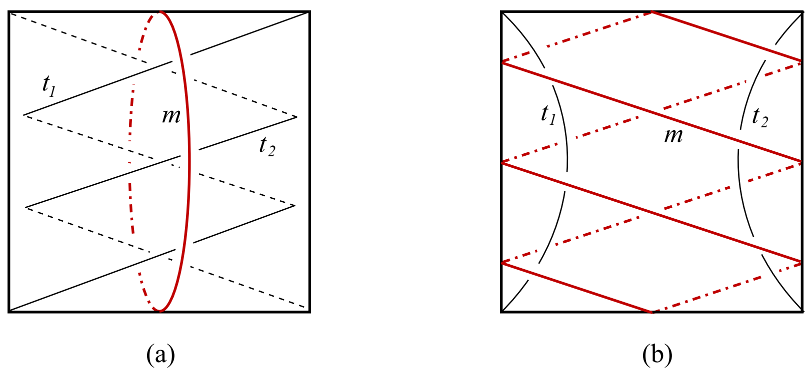
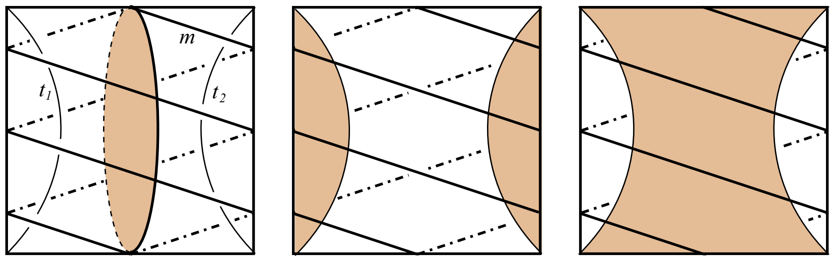


Figure 2.1

Let E_0 be an embedded disk in B that separates the two arcs of t and intersects m in $2q$ points, see Figure 2.2(a). Let E_i $(i = 1, 2)$ be embedded disks with $\partial E_i = t_i \cup \alpha_i$, where α_i is an arc on ∂B intersecting m at q points. See Figure 2.2(b). These are chosen to be disjoint from each other. Now let E_3 be the disk in Figure 2.2(c), which intersects E_0 in one arc but has interior disjoint from E_1 and E_2, and we have $\partial E_3 = t_1 \cup \beta_1 \cup t_2 \cup \beta_2$, where β_1, β_2 are arcs on ∂B, each intersecting m at $|\bar{p}|$ points. Thus $|\partial E_3 \cap m| = 2|\bar{p}|$. Note that E_3 is unique up to isotopy when $q > 2$, and there are two such E_3 when $p/q = 1/2$.



(a) (b) (c)

Figure 2.2

We call E_0, E_1, E_2, E_3 the *standard disks* for (B,t). We will also use the same notation E_i to denote the disk $E_i \cap E(t)$ when dealing with disks in $E(t)$.

An embedded disk D in $E(t)$ is *tight* if it intersects m and $A(t)$ minimally up to isotopy. D is an (r,s) disk if $|\partial D \cap A(t)| = r$ and $|\partial D \cap m| = s$. See Definition 3.1 for general definitions of tight curves and tight immersed disks. We need results which show that certain immersed disks are standard. As a warm up, we have the following lemma, which says that the standard disks are the standard models of tight (r,s) disks in $E(t)$ when $r \leq 1$, or $r = 2$ and s is minimal. The proof is a standard innermost circle outermost arc argument by considering the intersection of D with $E_1 \cup E_2$, and is omitted. Certain version of (2) and (3) is also true for immersed disks when changing isotopy to homotopy. See Lemmas 3.5 and 3.6.

Lemma 2.3 *Suppose (B,t,m) is a p/q rational tangle. Let $\bar{p}$ be as in Definition 2.2.*

(1) If D is am embedded tight $(0,s)$ disk in $E(t)$ then $s = 2q$ and D is P-isotopic to the standard disk E_0.

(2) If D is an embedded tight $(1,s)$ disk in $E(t)$ then $s = q$ and D is P-isotopic to the standard disk E_1 or E_2.

(3) If D is an embedded tight $(2,s)$ disk then $s \geq 2|\bar{p}|$, and if $s = 2|\bar{p}|$ then D is P-isotopic to a standard disk E_3.

(4) Standard disk of type E_i are unique up to isotopy unless $q = 2$ and $i = 3$, in which case there are exactly two such isotopy classes.

Denote by $|X|$ the number of components of X. Note that if X is an immersed curve or surface in a manifold M then it may have intersection between different components, in which case $|X|$ denotes the number of components before immersion, not the number of the components of its image in M. Thus two components of an immersed surface which intersect in M are still considered as different components when counting $|X|$.

3 Immersed disks in tangle spaces

Definition 3.1 (1) A (possibly non-simple) closed curve C on $\partial E(t)$ is a *tight curve* if each component of $C \cap A(t)$, $C \cap P_1(t)$ and $C \cap P_2(t)$ is nontrivial.

(2) A (possibly non-closed) curve C is an (r,s)*-curve* (or a curve of type (r,s)) if $|\partial D \cap A(t)| = r$, and $|\partial D \cap m| = s$.

(3) A disk D in $E(t)$ is an (r,s)*-disk* (or a disk of type (r,s)) if $\partial D \cap \partial E(t)$ is a (r,s)-curve on $\partial E(t)$. It is a *tight disk* if $\partial D \subset \partial E(t)$ is tight.

Note that if C is a tight curve on $\partial E(t)$ then each arc component of $C \cap A(t)$ is homotopic to an embedded arc. However, $C \cap P_i(t)$ may contain arcs which have self intersections that cannot be removed by homotopy.

Lemma 3.2 *(1) Suppose C is a tight curve on $F = \partial E(t)$. Then it has minimal intersection with both m and $A(t)$ up to homotopy.*

(2) Any curve C on F is homotopic to a tight curve C', which is unique up to P-homotopy. In particular, if two tight curves are homotopic then they are P-homotopic.

(3) Suppose C, C' are tight curves on $\partial E(t)$. If there are arcs $\alpha \subset C$ and $\beta \subset C'$ such that $\partial\alpha = \partial\beta$ and $\alpha \cup \beta$ is a trivial loop on $\partial E(t)$, then $|\alpha \cap m| = |\beta \cap m|$ and $|\alpha \cap A(t)| = |\beta \cap A(t)|$.

Proof. (1) If C is homotopic to C_1 which has fewer intersection with m, say, then the homotopy is a map φ from an annulus $H = S^1 \times I$ to $\partial E(t)$. Since $m \cup \partial A(t)$ is embedded, by transversality $\gamma = \varphi^{-1}(m \cup \partial A(t))$ is an embedded 1-manifold on H. By a homotopy we may assume γ has no trivial loops. Since $|C \cap m| > |C_1 \cap m|$, γ contains a trivial arc on H with both endpoints on C. An outermost such arc in γ then cuts off a disk D which gives rise to a homotopy from an arc α of C to an arc on m or $\partial A(t)$, and the image of the interior of D is disjoint from $m \cup \partial A(t)$ and hence is mapped into some $P_i(t)$ or $A(t)$. It follows that α is a trivial arc on $P_i(t)$ or $A(t)$, contradicting the assumption that C is tight.

(2) It is clear that any curve is homotopic to a tight curve. We only need to prove the uniqueness. Let $\varphi : H \to \partial E(t)$ be a homotopy from C_1 to C_2 and assume C_i are tight. Consider $\gamma = \varphi^{-1}(m \cup \partial A(t))$. As in the proof of (1) we may assume φ has no trivial loops, and the tightness of C_i implies that there is no trivial arcs, hence each component of γ is an essential arc.

Deform φ so that γ is a product $X \times I \subset S^1 \times I = H$, where X is a finite set in S^1. Clearly φ is now a P-homotopy since it maps each $z \times I$ to an arc in some $P_j(t)$ or $A(t)$.

(3) Let $\varphi : D \to \partial E(t)$ be a map with ∂D mapped to $\alpha \cup \beta$, chosen to have minimal intersection with $m \cup \partial A(t)$. Then $\varphi^{-1}(m \cup \partial A(t))$ has no loops, and each arc must have one end on α and the other on β as otherwise we can get a contradiction as above. It follows that $|\alpha \cap \gamma| = |\beta \cap \gamma|$ for each component γ of $m \cup \partial A(t)$. $\square$

Lemma 3.3 *Suppose that (B, t, m) is a p/q rational tangle with $q \geq 2$, and D is a nontrivial $(0, s)$ disk in $E(t)$. Then $s \geq 2q$.*

Proof. Let E_0 be the standard disk separating t_1, t_2, as defined in Section 2. Then E_0 cuts the tangle space $E(T)$ into two solid tori V_1, V_2, and ∂E_0 cuts the 4-punctured disk $\partial B \cap E(t)$ into two twice punctured disks Q_1, Q_2. We may assume that D is tight, and $D \cap E_0$ consists of arcs, each of which is embedded in E_0.

Assume to the contrary that $s < 2q$. Among all such disks, choose D so that $k = |\partial D \cap \partial E_0|$ is minimal. First assume $k > 0$ and let α be an arc component of $D \cap E_0$, which cuts E_0 into D'_1 and D'_2. Without loss of generality we may assume that $|D'_1 \cap m| \leq q$. Cutting D along α and pasting two copies of D'_1, we obtain two disks D_1, D_2. Since $|\partial D_1 \cap m| + |\partial D_2 \cap m| = |\partial D \cap m| + 2|\partial D'_1 \cap m| < 4q$, one of the D_i, say D_1, has $|\partial D_1 \cap m| < 2q$. Since D_1 can be perturbed to have fewer intersection with E_0, by our choice of D the curve ∂D_1 must be trivial. Write $\partial D_1 = \alpha \cup \beta$ with $\alpha \subset \partial D$ and $\beta \subset \partial E_0$. Since both D and E_0 are tight, by Lemma 3.2(3) we have $|\alpha \cap m| = |\beta \cap m|$. We can now homotope D so that α is deformed to β and then push off E_0. This reduces $|\partial D \cap \partial E_0|$ without changing $|\partial D \cap m|$, contradicting the choice of D.

We now assume that $\partial D \cap \partial E_0 = \emptyset$, so $D \subset V_1$, say. Note that m intersects Q_1 in q arcs $\alpha_1, ..., \alpha_q$, each cutting Q_1 into a surface $F_i \subset Q_1$ which is the union of two longitudinal annuli. If ∂D is disjoint from some α_i then $\partial D \subset F_i$, so ∂D would be null homotopic on F_i because F_i is longitudinal while ∂D is null homotopic in V_1, which contradicts the assumption that D is nontrivial. Hence $|\partial D \cap \alpha_i| > 0$. Let β_i be an arc on E_0 with $\partial \beta_i = \partial \alpha_i$. If $|\partial D \cap \alpha_i| = 1$ then we would have two closed curves ∂D and $\alpha_i \cup \beta_i$ on the annulus $Q_1 \cup E_0$ intersecting transversely at a single point, which is absurd. Therefore $|\partial D \cap \alpha_i| \geq 2$ for each i, hence $s = |\partial D \cap m| \geq 2q$. $\square$

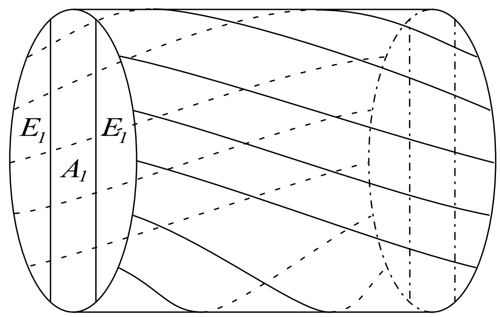


Figure 3.1

Lemma 3.4 *Suppose that (B, t, m) is a p/q rational tangle. Let $\bar{p}$ be as in Definition 2.2. If D is a nontrivial (r, s) disk in $E(t)$, then $|s| \geq q$ if r is odd, and $s \geq 2|\bar{p}|$ if r is even.*

Proof. We may assume that D is tight. Consider the standard disks E_1, E_2 defined in Section 2. Let $E = E_1 \cup E_2$. We proceed by induction on $(r, |D \cap E|)$. By Lemma 3.3 the result is true if $r = 0$. Cutting $E(t)$ along E produces a 3-ball on which $P(t)$ becomes an annulus F and m becomes a set of $2q$ essential arcs on F. See Figure 3.1 for the case $q = 5$ and $\bar{p} = 2$. Each boundary component ∂_i of F consists of 4 arcs, $\partial_i = a_i' \cup b_i' \cup a_i'' \cup b_i''$, where a_i', a_i'' are from $\partial A_i(t)$ and b_i', b_i'' are copies of $\partial E_i \cap P(t)$. From Figure 3.1 it is easy to see that the condition $2|\bar{p}| \leq q$ in the definition of $\bar{p}$ implies that any arc γ with endpoints on $a_1' \cup a_1'' \cup a_2' \cup a_2''$ either is rel ∂ homotopic to an arc on one of the a_i' or a_i'' and hence is trivial on $P(t)$, or it intersects m at least $|\bar{p}|$ times. Moreover, if γ has both endpoints on the same boundary component of F then it intersects m at least q times. If r is odd then there is at least one arc with both endpoints on the same component of ∂F, which intersects m at least q times; if r is even then there are at least two arcs, so they intersects m at least $2|\bar{p}|$ times. Hence the result is true if $|D \cap E| = 0$.

Now suppose $|D \cap E| > 0$ and let α be an arc component of $D \cap E_i$. Then α cuts D into D', D'' and it cuts E_i into E_i', E_i'', with E_i'' the one containing the arc on $A(t)$. The four disks $D_1 = D' \cup E_i'$, $D_2 = D' \cup E_i''$, $D_3 = D'' \cup E_i'$, and $D_4 = D'' \cup E_i''$ are nontrivial. For if D_1 is trivial, say, then we can homotope D' to E_i' and then push off E to reduce $|D \cap E|$; by Lemma 3.2(3) this will not change $|D \cap m|$ and $|D \cap A(t)|$ and hence is a contradiction to the minimality of $|D \cap E|$.

Suppose D', D'', E_i', E_i'' are of types (r_1, s_1), (r_2, s_2), $(0, s_3)$, $(1, s_4)$, respectively. Then $r = r_1 + r_2$, $s = s_1 + s_2$, and $s_3 + s_4 = q$. The type of each

D_i is the sum of the types of the corresponding subdisks, so the four disks D_i are of the following types.

D_1 : $(r_1,\, s_1+s_3)$
D_2 : $(r_1+1,\, s_1+s_4)$
D_3 : $(r_2,\, s_2+s_3)$
D_4 : $(r_2+1,\, s_2+s_4)$

First assume r is odd. Then we may assume without loss of generality that r_1 and r_2+1 are odd. This implies that $r_1, r_2+1 \le r$. Apply induction to D_1 and D_4 (note that they can be deformed to have fewer intersection with E), we have $s_1+s_3 \ge q$, $s_2+s_4 \ge q$. Adding these together gives $s+q = s_1+s_2+s_3+s_4 \ge 2q$, hence $s \ge q$.

Now assume r is even. If r_1, r_2 are odd then $r_i+1 \le r$, so by induction we have $s_1+s_3 \ge q$, $s_1+s_4 \ge 2|\bar{p}|$, $s_2+s_3 \ge q$, and $s_2+s_4 \ge 2|\bar{p}|$. Adding these together gives $2s+2q = 2(s_1+s_2+s_3+s_4) \ge 2q+4|\bar{p}|$, hence $s \ge 2|\bar{p}|$. If both r_i are even and nonzero then a similar argument as above applies. So we now assume that $r_1 = 0$ and $r_2 = r \ge 2$. Then we can apply induction to D_2 and D_3 to get $s_1+s_4 \ge q$ and $s_2+s_3 \ge 2|\bar{p}|$. Add these together gives $q+s = s_1+s_2+s_3+s_4 \ge q+2|\bar{p}|$, hence $s \ge 2|\bar{p}|$, as required. □

Suppose F is an immersed surface represented by $\varphi : \tilde{F} \to M$. Let S be an embedded surface in M. Then $C = \varphi^{-1}(S)$ is an embedded 1-manifold on $\tilde{F}$, and $\varphi(C)$ is an immersed 1-manifold on S. As in the embedded case, we use $C = F \cap S$ to denote both the 1-manifold $C = \varphi^{-1}(S)$ on $\tilde{F}$ and the immersed curve $\varphi : C \to S$ on S. To simplify notation we will not distinguish $\tilde{F}$ and F and simply refer $C \subset \tilde{F}$ as $C \subset F$. Thus the curves C above is still considered embedded on F even though it may have self intersection when considering F as a subset of M.

The following two lemmas show that immersed tight $(1,q)$ disks and $(2, 2|\bar{p}|)$ disks are standard up to P-homotopy.

Lemma 3.5 *Suppose D is an immersed tight $(2, 2|\bar{p}|)$ disk in $E(p/q)$. Then it is P-homotopic to a standard disk E_3 defined in Section 2.*

Proof. By Lemma 3.2(2), if two tight curves are homotopic then they are P-homotopic; hence we need only show that D is homotopic to some E_3.

If $|D \cap E| = 0$ then D lies on the 3-ball shown in Figure 3.1, with ∂D consisting of one arc on each $A_i(t)$ and two arcs c_1, c_2 on the annulus F obtained by cutting $P(t)$ along E. From Figure 3.1 we can see that the minimal intersection number between m and a nontrivial arc on F is $|\bar{p}|$, and there are exactly two such arc if $q > 2$, or four such arcs if $q = 2$. These are on the boundary of the standard disks of type E_3. The assumption that D is a $(2, 2|\bar{p}|)$ disk implies that $|c_i \cap m| = |\bar{p}|$. Hence we can deform c_i by a homotopy of (F, m) so that ∂D matches the boundary of a standard disk E_3. Since $E(t)$ is irreducible, a further P-homotopy deforms D to E_3.

Now assume $|D \cap E| > 0$. Examine the proof of Lemma 3.4. If one of ∂D_i $(i = 1, 2, 3, 4)$ is trivial then we may reduce $|D \cap E|$ by P-homotopy and the result follows by induction, so we may assume that they are all nontrivial.

We may assume $s_1 \leq s_2$, so $s_1 + s_2 = 2|\bar{p}| \leq q$ implies $s_1 < q$. Recall that $s_3 + s_4 = q$. By the proof of Lemma 3.4, if $r_1 = 0$ then D_1 would be a $(0, s_1 + s_3)$ disk, but since $s_1 < q$ and $s_3 \leq q$, we would have $s_1 + s_3 < 2q$, contradicting Lemma 3.3. Therefore $r_1 = r_2 = 1$. Since D_1 is a $(1, s_1 + s_3)$ disk, by Lemma 3.4 we have $s_1 + s_3 \geq q$, so $s_3 + s_4 = q$ implies $s_1 \geq s_4$. Similarly we have $s_2 \geq s_4$. Since D_2 is of type $(2, s_1 + s_4)$, we have $2|\bar{p}| = s_1 + s_2 \geq s_1 + s_4 \geq 2|\bar{p}|$, where the last inequality follows from Lemma 3.4. Hence we must have $s_2 = s_4$. By symmetry we have $s_1 = s_4$. Thus both D_2 and D_4 are $(2, 2|\bar{p}|)$ disks, so by induction they are P-homotopic to E_3. On the other hand, by construction one of the D_2, D_4 has the property that it has both arcs of $D_i \cap A(t)$ on the same component of $A(t)$, which is a contradiction because E_3 has one edge on each of $A_1(t)$ and $A_2(t)$. □

Lemma 3.6 *Suppose D is a tight $(1, q)$ disk. Then it is P-homotopic to the standard disk E_1 or E_2.*

Proof. Let $D_1, ..., D_4$ and r_i, s_i be as defined in the proof of Lemma 3.4. The result is clear when $|D \cap E| = 0$, and it follows by induction if one of the D_i is trivial, so assume $|D \cap E| > 0$ and ∂D_i is nontrivial for $i = 1, 2, 3, 4$. Since $r = r_1 + r_2 = 1$, we may assume $r_1 = 0$. The disk D_1 is of type $(0, s_1 + s_3)$, so by Lemma 3.3 we must have $s_1 + s_3 \geq 2q$. On the other hand, we have $s_1 + s_2 = s_3 + s_4 = q$, so $s_2 = s_4 = 0$. Now D_4 is a $(2, 0)$ disk, contradicting Lemma 3.4. □

We note that a statement similar to Lemmas 3.5 and 3.6 is not true for $(0, 2q)$ disks. A non-standard $(0, 2q)$ disk can be formed by winding E_0 around $A_i(t)$. In other words, we can take E_0 and n copies of A_1 or A_2 (but not both) and do cut and paste to form a disk which is still an immersed $(0, 2q)$ disk. We call such a surface an *n-winding disk* if n is the minimal number of tubes required in the above construction. The curve m cuts the boundary of a $(0, 2q)$ disk D into $2q$ arcs. We leave it to the reader to verify that if D is an n-winding disk then two of those $2q$ arcs must have self intersection if $n > 0$.

Lemma 3.7 *Suppose D is a tight $(0, 2q)$ disk in $E(p/q)$. Then D is P-homotopic to an n-winding disk for some n. In particular, if each component of $\partial D \cap P(t)$ is an embedded arc then D is P-homotopic to E_0.*

Proof. First assume $D \cap E = \emptyset$. Cut $E(t)$ along E produces a $D^2 \times I$, where $P(t)$ becomes an annulus F and m becomes a set of $2q$ parallel essential arcs on F, so it cuts F into $2q$ squares. See Figure 3.1. Since $|\partial D \cap m| = 2q$ and D is tight, m cuts ∂D into $2q$ arcs, each of which is nontrivial and hence connects one component of m on F to another. It follows that each square contains exactly one arc of ∂D, which can be straightened inside of the square. Therefore ∂D is homotopic to the core of F, and D is homotopic to E_0. By Lemma 3.2(2) D is P-homotopic to E_0.

Now assume $|D \cap E| > 0$. Consider an arc $\alpha \subset D \cap E$ which is outermost on D. Let D', D'', E', E'' and $D_1, ..., D_4$ be as in the proof of Lemma 3.4, with D' an outermost disk on D. Then D_1 is a $(0, s_1 + s_3)$ disk and D_3 is a $(0, s_2 + s_3)$ disk, so we have $s_1 + s_3 \geq 2q$ and $s_2 + s_3 \geq 2q$. Since $s_1 + s_2 = 2q$ and $s_3 \leq s_3 + s_4 = q$, we must have $s_1 = s_2 = s_3 = q$ and $s_4 = 0$. It follows that there are exactly two outermost disks on D, and all other components Q of D cut along $D \cap E$ are bigons in the sense that it intersects E in exactly two arcs, and $Q \cap m = \emptyset$. The union of Q with two disks on E form a $(2, 0)$ disk, so by Lemma 3.4 it must be trivial. In particular all components of $D \cap E$ are on the same disk E_1, say. It is now easy to see that each Q is a tube (i.e. an annulus parallel to $A_1(t)$) cut open, hence D is an n-winding disk. If $n > 0$ then by the above some component of $\partial D \cap P(t)$ must have self intersection. Therefore if each component of $\partial D \cap P(t)$ is embedded then we must have $n = 0$ and hence D is homotopic to E_0. □

4 Immersed surfaces in essential position

Recall that an embedded orientable surface F in a 3-manifold is *essential* if it is incompressible, ∂-incompressible, and no component of F is boundary parallel.

Lemma 4.1 *Let F be an embedded orientable essential surface in a 3-manifold M. Then no immersed essential arc α on F is rel $\partial\alpha$ homotopic to an arc on ∂M.*

Proof. Suppose α is rel $\partial\alpha$ homotopic to an arc β on ∂M and let D be a null-homotopy disk bounded by $\alpha \cup \beta$. If M is reducible then since F is essential we may assume it is disjoint from a reducing sphere S, and we can then modify D if necessary to make it disjoint from S. Therefore by decomposing along S we may assume that M is irreducible. Similarly, since F is incompressible and ∂-incompressible and M is irreducible, F can be isotoped and D can be modified to be disjoint from any ∂-reducing sphere, hence by cutting along ∂-reducing disks if necessary we may also assume that M is ∂-irreducible.

Let $2M$ be the double of M along ∂M, and let $2F$ be the double of F along $\partial F = F \cap \partial M$. Then the double of α is an essential curve on $2F$ which is null homotopic in $2M$. Thus $2F$ is not π_1-injective and hence is compressible. Let D is a compressing disk of $2F$ in $2M$ such that $|D \cap \partial M|$ is minimal. Since F and ∂M are incompressible in M, by an innermost circle argument one can show that $D \cap \partial M$ has no circle component, and we must have $D \cap \partial M \neq \emptyset$ because D cannot be a compressing disk of F. An outermost arc of $D \cap \partial M$ then cuts off a ∂-compressing disk of F in M. □

Lemma 4.2 *Let F be an embedded orientable essential surface in a 3-manifold M. Suppose $\rho : \tilde{M} \to M$ is a finite cover. Then the surface $\tilde{F} = \rho^{-1}(F)$ is essential in $\tilde{M}$.*

Proof. Let $\tilde{F}_1$ be a component of $\tilde{F}$ and let F_1 be the corresponding component in F. If $\tilde{F}_1$ is boundary parallel then it cuts off a regular neighborhood of a boundary component $\tilde{T}$ of $\tilde{M}$, which projects to a regular neighborhood of a boundary component T of M, hence F_1 would also be boundary parallel, contradicting the assumption that F is essential.

A compressing disk of $\tilde{F}_1$ would map to an immersed disk in M with boundary an essential curve on F_1, which contradicts the fact that an incompressible surface is π_1-injective. Therefore $\tilde{F}$ is incompressible in $\tilde{M}$.

If $\tilde{F}$ is ∂-compressible then a ∂-compressing disk of $\tilde{F}$ in $\tilde{M}$ projects to a disk in M whose boundary consists of an essential arc on F and an arc on ∂M, which contradicts Lemma 4.1. □

Definition 4.3 Suppose L is a link in a 3-manifold M and F is an embedded essential surface in $E(L) = M - \text{Int}N(L)$ with nonempty non-meridional boundary slope on each boundary component of $E(L)$. Let $\hat{S}$ be an immersed surface in M, and let $S = \hat{S} \cap E(L)$. Then $\hat{S}$ is said to be *in essential position with respect to* F if $\hat{S} \cap L \neq \emptyset$, each arc component of $S \cap F$ is essential on both S and F, and each circle component is nontrivial on S.

The following lemma is essentially due to Gabai [Ga].

Lemma 4.4 (Thin Position Lemma) *Suppose L is a link in S^3, and F an embedded essential surface in $E(L) = S^3 - \text{Int}N(L)$ with nonempty non-meridional boundary slopes on each boundary component of $E(L)$. Then there exist an embedded sphere $\hat{S}$ in S^3 which is in essential position with respect to F.*

Proof. When L is a knot this is [Ga, Lemma 4.4]. If L splits, one can use the incompressibility of F to find a splitting sphere disjoint from F, and proceed by induction to find $\hat{S}$ on the side containing F; hence we may assume that L is non-split. In this case the proof is the same as in [Ga, Lemma 4.4]. Put L in thin position and let $\hat{S}$ be a thin level surface. Since L is non-split, $\hat{S}$ has nonempty intersection with L. Isotoping F properly and using the argument in [Ga, Lemma 4.4] one can show that $\hat{S}$ can be chosen so that it has neither high disk nor low disk, in which case the arc components of $S \cap F$ are essential on both F and S. Since F is incompressible, any circle of $F \cap S$ which is trivial on S bounds a disk on F, so one can modify S by cut and paste to get rid of those curves. We refer the reader to the proof of [Ga, Lemma 4.4] for details. □

The following can be considered as an immersed version of the thin position lemma. It works for links in manifolds with finite fundamental group.

Lemma 4.5 (Immersed Thin Position Lemma) *Suppose M is a closed irreducible 3-manifold with finite fundamental group, L a link in M, and F an embedded essential surface in $E(L) = M - \text{Int}N(L)$ with nonempty non-meridional boundary slopes on each boundary component of $E(L)$. Then there exists an immersed sphere Q' in M which is in essential position with respect to F.*

Proof. The universal cover $\tilde{M}$ of M is a simply connected closed 3-manifold and hence by the Poincare conjecture proved by Perelman [Pr], it is an S^3. The lifting of L is a link $\tilde{L}$ in $\tilde{M}$, and the lifting of F is a surface $\tilde{F}$ in $E(\tilde{L}) = S^3 - \text{Int}N(\tilde{L})$. Since F is an essential embedded surface in $E(L)$, by Lemma 4.2 $\tilde{F}$ is an essential embedded surface in $\tilde{M} = E(\tilde{L})$. By the Thin Position Lemma above, there is a sphere $\tilde{Q}'$ in S^3 which is in essential position with respect to $\tilde{F}$. Let Q' be the projections of $\tilde{Q}'$ in M, let $\tilde{Q} = \tilde{Q}' \cap E(\tilde{L})$, and let $Q = Q' \cap E(L)$. The preimage of an arc in $Q \cap F$ may consist of several arcs, but at least one of them is in $\tilde{Q} \cap \tilde{F}$, and an inessential arc would lift to an inessential arc. Therefore all arcs of $Q \cap F$ are essential on both F and Q. Similarly all circle components of $Q \cap F$ are essential on Q. □

5 Elementary surfaces

We will always use K_r to denote the core of the Dehn filling solid torus in the surgered manifold $K(r)$. Given an immersed surface F in $E(K \cup C)$ with each boundary component either a meridional curve on $\partial N(C)$ or a curve of slope r on $\partial N(K)$, denote by $\hat{F}$ the closed surface obtained by attaching a meridian disk of $N(K_r \cup C)$ to each boundary component of F.

Suppose $K = K(r_1, r_2, r_3)$ is a Montesinos knot of length 3, where $r_i = p_i/q_i$ and $q_i \geq 2$. Let $L = K \cup C$, where C is the axis of K as shown in Figure 5.1. Let $E(L) = S^3 - \mathrm{Int}N(L)$ be the exterior of L. Denote by V the solid torus $S^3 - \mathrm{Int}N(C)$. There are three disks D_1, D_2, D_3 cutting the pair (V, K) into three rational tangles (B_i, t_i, m_i) of slope r_i. Let P_i be the twice punctured disk $D_i \cap E(L)$, and let $P = P_1 \cup P_2 \cup P_3$. Then $P = P_1 \cup P_2 \cup P_3$ cuts $E(K \cup C)$ into three tangle spaces $E(t_1)$, $E(t_2)$, $E(t_3)$. The surface $\partial E(t_i)$ is the union of P_i, P_{i+1} and three annuli $A_0(t_i)$, $A_1(t_i)$, $A_2(t_i)$, where $A_0(t_i) = \partial E(t_i) \cap \partial N(C)$ and $A_1(t_i) \cup A_2(t_i)$ are the ones on $\partial N(K)$.

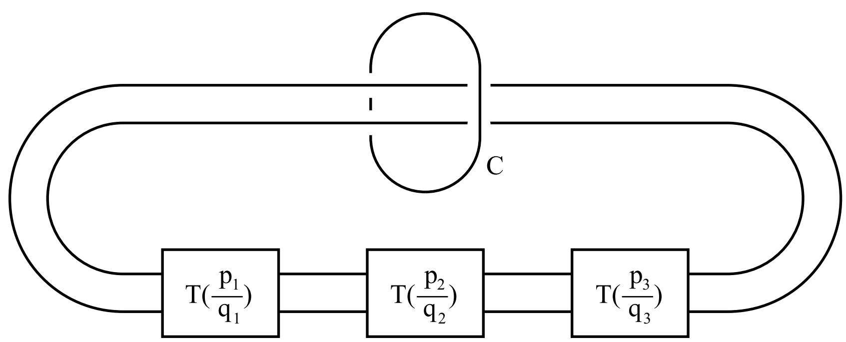


Figure 5.1

Definition 5.1 An immersed surface F in $E(K \cup C)$ is an *elementary surface* if the following holds.

(1) Each component of $F \cap E(t_i)$ is a tight disk of type $(0, 2q_i)$, $(1, q_i)$ or $(2, 2|\bar{p}_i|)$;

(2) at least one $F \cap E(t_i)$ has no disk of type $(0, 2q_i)$;

(3) if $q_i = 2$ then all disks of type $(2, 2|\bar{p}_i|)$ in $E(t_i)$ are homotopic to each other; and

(4) type $(0, 2q_i)$ and type $(2, 2|\bar{p}_i|)$ disks do not appear simultaneously in any $E(t_i)$.

Lemma 5.2 *Suppose K is hyperbolic, and there is an* embedded *closed torus or Klein bottle $\hat{F}$ in $K(r)$ such that $F = \hat{F} \cap E(K \cup C)$ is an elementary surface. Then $K(r)$ is toroidal.*

Proof. By considering the boundary of a regular neighborhood of F if necessary, we may assume F is orientable and hence is a punctured torus. Since F is embedded and $F \cap E(t_i)$ is the union of standard disks in $E(t_i)$ for all i, the surface $\hat{F}$ is a candidate surface as defined in [HO]. Since $\hat{F}$ is a torus, by [Wu3, Lemma 7.1] the knot K and the boundary slope r of $\hat{F}$ are among those listed in [Wu3, Theorem 1.1]. By [Wu3, Theorem 1.2] $\hat{F}$ is incompressible and hence $K(r)$ is toroidal. $\square$

The main result of this section is Proposition 5.6, which shows that the above lemme is still true for immersed $\hat{F}$.

Lemma 5.3 *Suppose F is an immersed elementary surface in $E(K \cup C)$.*
(1) For each i, F can be deformed so that $F \cap E(t_i)$ is embedded.
(2) F can be deformed so that $F \cap P$ is embedded.

Proof. By Lemmas 3.5–3.7 each component of $F \cap E(t_i)$ can be deformed to a standard disk or an n-winding disk. By condition (2) in Definition 5.1, up to relabeling we may assume that there is no disk of type $(0, 2q_1)$ and hence no n-winding disk in $E(t_1)$ for $n > 0$. Since standard disks are embedded, each arc of $F \cap P_1$ and $F \cap P_2$ is embedded up to homotopy. Since K is a knot, at most one of the q_j is even, hence without loss of generality we may assume that q_2 is odd. It means that a strand of t_2 has one endpoint on each of P_2 and P_3. If $F \cap E(t_2)$ contains an n-winding disk D for some $n > 0$ then $D \cap P_2$ would be an arc which is not homotopic to an embedded arc, a contradiction. Therefore $E(t_2)$ has no such disk, which implies that each component of $F \cap E(t_2)$ is also homotopic to a standard disk. Similarly, since each arc in $F \cap (P_1 \cup P_3)$ is homotopic to an embedded arc, there is no n-winding disk in $E(t_3)$ for any $n > 0$. Therefore each component of $F \cap E(t_j)$ is homotopic to a standard disk for all j.

By definition, standard disks can be homotoped in $E(t_j)$ to be disjoint from each other except that (i) a disk of type E_0 intersects a disk of type E_3 essentially, and (ii) when $q_j = 2$ there are two possible homotopy classes of type E_3 which intersect each other essentially. However, these have been excluded by conditions (3) and (4) in Definition 5.1. Therefore $F \cap E(t_j)$ can be deformed to be a set of mutually disjoint standard disks. This proves (1).

It may not be possible to do the above simultaneously for all the three tangles, but it implies that each $F \cap P_i$ can be deformed to be embedded, hence we can homotope F so that $F \cap P$ is embedded, as stated in (2). □

Lemma 5.4 *Let P be a twice punctured disk, and P_0, P_1 the two boundary copies of P in $P \times I$. Suppose D_1 is an embedded disk in $P \times I$ intersecting each P_i at a single essential arc a_i. Let D_2 be an immersed disk in $P \times I$ with $b_i = D_2 \cap P_i = a_i$ for $i = 0, 1$. Then D_2 is homotopic to D_1 rel $b_0 \cup b_1$.*

Proof. By an isotopy of $P \times I$ we may assume that D_1 is a product disk $\alpha \times I$. Let β_1, β_2 be two arcs on P such that α, β_1, β_2 are mutually disjoint, mutually non-parallel, and $\beta_1 \cup \beta_2$ cuts P into a disk. We may write $\partial D_1 = a_0 \cup a_2 \cup a_1 \cup a_3$ and $\partial D_2 = b_0 \cup b_2 \cup b_1 \cup b_3$, so that $\partial a_2 = \partial b_2$, $\partial a_3 = \partial b_3$, and a_2, a_3, b_2, b_3 are arcs on $\partial P \times I$. Put $Q_i = \beta_i \times I \subset P \times I$ and let $Q = Q_1 \cup Q_2$. Homotope D_2 rel $b_0 \cup b_1$ so that it has minimal intersection with Q. Let $\varphi : D \to D_2$ be the immersion. Then $\varphi^{-1}(Q)$ is an embedded 1-manifold on D. Using the fact that the arc α, β_1, β_2 are essential mutually non-homotopic proper arcs in P one can apply an innermost circle outermost arc argument to show that $D_2 \cap Q = \emptyset$. Since Q cuts $\partial P \times I$ into disks, we see that $a_2 \cup b_2$ and $a_3 \cup b_3$ are trivial loops, hence up to homotopy rel $b_0 \cup b_1$ we can deform ∂D_2 to ∂D_1. Since $P \times I$ is irreducible, one can further deform D_2 to D_1 by a homotopy rel ∂D_2. □

We now consider a set of *oriented* loops C on a torus T with oriented meridian-longitude pair (μ, λ). Then C represents some $a\mu + b\lambda$ in $H_1(T)$. Define b/a to be the *slope* of C. Note that C is not required to be embedded. If C has a crossing as in Figure 5.2(a), we can change it to that in Figure 5.2(b). This operation is called *smoothing* a crossing.

Figure 5.2

Lemma 5.5 *Let C be a set of oriented immersed curves on a torus T. Let C' be obtained from C by smoothing crossings. Then C and C' have the same slope on T.*

Proof. This is obvious since the curves in Figure 5.2(a) and 5.2(b) represents the same homology class in $H_1(T)$. $\square$

Proposition 5.6 *Suppose $K = K(p_1/q_1,\, p_2/q_2,\, p_3/q_3)$ is hyperbolic and $\hat{F}$ is an immersed π_1-injective torus in $K(r)$. If $F = \hat{F} \cap E(K \cup C)$ is an elementary surface, then $K(r)$ is toroidal.*

Proof. As before, let $P = P_1 \cup P_2 \cup P_3$ be the three twice punctured disks that cut $E(K \cup C)$ into $E(t_1) \cup E(t_2) \cup E(t_3)$. Let $W_i = P_i \times I$ be a collar of P_i. Then we can rewrite $E(K \cup C)$ as the union $(\cup W_i) \cup_R (\cup E(t_i))$, where W_i and $E(t_j)$ have mutually disjoint interiors, and $R = (\cup W_i) \cap (\cup E(t_i))$ is the union of 6 twice punctured disks.

Since F is elementary, by Lemma 5.3(2) we may assume that $F \cap P$ is embedded, so $F \cap W_i$ consists of mutually disjoint product disks. Since the $E(t_i)$ are mutually disjoint, by Lemma 5.3(1), we can deform $F_i = F \cap E(t_i)$ in $E(t_i)$ so that it is embedded. Note that this can be done by a homotopy of the pair $(E(t_i), R \cap E(t_i))$, so they can be combined and then extended into a small neighborhood of R in $\cup W_j$ to form a homotopy of F. After this homotopy $F \cap W_i$ may no longer be embedded; however, since we started with product disks and the homotopy maps each component of R to itself, each component of $F \cap W_i$ is still homotopic to a product disk.

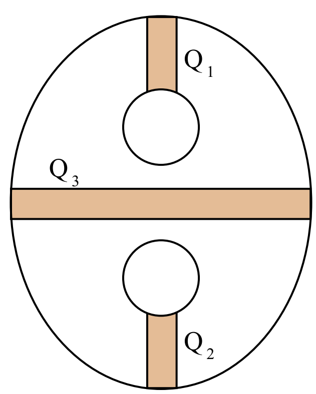


Figure 5.3

Let P_i', P_i'' be the two copies of P_i on ∂W_i. Since $F \cap E(t_i)$ is a union of mutually disjoint standard disks, we may assume that $F \cap P_i'$ consists of embedded essential arcs in the small disks $Q_1 \cup Q_2 \cup Q_3$ shown in Figure 5.3. Since each component D of $F \cap W_i$ is a disk, the two arcs $D \cap P_i'$ and $D \cap P_i''$

must lie in $Q_j \times 0$ and $Q_j \times 1$ for the same j because two essential arcs in different Q_j are not homotopic to each other. Let E be an embedded disk in $Q_j \times I$ such that $E \cap P_i' = D \cap P_i'$ and $E \cap P_i'' = D \cap P_i''$. By Lemma 5.4 D is rel $D \cap (P_i' \cup P_i'')$ homotopic to E. Therefore up to homotopy we may assume that each component D of $F \cap W_i$ is a flat embedded disk in $Q_j \times I$, so different components D', D'' of $F \cap W_i$ are either disjoint or intersect at a single arc, as shown in Figure 5.4(a).

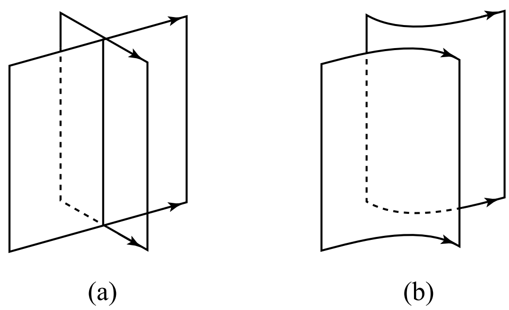


Figure 5.4

Orient ∂F so that all arcs of $\partial F \cap W_i$ runs monotonically from P_i' to P_i''. We can then perform a double edge smoothing of D', D'' to obtain two new disks D_1', D_1'', as shown in Figure 5.4(b). After doing this for all double curves, the surface $F \cap W_i$ becomes a set of mutually disjoint embedded disks, and the original surface F now becomes an embedded surface F' in $E(K \cup C)$. By definition F' is an elementary surface. Note that the double curve smoothing does not change the Euler characteristic of a surface. By Lemma 5.5 F' has the same boundary slope as that of F. Since both $\partial F'$ and ∂F are embedded, we see that they have the same number of boundary curves. Let $\hat{F}'$ be the closed surface obtained by adding meridian disks of $N(K_r \cup C)$ to $\partial F'$. The above implies that $\chi(\hat{F}') = \chi(\hat{F}) = 0$, hence $\hat{F}'$ is a torus or Klein bottle, and it is embedded. It now follows from Lemma 5.2 that $K(r)$ is toroidal. □

6 Intersection graphs

We continue using the notations in Section 5, so $K = K(p_1/q_1,\ p_2/q_2,\ p_3/q_3)$, $L = K \cup C$, and $P = P_1 \cup P_2 \cup P_3$, cutting $E(L)$ into three tangle spaces $E(t_1), E(t_2), E(t_3)$. Let $K(r)$ be the manifold obtained by Dehn surgery along a slope r on K. Denote by K_r the core of the Dehn filling solid torus in $K(r)$, and let $L_r = K_r \cup C$.

Lemma 6.1 *P is an essential surface in $E(L)$.*

Proof. If P is compressible then the compressing disk would separate the two strings of t and t would be a trivial tangle, contradicting the assumption that $q_i \geq 2$. It is well known that if ∂M is a set of tori then any connected incompressible surface in M is also ∂-incompressible unless it is an annulus. Hence P is also ∂-incompressible. Since each component of P has nonempty boundary, this also implies that no component of P is boundary parallel. $\square$

Lemma 6.2 *Suppose $M = K(r)$ is Seifert fibered. Then there is an immersed torus or sphere $\hat{Q}$ in M which is in essential position with respect to P.*

Proof. By Lemma 6.1 P is an embedded essential surface in $E(L_r) = E(L)$. Therefore the result follows from Lemma 4.5 if $\pi_1(K(r))$ is finite, and the surface $\hat{Q}$ is a sphere in this case.

Now assume $\pi_1(K(r))$ is infinite. By [Sc] a Seifert fibered manifold admits one of six 3-manifold geometries of Thurston. Since $K(r)$ is irreducible by [Wu1], it does not admit $S^2 \times R$ geometry, and since $\pi_1(K(r))$ is infinite it does not admit S^3 geometry. Hence the orbifold X of $K(r)$ must be euclidean or hyperbolic and therefore has infinite orbifold fundamental group as defined in [Sc]. The torus $\hat{Q}$ in $K(r)$ that projects to a curve on X with infinite order in the orbifold fundamental group is then an immersed torus in $K(r)$ which is π_1-injective in $K(r)$.

Let $Q = \hat{Q} \cap E(L_r)$. Choose $\hat{Q}$ to have minimal intersection with L_r and then homotoped so that $(|\partial Q \cap \partial P|, |Q \cap P|)$ is minimal in lexicographic order. In particular, ∂Q intersects ∂P minimally. We need to show that $\hat{Q}$ is in essential position with respect to P. Since P is essential, we may get rid of loops in $Q \cap P$ which is trivial on Q. We need to show that all arc components α on $P \cap Q$ are nontrivial on both P and Q. Note that α is embedded on Q but may be immersed on P.

Assume α is inessential on both P and Q. Then it is rel $\partial\alpha$ homotopic to an arc β on ∂Q and an arc γ on ∂P, so the incompressibility of $\partial N(K)$ implies that β is homotopic to γ and hence there is a homotopy to reduce $|\partial Q \cap \partial P|$, contradicting its minimality.

Now assume α is inessential on P but essential on Q. Since α is trivial on P, we may deform Q near P to make α embedded on P. One can then push

$\hat{Q}$ through the disk on P cut off by α to reduce $|\hat{Q} \cap L_r|$ by 2, contradicting its minimality.

If α is essential on P but inessential on Q then it cuts off a disk D on Q and hence is rel $\partial\alpha$ homotopic to the arc $\partial D - \text{Int}\alpha \subset \partial E(L_r)$. Since P is essential by Lemma 6.1, this contradicts Lemma 4.1. $\square$

Let $\hat{Q}$ be as in Lemma 6.2 and put $Q = \hat{Q} \cap E(L_r)$. Since P is essential and embedded, $P \cap Q$ is an embedded compact 1-manifold in Q. However it may not be embedded in P because Q is immersed in $E(L)$. Since $\hat{Q}$ is in essential position with respect to P, no component of $Q \cap P$ is a trivial circle or trivial arc on Q.

Define a *generalized graph* to be a graph with possibly some valence 2 vertices removed from the vertex list. Thus it is a graph except that some loop components may not have vertices on it.

Definition 6.3 Consider the disks of $\hat{Q} - \text{Int}Q = \hat{Q} \cap N(L_r)$ as fat vertices, the arc components of $P \cap Q$ as edges, and the circles of $P \cap Q$ as loops. This produces a generalized graph G on $\hat{Q}$, called the *intersection graph* of $\hat{Q}$ and P.

There are two types of vertices on G. A component of $\hat{Q} \cap N(C)$ is called a *small vertex*, and a component of $\hat{Q} \cap N(K)$ is called a *large vertex*. Since a meridian of C intersects P at three points, one on each P_i, we see that each small vertex has valence 3. Denote by $\Delta = \Delta(\mu, r)$ the minimal intersection number between a meridian μ of K and the surgery slope r. Then the surgery slope r intersects ∂P minimally at 6Δ points, so each large vertex has valence 6Δ, with 2Δ endpoints on each P_i. In particular, if r is an integer slope, we have $\Delta = 1$, in which case the valence of each large vertex is 6.

Write the boundary of the tangle space $E(t_i)$ as $\partial E(t_i) = P_i \cup P_{i+1} \cup A_i \cup A_i' \cup A_i''$, where P_i are the twice punctured disk $D_i \cap E(L)$ defined above, $A_i = \partial N(C) \cap E(t_i)$, and A_i', A_i'' are the two annuli $\partial N(K) \cap E(t_i)$. Since $\hat{Q}$ is in essential position with P, we have the following lemma.

Lemma 6.4 *Let σ be a face of G lying in $E(t_i)$. Then $\partial\sigma$ intersects each of the above five subsurfaces of $\partial E(t_i)$ in essential arcs and hence is a set of tight curves. In particular, each disk face of G is an essential disk in some $E(t_i)$.*

An arc of a face σ on the boundary of a fat vertex v is called a *corners* of σ at v. Note that when shrinking each fat vertex to a single point a corner becomes a vertex on the boundary of the face σ. A corner at v is *large* or *small* according to whether v is a large vertex or a small vertex. Thus a large corner lies on A_i' or A_i'' for some i while a small corner is on the other annulus A_i and hence intersects m_i at a single point. Therefore a disk face σ of G is of type (r, s) if and only if it has r large corners and s small corners. The results in Section 3 now apply to the disk faces of G. In particular, an (r, s) face in $E(t_i)$ has $s \geq 2q_i$ if $r = 0$, $s \geq q_i$ if r is odd, and $s \geq 2|\bar{p}_i|$ if r is even.

7 Euler number of an angled surface

An *angled surface* is a compact surface σ with a set of points $V = (v_1, ..., v_n)$ on $\partial\sigma$ called vertices or corners, and an angle α_i assigned to each corner v_i, with $0 \leq \alpha_i < \pi$. When σ is a disk, it is a polygon or n-gon. We will always use $\bar{\alpha}$ to denote the external angle $\bar{\alpha} = \pi - \alpha$.

Definition 7.1 The *angled Euler number* of an angled surface σ with corner angles $\alpha_1, ..., \alpha_n$ is defined as

$$e(\sigma) = \chi(\sigma) - \frac{1}{2\pi} \sum \bar{\alpha}_i$$

Recall that a generalized graph is a graph on which some of the loops may not have vertices.

Lemma 7.2 *Let G be a generalized graph on a closed surface F, cutting it into angled surfaces $\sigma_1, ..., \sigma_m$, such that the sum of angles around each vertex is at least 2π. Then $\chi(F) \leq \sum e(\sigma_i)$, and equality holds if and only if the sum of angles around each vertex is 2π.*

Proof. Note that adding vertices to loops with an angle π at each corner of the new vertices will not change the angled Euler number of the faces. Therefore by adding such vertices if necessary we may assume that G is a genuine graph. We assume that the sum of the angles around each vertex of G is exactly 2π. The proof for the other case is similar.

Let n_i be the number of vertices on $\partial\sigma_i$, let v_{ij} be the vertices on $\partial\sigma_i$, and let α_{ij} be the angles at v_{ij}. Note that n_i is also the number of edges on $\partial\sigma_i$. Denote by E and V the number of edges and vertices of G, respectively.

Then

$$\begin{aligned}
\sum_i e(\sigma_i) &= \sum_i [\chi(\sigma_i) - \frac{1}{2\pi}\sum_j (\pi - \alpha_{ij})] \\
&= \sum_i [\chi(\sigma_i) - \frac{1}{2}n_i + \sum_j \frac{\alpha_{ij}}{2\pi}] \\
&= \sum \chi(\sigma_i) - \sum \frac{n_i}{2} + \frac{\sum \alpha_{ij}}{2\pi} \\
&= \sum \chi(\sigma_i) - E + V \\
&= \chi(F) \qquad \square
\end{aligned}$$

A face σ is said to be spherical, euclidean or hyperbolic according to whether $e(\sigma)$ is positive, zero or negative, respectively. When the angles are nonzero, one can show that σ has a corresponding geometric structure with geodesic boundary edges and an angle of α_i at corner v_i. Thus for example if all σ_i are euclidean or hyperbolic with at least one σ_i hyperbolic, and if the sum of angles at each vertex is at least 2π, then the surface F is a hyperbolic surface.

8 Proof of the main theorems

Suppose $K(r)$ is an atoroidal Seifert fiber space. By Lemma 6.2 there is an immersed sphere or a torus $\hat{F}$ in $K(r)$ which is in essential position with respect to $P = P_1 \cup P_2 \cup P_3$. Let G be the generalized graph on $\hat{F}$ as defined in Section 6. Let α_i and β_i be the angles of large corners and small corners of $F \cap E(t_i)$, respectively. We would like to show that if α_i, β_i can be chosen to satisfy certain conditions then $K(r)$ must be toroidal.

As before, denote by $\bar{\alpha}_i = \pi - \alpha_i$ and $\bar{\beta}_i = \pi - \beta_i$. Recall that $\bar{p}_i$ is the mod q_i inverse of $-p_i$, as in Definition 2.2. Let μ be the meridional slope of K.

Theorem 8.1 *Suppose $K = K(p_1/q_1, p_2/q_2, p_3/q_3)$ is a hyperbolic Montesinos knot with $q_i \geq 2$. Then $K(r)$ is not an atoroidal Seifert fibered manifold if there are angles $\pi \geq \bar{\alpha}_i > 0$ and $\pi > \bar{\beta}_i > 0$ satisfying the following conditions.*

(1) $\bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\alpha}_3 \leq 2\pi$;

(2) $\bar{\beta}_1 + \bar{\beta}_2 + \bar{\beta}_3 \leq \pi$;

(3) $\bar{\alpha}_i + q_i \bar{\beta}_i \geq 2\pi$*;*
(4) $\bar{\alpha}_i + |\bar{p}_i| \bar{\beta}_i \geq \pi$*;*
(5) if $q_i = 2$ *then* $\bar{\alpha}_i + \bar{\beta}_i > \pi$*.*

Proof. Assume to the contrary that $K(r)$ is an atoroidal Seifert fibred manifold. Let $\hat{F}$ and F be as above, with α_i and β_i the large corners and small corners of $F \cap E(t_i)$, respectively. Conditions (1) and (2) can be rewritten as $\sum 2\alpha_i \geq 2\pi$ and $\sum \beta_i \geq 2\pi$, which mean that the sum of the angles around each vertex of G is at least 2π. Conditions (3) and (4) say that the sum of external angles of a face σ of type $(1, q)$ or $(2, 2|\bar{p}_i|)$ is at least 2π, so by definition we have $e(\sigma) \leq 0$. We want to show that $e(\sigma) \leq 0$ for all other faces as well.

By Lemmas 3.3 and 3.4 a disk face of type (r, s) in $E(t_i)$ is of one of the following type.

(a) $r = 0$ and $s \geq 2q_i$;
(b) r is odd and $s \geq q_i$;
(c) $r \geq 2$ is even and $s \geq 2|\bar{p}_i|$.

Let σ_j be a face of type $j = a$, b, or c above in $E(t_i)$. Then

$$\begin{aligned} e(\sigma_a) &= 1 - \frac{1}{2\pi} s\bar{\beta}_i \leq \frac{2}{2\pi}(\pi - q_i\bar{\beta}_i) \leq \frac{\bar{\alpha}_i - \pi}{\pi} \leq 0 \\ e(\sigma_b) &= 1 - \frac{1}{2\pi}(r\bar{\alpha}_i + s\bar{\beta}_i) \leq 1 - \frac{1}{2\pi}(\bar{\alpha}_i + q_i\bar{\beta}_i) \leq 0 \\ e(\sigma_c) &= 1 - \frac{1}{2\pi}(r\bar{\alpha}_i + s\bar{\beta}_i) \leq 1 - \frac{1}{2\pi}(2\bar{\alpha}_i + 2|\bar{p}_i|\bar{\beta}_i) \leq 0 \end{aligned}$$

By definition all non-disk faces σ of G have $e(\sigma) \leq 0$. Thus all faces σ of G have $e(\sigma) \leq 0$. Since $\chi(\hat{F}) = \sum e(\sigma) \leq 0$ by Lemma 7.2, the surface $\hat{F}$ cannot be a sphere, so it must be a torus, and $e(\sigma) = 0$ for all faces σ. If there is a non-disk face then the outermost one has some corner on it and hence has $e(\sigma) < 0$, which is a contradiction. Therefore all σ are disk faces with $e(\sigma) = 0$. This implies that all the above inequalities must be equalities. Since $\bar{\alpha}_i, \bar{\beta}_i > 0$, checking the above inequalities gives

(a) If σ_a exists in $E(t_i)$ then $\bar{\alpha}_i = \pi$, and $s = 2q_i$, so σ_a is a $(0, 2q_i)$ face.
(b) If σ_b exists in $E(t_i)$ then $r = 1$ and $s = q_i$, so σ_b is a $(1, q_i)$ face.
(c) If σ_c exists in $E(t_i)$ then $r = 2$ and $s = 2|\bar{p}_i|$, so σ_c is a $(2, 2|\bar{p}_i|)$ face.

These implies that F satisfies condition (1) in Definition 5.1. Condition (1) above implies that $\bar{\alpha}_i < \pi$ for some i, hence (a) above implies that there is no face of type $(0, 2q_i)$ in $F \cap E(t_i)$ for some i, so condition (2) in Definition 5.1 holds. When $q_i = 2$ we have $\bar{p}_i = 1$, and by condition (5) above we have $\bar{\alpha}_i + \bar{\beta}_i > \pi$, so by the calculation above we would have $e(\sigma_c) < 0$, which is a contradiction. Therefore there is no disk of type $(2,2) = (2, 2|\bar{p}_i|)$, which gives (3) in Definition 5.1. Similarly if $F \cap E(t_i)$ has a disk of type $(0, 2q_i)$ then $e(\sigma_a) = 0$ gives $\bar{\alpha}_i = \pi$, and again we have $\bar{\alpha}_i + |\bar{p}_i|\bar{\beta}_i > \pi$, hence there is no disk of type $(2, 2|p_i|)$, which verifies condition (4) of Definition 5.1. Therefore F is an elementary surface. It now follows from Proposition 5.6 that $K(r)$ is toroidal, a contradiction. □

Proof of Theorem 1.1. Let $K = K(p_1/q_1, p_2/q_2, p_3/q_3)$, with $2 \leq q_1 \leq q_2 \leq q_3$. Since K is a knot, at most one q_i is even. Hence the condition $\sum \frac{1}{q_i - 1} \leq 1$ implies that either $q_i \geq 4$ for all i, or $q_1 = 3$ and $q_2, q_3 \geq 5$, or $q_1 = 3$, $q_2 = 4$ and $q_3 \geq 7$.

If $q_i \geq 4$, let $\bar{\alpha}_1 = \bar{\alpha}_2 = \bar{\alpha}_3 = \frac{2\pi}{3}$, and $\bar{\beta}_1 = \bar{\beta}_2 = \bar{\beta}_3 = \frac{\pi}{3}$.

If $q_1 = 3$ and $q_2, q_3 \geq 5$, let $(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3) = (\frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{4})$, and $(\bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3) = (\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4})$.

If $q_1 = 3$, $q_2 = 4$ and $q_3 \geq 7$, let $(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3) = (\frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6})$, and $(\beta_1, \beta_2, \beta_3) = (\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6})$.

One can easily check that the conditions (1)–(3) of Theorem 8.1 are satisfied in each of the above cases. Note that in all cases we have $\bar{\alpha}_i + \bar{\beta}_i = \pi$, so condition (4) holds because $|\bar{p}_i| \geq 1$. Conditions (5) holds trivially since $q_i > 2$. Theorem 1.1 now follows from Theorem 8.1. □

By Theorem 1.1 if a Montesinos knot K of length 3 admits atoroidal Seifert fibered surgery then $K = K(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3})$, such that either $q_1 = 2$, or $(q_1, q_2) = (3, 3)$, or $(q_1, q_2, q_3) = (3, 4, 5)$. The following theorem gives further restrictions on p_i. In this theorem it is not assumed that $q_2 \leq q_3$, so q_2 in (4) below may be larger than q_3. Two knots K, K' are equivalent if K is isotopic to K' or its mirror image.

Theorem 8.2 *Suppose a Montesinos knot K of length 3 admits an atoroidal Seifert fibered surgery. Then K is equivalent to one of the following knots.*

(1) $K(1/3,\ \pm 1/4,\ p_3/5)$ and $p_3 \equiv \pm 1$ mod 5;

(2) $K(1/3,\ \pm 1/3,\ p_3/q_3)$ and $|\bar{p}_3| \leq 2$;

(3) $K(1/2,\ 2/5,\ p_3/q_3)$, $q_3 = 5$ or 7, and $|\bar{p}_3| > 1$;

(4) $K(1/2,\ 1/q_2,\ p_3/q_3)$, $q_2 \geq 5$ and $|\bar{p}_3| \leq 2$;

(5) $K(1/2,\ 1/3,\ p_3/q_3)$ and $|\bar{p}_3| \leq 6$.

Proof. By Theorem 1.1, we have $(q_1, q_2, q_3) = (3, 4, 5)$, $(3, 3, q_3)$ or $(2, q_2, q_3)$. It is easy to see that if K is not equivalent one of those listed in the theorem then it is listed in one of the following five cases. In each case we will list the angles $\bar{\alpha}_i, \bar{\beta}_i$. These satisfy $\sum \bar{\alpha}_i = 2\pi$, $\sum \bar{\beta}_i = \pi$, and $\alpha_1 = \pi$ when

$q_1 = 2$, so conditions (1), (2) and (5) in Theorem 8.1 hold. We leave it to the reader to check that they satisfy conditions (3) $\bar\alpha_i + q_i\bar\beta_i \geq 2\pi$ and (4) $\bar\alpha_i + |\bar p_i|\bar\beta_i \geq \pi$ in Theorem 8.1.

CASE 1. *$(q_1, q_2, q_3) = (3, 4, 5)$, and $|\bar p_3| = 2$.*
Let $(\bar\alpha_1, \bar\alpha_2, \bar\alpha_3) = (\pi, \frac{2\pi}{3}, \frac{\pi}{3})$, and $(\bar\beta_1, \bar\beta_2, \bar\beta_3) = (\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$.

CASE 2. *$q_1 = q_2 = 3$, and $|\bar p_3| \geq 3$.*
We must have $q_3 \geq 7$ because $2|\bar p_3| \leq q_3$ and $\bar p_3$ is coprime with q_3. Let $(\bar\alpha_1, \bar\alpha_2, \bar\alpha_3) = (\frac{7\pi}{8}, \frac{7\pi}{8}, \frac{\pi}{4})$ and $(\bar\beta_1, \bar\beta_2, \bar\beta_3) = (\frac{3\pi}{8}, \frac{3\pi}{8}, \frac{\pi}{4})$.

CASE 3. *$q_1 = 2$, and $|\bar p_i| > 1$ for $i = 2, 3$.*
Note that q_2, q_3 must be odd since K is a knot. By definition of $|\bar p_i|$ we have $q_2, q_3 \geq 5$. If $q_2, q_3 \geq 7$, let $(\bar\alpha_1, \bar\alpha_2, \bar\alpha_3) = (\pi, \frac{\pi}{2}, \frac{\pi}{2})$ and $(\bar\beta_1, \bar\beta_2, \bar\beta_3) = (\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4})$. If $q_2 = 5$ and $q_3 \geq 9$, let $(\bar\alpha_1, \bar\alpha_2, \bar\alpha_3) = (\pi, \frac{\pi}{3}, \frac{2\pi}{3})$ and $(\bar\beta_1, \bar\beta_2, \bar\beta_3) = (\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6})$. Both cases can be excluded by Theorem 8.1. Therefore up to equivalence we have $q_2 = 5$ and $q_3 = 5$ or 7, as in (3).

CASE 4. *$q_1 = 2$, $|\bar p_2| = 1$, $q_2 \geq 5$, $|\bar p_3| \geq 3$.*
By convention we have $2|p_i| \leq q_i$ for $i = 1, 2$, hence $p_2 = \pm 1$. The condition $|\bar p_3| \geq 3$ implies that $q_3 \geq 7$. Let $(\bar\alpha_1, \bar\alpha_2, \bar\alpha_3) = (\pi, \frac{3\pi}{4}, \frac{\pi}{4})$ and $(\bar\beta_1, \bar\beta_2, \bar\beta_3) = (\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4})$.

CASE 5. *$q_1 = 2$, $q_2 = 3$, $|\bar p_3| \geq 7$.*
We have $q_3 \geq 15$. Let $(\bar\alpha_1, \bar\alpha_2, \bar\alpha_3) = (\pi, \frac{7\pi}{8}, \frac{\pi}{8})$ and $(\bar\beta_1, \bar\beta_2, \bar\beta_3) = (\frac{\pi}{2}, \frac{3\pi}{8}, \frac{\pi}{8})$.

In all cases above, we see that $\bar\alpha_i, \bar\beta_i$ satisfy all conditions of Theorem 8.1, therefore by that theorem $K(r)$ is not an atoroidal Seifert fibred manifold, a contraction. □

Remark 8.3 The first two tangles of the knots in the above theorem are very simple. The small values of $|\bar p_3|$ implies that the third tangles of those knots are also relatively simple. For example if $\bar p_3 = 2$ and if we write $p_3 = nq_3 + m$ with $2|m| < q_3$, then $p_3\bar p_3 \equiv m\bar p_3 \equiv -1 \mod q_3$ implies that $q_3 = \pm(m\bar p_3 + 1) = \pm 2m \pm 1$, so we have a partial fraction decomposition $p_3/q_3 = n \pm 1/(2 \pm 1/m)$. The results above will be used in [Wu4] to reduce the classification problem of Seifert fibered surgeries on Montesinos knots to that on a few specific families of knots.

References

[BW] M. Brittenham and Y-Q. Wu, *The classification of exceptional Dehn surgeries on 2-bridge knots*, Comm. Anal. Geom. **9** (2001), 97–113.

[De] C. Delman, *Constructing essential laminations which survive all Dehn surgeries*, preprint.

[Ga] D. Gabai, *Foliations and the topology of* 3*-manifolds III*, J. Diff. Geom. **26** (1987), 479–536.

[FIKMS] D. Futer, M. Ishikawa, Y. Kabaya, T. Mattman and K. Shimokawa, *Finite surgeries on three-tangle pretzel knots*, Algebr. Geom. Topol. **9** (2009), 743–771.

[GW] C. Gordon and Y-Q. Wu, *Annular Dehn fillings*, Comment. Math. Helv. **75** (2000), 430–456.

[HO] A. Hatcher and U. Oertel, *Boundary slopes for Montesinos knots*, Topology **28** (1989), 453–480.

[HT] A. Hatcher and W. Thurston, *Incompressible surfaces in 2-bridge knot complements*, Invent. Math. **79** (1985), 225–246.

[IJ1] K. Ichihara, I. Jong, *Cyclic and finite surgeries on Montesinos knots*, Alg. Geom. Topol. **9** (2009), 731–742.

[IJ2] ——, *Toroidal Seifert fibered surgeries on Montesinos knots*, arXiv:Math.GT/1003.3517v2.

[Ma] T. Mattman, *Cyclic and finite surgeries on pretzel knots*, J. Knot Theory Ram. **11** (2002), 891–902.

[Ni] Y. Ni, *Knot Floer homology detects fibred knots*, Invent. Math. **170** (2007), 577–608.

[Pr] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, ArXiv:math.DG/0211159.

[Sc] P. Scott, *The geometries of* 3*-manifolds*, Bull. London Math. Soc. **15** (1983), 401–487.

[Wa] L. Watson, *Surgery obstructions from Khovanov homology*, preprint.

[Wu1] Y-Q. Wu, *Dehn surgery on arborescent knots*, J. Diff. Geom. **42** (1996), 171–197.

[Wu2] ——, *Exceptional Dehn surgery on large arborescent knots*, Pac. J. Math. (to appear).

[Wu3] ——, *The classification of toroidal Dehn surgeries on Montesinos knots*, preprint.

[Wu4] ——, *Persistently laminar branched surfaces*, preprint.

Department of Mathematics, University of Iowa, Iowa City, IA 52242
Email: *wu@math.uiowa.edu*